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Cosmological and astrophysical aspects in $f(\mathcal{Q})$ gravity

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Preface

General Relativity is one of the pillars of modern theoretical physics: it describes gravity through geometry. Over the years, several alternative theories have been developed to answer some of the questions left open by Einstein's theory. One of the most fruitful approaches is to separate the causal structure of spacetime from its geodesic structure. This separation is obtained by considering the metric and connection as independent fields. The resulting geometrical framework is the so-called metric-affine approach to gravity. Three main geometrical attributes are associated with a given connection: the curvature, the torsion, and the nonmetricity tensors. They represent the rotation of a vector parallel transported along a closed curve, the failure of the infinitesimal parallelogram formed when two infinitesimal vectors are parallel transported along each other to close, and the change of length of a vector parallel transported along a generic curve, respectively.

This thesis focuses on Symmetric Teleparallel Gravity and its generalization called $f(\mathcal{Q})$ gravity. They are alternative theories of gravity in which both curvature and torsion are zero and where only metric and nonmetricity tensors are involved in the description of the gravitational interaction. We investigate both the cosmological and astrophysical aspects of $f(\mathcal{Q})$ gravity to assess the possible improvements brought by this theory compared to the alternative ones that already exist. The main points of our study can be summarized as follows.

First, we present a reconstruction algorithm for cosmological models. We specifically focus on Bianchi type-I and Friedmann-Lemaître-Robertson-Walker spacetimes, obtaining exact solutions that might have application in a variety of scenarios such as spontaneous isotropization of Bianchi type-I models, dark energy, and inflation as well as pre-Big Bang cosmologies.

After that, using the $1+3$ covariant formalism, we investigate the effect of nonmetricity on the universe's dynamics. Then, using the Dynamical System Approach, we analyze the evolution of Bianchi type-I cosmologies. We consider several models of the function $f(\mathcal{Q})$, each manifesting isotropic eras of the universe, whether transitional or not. In one case, in addition to the qualitative analysis provided by the dynamical system method, we also obtain analytical solutions, showing agreement with the previously reconstructed results.

We also apply the $1+1+2$ formalism, where preferred directions are chosen for time and space. Thanks to this formalism, we can introduce static and Locally Rotationally Symmetric spacetimes. Moreover, we show how nonmetricity affects all kinematic quantities involved in the covariant $1+1+2$ decomposition. We apply the resulting geometrical framework to study spherically symmetric solutions in the context of $f(\mathcal{Q})$ gravity in vacuum. We obtain explicit solutions and sufficient conditions for the existence of Schwarzschild-de Sitter type spacetimes.

Finally, we investigate $f(\mathcal{Q})$ gravity coupled with spinor fields of spin-1/2. We present a tetrad-affine approach. After deriving the field equations, the conservation law of the spin density ensures the vanishing of the antisymmetric part of the Einstein-like equations, just as it happens in theories with torsion and metricity. We show that spinors are unaffected by the presence of the nonmetricity. We then focus on Bianchi type-I cosmological models, proposing a general procedure to solve the corresponding field equations in the coincident gauge. We provide analytical solutions in the case of gravitational Lagrangian functions of the kind $f(\mathcal{Q}) = \alpha\mathcal{Q}^n$. At late times, such solutions are seen to isotropize, and depending on the value of the exponent n, they can undergo an accelerated expansion of the spatial scale factors.

List of publications

- **Fabrizio Esposito**, Sante Carloni, Roberto Cianci, Stefano Vignolo, *Reconstructing isotropic and anisotropic $f(Q)$ cosmologies*, Phys. Rev. D, **105**, 084061 (2022) [Ch. 5];
- **Fabrizio Esposito**, Sante Carloni, Stefano Vignolo, *Bianchi type-I cosmological dynamics in $f(Q)$ gravity: a covariant approach*, Class. Quant. Grav., **39**, 235014 (2022) [Chs. 6 and 7];
- **Fabrizio Esposito**, Sante Carloni, Stefano Vignolo, *Static and LRS spacetimes of type II in $f(Q)$ gravity*, arXiv:2311.17669 (2023) [Ch. 8];
- Stefano Vignolo, Sante Carloni, **Fabrizio Esposito**, Roberto Cianci, Luca Fabbri, *Spinor fields in $f(Q)$-gravity*, Class. Quant. Grav., **39**, 015009 (2022) [Ch. 9];

Contents

List of abbreviations

GR	General Relativity
ΛCDM	Λ Cold Dark Matter
TG	Teleparallel Gravity
STG	Symmetric Teleparallel Gravity
TEGR	Teleparallel Equivalent of General Relativity
STEGR	Symmetric Teleparallel Equivalent of General Relativity
FLRW	Friedmann-Lemaître-Robertson-Walker
BI	Bianchi type-I
DSA	Dynamical System Approach
CMB	Cosmic Microwave Background
l(r).h.s.	left(right) hand side

Notation and conventions

g_{ab}	Metric tensor
g	Determinant of the metric tensor
η_{ab}	Minkowski metric tensor
δ^a_b	Kronecker delta
$\Gamma_{ab}{}^c$	Affine connection
$\tilde{\Gamma}_{ab}{}^c$	Levi-Civita connection
∇_a	Covariant derivative with respect to the affine connection
∂_a	Partial derivative
Q_{abc}	Nonmetricity tensor
$\mathcal{Q}$	Nonmetricity scalar
$T_{ab}{}^c$	Torsion tensor
T	Torsion scalar
$N_{ab}{}^c$	Distortion tensor
$K_{ab}{}^c$	Contortion tensor
$L_{ab}{}^c$	Disformation tensor
$R^a{}_{bcd}$	Riemann tensor
R_{ab}	Ricci tensor
R	Ricci scalar
C_{abcd}	Weyl tensor
ϵ_{abcd}	Levi-Civita symbol
ε_{abcd}	Levi-Civita tensor
$\mathcal{L}_{\mathbf{X}}$	Lie derivative with respect to the vector field $\mathbf{X}$
Ψ_{ab}	Energy-momentum tensor
$\Delta^{ab}{}_c$	Hypermomentum tensor
ρ	Energy density of a fluid
p	Pressure of a fluid
h_{ab}	Transverse metric
$\mathbb{R}$	Set of real numbers
$GL(n,\mathbb{R})$	General linear group of degree n over $\mathbf{R}$
$SO(n)$	Special orthogonal group in dimension n

- Throughout the thesis we will mostly use the natural units $c = 8\pi G_N = 1$.
- The metric signature $(-,+,+,+)$ is used. An exception is Chapter 9 in which instead we use the signature $(+,-,-,-)$.
- We denote by a tilde all quantities related to the Levi-Civita connection, e.g., $\tilde{\nabla}_a$ is the covariant derivative with respect to the Levi-Civita connection.
- Symmetrization of indices is given by

  $$T_{(a_1\cdots a_n)} = \frac{1}{n!}\sum_{\sigma} T_{\sigma(a_1)\cdots\sigma(a_n)},$$

  whereas antisymmetrization by

  $$T_{[a_1\cdots a_n]} = \frac{1}{n!}\sum_{\sigma} \mathrm{sgn}(\sigma) T_{\sigma(a_1)\cdots\sigma(a_n)},$$

  where the sum is over all possible permutations, and sgn() is the sign function.
- Commutation and anticommutation relations for geometrical objects are,

  $$[A,B] = AB - BA,$$

  and,

  $$\{A,B\} = AB + BA,$$

  respectively.

Introduction

Gravity is one of the fundamental forces in nature, along with electromagnetism, weak nuclear force, and strong nuclear force.

Ancient Greek philosopher Aristotle made an early attempt to explain gravity by dividing the universe into terrestrial and celestial spheres. He believed that all objects naturally move toward their "natural place" and was the first to suggest that objects with different masses fall at different rates. However, this idea was later disproved by Galilei's experiments between the 16th and 17th centuries. Galilei used inclined planes, pendulums, and telescopes to demonstrate that all objects accelerate uniformly toward the Earth if we neglect friction forces.

In 1687, Sir Isaac Newton proposed a comprehensive theory of gravity in his treatise "Philosophiae Naturalis Principia Mathematica." Newton formulated the inverse-square law of universal gravitation and introduced the crucial idea that space and time are absolute entities, providing a non-dynamical background for all physical phenomena. Newton also assumed that gravitational and inertial masses are equivalent, a hypothesis known nowadays as the Weak Equivalence Principle.

The failure of Newtonian gravitation to explain the excess in the precession of Mercury's orbit, together with the incompatibility of the Special Theory of Relativity with Newton's theory, led Einstein to formulate General Relativity (GR) in 1915 [1]. Unlike Newton's notion of gravity as a force, GR proposed that objects fall towards the Earth due to the curvature of spacetime. The gravitational field is described in terms of the metric tensor g_{ab} and the curvature of spacetime is connected to the matter distribution. Curvature and metric are related to each other by the Levi-Civita connection.

At present, GR has passed numerous observational tests, including the so-called classic tests (i.e. the perihelion precession, the gravitational deflection of light and the gravitational redshift [2–6]) proposed by Einstein himself, and more recently the direct detection of gravitational waves in 2015 [7, 8]. However, Einstein's theory is not free from open issues.

A first shortcoming is the prediction of singularities, such as those found in black holes or at the beginning of the universe in the Big Bang theory, in which curvature and matter density become infinite. The presence of singularities challenges the physicality of the theory, and it is believed that their ultimate understanding may lie in a quantum theory of gravity. At the same time, GR has yet to be successfully integrated with Quantum Field Theory, mainly because Quantum Field Theory assumes a non-dynamic spacetime, while GR treats spacetime as a dynamic quantity [9].

From a cosmological point of view, the recent discovery of Cosmic Acceleration, i.e. the present phase of accelerated expansion that the universe seems to be currently undergoing, has led to the idea (and indeed the problem) of the "Dark universe." Indeed, in the Concordance model, the dominance of an unclustered fluid with negative pressure, known as dark energy, driving the accelerated expansion is assumed [10, 11]. A candidate for dark energy is the cosmological constant Λ. However, the Concordance model fails to explain why the inferred value of Λ is much lower than the typical value of the vacuum energy density predicted by the Standard Model of particle physics [12]. In addition to dark energy, GR does not offer a framework for the existence of the so-called dark matter, a type of unknown non-interacting matter thought to govern astrophysical structures such as galaxies, galaxy clusters, etc. The study of the rotation curves of galaxies provides simple evidence of the existence of dark matter. According to Kepler's second law, the velocity of objects orbiting the center of spiral galaxies should decrease with increasing distance from the center. However, experimental observations show that the orbital velocity does not decrease with distance, but remains more or less constant [13–15]. The existence of dark matter is why the Concordance model is often called the Λ Cold Dark Matter (ΛCDM) model.

One of the main problems of the ΛCDM model is the so-called coincidence problem. It refers to the observed fact that the energy densities of dark matter and dark energy are currently roughly the same. This fact implies a fine-tuning problem since these two components of the universe must have evolved differently, being different in nature, and since we expect dark energy to be negligible at the time of the formation of cosmic structures.

The shortcomings of GR in accounting for the phenomena mentioned above have led the research community to study extensions and modifications of Einstein's theories. Over the years, these theories of gravitation have been developed to incorporate gravity with the other fundamental forces and overcome the limits of GR [16–18].

One of the most common approaches to extending gravity is adding degrees of freedom by introducing additional fields that describe the gravitational interaction. The most famous are scalar-tensor theories, in which a scalar field mediates gravity in addition to the metric [19–22]. The prototype of these theories is the so-called Brans-Dicke theory, which, however, is a particular case of the Horndenski theory, the most general scalar-tensor theory in four dimensions with at most second derivatives of the scalar field giving second-order equations of motion [23–25]. On the other hand, more complicated objects might be introduced in the theory, like in the case of bimetric theories [26–28] in which an additional (dynamic or non-dynamic) second-order tensor is present. The Minkowski metric gives the simplest non-dynamic example of such tensor, which represents the choice made by Rosen in the first description of bimetric theory conceived [29]. The geodesic motion of particles is still governed by the usual metric g_{ab}, but in the field equations, we have both metrics. It is also worth mentioning the relativistic gravitation theory for the MOND paradigm by Bekenstein, which accounts for a vector and a scalar field other than metric, so it is an example of the so-called tensor-vector-scalar theories (TeVeS) [30–32]

Another type of GR extension is the approach in which the field equations are higher than second-order. The classic example is $f(R)$ gravity, where a generic function of the Ricci scalar is considered [33–35]. With a redefinition of the significant parameters, one can connect the $f(R)$ theory and the scalar-tensor one. In particular, $f(R)$ gravity can be connected to Brans-Dicke theory, showing that despite their apparent complexity, $f(R)$ theories possess only an additional scalar degree of freedom with respect to GR. One can also consider theories in which contractions of Ricci and Riemann tensors appear in the action, like, e.g., Gauss-Bonnet invariant. In four dimensions, the Gauss-Bonnet term corresponds to a boundary term, but this is not true if the invariant is non-minimally coupled to other fields or in higher dimensions [36–38].

In this regard, an interesting approach consists of considering a spacetime of dimensions greater than four, as in the Kaluza-Klein theory developed between the 1910s and 1920s to unify gravitation and electromagnetism [39, 40], and representing the precursor of the formulation of String theory [41, 42]. Among the modern theories that consider high-dimensional spacetimes, the most famous is Dvali-Gabadadze-Porrati (DPG) gravity [43], which is based on the concept of braneworld, in which the universe is confined to four-dimensional hypersurfaces, known as brane, encompassed in a higher-dimensional spacetime called bulk [44–46].

So far, we have mentioned only purely metric theories of gravity where the affine structure of spacetime is the Levi-Civita one. However, metric tensor and affine connection can be treated as independent of each other. In this case, one disconnects the causal structure of spacetime and the idea of parallel transport, which are associated with metric and connection, respectively, obtaining a much richer theory than GR.

When considered from the variational point of view, there are two different approaches in considering metric and connection as separate geometric entities: Palatini's method and the metric-affine formalism [16, 17, 47, 48]. The main difference is that in the Palatini approach, the matter Lagrangian is independent of connection, whereas in the metric-affine one, both gravity and matter depend on it. If we apply Palatini's method to GR, the variation with respect to connection constrains the connection itself to be that of Levi-Civita. However, this is a mere coincidence due to the use of the Einstein-Hilbert action. In general, for other theories, we would obtain different results. In the metric-affine approach, the resulting theory is different, even in the case of Einstein-Hilbert Lagrangian.

These alternative ways to geometrize gravity were explored right after the formulation of GR, and they led Weyl to develop a theory with a symmetric and nonmetric connection, which aimed to unify gravity and electromagnetism [49]. The adjective "nonmetric" indicates that the metric tensor g_{ab} is not preserved under parallel transport, implying that the inner product between vectors changes when transported along a given curve. From the geometric point of view, all this is expressed by the nonmetricity tensor Q_{abc}. Another interesting geometric approach was considered by Cartan, who developed a theory where the connection is not symmetric but metric compatible [50], thus introducing degrees of freedom connected to the torsion tensor ${T_{ab}}^{c}$. Today, we know that torsion can be associated with a purely quantum property of matter, i.e., the spin. This connection led to the development of the so-called Einstein-Cartan-Sciama-Kibble theory [51–53] and its several extensions [54–59], which allow introducing spin degrees of freedom in a relativistic geometric framework.

Both torsion and nonmetricity contain enough degrees of freedom to describe spacetime geometry entirely. Based on this fact, Teleparallel theories were introduced. They are metric-affine theories of gravity characterized by zero curvature, hence their name. Einstein initially introduced teleparallelism in an attempt to unify gravitation and electromagnetism [60]. The idea was based on using tetrad fields since they possess more degrees of freedom than the metric. However, the Lorentz invariance of the theory limits the additional degrees of freedom. Today, we can recognize two main branches in teleparallel theories, which are identified by Teleparallel Gravity (TG) and Symmetric Teleparallel Gravity (STG). Teleparallel Gravity is a gravitational gauge theory for the translation group that assumes spacetime is flat and describes gravitation through the torsion and the Weitzenböck connection [61–63]. Also, in TG, the tetrads are the dynamical fields, whereas torsion is the field strength. On the other hand, STG assumes a torsion-free and curvature-free connection, but with nonmetricity that differs from zero [64–66]. Although these theories are geometric descriptions of gravity different from GR, among them, we can consider theories that have field equations equivalent to those of GR. These are called Teleparallel Equivalent of General Relativity (TEGR) and Symmetric Teleparallel Equivalent of General Relativity (STEGR). Teleparallel theories have been extended and modified to include new parameters. Significant examples are given by the $f(T)$ (where T is the torsion scalar) and $f(\mathcal{Q})$ (where $\mathcal{Q}$ is the nonmetricity scalar) theories [67–73].

Within the $f(\mathcal{Q})$ gravity framework, this thesis aims to investigate how nonmetricity affects spacetime dynamics at both cosmological and astrophysical levels. To this end, different mathematical methods are used to derive the field equations and their resolution.

Chapter 1 describes the metric-affine structure with which the differential manifold representing spacetime is endowed. Special attention is given to nonmetricity, torsion, and curvature, which constitute the three geometric quantities that characterize an affine connection. In addition to the covariant derivative, the Lie derivative is also employed, which is a valuable tool for finding cosmological and astrophysical solutions, given its connection with the symmetries of the metric.

In Ch. 2, we review the basic principles and concepts that led to the formulation of GR, showing how the field equations can be derived by a logical process of comparison with Newtonian gravity as well as by variational principle through Einstein-Hilbert action. Next, we analyze the extension of GR given by the $f(R)$ theory, whose paradigm led to the introduction of $f(\mathcal{Q})$ gravity. In applying the metric-affine approach, we note how the projection invariance of the Ricci scalar implies stringent constraints on the hypermomentum, i.e. the variation of the matter Lagrangian with respect to connection, and we show some solutions to this issue that have been proposed in the literature.

The STEGR theory and its extensions are described in Ch. 3, focusing in particular on showing step-by-step how the field equations are derived since the literature is not always clear about this aspect. In the STEGR, since both curvature and torsion are zero, a particular choice of coordinates exists where the connection is zero. This coordinate choice constitutes the so-called "coincident gauge" and is widely used in literature. The coincident gauge will be employed in the following chapters.

In Ch. 4, we review how the concepts of homogeneity and isotropy imply the Friedmann-Lemaître-Robertson-Walker (FLRW) metric through the use of the Killing equations. In addition, we have that the energy-momentum tensor must necessarily be that of a perfect fluid. By relaxing the concept of isotropy, we can consider possible anisotropic spaces according to the Bianchi classification. This thesis uses the Bianchi type-I (BI) metric several times, which is a straightforward generalization of the flat FLRW metric. Using the reconstruction method and the coincident gauge, in Ch. 5, we obtain exact solutions for both BI and flat FLRW metric in $f(\mathcal{Q})$ gravity, thereby having a first glimpse of how nonmetricity affects the evolution of the universe.

In Ch. 6, we apply the $1+3$ formalism to a spacetime endowed with nonmetricity but without torsion. The formalism is based on splitting spacetime into a congruence of timelike curves, representing the observers, and three-dimensional spacelike hypersurfaces. This splitting allows the nonmetricity tensor to be decomposed into its components along the congruence and in the orthogonal hypersurfaces, and, in turn, this decomposition highlights the influence of nonmetricity in the kinematic aspects that characterize spacetime. Using the $1+3$ formalism, we derive the covariant cosmological equations for the BI model in the $f(\mathcal{Q})$ gravity. These equations are then used in Ch. 7 in which we give a semi-quantitative dynamical analysis through the Dynamical Systems Approach (DSA). Such analysis is performed by choosing several functions $f(\mathcal{Q})$ widely used in the literature. Whenever possible, a comparison is made with the results obtained by the reconstruction method to understand the compatibility between the two approaches.

Chapter 8 is devoted to the study of static and Locally Rotationally Symmetric (LRS) solutions. We adopt the $1+1+2$ formalism, where a congruence of spacelike curves is introduced to identify a preferred direction in space. As in the case of the $1+3$ approach, the decomposition helps understand the roles of the different components of the non-metricity tensor in the LRS spacetimes. Indeed, the $1+1+2$ decomposition allows us to recast the gravitational field equations as a set of scalar differential equations, simplifying their solution considerably. Using these equations, we could show that to have a Schwarzschild-de Sitter type metric, certain conditions must be imposed on the function $f(\mathcal{Q})$, the nonmetric scalar, or the scalar components of the nonmetricity tensor.

The last Ch. 9 is dedicated to the study of the coupling between gravitation and spinors in the context of $f(\mathcal{Q})$ gravity. The unique feature of the proposed approach is that the dynamical variables are the tetrad field and the affine connection, not the spin connection as it usually happens. Spinors appear to be unaffected by the presence of nonmetricity. The theory is applied to BI cosmologies, providing an exact solution that isotropizes at a late time.

To conclude, in Ch. 10 we discuss final remarks on the results obtained in this thesis.

The material in Chs. 5, 6, 7 and 8 is based on the works [74], [75], and [76] for which I was responsible for both its drafting and all calculations. The other authors gave me valuable insights into the general considerations of the results shown. On the other hand, Ch. 9 is based on the paper [77], in which I contributed to part of the calculations and the overall analysis of the results.

I
Gravity as geometry

1
Geometric preliminaries

This chapter provides mathematical concepts and definitions that are useful for developing gravitational theories in a geometric framework. In this way, we will be able to understand what differentiates General Relativity (GR) from other theories that rely not only on the metric tensor but also on geometric quantities such as torsion and nonmetricity [53, 73, 78–84].

1.1 Metric-affine geometry

Spacetime is described by a 4-dimensional differentiable manifold **M** endowed with a metric

$$\mathbf{g} = g_{ab}\mathrm{d}x^a\mathrm{d}x^b, \qquad g_{ab} = g_{ba}, \tag{1.1}$$

a rank-2 symmetric covariant tensor, which defines a scalar product on the manifold between any two vectors $\mathbf{v} = v^a\partial_a$ and $\mathbf{w} = w^a\partial_a$,

$$\mathbf{g}(v, w) = g_{ab}v^a w^b, \tag{1.2}$$

and consequently the length of a vector,

$$l^2 = g_{ab}v^a v^b. \tag{1.3}$$

According to the sign of $g_{ab}v^a v^b$, we distinguish timelike vectors $l^2 < 0$, spacelike vectors $l^2 > 0$, and lightlike vectors $l^2 = 0$. In view of this, we can define the line element $\mathrm{d}s^2$ as the length of an infinitesimal displacement $\mathrm{d}\mathbf{x}$,

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b. \tag{1.4}$$

In the case where $\mathrm{d}s^2 < 0$, the line element represents the so-called proper time t, that is the time measured by a clock following the curve to which the infinitesimal displacements $\mathrm{d}\mathbf{x}$ are tangent.

The metric is assumed to be non-degenerate, $g = det(g_{ab}) \neq 0$, therefore we can define the tensor g^{ab} such that $g^{ab}g_{bc} = \delta^a_c$, where $\delta^a{}_c$ is the Kronecker delta. In matrix representation, g^{ab} is nothing more than the inverse matrix of g_{ab}. With the introduction of the metric, at each point x of the manifold, we can establish an isomorphism between the tangent space $\boldsymbol{V}_x$ and the dual one $\boldsymbol{V}^*_x$. Locally, it is described by the procedure of raising and lowering the indices:

$$g_{ab}v^b = v_a. \tag{1.5}$$

In addition to the metric, the manifold can be endowed with an affine structure induced by the connection $\Gamma_{ab}{}^c$. The connection is a geometric object that establishes a rule that maps a vector v^a at a point x, into a vector w^a at an infinitely near point $x + dx$:

$$\delta v^c = w^c - v^c = -\Gamma_{ab}{}^c v^b dx^a. \tag{1.6}$$

Under a coordinate change $\bar{x} : x^a \to \bar{x}^b(x^a)$, $\Gamma_{ab}{}^c$ transforms as follows

$$\bar{\Gamma}_{ab}{}^c(\bar{x}) = \frac{\partial \bar{x}^c}{\partial x^d}\frac{\partial^2 x^d}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial \bar{x}^c}{\partial x^d}\Gamma_{ef}{}^d(x)\frac{\partial x^e}{\partial \bar{x}^a}\frac{\partial x^f}{\partial \bar{x}^b}, \tag{1.7}$$

so it is not a tensor quantity. A generalization of the Euclidean concept of directional derivative can be introduced using the affine connection. It is called covariant derivative ∇, and applied to an arbitrary tensor $\mathbb{T}$ gives

$$\begin{aligned}\nabla_c \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} = {} & \partial_c \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} + \Gamma_{cd}{}^{a_1}\mathbb{T}^{d\cdots a_n}{}_{b_1\cdots b_m} + \cdots + \Gamma_{cd}{}^{a_n}\mathbb{T}^{a_1\cdots d}{}_{b_1\cdots b_m} + \\ & - \Gamma_{cb_1}{}^d \mathbb{T}^{a_1\cdots a_n}{}_{d\cdots b_m} - \cdots - \Gamma_{cb_n}{}^d \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots d}\,.\end{aligned} \tag{1.8}$$

The covariant derivative is linear,

$$\nabla_c \left(\alpha v^a + \beta w^a\right) = \alpha \nabla_c v^a + \beta \nabla_c w^a \qquad \forall\, \alpha, \beta \in \mathbb{R}, \tag{1.9}$$

it satisfies the Leibniz rule,

$$\begin{aligned}\nabla_c \left(\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}\bar{T}^{d_1\cdots d_n}{}_{e_1\cdots e_m}\right) = {} & \left(\nabla_c \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}\right)\bar{\mathbb{T}}^{d_1\cdots d_n}{}_{e_1\cdots e_m} + \\ & + \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}\left(\nabla_c \bar{\mathbb{T}}^{d_1\cdots d_n}{}_{e_1\cdots e_m}\right),\end{aligned} \tag{1.10}$$

and applied to an arbitrary function $f \in C^\infty$, it is equal to the partial derivative,

$$\nabla_a f = \partial_a f. \tag{1.11}$$

From Eqs. (1.7) and (1.8), it is evident that the covariant derivative of a tensor is still a tensor. On the other hand, if $\mathbb{T}$ is a generic tensor density of weight w, its covariant derivative is given by

$$\begin{aligned}\nabla_c \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} = {} & \partial_c \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} + \Gamma_{cd}{}^{a_1}\mathbb{T}^{d\cdots a_n}{}_{b_1\cdots b_m} + \cdots + \Gamma_{cd}{}^{a_n}\mathbb{T}^{a_1\cdots d}{}_{b_1\cdots b_m} + \\ & - \Gamma_{cb_1}{}^d \mathbb{T}^{a_1\cdots a_n}{}_{d\cdots b_m} - \cdots - \Gamma_{cb_n}{}^d \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots d} + w\Gamma_{cd}{}^d \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}\,.\end{aligned} \tag{1.12}$$

Using the connection and the associated covariant derivative, we can also define the notion of parallel transport along a given curve. To this end, let $\gamma(\lambda)$ be a curve in $\mathbf{M}$ with tangent vector $u^a = \mathrm{d}x^a/\mathrm{d}\lambda$, we say that a tensor field $\mathbb{T}$ is parallel transported with respect to $\Gamma_{ab}{}^c$ along γ if

$$u^c \nabla_c \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} = 0. \tag{1.13}$$

Moreover, if the following relation holds

$$u^a \nabla_a u^b = 0, \tag{1.14}$$

the curve $\gamma(\lambda)$ is called autoparallel with respect to $\Gamma_{ab}{}^c$.

1.2 Nonmetricity and torsion

Given two vectors v^a and w^a which are parallel transported along a curve γ, their scalar product is not in general preserved,

$$u^c \nabla_c \left(g_{ab} v^a w^b \right) = v^a w^b u^c \nabla_c g_{ab} + w_a u^c \nabla_c v^a + v_a u^c \nabla_c w^a = v^a w^b u^c Q_{cab}. \tag{1.15}$$

The tensor Q_{cab} is called nonmetricity,

$$Q_{cab} = \nabla_c g_{ab} \qquad Q_{cab} = Q_{cba}, \tag{1.16}$$

and there are two independent vectors associated to it,

$$q_c = g^{ab} Q_{cab} \quad \text{and} \quad Q_b = g^{ca} Q_{cab}. \tag{1.17}$$

Nonmetricity represents the failure of the connection to covariantly preserve the metric and the scalar product of parallel transported vectors, so the norm of a vector is not conserved either. Another consequence is that the process of raising and lowering indices does not commute with the covariant derivation. For example, if we want to lower the index in $\nabla_a v^b$, we have

$$g_{bc} \nabla_a v^c = \nabla_a v_b - Q_{abc} v^c. \tag{1.18}$$

In view of this, we must be careful in defining tensors that are the covariant derivative of covariant or contravariant objects. An example is represented by the nonmetricity itself. Let us explicitly calculate $Q_a{}^{bc}$,

$$\begin{aligned} Q_c{}^{ab} =& g^{ad} g^{be} Q_{cde} = g^{ad} g^{be} \nabla_c g_{de} = \nabla_c \left(g^{ad} g^{be} g_{de} \right) - g_{de} g^{be} \nabla_c g^{ad} - g_{de} g^{ad} \nabla_c g^{be} = \\ =& \nabla_c g^{ab} - \nabla_c g^{ab} - \nabla_c g^{ab} = -\nabla_c g^{ab}. \end{aligned} \tag{1.19}$$

If we had defined nonmetricity with the totally contravariant metric instead, we would have had $Q_c{}^{ab} = \nabla_c g^{ab}$, which differs from Eq. (1.19) by a minus sign.

From the definition (1.16),

$$Q_{cab} = \partial_c g_{ab} - \Gamma_{ca}{}^d g_{db} - \Gamma_{cb}{}^d g_{ad}, \tag{1.20}$$

and subsequent permutations,

$$Q_{abc} = \partial_a g_{bc} - \Gamma_{ab}{}^d g_{dc} - \Gamma_{ac}{}^d g_{bd}, \tag{1.21}$$

$$Q_{bca} = \partial_b g_{ca} - \Gamma_{bc}{}^d g_{da} - \Gamma_{ba}{}^d g_{cd}, \tag{1.22}$$

subtracting Eq. (1.20) from the sum of Eq. (1.21) and (1.22), we derive

$$Q_{abc} + Q_{bca} - Q_{cab} = \partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab} - 2\Gamma_{(ab)}{}^d g_{dc} - 2\Gamma_{[ac]}{}^d g_{bd} - 2\Gamma_{[bc]}{}^d g_{da}. \tag{1.23}$$

From the last expression, we obtain the connection decomposition

$$\Gamma_{ab}{}^c = \tilde{\Gamma}_{ab}{}^c + \frac{1}{2} \left(T_{ab}{}^c + T^c{}_{ab} + T^c{}_{ba} \right) + \frac{1}{2} \left(Q^c{}_{ab} - Q_{ab}{}^c - Q_{ba}{}^c \right). \tag{1.24}$$

We have introduced the Levi-Civita connection[1],

$$\tilde{\Gamma}_{ab}{}^{c} = \frac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right) \qquad \tilde{\Gamma}_{ab}{}^{c} = \tilde{\Gamma}_{ba}{}^{c}, \tag{1.25}$$

the only symmetric and metric compatible connection, i.e., so that $\nabla_c g_{ab} = 0$, and the torsion $T_{ab}{}^{c}$,

$$T_{ab}{}^{c} = \Gamma_{ab}{}^{c} - \Gamma_{ba}{}^{c}, \qquad T_{ab}{}^{c} = -T_{ba}{}^{c}, \tag{1.26}$$

the antisymmetric part of the connection. From Eq. (1.7), we prove that torsion is a tensor,

$$\bar{T}_{ab}{}^{c} = \bar{\Gamma}_{ab}{}^{c} - \bar{\Gamma}_{ba}{}^{c} = \frac{\partial \bar{x}^c}{\partial x^d} \frac{\partial x^e}{\partial \bar{x}^a} \frac{\partial x^f}{\partial \bar{x}^b} \left(\Gamma_{ef}{}^{d} - \Gamma_{fe}{}^{d} \right). \tag{1.27}$$

Geometrically, torsion measures the non-closure of the parallelogram formed when two infinitesimal vectors, such as v^a and w^a, are parallel transported along each other. The failure to construct the parallelogram is represented by the vector $T_{ab}{}^{c} v^a w^b$.

In decomposition (1.24), it is useful to define the distortion tensor,

$$N_{ab}{}^{c} = K_{ab}{}^{c} + L_{ab}{}^{c}, \tag{1.28}$$

where $K_{ab}{}^{c}$ is the contortion tensor,

$$K_{ab}{}^{c} = \frac{1}{2} \left(T_{ab}{}^{c} + T^{c}{}_{ab} + T^{c}{}_{ba} \right), \tag{1.29}$$

which is antisymmetric in the last two indices,

$$K_{acb} = \frac{1}{2} \left(T_{acb} + T_{bac} + T_{bca} \right) = \frac{1}{2} \left(-T_{cab} - T_{abc} - T_{cba} \right) = -K_{abc}, \tag{1.30}$$

whereas $L_{ab}{}^{c}$ is the disformation tensor,

$$L_{ab}{}^{c} = \frac{1}{2} \left(Q^{c}{}_{ab} - Q_{ab}{}^{c} - Q_{ba}{}^{c} \right), \tag{1.31}$$

which is instead symmetric in the first two indices, $L_{ab}{}^{c} = L_{ba}{}^{c}$.

1.2.1 Levi-Civita tensor

The Levi-Civita tensor is the tensor constructed from the Levi-Civita symbol ϵ^{abcd} and the determinant of the metric g, which are both tensor densities of weight -1 and -2, respectively,

$$\varepsilon_{abcd} = \sqrt{-g} \epsilon_{abcd}, \qquad \varepsilon^{abcd} = \frac{1}{\sqrt{-g}} \epsilon^{abcd}, \tag{1.32}$$

where

$$\epsilon^{mnpq} g_{ma} g_{nb} g_{pc} g_{qd} = -g \epsilon_{abcd}, \tag{1.33}$$

and it is characterized by the following properties,

$$\varepsilon_{abcd} = \varepsilon_{[abcd]}, \tag{1.34}$$

[1] Throughout the thesis, we will denote by a tilde all quantities related to the Levi-Civita connection.

$$\varepsilon_{abcd}\varepsilon^{efgh} = -24\delta^{e}{}_{[a}\delta^{f}{}_{b}\delta^{g}{}_{c}\delta^{h}{}_{d]}, \tag{1.35}$$

$$\varepsilon_{abcd}\varepsilon^{aefg} = -6\delta^{e}{}_{[b}\delta^{f}{}_{c}\delta^{g}{}_{d]}, \tag{1.36}$$

$$\varepsilon_{abcd}\varepsilon^{abef} = -4\delta^{e}{}_{[c}\delta^{f}{}_{d]}, \tag{1.37}$$

$$\varepsilon_{abcd}\varepsilon^{abce} = -6\delta^{e}{}_{d}, \tag{1.38}$$

$$\varepsilon_{abcd}\varepsilon^{abcd} = -24. \tag{1.39}$$

Its covariant derivative is equal to

$$\begin{aligned}\nabla_e\varepsilon_{abcd} =& \partial_e\varepsilon_{abcd} - \Gamma_{ea}{}^{h}\varepsilon_{hbcd} - \Gamma_{eb}{}^{h}\varepsilon_{ahcd} - \Gamma_{ec}{}^{h}\varepsilon_{abhd} - \Gamma_{ed}{}^{h}\varepsilon_{abch} = \partial_e\varepsilon_{abcd} - \Gamma_{eh}{}^{h}\varepsilon_{abcd} = \\ =& \left(\frac{1}{\sqrt{-g}}\partial_e\sqrt{-g}\right)\varepsilon_{abcd} + \left(\frac{1}{2}q_e - \frac{1}{2}g^{ab}\partial_e g_{ab}\right)\varepsilon_{abcd} = \frac{1}{2}q_e\varepsilon_{abcd}.\end{aligned} \tag{1.40}$$

Therefore, it is covariantly preserved only in the absence of nonmetricity. In Eq. (1.40) we used the relation

$$\frac{1}{\sqrt{-g}}\partial_e\sqrt{-g} = \frac{1}{2}g^{ab}\partial_e g_{ab}. \tag{1.41}$$

1.3 Curvature

From the commutator of two covariant derivatives applied to a vector v^a,

$$[\nabla_c, \nabla_d]\, v^a = \nabla_c\nabla_d v^a - \nabla_d\nabla_c v^a = R^{a}{}_{bcd}v^b - T_{cd}{}^{b}\nabla_b v^a, \tag{1.42}$$

we define the Riemann tensor, or curvature tensor, for the connection $\Gamma_{ab}{}^{c}$,

$$R^{a}{}_{bcd} = \partial_c\Gamma_{db}{}^{a} - \partial_d\Gamma_{cb}{}^{a} + \Gamma_{ce}{}^{a}\Gamma_{db}{}^{e} - \Gamma_{de}{}^{a}\Gamma_{cb}{}^{e}, \tag{1.43}$$

which is antisymmetric in the last two indices,

$$R^{a}{}_{bcd} = -R^{a}{}_{bdc}, \tag{1.44}$$

and satisfies the two Bianchi identities:

- First Bianchi identity,
$$R^{a}{}_{[bcd]} = \nabla_{[b}T_{cd]}{}^{a} - T_{[cd}{}^{e}T_{b]e}{}^{a}; \tag{1.45}$$
- Second Bianchi identity,
$$\nabla_{[e|}R^{a}{}_{b|cd]} = -R^{a}{}_{bf[e}T_{cd]}{}^{f}. \tag{1.46}$$

Similar relations are obtained for a covector w_a,

$$[\nabla_c, \nabla_d]\, w_b = -R^{a}{}_{bcd}w_a - T_{cd}{}^{a}\nabla_a w_a, \tag{1.47}$$

and an arbitrary tensor $\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}$,

$$\begin{aligned}[\nabla_c, \nabla_d]\, \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} =& R^{a_1}{}_{ecd}\mathbb{T}^{e\cdots a_n}{}_{b_1\cdots b_m} + \cdots - R^{e}{}_{b_1cd}\mathbb{T}^{a_1\cdots a_n}{}_{e\cdots b_m} + \cdots + \\ & - T_{cd}{}^{e}\nabla_e\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}.\end{aligned} \tag{1.48}$$

Equations (1.42), (1.47), and (1.48) are called Ricci identities.

From a geometric point of view, curvature measures the rotations of a vector when parallel transported along a closed curve. For example, if we consider a parallelogram built by two vectors r^a and s^a, and we perform the parallel transport of a vector v^a along it, we obtain that the difference of v^a after and before the loop is

$$\delta v^a = R^a{}_{bcd} v^b r^c s^d. \tag{1.49}$$

Furthermore, we can demonstrate that the relative acceleration of two infinitely near curves is deeply related to Riemann and torsion tensors. Let $\gamma_s(\lambda)$ be a family of curves, with $s \in \mathbb{R}$ the parameter that identifies a curve of the family, $u^a = \partial x^a/\partial\lambda$ the vector field tangent to the family of curves, and $w^a = \partial x^a/\partial s$ the vector field which gives the infinitesimal displacement at a fixed value of λ from a curve to another. Since the following relation for the torsion tensor holds

$$u^a \nabla_a w^b - w^a \nabla_a u^b = -T_{ac}{}^b u^a w^c, \tag{1.50}$$

the relative acceleration of two curves in the family is equal to

$$\begin{aligned} a^b =& u^a \nabla_a \left(u^c \nabla_c w^b \right) = u^a \nabla_a \left(w^c \nabla_c u^b \right) - u^a \nabla_a \left(T_{cd}{}^b u^c w^d \right) = \\ =& \left(u^a \nabla_a w^c \right) \nabla_c u^b + u^a w^c \nabla_a \nabla_c u^b - u^a \nabla_a \left(T_{cd}{}^b u^c w^d \right) = \\ =& \left(u^a \nabla_a w^c \right) \nabla_c u^b + u^a w^c \nabla_c \nabla_a u^b + u^a w^c R^b{}_{dac} u^d + \\ & - u^a w^c T_{ac}{}^d \nabla_d u^b - u^a \nabla_a \left(T_{cd}{}^b u^c w^d \right) = \\ =& \left(w^a \nabla_a v^c \right) \nabla_c u^b - T_{ad}{}^c u^a w^d \nabla_c u^b + u^a w^c \nabla_c \nabla_a u^b + u^a w^c R^b{}_{dac} u^d + \\ & - u^a w^c T_{ac}{}^d \nabla_d u^b - u^a \nabla_a \left(T_{cd}{}^b u^c w^d \right) = \\ =& w^c \nabla_c \left(u^a \nabla_a u^b \right) + R^b{}_{dac} u^d u^a w^c - u^a \nabla_a \left(T_{cd}{}^b u^c w^d \right). \end{aligned} \tag{1.51}$$

Specifically, there is a direct proportionality between the relative acceleration and the Riemann tensor when the curves of the family are autoparallel and the torsion is null.

Riemann tensor has three independent contractions:

- Ricci tensor,
$$R_{bd} = R^a{}_{bad}; \tag{1.52}$$

- the contraction of the second and third index,
$$\bar{R}_{ad} = R_a{}^b{}_{bd}; \tag{1.53}$$

- homothetic curvature,
$$\check{R}_{cd} = R^a{}_{acd}, \qquad \check{R}_{cd} = -\check{R}_{dc}. \tag{1.54}$$

The contraction of Eqs. (1.52) and (1.53) with the metric gives the Ricci scalar,

$$R = g^{ab} R_{ab} = -g^{ab} \bar{R}_{ab}. \tag{1.55}$$

Using Eq. (1.24), we have the decomposition of Riemann tensor,

$$R^a{}_{bcd} = \tilde{R}^a{}_{bcd} + \tilde{\nabla}_c N_{db}{}^a - \tilde{\nabla}_d N_{cb}{}^a + N_{cp}{}^a N_{db}{}^p - N_{dp}{}^a N_{cb}{}^p, \tag{1.56}$$

where $\tilde{R}^a{}_{bcd}$ and $\tilde{\nabla}$ are the curvature and the covariant derivative of the Levi-Civita connection $\tilde{\Gamma}_{ab}{}^c$, respectively. Notice that $\tilde{R}_{abcd}$ is antisymmetric in the first two indices, therefore we can derive a simple relation between the homothetic curvature and nonmetricity tensor,

$$\check{R}_{cd} = \tilde{\nabla}_c L_{da}{}^a - \tilde{\nabla}_d L_{ca}{}^a = -\tilde{\nabla}_{[c} q_{d]} = -\partial_{[c} q_{d]}, \tag{1.57}$$

emphasizing the purely nonmetric nature of $\check{R}_{cd}$. From the decomposition of the Ricci scalar,

$$\begin{aligned} R =& \tilde{R} + \frac{1}{4}Q_{cab}Q^{cab} - \frac{1}{2}Q_{cab}Q^{abc} - \frac{1}{4}q_a q^a + \frac{1}{2}q_a Q^a + \frac{1}{4}T_{abc}T^{abc} - \frac{1}{2}T_{cab}T^{abc} - T_a T^a + \\ &+ Q_{abc}T^{abc} - q_a T^a + Q_a T^a + \tilde{\nabla}_a\left(q^a - Q^a + 2T^a\right) = \\ =& \tilde{R} + \mathcal{Q} + T + Q_{abc}T^{abc} - q_a T^a + Q_a T^a + \tilde{\nabla}_a\left(q^a - Q^a + 2T^a\right), \end{aligned} \tag{1.58}$$

with $T_a = T_{ab}{}^b$, we define both the nonmetricity scalar,

$$\mathcal{Q} = \frac{1}{4}Q_{cab}Q^{cab} - \frac{1}{2}Q_{cab}Q^{abc} - \frac{1}{4}q_a q^a + \frac{1}{2}q_a Q^a, \tag{1.59}$$

and the torsion scalar,

$$T = \frac{1}{4}T_{abc}T^{abc} - \frac{1}{2}T_{cab}T^{abc} - T_a T^a. \tag{1.60}$$

Decomposition (1.58) will assist us in understanding the equivalence between GR and STEGR (see Ch. 3).

1.3.1 Irreducible decomposition of Riemann tensor: the Weyl tensor

Let us split the Riemann tensor into its antisymmetric and symmetric parts in the first two indices [85, 86],

$$R_{abcd} = R_{[ab]cd} + R_{(ab)cd} = W_{abcd} + Z_{abcd}, \tag{1.61}$$

where

$$W_{abcd} = R_{[ab]cd}, \qquad \text{and} \qquad Z_{abcd} = R_{(ab)cd}. \tag{1.62}$$

The antisymmetric part can be decomposed as follows,

$$W_{abcd} = \sum_{n=1}^{6} W^{(n)}_{abcd}, \tag{1.63}$$

$$W^{(1)}_{abcd} = W_{abcd} - \sum_{n=2}^{6} W^{(n)}_{abcd}, \tag{1.64}$$

$$W^{(2)}_{abcd} = \frac{1}{2}\left(W_{abcd} - W_{cdab}\right) - \left(R_{[mn]} - \bar{R}_{[mn]}\right)\delta^m_{[a} g_{b][c}\delta^n_{d]}, \tag{1.65}$$

$$W^{(3)}_{abcd} = W_{[abcd]} = R_{[abcd]}, \tag{1.66}$$

$$W^{(4)}_{abcd} = -\left(R_{(mn)} - \bar{R}_{(mn)}\right)\delta^m_{[a}g_{b][c}\delta^n_{d]} - \frac{1}{2}g_{c[a}g_{b]d}R, \tag{1.67}$$

$$W^{(5)}_{abcd} = \left(R_{[mn]} - \bar{R}_{[mn]}\right)\delta^m_{[a}g_{b][c}\delta^n_{d]}, \tag{1.68}$$

$$W^{(6)}_{abcd} = \frac{1}{6}g_{c[a}g_{b]d}R. \tag{1.69}$$

On the other hand, the symmetric part is decomposed as

$$Z_{abcd} = \sum_{n=1}^{5} Z^{(n)}_{abcd}, \tag{1.70}$$

$$Z^{(1)}_{abcd} = Z_{abcd} - \sum_{n=2}^{5} Z^{(n)}_{abcd}, \tag{1.71}$$

$$\begin{aligned} Z^{(2)}_{abcd} =& \frac{1}{2}\left(Z_{abcd} - Z_{c(ab)d} + Z_{d(ab)c}\right) + \\ &- \frac{1}{4}\left(R_{[mn]} + \bar{R}_{[mn]} - \check{R}_{mn}\right)\left(2\delta^m_{(a}g_{b)[c}\delta^n_{d]} - g_{ab}\delta^m_{[c}\delta^n_{d]}\right), \end{aligned} \tag{1.72}$$

$$Z^{(3)}_{abcd} = \frac{1}{6}\left(R_{[mn]} + \bar{R}_{[mn]} - \frac{1}{2}\check{R}_{mn}\right)\left(4\delta^m_{(a}g_{b)[c}\delta^n_{d]} - g_{ab}\delta^m_{[c}\delta^n_{d]}\right), \tag{1.73}$$

$$Z^{(4)}_{abcd} = \frac{1}{2}\left(R_{(mn)} + \bar{R}_{(mn)}\right)\delta^m_{(a}g_{b)[c}\delta^n_{d]}, \tag{1.74}$$

$$Z^{(5)}_{abcd} = \frac{1}{4}g_{ab}\check{R}_{cd}. \tag{1.75}$$

We call Weyl tensor the following expression,

$$C_{abcd} = W^{(1)}_{abcd} + Z^{(1)}_{abcd}, \tag{1.76}$$

which is the trace-free component of the Riemann tensor, and it has the same symmetries. In a pure metric theory, the Weyl tensor is defined by the Ricci decomposition:

$$\tilde{C}_{abcd} = \tilde{R}_{abcd} + \frac{1}{2}\left(g_{ad}\tilde{R}_{bc} - g_{ac}\tilde{R}_{bd} + g_{bc}\tilde{R}_{ad} - g_{bd}\tilde{R}_{ac}\right) + \frac{1}{6}\tilde{R}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right). \tag{1.77}$$

The Weyl tensor (1.77) is also called conformal tensor since, if we raise the first index, $C^a{}_{bcd}$, it is invariant under conformal transformations of the metric, that is, transformations that preserve the metric up to a scale factor [82],

$$g \quad \longrightarrow \quad e^{2\sigma}g. \tag{1.78}$$

In the case of $\tilde{C}_{abcd}$ equal to zero, the metric is said to be conformally flat. An example is the Friedmann-Lemaître-Robertson-Walker metric given in Sec. 4.1.

1.4 Geodesics

A geodesic is the generalization of the Euclidean concept of a straight line as the shortest path between two fixed points. Given a curve $\gamma(\lambda)$, with endpoints A and B, and tangent vector $u^a = \mathrm{d}x^a/\mathrm{d}\lambda$, the geodesic is the extremal of the length functional

$$l_{AB} = \int_A^B \sqrt{\pm g_{ab}u^a u^b}\mathrm{d}\lambda. \tag{1.79}$$

Considering, for example, a timelike curve $g_{ab}u^a u^b < 0$, we have

$$\begin{aligned}
0 =&\delta l_{AB} = \delta \int_A^B \sqrt{-g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}}\mathrm{d}\lambda = \\
=& -\int_A^B \frac{1}{2}\left(-g_{de}\frac{\mathrm{d}x^d}{\mathrm{d}\lambda}\frac{\mathrm{d}x^e}{\mathrm{d}\lambda}\right)^{-\frac{1}{2}}\left(\delta x^c\partial_c g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} + 2g_{ab}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\delta x^a\right)\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda = \\
=& -\int_A^B \frac{1}{2}\left(-g_{de}\frac{\mathrm{d}x^d}{\mathrm{d}\lambda}\frac{\mathrm{d}x^e}{\mathrm{d}\lambda}\right)^{-\frac{1}{2}}\left[\delta x^c\partial_c g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} - 2g_{ab}\delta x^a\frac{\mathrm{d}^2x^b}{\mathrm{d}\lambda^2} - 2\delta x^a\partial_c g_{ab}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} + \right. \\
&\left. - \left(-g_{mn}\frac{\mathrm{d}x^m}{\mathrm{d}\lambda}\frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-g_{pq}\frac{\mathrm{d}x^p}{\mathrm{d}\lambda}\frac{\mathrm{d}x^q}{\mathrm{d}\lambda}\right)g_{ab}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\delta x^a\right]\mathrm{d}\lambda = \\
=& \int_A^B \left[-g_{de}\frac{\mathrm{d}x^d}{\mathrm{d}\lambda}\frac{\mathrm{d}x^e}{\mathrm{d}\lambda}\right]^{-\frac{1}{2}}\left[g_{ab}\frac{\mathrm{d}^2x^b}{\mathrm{d}\lambda^2} + \partial_c g_{ab}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} - \frac{1}{2}\partial_a g_{cb}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} + \right. \\
&\left. + \left(-g_{mn}\frac{\mathrm{d}x^m}{\mathrm{d}\lambda}\frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-g_{pq}\frac{\mathrm{d}x^p}{\mathrm{d}\lambda}\frac{\mathrm{d}x^q}{\mathrm{d}\lambda}\right)g_{ab}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\right]\delta x^a\mathrm{d}\lambda,
\end{aligned} \tag{1.80}$$

where we imposed $\delta x^a(A) = \delta x^b(B) = 0$. Multiplying the integrand by the g^{da}, we obtain the desired geodesic equation,

$$\begin{aligned}
&\frac{\mathrm{d}^2x^d}{\mathrm{d}\lambda^2} + \frac{1}{2}g^{da}\left(\partial_b g_{ca} + \partial_c g_{ba} - \partial_a g_{bc}\right)\frac{\mathrm{d}x^c}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} + \\
&\qquad + \left(-g_{mn}\frac{\mathrm{d}x^m}{\mathrm{d}\lambda}\frac{\mathrm{d}x^n}{\mathrm{d}\lambda}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-g_{pq}\frac{\mathrm{d}x^p}{\mathrm{d}\lambda}\frac{\mathrm{d}x^q}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}x^d}{\mathrm{d}\lambda} = 0,
\end{aligned} \tag{1.81}$$

Of course, the same result can be obtained using the Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial u^a} - \frac{\partial L}{\partial x^a} = 0, \tag{1.82}$$

with $L = \sqrt{-g_{ab}u^a u^b}$.

If the tangent vector u^a has constant length, as in absence of nonmetricity, the geodesic equation (1.81) is equal to

$$\frac{\mathrm{d}^2x^d}{\mathrm{d}\lambda^2} + \frac{1}{2}g^{da}\left(\partial_b g_{ca} + \partial_c g_{ba} - \partial_a g_{bc}\right)\frac{\mathrm{d}x^c}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = \frac{\mathrm{d}^2x^d}{\mathrm{d}\lambda^2} + \tilde{\Gamma}_{bc}{}^{d}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0, \tag{1.83}$$

that is,

$$u^a\tilde{\nabla}_a u^d = 0. \tag{1.84}$$

Therefore, when we have zero torsion and nonmetricity, autoparallel curves (1.14) are geodesics.

1.5 Lie derivative and Killing vectors

Let $\mathbf{X} = X^a\partial_a$ be a vector field, the Lie derivative with respect to $\mathbf{X}$ is an operator, $\mathfrak{L}_{\mathbf{X}}$, that maps tensor to tensor of the same manifold as follows

$$\begin{aligned}\mathfrak{L}_{\mathbf{X}}\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} =& X^c\partial_c\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} - \mathbb{T}^{c\cdots a_n}{}_{b_1\cdots b_m}\partial_c X^{a_1} + \cdots + \\ &+ \mathbb{T}^{a_1\cdots a_n}{}_{c\cdots b_m}\partial_{b_1}X^c + \cdots,\end{aligned} \tag{1.85}$$

and it satisfies the properties [82]:

- let $\mathbf{Y}$ be a vector field, the Lie derivative is linear, that is $\forall\, \alpha, \beta \in \mathbb{R}$

$$\mathfrak{L}_{\mathbf{X}}\left(\alpha\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} + \beta\bar{\mathbb{T}}^{a_1\cdots a_n}{}_{b_1\cdots b_m}\right) = \alpha\mathfrak{L}_{\mathbf{X}}\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} + \beta\mathfrak{L}_{\mathbf{X}}\bar{\mathbb{T}}^{a_1\cdots a_n}{}_{b_1\cdots b_m}, \tag{1.86}$$

$$\mathfrak{L}_{\alpha\mathbf{X}+\beta\mathbf{Y}}\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} = \alpha\mathfrak{L}_{\mathbf{X}}\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} + \beta\mathfrak{L}_{\mathbf{Y}}\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}; \tag{1.87}$$

- Leibniz rule,

$$\begin{aligned}\mathfrak{L}_{\mathbf{X}}\left(\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}\bar{\mathbb{T}}^{d_1\cdots d_n}{}_{e_1\cdots e_m}\right) =& \left(\mathfrak{L}_{\mathbf{X}}\mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}\right)\bar{\mathbb{T}}^{d_1\cdots d_n}{}_{e_1\cdots e_m} + \\ &+ \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}\left(\mathfrak{L}_{\mathbf{X}}\bar{\mathbb{T}}^{d_1\cdots d_n}{}_{e_1\cdots e_m}\right);\end{aligned} \tag{1.88}$$

- let $\mathbf{Y}$ be a vector field,

$$\mathfrak{L}_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}] = \mathbf{X}(\mathbf{Y}) - \mathbf{Y}(\mathbf{X}); \tag{1.89}$$

- let f be an arbitrary function,

$$\mathfrak{L}_{\mathbf{X}}f = \mathbf{X}(f) = X^a\partial_a f. \tag{1.90}$$

Lie derivative is closely related to the one-parameter transformation groups: if a tensor is invariant under the action of a one-parameter transformation group, the Lie derivative with respect to the infinitesimal generator is zero. A simple example is given by isometries, transformations such as $y : x^a \to y^b(x^a)$ that preserve the metric:

$$g_{ab}(x) = \frac{\partial y^c}{\partial x^a}\frac{\partial y^d}{\partial x^b}g_{cd}(y). \tag{1.91}$$

Considering the infinitesimal transformation

$$y^a = x^a + \epsilon X^a, \tag{1.92}$$

with $\epsilon \to 0$ and $\mathbf{X}$ the vector field that generates the transformation, the metric becomes,

$$\begin{aligned}g_{ab}(x) =& \left(\delta^c_a + \epsilon\frac{\partial X^c}{\partial x^a}\right)\left(\delta^d_b + \epsilon\frac{\partial X^d}{\partial x^b}\right)g_{cd}(y) \\ \approx& g_{ab}(x) + \epsilon X^c\partial_c g_{ab}(x) + \epsilon\partial_a X^c g_{cb}(x) + \epsilon\partial_b X^c g_{ac}(x),\end{aligned} \tag{1.93}$$

where we stopped at the first order in ϵ. Therefore, for the metric to be invariant under the transformation (1.92), it must be

$$X^c\partial_c g_{ab}(x) + \partial_a X^c g_{cb}(x) + \partial_b X^c g_{ac}(x) = 0, \tag{1.94}$$

which is exactly the definition of the Lie derivative, $\mathfrak{L}_{\mathbf{X}} g_{ab} = 0$. Equation (1.94) is called Killing equation and the infinitesimal generator of the isometry $\mathbf{X}$ is called Killing vector field. If we write $X^c = g^{cd} X_d$, we have

$$\tilde{\nabla}_{(a} X_{b)} = 0. \tag{1.95}$$

This equation allows us to completely determine the Killing vectors from given values of X_a and $\tilde{\nabla}_a X_b$ at some point P of the manifold [78, 87]. Indeed, the Ricci identity (1.47) and the first Bianchi identity for the Levi-Civita Riemann tensor,

$$\tilde{R}^a{}_{bcd} + \tilde{R}^a{}_{cdb} + \tilde{R}^a{}_{dbcd} = 0, \tag{1.96}$$

together with Eq. (1.95) yield the following relation

$$\tilde{\nabla}_b \tilde{\nabla}_c X_d = \tilde{R}^a{}_{bcd} X_a. \tag{1.97}$$

Therefore, given X_a and $\tilde{\nabla}_a X_b$ at a given point P, we can determine the second derivative of the Killing vector at P, and its successive derivatives by Eq. (1.97). Hence, all the derivatives of X_a in P are a linear combination of $X_a(P)$ and $\tilde{\nabla}_a X_b(P)$. So, we can write a generic Killing vector as

$$X_a(x) = A_a{}^b(x) X_b(P) + B_a{}^{bc}(x) \tilde{\nabla}_b X_c(P), \tag{1.98}$$

where $A_a{}^b$ and $B_a{}^{bc}$ are functions independent of the initial values $X_a(P)$ and $\tilde{\nabla}_a X_b(P)$. Because of that, they are the same for all Killing vectors. In this way, we proved that the Killing vectors are well established by Eq. (1.95).

Killing vectors form a vector space since the Killing equations (1.94) are linear in X. Moreover, they also form a Lie algebra, that is, a vector space closed to commutation. To prove this we need to use the Jacobi identity[2]

$$[[\mathbf{X}, \mathbf{Y}], \mathbf{Z}] + [[\mathbf{Z}, \mathbf{X}], \mathbf{Y}] + [[\mathbf{Y}, \mathbf{Z}], \mathbf{X}] = 0, \tag{1.99}$$

and to demonstrate the well-known property of the Lie derivative,

$$\mathfrak{L}_{[\mathbf{X},\mathbf{Y}]} = \mathfrak{L}_{\mathbf{X}} \mathfrak{L}_{\mathbf{Y}} - \mathfrak{L}_{\mathbf{Y}} \mathfrak{L}_{\mathbf{X}}. \tag{1.100}$$

First, we apply Eq. (1.100) to a generic function f,

$$(\mathfrak{L}_{\mathbf{X}} \mathfrak{L}_{\mathbf{Y}} - \mathfrak{L}_{\mathbf{Y}} \mathfrak{L}_{\mathbf{X}}) f = \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)) = [\mathbf{X}, \mathbf{Y}](f) = \mathfrak{L}_{[\mathbf{X},\mathbf{X}]} f, \tag{1.101}$$

and to a vector field $\mathbf{Z}$,

$$\left(\mathfrak{L}_{[\mathbf{X},\mathbf{Y}]} - \mathfrak{L}_{\mathbf{X}} \mathfrak{L}_{\mathbf{Y}} + \mathfrak{L}_{\mathbf{Y}} \mathfrak{L}_{\mathbf{X}}\right) \mathbf{Z} = [[\mathbf{X}, \mathbf{Y}], \mathbf{Z}] - [\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{X}, \mathbf{Z}]] = 0, \tag{1.102}$$

which is zero because of the Jacobi identity. Now, let us show that the r.h.s. of Eq. (1.100) satisfies the Leibniz rule,

$$\begin{aligned}
&(\mathfrak{L}_{\mathbf{X}} \mathfrak{L}_{\mathbf{Y}} - \mathfrak{L}_{\mathbf{Y}} \mathfrak{L}_{\mathbf{X}}) \left(\mathbb{T}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} \bar{\mathbb{T}}^{d_1 \cdots d_n}{}_{e_1 \cdots e_m}\right) = \\
&= \mathfrak{L}_{\mathbf{X}} \left[\left(\mathfrak{L}_{\mathbf{Y}} \mathbb{T}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m}\right) \bar{\mathbb{T}}^{d_1 \cdots d_n}{}_{e_1 \cdots e_m} + \mathbb{T}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} \left(\mathfrak{L}_{\mathbf{Y}} \bar{\mathbb{T}}^{d_1 \cdots d_n}{}_{e_1 \cdots e_m}\right)\right] + \\
&\quad - \mathfrak{L}_{\mathbf{Y}} \left[\left(\mathfrak{L}_{\mathbf{X}} \mathbb{T}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m}\right) \bar{\mathbb{T}}^{d_1 \cdots d_n}{}_{e_1 \cdots e_m} + \mathbb{T}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} \left(\mathfrak{L}_{\mathbf{X}} \bar{\mathbb{T}}^{d_1 \cdots d_n}{}_{e_1 \cdots e_m}\right)\right] = \\
&= \left[(\mathfrak{L}_{\mathbf{X}} \mathfrak{L}_{\mathbf{Y}} - \mathfrak{L}_{\mathbf{Y}} \mathfrak{L}_{\mathbf{X}}) \mathbb{T}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m}\right] \bar{\mathbb{T}}^{d_1 \cdots d_n}{}_{e_1 \cdots e_m} + \\
&\quad + \mathbb{T}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} \left[(\mathfrak{L}_{\mathbf{X}} \mathfrak{L}_{\mathbf{Y}} - \mathfrak{L}_{\mathbf{Y}} \mathfrak{L}_{\mathbf{X}}) \bar{T}^{d_1 \cdots d_n}{}_{e_1 \cdots e_m}\right].
\end{aligned} \tag{1.103}$$

[2] To prove Eq. (1.99), it is sufficient to apply the l.h.s. to an arbitrary function f.

The last thing we have left to prove is that Eq. (1.100) is true when it acts on a dual field w_a. In order to do this we consider the contraction of w_a with a vector field v^a, which forms a scalar,

$$0 = \left(\mathfrak{L}_{[\mathbf{X},\mathbf{Y}]} - \mathfrak{L}_{\mathbf{X}}\mathfrak{L}_{\mathbf{Y}} + \mathfrak{L}_{\mathbf{Y}}\mathfrak{L}_{\mathbf{X}}\right) w_a v^a = v^a \left(\mathfrak{L}_{[\mathbf{X},\mathbf{Y}]} - \mathfrak{L}_{\mathbf{X}}\mathfrak{L}_{\mathbf{Y}} + \mathfrak{L}_{\mathbf{Y}}\mathfrak{L}_{\mathbf{X}}\right) w_a. \tag{1.104}$$

Thus, Eq. (1.100) is true for any arbitrary tensor. Hence, we can show that the commutator of two Killing fields is a Killing field,

$$\mathfrak{L}_{[\mathbf{X},\mathbf{Y}]} g_{ab} = \left(\mathfrak{L}_{\mathbf{X}}\mathfrak{L}_{\mathbf{Y}} - \mathfrak{L}_{\mathbf{Y}}\mathfrak{L}_{\mathbf{X}}\right) g_{ab} = 0. \tag{1.105}$$

The fact that the Killing fields form a Lie algebra will be an important aspect in the analysis of homogeneous spaces (see Ch. 4).

Finally, Eq. (1.85) can be recast using both the definition of covariant derivative (1.8) and the connection decomposition (1.24),

$$\begin{aligned}\mathfrak{L}_{\mathbf{X}} \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} =& X^c \nabla_c \mathbb{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m} - \mathbb{T}^{c\cdots a_n}{}_{b_1\cdots b_m} \nabla_c X^{a_1} + \cdots + \mathbb{T}^{a_1\cdots a_n}{}_{c\cdots b_m} \nabla_{b_1} X^c + \\ &+ \cdots + \mathbb{T}_{cd}{}^{a_1} \mathbb{T}^{c\cdots a_n}{}_{b_1\cdots b_m} X^d + \cdots + \mathbb{T}_{cb_1}{}^{d} X^c \mathbb{T}^{a_1\cdots a_n}{}_{d\cdots b_m} + \cdots,\end{aligned} \tag{1.106}$$

providing a new geometric interpretation of the Lie derivative. Let $\mathbf{Y} = Y^a \partial_a$ a vector parallel transported along $\mathbf{X}$, and vice versa, the Lie derivative with respect to $\mathbf{X}$ is equal to

$$\mathfrak{L}_{\mathbf{X}} Y^a = X^c \nabla_c Y^a - Y^c \nabla_c X^a + T_{cd}{}^a Y^c X^d = T_{cd}{}^a Y^c X^d, \tag{1.107}$$

i.e., it measures the failure of the closure of the parallelogram given by the vector $T_{cd}{}^a Y^c X^d$.

2

General Relativity and metric-affine formulation

Making use of the geometric framework described above, in this chapter, we discuss how gravity is described in the context of General Relativity (GR), the first geometric theory of gravitation. Thereafter, we focus on theories that are modifications or extensions of GR itself.

2.1 General Relativity

General Relativity was formulated by Einstein in the early twentieth century, and it is based on what we refer to as Einstein's Equivalence Principle (EEP). It states that [2]:

- Weak Equivalence Principle (WEP) holds, that is "*if an uncharged test body is placed at an initial event*[1] *in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition*";
- *the outcome of any local nongravitational test experiment is independent of the velocity of the freely falling apparatus* (Local Lorentz Invariance);
- *the outcome of any local nongravitational test experiment is independent of where and when in the universe it is performed* (Local Position Invariance).

WEP means that in a local system of reference, it is impossible to distinguish the effects of a gravitational field from those due to uniformly accelerated frames just by using the observation of free-falling particles. On the other hand, by local nongravitational test experiment, we refer to an experiment performed in a free-falling laboratory small enough to avoid inhomogeneities and in which self-gravitational effects are negligible.

It is possible to demonstrate that if a gravitation theory satisfies the EEP, then the following postulates are valid (more details are given in [2]):

- *the spacetime is endowed with a metric tensor* g_{ab};

[1] We refer to "event" as an arbitrary point of spacetime.

- *the world lines of test bodies are geodesics of the metric tensor;*
- *there always exists a local free-falling frame, called local inertial frame, where the nongravitational laws of physics are those of special relativity.*

The last postulate implies that in a local inertial frame

$$g_{ab} = \eta_{ab}, \qquad \partial_c g_{ab} = 0, \tag{2.1}$$

where $\eta_{ab} \equiv (-1, 1, 1, 1)$ is the Minkowski metric.

Another cornerstone of GR is that equations preserve their form under an arbitrary coordinate transformation. This statement is called the Principle of General Covariance. The significance of this principle lies in the fact that, because of its general covariance, a physical equation will be true in the presence of a gravitational field if it is true in the absence of it. Thus, it relates to the postulate of the existence of the local inertial frame.

Now, let us show that the free-fall motion of a test particle is given by the geodesic equation [78]. In a local inertial frame, the equation of motion is that of a straight line,

$$\frac{\mathrm{d}^2 y^a}{\mathrm{d}s^2} = 0, \tag{2.2}$$

where

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}y^a\mathrm{d}y^b \tag{2.3}$$

is the line element of spacetime. Supposing the free-falling coordinates y^a are a function of an arbitrary coordinate system x^a, i.e, $y^a = y^a(x^b)$, we have

$$0 = \frac{\mathrm{d}^2 y^a}{\mathrm{d}s^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial y^a}{\partial x^b}\frac{\mathrm{d}x^b}{\mathrm{d}s}\right) = \frac{\partial y^a}{\partial x^b}\frac{\mathrm{d}^2 x^b}{\mathrm{d}s^2} + \frac{\partial^a y^a}{\partial x^b \partial x^c}\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s}. \tag{2.4}$$

Multiplying by $\partial x^d/\partial y^a$, Eq. (2.4) becomes

$$\frac{\mathrm{d}^2 x^d}{\mathrm{d}s^2} + \left(\frac{\partial x^d}{\partial y^a}\frac{\partial^2 y^a}{\partial x^b \partial x^c}\right)\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s} = 0. \tag{2.5}$$

It is simple to derive that the term in parentheses represents exactly the Levi-Civita connection (1.25). Recalling how the metric transforms under a change of coordinate system,

$$g_{ab} = \frac{\partial y^c}{\partial x^a}\frac{\partial y^d}{\partial x^b}\eta_{cd}, \tag{2.6}$$

and differentiating with respect to x^h, we obtain

$$\begin{aligned}
\frac{\partial g_{ab}}{\partial x^h} =& \frac{\partial^2 y^c}{\partial x^h \partial x^a}\frac{\partial y^d}{\partial x^b}\eta_{cd} + \frac{\partial y^c}{\partial x^a}\frac{\partial^2 y^d}{\partial x^h \partial x^b}\eta_{cd} = \\
=& \left(\frac{\partial^2 y^c}{\partial x^h \partial x^a}\frac{\partial x^p}{\partial y^c}\right)\frac{\partial y^q}{\partial x^p}\frac{\partial y^d}{\partial x^b}\eta_{qd} + \frac{\partial y^c}{\partial x^a}\frac{\partial y^q}{\partial x^p}\left(\frac{\partial^2 y^d}{\partial x^h \partial x^b}\frac{\partial x^p}{\partial y^d}\right)\eta_{cq} = \\
=& \left(\frac{\partial^2 y^c}{\partial x^h \partial x^a}\frac{\partial x^p}{\partial y^c}\right)g_{pb} + \left(\frac{\partial^2 y^d}{\partial x^h \partial x^b}\frac{\partial x^p}{\partial y^d}\right)g_{ap}.
\end{aligned} \tag{2.7}$$

From subsequent permutations of the indices,

$$\frac{\partial g_{bh}}{\partial x^a} = \left(\frac{\partial^2 y^c}{\partial x^a \partial x^b}\frac{\partial x^p}{\partial y^c}\right) g_{ph} + \left(\frac{\partial^2 y^d}{\partial x^a \partial x^h}\frac{\partial x^p}{\partial y^d}\right) g_{bp}, \tag{2.8}$$

$$\frac{\partial g_{ha}}{\partial x^b} = \left(\frac{\partial^2 y^c}{\partial x^h \partial x^b}\frac{\partial x^p}{\partial y^c}\right) g_{pa} + \left(\frac{\partial^2 y^d}{\partial x^b \partial x^a}\frac{\partial x^p}{\partial y^d}\right) g_{hp}, \tag{2.9}$$

subtracting Eq. (2.7) from the sum of Eq. (2.8) and Eq. (2.9), and multiplying by g^{eh}, finally we find

$$\left(\frac{\partial^2 y^c}{\partial x^b \partial x^b}\frac{\partial x^e}{\partial y^c}\right) = \frac{1}{2} g^{eh}\left(\partial_a g_{bh} + \partial_b g_{ah} - \partial_h g_{ab}\right), \tag{2.10}$$

that is

$$\tilde{\Gamma}_{ab}{}^{e} = \frac{\partial^2 y^c}{\partial x^a \partial x^b}\frac{\partial x^e}{\partial y^c}. \tag{2.11}$$

Therefore, Eq. (2.5) is equal to the geodesic equation (1.83),

$$\frac{\mathrm{d}^2 x^d}{\mathrm{d}s^2} + \tilde{\Gamma}_{bc}{}^{d}\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s} = 0. \tag{2.12}$$

The connection $\tilde{\Gamma}_{bc}{}^{d}$ represents the gravitational force applied on the test particle, or in the absence of a gravitational field the apparent force that appears when performing a transformation from an inertial to an arbitrary frame.

According to this result, in GR the connection of the spacetime is assumed to be the Levi-Civita one, without torsion and nonmetricity. Therefore, the metric is preserved by the covariant derivative,

$$\tilde{\nabla}_c g_{ab} = 0, \tag{2.13}$$

and the Riemann tensor has the following expression

$$\tilde{R}^a{}_{bcd} = \partial_c \tilde{\Gamma}_{db}{}^{a} - \partial_d \tilde{\Gamma}_{cb}{}^{a} + \tilde{\Gamma}_{ce}{}^{a}\tilde{\Gamma}_{db}{}^{e} - \tilde{\Gamma}_{de}{}^{a}\tilde{\Gamma}_{cb}{}^{e}, \tag{2.14}$$

with the properties:

- antisymmetry in the last two indices,
$$\tilde{R}_{abcd} = -\tilde{R}_{abdc}; \tag{2.15}$$
- antisymmetry in the first two indices,
$$\tilde{R}_{abcd} = -\tilde{R}_{bacd}; \tag{2.16}$$
- symmetry by exchanging the pair of the first two indices with the pair of the last two ones,
$$\tilde{R}_{abcd} = \tilde{R}_{cdab}; \tag{2.17}$$
- First Bianchi identity,
$$R^a{}_{[bcd]} = 0; \tag{2.18}$$
- Second Bianchi identity,
$$\nabla_{[e|}R^a{}_{b|cd]} = 0. \tag{2.19}$$

In addition, $R_{ab} = \bar{R}_{ab}$, and $\check{R}_{ab} = 0$. Through the contractions of the Riemann tensor we can construct the so-called Einstein tensor,

$$G_{ab} = \tilde{R}_{ab} - \frac{1}{2} g_{ab} \tilde{R}, \tag{2.20}$$

which satisfies the contracted Bianchi identity,

$$\tilde{\nabla}_a G^a{}_b = 0. \tag{2.21}$$

From Eq. (1.51), we have for two infinitely close geodesics the proportionality between the relative acceleration and the Riemann tensor,

$$a^a = \tilde{R}^a{}_{bcd} u^b u^c w^d, \tag{2.22}$$

which manifests the fact that in GR gravity is mediated by the curvature of spacetime.

2.1.1 Field Equations

One of the starting points followed by Einstein in describing GR through field equations was the comparison with the Poisson equation of Newtonian gravity,

$$\partial_a \partial^a \phi = 4\pi G_N \rho, \tag{2.23}$$

where ϕ is the gravitational potential, G_N is the Newtonian gravitational constant, and ρ is the mass density of matter (we set c=1) [78, 80]. In analogy with the Poisson equation, we are looking for a tensor that is linear in second-order derivatives of the metric, which is the gravitational field. However, the only tensors that satisfy this requirement are the Riemann tensor and its contraction. In addition, in Newtonian gravity, the acceleration of two near particles is proportional to $w^c \partial_c \partial^a \phi$, with w^d the separation vector of the particles, while we showed in Eq. (2.22) that in GR it is proportional to $\tilde{R}^a{}_{bcd} u^b u^d w^c$, with u^a the tangent vector of the geodesics, which correspond to the 4-velocity of the particles. Therefore, a correspondence between these two quantities can be proposed,

$$\tilde{R}^a{}_{bcd} u^b u^c \longleftrightarrow \partial_d \partial^a \phi. \tag{2.24}$$

Moreover, in GR the properties of matter distribution are described by the energy-momentum tensor Ψ_{ab}, and the conservation equations[2]

$$\tilde{\nabla}_a \Psi^{ab} = 0. \tag{2.25}$$

A correspondence with ρ can be given by

$$\Psi_{ab} u^a u^b \longleftrightarrow \rho. \tag{2.26}$$

Having in mind Eq. (2.23), Einstein was originally driven to postulate the following field equations

$$\tilde{R}_{ab} = 4\pi G_N \Psi_{ab}. \tag{2.27}$$

[2]Notice that the introduction of an energy-momentum tensor and Eq. (2.25) cannot be justified with the postulates of GR alone, but it results from the choice of having equations that express the conservation of the total energy [1].

However, due to Eq. (2.25), we obtain

$$\tilde{\nabla}_a \tilde{R}^{ab} = 0. \tag{2.28}$$

This equation represents a severe constraint since the identity (2.21) implies $\tilde{\nabla}_a \tilde{R} = 0$. Hence, the scalar $\tilde{R}$ would be constant everywhere. The most natural choice was to use the Einstein tensor to replace the Ricci tensor, in order to have the well-known Einstein equations,

$$G_{ab} = 8\pi G_N \Psi_{ab}. \tag{2.29}$$

The same equations can be derived by the Einstein-Hilbert action,

$$A_{E-H} = A_G + A_M = \frac{1}{16\pi G_N} \int \tilde{R} \sqrt{-g} \mathrm{d}^4x + \int \mathcal{L}_m(g_{ab}, \psi) \sqrt{-g} \mathrm{d}^4x, \tag{2.30}$$

where A_G and A_m are the gravitational and matter actions, respectively, $\sqrt{-g}\mathrm{d}^4x$ is the invariant volume, and $\mathcal{L}_m$ the matter Lagrangian density, with ψ representing the field that describe the ordinary matter. The variation with respect to the metric tensor yields

$$\begin{aligned}
\delta A_G =& \frac{1}{16\pi G_N} \int \left[\delta\sqrt{-g}\tilde{R} + \sqrt{-g}\delta g^{ab} \tilde{R}_{ab} + \sqrt{-g} g^{ab} \delta \tilde{R}_{ab} \right] \mathrm{d}^4x = \\
=& \frac{1}{16\pi G_N} \int \Bigg[\sqrt{-g}\delta g^{ab} \left(\tilde{R}_{ab} - \frac{1}{2} g_{ab} \tilde{R} \right) + \\
& + \sqrt{-g} \tilde{\nabla}_c \left(g^{ab} \delta \tilde{\Gamma}_{ba}{}^{c} - g^{cb} \delta \tilde{\Gamma}_{ab}{}^{a} \right) \Bigg] \mathrm{d}^4x = \\
=& \frac{1}{16\pi G_N} \int \Bigg[\sqrt{-g}\delta g^{ab} \left(\tilde{R}_{ab} - \frac{1}{2} g_{ab} \tilde{R} \right) + \\
& + \partial_c \left(\sqrt{-g} g^{ab} \delta \tilde{\Gamma}_{ba}{}^{c} - \sqrt{-g} g^{cb} \delta \tilde{\Gamma}_{ab}{}^{a} \right) \Bigg] \mathrm{d}^4x,
\end{aligned} \tag{2.31}$$

where we used the relation

$$\delta \tilde{R}_{ab} = \tilde{\nabla}_c \left(\delta \tilde{\Gamma}_{ba}{}^{c} - \delta^c_a \tilde{\Gamma}_{db}{}^{d} \right), \tag{2.32}$$

and that from Eq. (1.12), considering the Levi-Civita covariant derivative of a vector density, we have

$$\tilde{\nabla}_c \left(\sqrt{-g} v^c \right) = \partial_c \left(\sqrt{-g} v^c \right) + \tilde{\Gamma}_{cd}{}^{c} \left(\sqrt{-g} v^c \right) - \tilde{\Gamma}_{dc}{}^{c} \left(\sqrt{-g} v^c \right) = \partial_c \left(\sqrt{-g} v^c \right). \tag{2.33}$$

In the last line of Eq. (2.31), the second term is a total derivative, and, by the divergence theorem, it results in a boundary term. Usually, this boundary term is nonzero since it depends on δg^{ab} and its first derivative $\partial_c \delta g^{ab}$, which in general does not vanish on the boundary [88, 89]. To avoid this problem, it is necessary to add to the Einstein-Hilbert action an additional term that cancels the boundary one out[3]. However, we assume that the boundary term is null, and thus we obtain the Einstein equations[4],

$$\tilde{R}_{ab} - \frac{1}{2} g_{ab} \tilde{R} = 8\pi G_N \Psi_{ab}, \tag{2.34}$$

with the energy-momentum tensor

$$\Psi_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} \mathcal{L}_m \right)}{\delta g^{ab}}. \tag{2.35}$$

[3] An explicit evaluation is given in Appendix A.

[4] According to Lovelock's theorem [16, 90], the only field equations of second order that can be derived by an action involving the metric tensor and its first and second derivatives are represented by Eq. (2.34).

2.2 Alternative and extended theories of gravity

Despite the great successes of Einstein's theory, some shortcomings of GR have emerged over the years. For example, in cosmology, no known matter source can generate the accelerated expansion phase that we measure. The currently most widely accepted model for cosmology, the ΛCDM model, assumes that the accelerated expansion is due to the cosmological constant Λ. However, the observed value of Λ disagrees with the theoretical prediction. This result led the community to consider more general fluids, generally known as dark energy, with the same key property as Λ, that is, negative pressure. At the quantum level, the main conceptual problem is that in GR the metric is the field describing both the dynamics of gravity and the spacetime background, while quantum theories are formulated on a fixed background.

Alternative and extended theories of gravity have been developed [16–18] to unify gravity with the other three fundamental forces and/or to overcome the open questions left by GR. They can be characterized by extra tensor fields that mediate the gravity besides metric tensor (e.g., scalar-tensor theories [21, 22], bimetric theories [26–29], Einstein-Aether theory [91, 92] and so on), as well as extra spatial/temporal dimensions of spacetime (e.g, Kaluza-Klein theory [39, 40], Einstein-Gauss-Bonnet gravity [93], DGP gravity [43] and so on) or higher-order derivative field equations (e.g., $f(R)$ theories [34, 35], Hořava-Lifschitz gravity [33, 94] and so on). Theories and approaches involving torsion and nonmetricity may also be considered, as we will see below. In this section, we will analyze $f(R)$ theories, which will be a guideline to introduce the $f(\mathcal{Q})$ gravity, the leading theory of this thesis.

2.2.1 $f(R)$ theories

The $f(R)$ theories [34, 35] are based on the generalization of the Einstein-Hilbert action considering a generic function of the Ricci scalar $\tilde{R}$,

$$A = \frac{1}{16\pi G_N}\int f(\tilde{R})\sqrt{-g}\mathrm{d}^4x + \int \mathcal{L}_m(g_{ab},\psi)\sqrt{-g}\mathrm{d}^4x \tag{2.36}$$

The variation with respect to the metric tensor of the matter Lagrangian is given by Eq. (2.35), whereas from the variation of the gravitational one we obtain

$$\begin{aligned}\delta\int f\sqrt{-g}\mathrm{d}^4x &= \int\left[\delta\left(\sqrt{-g}\right)f + \sqrt{-g}\delta f\right]\mathrm{d}^4x = \\ &= \int\left[-\frac{1}{2}g_{ab}\delta g^{ab} f + f'\tilde{R}_{ab}\delta g^{ab} + f' g^{ab}\delta\tilde{R}_{ab}\right]\sqrt{-g}\mathrm{d}^4x\end{aligned} \tag{2.37}$$

where $f' = \mathrm{d}f/\mathrm{d}\tilde{R}$, that is, it denotes the first derivative of $f(R)$ with respect to the argument. Let us focus on the variation of the Ricci tensor. Using the relation (2.32) and

$$\delta\tilde{\Gamma}_{ab}{}^{c} = \frac{1}{2}g^{cd}\left[2\tilde{\nabla}_{(a}\delta g_{b)d} - \tilde{\nabla}_d\delta g_{ab}\right], \tag{2.38}$$

$$\delta\tilde{\Gamma}_{ab}{}^{a} = \frac{1}{2}g^{ac}\tilde{\nabla}_b\delta g_{ac}, \tag{2.39}$$

we find

$$\begin{aligned}\int f' g^{ab}\delta\tilde{R}_{ab}\sqrt{-g}\mathrm{d}^4x &= \int f' g^{ab}\nabla_c\left(\delta\tilde{\Gamma}_{ba}{}^{c} - \delta^c_a\tilde{\Gamma}_{db}{}^{d}\right)\sqrt{-g}\mathrm{d}^4x = \\ &= -\int \frac{1}{2}g^{ab}g^{pc}\tilde{\nabla}_c f'\left(2\tilde{\nabla}_a\delta g_{bp} - \tilde{\nabla}_p\delta g_{ab} - g_{ap}g^{dq}\tilde{\nabla}_b\delta g_{dq}\right)\sqrt{-g}\mathrm{d}^4x = \\ &= \int\left(-\tilde{\nabla}_a\tilde{\nabla}_b f' + g_{ab}\tilde{\nabla}_c\tilde{\nabla}^c f'\right)\delta g^{ab}\sqrt{-g}\mathrm{d}^4x.\end{aligned} \tag{2.40}$$

Therefore, the field equations due to the variation with respect to the metric are equal to

$$f'\tilde{R}_{ab} - \frac{1}{2}g_{ab}f - \tilde{\nabla}_a\tilde{\nabla}_b f' + g_{ab}\tilde{\nabla}_c\tilde{\nabla}^c f' = 8\pi G_N \Psi_{ab}, \tag{2.41}$$

which are fourth-order partial differential equations reducing to Einstein equations for $f(\tilde{R}) = \tilde{R}$. As in the case of the Einstein-Hilbert action, we have neglected the boundary term in the variation (2.37). More details on boundary terms in $f(R)$ theories can be found, for example, in [95].

From the trace of Eq. (2.41),

$$f'\tilde{R} - 2f + 3\tilde{\nabla}_a\tilde{\nabla}^a f' = 8\pi G_N \Psi, \tag{2.42}$$

with $\Psi = g^{ab}\Psi_{ab}$, we notice that in $f(R)$ theories the Ricci scalar and the trace of the energy-momentum tensor are related by a differential equation and not an algebraic one, which is a remarkable difference with respect to GR.

The energy-momentum conservation,

$$\tilde{\nabla}^a \Psi_{ab} = 0, \tag{2.43}$$

can be proven by using the identity

$$\left[\tilde{\nabla}_b\tilde{\nabla}_c\tilde{\nabla}^c - \tilde{\nabla}_c\tilde{\nabla}^c\tilde{\nabla}_b\right]\varphi = -R_{ab}\tilde{\nabla}^a\varphi, \tag{2.44}$$

with φ an arbitrary function, and verifying that the contraction of the covariant derivative with the l.h.s. of Eq. (2.41) is zero, i.e.,

$$\tilde{\nabla}^a\left[f'R_{ab} - \frac{1}{2}g_{ab}f - \tilde{\nabla}_a\tilde{\nabla}_b f' + g_{ab}\tilde{\nabla}_c\tilde{\nabla}^c f'\right] = 0. \tag{2.45}$$

Finally, we show that $f(R)$ gravity is mathematically equivalent to a scalar-tensor theory, in which, besides the metric tensor, we have a scalar field describing the gravity. In particular, it is equivalent to the Brans-Dicke theory, one of the prototypes of the extended theories of gravity. We prove this equivalence for completeness in our description of $f(R)$ theories, but it is not our interest to give an exhaustive description of scalar-tensor theories, a detailed discussion of which can be found in [19–25]. By introducing a scalar field χ, we can rewrite Eq. (2.36) in the following equivalent form,

$$A = \frac{1}{16\pi G_N}\int\left[f(\chi) + f'(\chi)(R-\chi)\right]\sqrt{-g}\mathrm{d}^4x + \int \mathcal{L}_m\sqrt{-g}\mathrm{d}^4x. \tag{2.46}$$

Variation with respect to χ gives

$$f''(\chi)(R-\chi) = 0, \tag{2.47}$$

which yields $R = \chi$ for $f''(R) \neq 0$. Now, defining a new field $\phi = f'(\chi)$ and the potential

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)), \tag{2.48}$$

Eq. (2.46) is recast as follows

$$A = \frac{1}{16\pi G_N}\int [\phi R - V(\phi)]\sqrt{-g}\mathrm{d}^4x + \int \mathcal{L}_m\sqrt{-g}\mathrm{d}^4x, \tag{2.49}$$

which corresponds to the Brans-Dicke action, i.e.,

$$A = \frac{1}{16\pi G_N}\int \left[\phi\tilde{R} - \frac{\omega}{\phi}g^{ab}\tilde{\nabla}_a\phi\tilde{\nabla}_b\phi - V(\phi)\right]\sqrt{-g}\mathrm{d}^4x + \int \mathcal{L}_m\sqrt{-g}\mathrm{d}^4x, \tag{2.50}$$

with parameter $\omega = 0$. In this representation, it is evident that $f(R)$ theories have just one extra scalar degree of freedom with respect to GR.

2.3 Metric-affine formalism

In this section, we introduce the metric-affine formalism, where metric and connection are two independent dynamical variables. The consequence is that the connection is characterized by torsion and nonmetricity as well. Furthermore, the matter Lagrangian density can be connection dependent, $\mathcal{L}_m(g_{ab}, \Gamma_{ab}{}^c, \psi)$. Consequently, we have to define the variation of $\mathcal{L}_m$ with respect to the connection,

$$\Delta^{ab}{}_c = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta\Gamma_{ab}{}^c}, \tag{2.51}$$

which is called hypermomentum tensor [47]. Henceforth, the metric-affine approach will be used throughout this thesis. As an introductory example, we will review the case of metric-affine $f(R)$ theories.

2.3.1 $f(R)$ theories in metric-affine formalism

In the metric-affine formalism, the Riemann tensor is expressed as a function of the full connection by Eq. (1.43) and it is independent of the metric tensor. Hence, $f(R)$ is a generic function of the full connection Ricci scalar [47] and the action is equal to

$$A = \frac{1}{16\pi G_N}\int f(R)\sqrt{-g}\mathrm{d}^4x + \int \mathcal{L}_m(g_{ab}, \Gamma_{ab}{}^c, \psi)\sqrt{-g}\mathrm{d}^4x. \tag{2.52}$$

The total variation of the gravitational action yields

$$
\begin{aligned}
\delta \int f \sqrt{-g} \mathrm{d}^4 x = & \int \left(f' R_{ab} - \frac{1}{2} f g_{ab} \right) \delta g^{ab} \sqrt{-g} \mathrm{d}^4 x + \\
& + \int \left\{ f' \left[g^{ab} \left(\nabla_c \delta_\Gamma \Gamma_{ba}{}^c - \nabla_b \delta_\Gamma \Gamma_{ca}{}^c \right) + g^{ab} T_{cb}{}^d \delta_\Gamma \Gamma_{da}{}^c \right] \right\} \sqrt{-g} \mathrm{d}^4 x = \\
= & \int \left(f' R_{ab} - \frac{1}{2} f g_{ab} \right) \delta g^{ab} \sqrt{-g} \mathrm{d}^4 x + \\
& + \int \left[\nabla_d \left(\sqrt{-g} f' g^{bd} \right) \delta^a_c - \nabla_c \left(\sqrt{-g} f' g^{ab} \right) + \right. \\
& \left. + f' \sqrt{-g} \left(g^{ab} T_{dc}{}^d - g^{bp} T_{dp}{}^d \delta^a_c + g^{bd} T_{cd}{}^a \right) \right] \delta_\Gamma \Gamma_{ab}{}^c \mathrm{d}^4 x + \\
& + \int \partial_c \left(\sqrt{-g} f' g^{ab} \delta_\Gamma \Gamma_{ba}{}^c - \sqrt{-g} f' g^{bc} \delta_\Gamma \Gamma_{ab}{}^a \right) \mathrm{d}^4 x .
\end{aligned}
\tag{2.53}
$$

We used the relation

$$
\delta_\Gamma R_{ab} = \nabla_c \delta_\Gamma \Gamma_{ba}{}^c - \nabla_b \delta_\Gamma \Gamma_{ca}{}^c + T_{cb}{}^d \delta_\Gamma \Gamma_{da}{}^c . \tag{2.54}
$$

It is worth noticing that there is no variation of the Ricci tensor with respect to the metric. The last line in Eq. (2.53) is a boundary term that we can neglect without further assumptions since in the metric-affine approach, it depends only on linear terms of $\delta\Gamma_{ab}{}^c$, which is a completely different scenario from those studied above. Then we obtain

$$
f' R_{(ab)} - \frac{1}{2} f g_{ab} = 8\pi G_N \Psi_{ab}, \tag{2.55}
$$

and

$$
\begin{aligned}
& \frac{1}{\sqrt{-g}} \left[\nabla_d \left(\sqrt{-g} f' g^{bd} \right) \delta^a_c - \nabla_c \left(\sqrt{-g} f' g^{ab} \right) \right] + \\
& + f' \sqrt{-g} \left(g^{ab} T_{dc}{}^d - g^{bp} T_{dp}{}^d \delta^a_c + g^{bd} T_{cd}{}^a \right) = 8\pi G_N \Delta^{ab}{}_c .
\end{aligned}
\tag{2.56}
$$

Contracting the indices b and c, we have

$$
\Delta^{ab}{}_b = 0, \tag{2.57}
$$

that is, the hypermomentum associated with the matter Lagrangian density is traceless. However, Eq. (2.57) is too stringent as a constraint to be a property of any form of matter, so we have an inconsistency in the field equations.

The reason for this inconsistency is found in projective invariance, which is a symmetry related to connection transformations that preserve the autoparallelism of a curve up to a reparametrization, thus leaving the paths followed by the test particles unchanged. Moreover, the Ricci scalar is also preserved under such transformations [34, 96, 97]. Let us consider the projective transformation

$$
\Gamma_{ab}{}^c \longrightarrow \Gamma_{ab}{}^c + \delta^c_b \xi_a, \tag{2.58}
$$

with ξ_b an arbitrary vector field. Under this transformation, the Ricci tensor transforms as

$$
R_{ab} \longrightarrow R_{ab} + 2\partial_{[a} \xi_{b]} . \tag{2.59}
$$

Therefore, the Ricci scalar is invariant under projective transformations since the metric is a symmetric tensor [5]. On the other hand, the matter Lagrangian is in general not invariant under projective transformations, so we have inconsistency in the action. Hence, we must break the invariance to have the correct field equations. There are several possibilities to achieve our purpose, such as choosing a non-symmetric metric or modifying the action by adding extra terms. The path we follow includes constraints on the connection that fix the four extra degrees of freedom due to the projective invariance. A first choice was made in [98], in which the vector component q_a, see Eq. (1.17), of the nonmetricity, was fixed by adding to the action the term

$$\int \lambda^a q_a \sqrt{-g}\mathrm{d}^4x, \tag{2.60}$$

where λ^a is a Lagrange multiplier. However, this proposal has no other solution than a linear function $f(R)$. The same result is obtained by choosing Q_a instead of q_a [99]. On the other hand, a consistent system of field equations is obtained by fixing the vector component T_a of the torsion [47]. The variations with respect to the metric and the connection are

$$\delta \int \lambda^a T_a \sqrt{-g}\mathrm{d}^4x = \int \left[\lambda_a T_b - \frac{1}{2}g_{ab}\lambda^c T_c\right] \delta g^{ab} \sqrt{-g}\mathrm{d}^4x, \tag{2.61}$$

and

$$\delta_\Gamma \int \lambda^a T_a \sqrt{-g}\mathrm{d}^4x = \int \left(\lambda^a \delta^b_c - \lambda^b \delta^a_c\right) \delta\Gamma_{ab}{}^c \sqrt{-g}\mathrm{d}^4x, \tag{2.62}$$

respectively, whereas the field equations are

$$f' R_{(ab)} - \frac{1}{2} f g_{ab} = 8\pi G_N \left(\Psi_{ab} - \lambda_{(a}T_{b)} + \frac{1}{2}g_{ab}\lambda^c T_c\right), \tag{2.63}$$

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\left[\nabla_d\left(\sqrt{-g}f' g^{bd}\right)\delta^a_c - \nabla_c\left(\sqrt{-g}f' g^{ab}\right)\right] + \\ &+ f'\sqrt{-g}\left(g^{ab}T_{dc}{}^d - g^{bp}T_{dp}{}^d\delta^a_c + g^{bd}T_{cd}{}^a\right) = 8\pi G_N\left(\Delta^{ab}{}_c - \lambda^a\delta^b_c + \lambda^b\delta^a_c\right), \end{aligned} \tag{2.64}$$

$$T_a = 0. \tag{2.65}$$

The contraction of indices b and c in Eq. (2.64) gives the value of the Lagrange multipliers so that the field equations are consistent,

$$\lambda^a = \frac{1}{3}\Delta^{ab}{}_b. \tag{2.66}$$

Furthermore, from the antisymmetric part in the upper indices of Eq. (2.64), we derive $T_{ab}{}^c = 0$ if $\Delta^{[ab]}{}_c = 0$. Similarly, the symmetric part provides $Q_{abc} = 0$ assuming $\Delta^{(ab)}{}_c = 0$. Both results imply that torsion and nonmetricity are connected to the antisymmetric and symmetric parts of the hypermomentum, respectively.

[5] Projective invariance implies that the metric-affine approach is subject to additional gauge freedom, due to considering the connection as an independent variable, which is not present in the metric one.

3 Symmetric Teleparallel Gravity and extensions

The first appearance of nonmetricity in theories of gravitation is due to Weyl in his attempt to unify gravity and electromagnetism [49]. However, nonmetric theories came into the limelight only in 1999 when the so-called Symmetric Teleparallel Gravity (STG) [64–66, 100] was proposed. In this theory, gravitation is strictly connected to the nonmetricity tensor and the related nonmetricity scalar $\mathcal{Q}$, while both curvature and torsion are set to zero. Among them, a theory that recently has gained great attention is the $f(\mathcal{Q})$ gravity [69], where the action of the gravitational field is described by a generic function of the nonmetricity scalar. In this chapter, we will focus on the description of the Symmetric Teleparallel Equivalent of General Relativity (STEGR), a STG theory whose metric field equations are equivalent to Einstein's, and its extensions, which are the main theories of this thesis.

3.1 Symmetric Teleparallel Equivalent of General Relativity

As mentioned above, in STEGR both curvature and torsion are zero, thus the nonmetricity satisfies the identity

$$\nabla_{[d}Q_{c]ab} = 0. \tag{3.1}$$

The action is equal to

$$A = \int \left[-\frac{1}{2}\sqrt{-g}\mathcal{Q} + \lambda_a{}^{bcd}R^a{}_{bcd} + \lambda_c{}^{ab}T_{ab}{}^c + \sqrt{-g}\mathcal{L}_m(g_{ab}, \Gamma_{ab}{}^c, \psi) \right] \mathrm{d}^4x, \tag{3.2}$$

where $\mathcal{Q}$ is the nonmetricity scalar (1.59),

$$\mathcal{Q} = \frac{1}{4}Q_{cab}Q^{cab} - \frac{1}{2}Q_{cab}Q^{abc} - \frac{1}{4}q_a q^a + \frac{1}{2}q_a Q^a, \tag{3.3}$$

while $\lambda_a{}^{bcd}$ and $\lambda_c{}^{ab}$ are Lagrange multipliers introduced to impose the vanishing of curvature and torsion. Henceforth, we set $c = 8\pi G_N = 1$.

Writing the Ricci scalar as in Eq. (1.58), that is,

$$\tilde{R} = -\mathcal{Q} - \tilde{\nabla}_a \left(q^a - Q^a\right), \tag{3.4}$$

allows us to highlight that the Lagrangian of STEGR and GR one are equivalent except for a total divergence term. Thereafter, we will prove that the metric field equations of the STEGR can be recast as Einstein equations as well.

The variation with respect to the Lagrangian multipliers gives the constraints

$$R^a{}_{bcd} = 0 \qquad \text{and} \qquad T_{ab}{}^c = 0, \tag{3.5}$$

while the one with respect to the metric g^{ab} yields

$$\delta A = \int \left[-\frac{1}{2}\sqrt{-g}\delta\mathcal{Q} + \frac{1}{4} g_{ab}\sqrt{-g}\mathcal{Q}\delta g^{ab} + \delta\left(\sqrt{-g}\mathcal{L}_m\right)\right] \mathrm{d}^4x. \tag{3.6}$$

Let us focus on the variation of the nonmetricity scalar. Foremost, we evaluate the variation of the nonmetricity tensor with respect to the metric,

$$\delta Q_{cab} = \nabla_c \delta g_{ab} = -\nabla_c\left(g_{ai}g_{bj}\delta g^{ij}\right) = -Q_{cai}g_{bj}\delta g^{ij} - Q_{cbj}g_{ai}\delta g^{ij} - g_{ai}g_{bj}\nabla_c\delta g^{ij}, \tag{3.7}$$

then we rewrite the nonmetricity scalar as follows

$$\mathcal{Q} = g^{ch}g^{ai}g^{bj}\left(\frac{1}{4}Q_{hij}Q_{cab} - \frac{1}{2}Q_{hij}Q_{(ab)c} - \frac{1}{4}Q_{hai}Q_{cbj} + \frac{1}{2}Q_{hai}Q_{bcj}\right). \tag{3.8}$$

Therefore, we have

$$\begin{aligned}
\delta\mathcal{Q} = &\left[\frac{1}{4}Q_{acd}Q_b{}^{cd} - \frac{1}{2}Q_{cda}Q^{cd}{}_b - \frac{1}{4}q_a q_b - \frac{1}{2}q^c Q_{cab} + \frac{1}{2}Q_{cab}q^c - \frac{1}{2}Q_{cab}Q^c\right]\delta g^{ab} + \\
&+ \left[-\frac{1}{2}Q^c{}_{ab} + Q_{ab}{}^c + \frac{1}{2}q^c g_{ab} - \frac{1}{2}Q^c g_{ab} - \frac{1}{2}q_a\delta^c_b\right]\nabla_c\delta g^{ab} = \\
= &\left[\frac{1}{4}Q_{acd}Q_b{}^{cd} - \frac{1}{2}Q_{cda}Q^{cd}{}_b + \frac{1}{2}Q_{cda}Q^{dc}{}_b + \frac{1}{2}Q_{cab}q^c - \frac{1}{4}q_a q_a - \frac{1}{2}Q^c Q_{cab}\right]\delta g^{ab} + \\
&+ 2P^c{}_{ab}\nabla_c\delta g^{ab}.
\end{aligned} \tag{3.9}$$

where we used the relation

$$\delta g_{cd} = -g_{ac}g_{bd}\delta g^{ab}. \tag{3.10}$$

In the last line of Eq. (3.9), we have defined the nonmetricity conjugate tensor

$$P^c{}_{ab} = -\frac{1}{4}Q^c{}_{ab} + \frac{1}{2}Q_{(ab)}{}^c + \frac{1}{4}q^c g_{ab} - \frac{1}{4}Q^c g_{ab} - \frac{1}{4}q_{(a}\delta^c_{b)}, \tag{3.11}$$

which is symmetric in the last two indices,

$$P^c{}_{ab} = P^c{}_{ba}, \tag{3.12}$$

and its contraction with Q_{cab} gives the opposite of nonmetricity scalar,

$$\mathcal{Q} = -Q_{cab}P^{cab}. \tag{3.13}$$

Moreover, the terms in brackets can be recast as follows

$$\begin{aligned}\frac{1}{4}Q_{acd}Q_b{}^{cd} - \frac{1}{2}Q_{cda}Q^{cd}{}_b + \frac{1}{2}Q_{cda}Q^{dc}{}_b + \\ +\frac{1}{2}Q_{cab}q^c - \frac{1}{4}q_a q_a - \frac{1}{2}Q^c Q_{cab} = 2Q^{cd}{}_a P_{cdb} - P_{acd}Q_b{}^{cd}.\end{aligned} \tag{3.14}$$

Returning to Eq. (3.6), we find

$$\begin{aligned}\delta A =& \int \left[-\frac{1}{2}\sqrt{-g}\left(2P^c{}_{ab}\nabla_c \delta g^{ab} + 2Q^{cd}{}_a P_{cdb}\delta g^{ab} - P_{acd}Q_b{}^{cd}\delta g^{ab} \right) + \right. \\ &\left. + \frac{1}{4}g_{ab}\sqrt{-g}\mathcal{Q}\delta g^{ab} + \delta\left(\sqrt{-g}\mathcal{L}_m\right) \right] \mathrm{d}^4x = \\ =& \int \left\{ \frac{1}{2}\sqrt{-g}\left[\frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}P^c{}_{ab}\right) - 2Q^{cd}{}_a P_{cdb} + P_{acd}Q_b{}^{cd} + \frac{1}{2}g_{ab}\mathcal{Q} \right]\delta g^{ab} + \right. \\ &\left. - \nabla_c\left(\sqrt{-g}P^c{}_{ab}\delta g^{ab}\right) + \delta\left(\sqrt{-g}\mathcal{L}_m\right) \right\} \mathrm{d}^4x.\end{aligned} \tag{3.15}$$

The term

$$\nabla_c\left(\sqrt{-g}P^c{}_{ab}\delta g^{ab}\right) \tag{3.16}$$

is a covariant derivative of a tensor density due to the presence of $\sqrt{-g}$ and for this reason it is a boundary term linear in δg^{ab},

$$\begin{aligned}\nabla_h\left(\sqrt{-g}P^c{}_{ab}\delta g^{ab}\right) =& \tilde{\nabla}_c\left(\sqrt{-g}P^c{}_{ab}\delta g^{ab}\right) + \\ &+ L_{dc}{}^d\sqrt{-g}P^c{}_{ab}\delta g^{ab} - L_{cd}{}^d\sqrt{-g}P^c{}_{ab}\delta g^{ab} = \\ =& \tilde{\nabla}_h\left(\sqrt{-g}P^c{}_{ab}\delta g^{ab}\right) = \partial_c\left(\sqrt{-g}P^c{}_{ab}\delta g^{ab}\right),\end{aligned} \tag{3.17}$$

thus we neglect it. We finally find the first set of field equations

$$\frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}P^c{}_{ab}\right) + \frac{1}{2}g_{ab}\mathcal{Q} + P_{acd}Q_b{}^{cd} - 2Q^{cd}{}_a P_{cdb} = \Psi_{ab}, \tag{3.18}$$

with

$$\Psi_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{ab}}. \tag{3.19}$$

Now, let us vary the action with respect to the connection $\Gamma_{ab}{}^c$,

$$\delta_\Gamma A = \int \left[-\frac{1}{2}\sqrt{-g}\delta_\Gamma\mathcal{Q} + \lambda_a{}^{bcd}\delta_\Gamma R^a{}_{bcd} + \lambda_c{}^{ab}\delta_\Gamma T_{ab}{}^c + \delta_\Gamma\left(\sqrt{-g}\mathcal{L}_m\right) \right] \mathrm{d}^4x \tag{3.20}$$

The variation of the nonmetricity scalar gives

$$\begin{aligned}\delta_\Gamma\mathcal{Q} =& -2P^{cab}\delta_\Gamma Q_{cab} = -2P^{cab}\delta_\Gamma\left(\partial_c g_{ab} - \Gamma_{ca}{}^p g_{pb} - \Gamma_{cb}{}^p g_{ap}\right) = \\ =& 2P^{cab}g_{pb}\delta_\Gamma\Gamma_{ca}{}^p + 2P^{cab}g_{ap}\delta_\Gamma\Gamma_{cb}{}^p = 4P^{ab}{}_c\delta_\Gamma\Gamma_{ab}{}^c.\end{aligned} \tag{3.21}$$

On the other hand, the variations of the Riemann and torsion tensors are equal to,

$$\begin{aligned}\lambda_a{}^{bcd}\delta_\Gamma R^a{}_{bcd} = &\left(\nabla_c\delta_\Gamma\Gamma_{db}{}^a - \nabla_d\delta_\Gamma\Gamma_{cb}{}^a\right)\lambda_a{}^{bcd} = \\ =&\nabla_c\left(\lambda_a{}^{bcd}\delta_\Gamma\Gamma_{db}{}^a\right) - \nabla_c\left(\lambda_a{}^{bcd}\right)\delta_\Gamma\Gamma_{db}{}^a - \nabla_d\left(\lambda_a{}^{bcd}\delta_\Gamma\Gamma_{cb}{}^a\right) + \\ &+ \nabla_d\left(\lambda_a{}^{bcd}\right)\delta_\Gamma\Gamma_{ib}{}^a = \\ =&\tilde{\nabla}_c\left(\lambda_a{}^{bcd}\delta_\Gamma\Gamma_{db}{}^a\right) + N_{cp}{}^c\lambda_a{}^{bpd}\delta_\Gamma\Gamma_{db}{}^a - \tilde{\nabla}_d\left(\lambda_a{}^{bcd}\delta_\Gamma\Gamma_{cb}{}^a\right) + \\ &- N_{dp}{}^d\lambda_a{}^{bcp}\delta_\Gamma\Gamma_{cb}{}^a - N_{cp}{}^c\lambda_a{}^{bpd}\delta_\Gamma\Gamma_{db}{}^a + N_{dp}{}^d\lambda_a{}^{bcp}\delta_\Gamma\Gamma_{cb}{}^a + \\ &- \nabla_c\left(\lambda_a{}^{bcd}\right)\delta_\Gamma\Gamma_{db}{}^a + \nabla_d\left(\lambda_a{}^{bcd}\right)\delta_\Gamma\Gamma_{cb}{}^a = \\ =&\tilde{\nabla}_c\left(\lambda_a{}^{bcd}\delta_\Gamma\Gamma_{db}{}^a\right) - \tilde{\nabla}_d\left(\lambda_a{}^{bcd}\delta_\Gamma\Gamma_{cb}{}^a\right) + 2\nabla_p\left(\lambda_h{}^{dcp}\right)\delta_\Gamma\Gamma_{cd}{}^h,\end{aligned} \tag{3.22}$$

and

$$\lambda_c{}^{ab}\delta_\Gamma T_{ab}{}^c = \lambda_c{}^{ab}\delta_\Gamma\left(\Gamma_{ab}{}^c - \Gamma_{ba}{}^c\right) = 2\lambda_c{}^{ab}\delta_\Gamma\Gamma_{ab}{}^c, \tag{3.23}$$

respectively. We used the fact that the Lagrangian multipliers are density tensors and antisymmetric in the last two indices. In the first line of Eq. (3.22), we omitted the torsion because of the constraints (3.5), whereas in the last line, the first two terms are boundary ones. Hence, the field equations derived by varying with respect to the connection are

$$\nabla_d\lambda_c{}^{bad} + \lambda_c{}^{ab} - \sqrt{-g}P^{ab}{}_c = \Delta^{ab}{}_c, \tag{3.24}$$

where

$$\Delta^{ab}{}_c = -\frac{1}{2}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta\Gamma_{ab}{}^c} \tag{3.25}$$

is the hypermomentum tensor. Due to the flatness and torsionless conditions, from Eq. (3.24) we can also derive the relation

$$\nabla_a\nabla_b\left(\sqrt{-g}P^{ab}{}_c + \Delta^{ab}{}_c\right) = 0. \tag{3.26}$$

Let us take a more detailed look at the consequences of the constraints (3.5). The flatness condition implies that the connection can be parameterized by general elements $\Lambda^a{}_b$ of $\mathbf{GL}(4,\mathbb{R})$,

$$\Gamma_{ab}{}^c = (\Lambda^{-1})^c{}_d\partial_a\Lambda^d{}_b. \tag{3.27}$$

In addition, from the vanishing of torsion, we have

$$(\Lambda^{-1})^c{}_d\partial_{[a}\Lambda^d{}_{b]} = 0, \tag{3.28}$$

and arbitrary coordinates $\xi^a = \xi(x^a)$ can be introduced, such that

$$\Gamma_{ab}{}^c = \frac{\partial x^c}{\partial\xi^d}\partial_a\partial_b\xi^d. \tag{3.29}$$

Comparison with Eq. (1.7) makes manifest that the connection in Eq. (3.29) is purely inertial since it differs from the trivial one by a generic coordinate change. Therefore, in STEGR the connection vanishes using an appropriate diffeomorphism. The gauge in which the connection is null is called coincident gauge. In this gauge, the nonmetricity tensor is just the partial derivative of the metric,

$$Q_{cab} = \partial_c g_{ab}, \tag{3.30}$$

being the covariant derivative equal to the partial one, and the Levi-Civita connection is the opposite of the disformation tensor,

$$\tilde{\Gamma}_{ab}{}^{c} = -L_{ab}{}^{c}. \tag{3.31}$$

The possibility to eliminate inertial effects by canceling the connection and the resulting great computational simplifications are some properties that make the theory fascinating. However, we will see that the coincident gauge is not the most viable choice in any situation (see Ch. 8).

It is worth noticing that from Eq. (3.26) we can fix the four degrees of freedom given by the four functions ξ^a that determine the connection Γ via Eq. (3.29). Once the connection is chosen, via choice of a gauge or Eq. (3.29), the only use of the equations (3.24) is to derive the Lagrange multipliers. However, the metric field equations (3.18) are independent of the multipliers, so their determination is irrelevant in the STEGR[1], which would not be in the case of TEGR [102, 103].

Now, we derive the energy-momentum conservation in STEGR. We start by raising an index in Eq. (3.18),

$$\frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}P^{ca}{}_{b}\right) + \frac{1}{2}\delta^a_b\mathcal{Q} + P^{a}{}_{cd}Q_b{}^{cd} = \Psi^a{}_b, \tag{3.32}$$

and considering its Levi-Civita divergence,

$$\frac{2}{\sqrt{-g}}\tilde{\nabla}_a\nabla_c\left(\sqrt{-g}P^{ca}{}_{b}\right) + \frac{1}{2}\delta^a_b\tilde{\nabla}_a\mathcal{Q} + \tilde{\nabla}_a\left(P^{a}{}_{cd}Q_b{}^{cd}\right) = \tilde{\nabla}_a\Psi^a{}_b. \tag{3.33}$$

Since the following relations hold,

$$\nabla_a\nabla_c\left(\sqrt{-g}P^{ca}{}_{b}\right) = \tilde{\nabla}_a\nabla_c\left(\sqrt{-g}P^{ca}{}_{b}\right) - L_{ab}{}^{d}\nabla_c\left(\sqrt{-g}P^{ca}{}_{d}\right), \tag{3.34}$$

$$\nabla_c\left(\sqrt{-g}P^{ca}{}_{d}\right) = \sqrt{-g}\left(\nabla_c P^{ca}{}_{d} - L_{cp}{}^{p}P^{ca}{}_{d}\right), \tag{3.35}$$

and

$$\tilde{\nabla}_a\left(P^{a}{}_{cd}Q_b{}^{cd}\right) = \nabla_a\left(P^{a}{}_{cd}Q_b{}^{cd}\right) - L_{aq}{}^{a}P^{q}{}_{cd}Q_b{}^{cd} + L_{ab}{}^{q}P^{a}{}_{cd}Q_q{}^{cd}, \tag{3.36}$$

by using the equations (3.26), we obtain

$$\begin{aligned}\tilde{\nabla}_a\Psi^a{}_b + \frac{2}{\sqrt{-g}}\nabla_a\nabla_c\Delta^{ac}{}_b =& \frac{1}{2}\tilde{\nabla}_b\mathcal{Q} + \nabla_a\left(P^{a}{}_{cd}Q_b{}^{cd}\right) + 2L_{ab}{}^{d}\nabla_c P^{ca}{}_{d} + \\ & - 2L_{ab}{}^{d}L_{cp}{}^{p}P^{ca}{}_{d} - L_{aq}{}^{a}P^{q}{}_{cd}Q_b{}^{cd} + L_{ab}{}^{q}P^{a}{}_{cd}Q_q{}^{cd}.\end{aligned} \tag{3.37}$$

Let us prove that the right-hand side of the above equation is zero. The sum of the first three terms is equal to

$$\frac{1}{2}\tilde{\nabla}_b\mathcal{Q} + \nabla_a\left(P^{a}{}_{cd}Q_b{}^{cd}\right) + 2L_{ab}{}^{d}\nabla_c P^{ca}{}_{d} = 2L_{ab}{}^{d}Q_{cdq}P^{caq} - \nabla_b P^{acd}Q_{acd} - \frac{1}{2}\nabla_b\mathcal{Q}, \tag{3.38}$$

[1]Because of the symmetries of the Lagrange multiplier, not all their independent components are completely determined by the field equations. However, as we pointed out in the text, their determination does not affect the metric equations, so having indeterminate components is not a problem [101].

where we used Eq. (3.1). From the explicit evaluation of

$$\begin{aligned}
-\nabla_b P^{acd} Q_{acd} = & -\nabla_b \left(g^{a,m} g^{c,n} g^{d,l} P_{acd} \right) Q_{acd} = \\
& = -\frac{1}{4} Q_b{}^{ac} Q_a{}^{dl} Q_{cdl} + Q_b{}^{ac} Q_a{}^{dl} Q_{dcl} - \frac{1}{2} Q_b{}^{ac} Q^d{}_a{}^l Q_{dcl} + \\
& + \frac{1}{2} Q_b{}^{ac} Q^d{}_a{}^l Q_{lcd} + \frac{1}{4} q^a Q_{acd} Q_b{}^{cd} - \frac{1}{4} q^a Q_{cad} Q_b{}^{cd} + \\
& - \frac{1}{4} Q^a Q_{acd} Q_b{}^{cd} - \frac{1}{2} q^a Q^c Q_{bac} + \frac{1}{4} q^a q^c Q_{bac} + \\
& + \frac{1}{4} Q^{acd} \nabla_b Q_{acd} - \frac{1}{2} Q^{acd} \nabla_b Q_{cda} + \frac{1}{4} q^a \nabla_b Q_a - \frac{1}{4} q^a \nabla_b q_a + \frac{1}{4} Q^c \nabla_b q_c,
\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}
-\frac{1}{2} \nabla_b \mathcal{Q} = & \frac{1}{4} Q_{bde} Q_{ac}{}^d Q^{ace} - \frac{1}{4} Q_{bde} Q_{ca}{}^d Q^{ace} - \frac{1}{4} Q_{bce} Q^c{}_{ad} Q^{aed} + \\
& + \frac{1}{8} Q_{bae} Q^a{}_{cd} Q^{ecd} + \frac{1}{4} Q^a q^c Q_{bac} - \frac{1}{8} q^a q^c Q_{bac} + \\
& - \frac{1}{4} Q^a \nabla_b q_a - \frac{1}{4} Q^{acd} \nabla_b Q_{acd} + \frac{1}{2} Q^{acd} \nabla_b Q_{cad} + \frac{1}{4} q^a \nabla_b q_a - \frac{1}{4} q^c \nabla_b Q_c,
\end{aligned} \tag{3.40}$$

we find

$$\begin{aligned}
-\nabla_b P^{acd} Q_{acd} - \frac{1}{2} \nabla_b \mathcal{Q} = & -\frac{1}{8} Q_b{}^{ac} Q_a{}^{de} Q_{cde} + \frac{1}{2} Q_b{}^{ac} Q_a{}^{de} Q_{dce} - \frac{1}{4} Q_b{}^{ac} Q^d{}_{ae} Q_{dce} + \\
& + \frac{1}{4} Q_b{}^{ac} Q^d{}_{ae} Q_{ecd} + \frac{1}{4} q^a Q_b{}^{cd} Q_{acd} - \frac{1}{4} q^a Q_b{}^{cd} Q_{cad} + \\
& - \frac{1}{4} Q^a Q_b{}^{cd} Q_{acd} - \frac{1}{4} q^a Q^c Q_{bac} + \frac{1}{8} q^a q^c Q_{bac}.
\end{aligned} \tag{3.41}$$

On the other hand, we have

$$\begin{aligned}
2L_{ab}{}^d Q_{cdq} P^{caq} - 2L_{ab}{}^d L_{cp}{}^p P^{ca}{}_d - L_{aq}{}^a P^q{}_{cd} Q_b{}^{cd} + L_{ab}{}^q P^a{}_{cd} Q_q{}^{cd} = & \\
= \frac{1}{8} Q_b{}^{ac} Q_a{}^{de} Q_{cde} - \frac{1}{2} Q_b{}^{ac} Q_a{}^{de} Q_{dce} + \frac{1}{4} Q_b{}^{ac} Q^d{}_{ae} Q_{dce} + & \\
- \frac{1}{4} Q_b{}^{ac} Q^d{}_{ae} Q_{ecd} - \frac{1}{4} q^a Q_b{}^{cd} Q_{acd} + \frac{1}{4} q^a Q_b{}^{cd} Q_{cad} + & \\
+ \frac{1}{4} Q^a Q_b{}^{cd} Q_{acd} + \frac{1}{4} q^a Q^c Q_{bac} - \frac{1}{8} q^a q^c Q_{bac}, &
\end{aligned} \tag{3.42}$$

which is exactly the right-hand side of Eq. (3.41) but with opposite sign. Hence, we obtain the following energy-momentum conservation equations,

$$\tilde{\nabla}_a \Psi^a{}_b + \frac{2}{\sqrt{-g}} \nabla_a \nabla_c \Delta^{ac}{}_b = 0. \tag{3.43}$$

They reduce to the usual ones, $\tilde{\nabla}_a \Psi^a{}_b = 0$, for $\Delta^{(ac)}{}_b = 0$. We can, of course, use Eqs. (3.43) and (3.33) to derive Eq. (3.26). Hence, for any solution of the metric field equations (3.18), if the energy-momentum conservation (3.43) holds, the equations (3.26) are necessarily satisfied.

Finally, we prove the equivalence between the Einstein field equations and Eq. (3.18). Due to the condition $R^a{}_{bcd} = 0$, the Levi-Civita Ricci tensor is equal to

$$\tilde{R}_{ab} = -\tilde{\nabla}_c L_{ab}{}^c + \tilde{\nabla}_b L_{ca}{}^c - L_{cd}{}^c L_{ab}{}^d + L_{bd}{}^c L_{ca}{}^d. \tag{3.44}$$

Moreover, because of the identities

$$\frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}P^c{}_{ab}\right) = 2\tilde{\nabla}_c P^c{}_{ab} - 2L_{ca}{}^d P^c{}_{db} - 2L_{cb}{}^d P^c{}_{da}, \tag{3.45}$$

and

$$L_{bd}{}^c L_{ca}{}^d - L_{cd}{}^c L_{ab}{}^d = -2L_{ca}{}^d P^c{}_{db} - 2L_{cb}{}^d P^c{}_{da} + P_{acd} Q_b{}^{cd} - 2Q^{cd}{}_a P_{cdb}, \tag{3.46}$$

Eq. (3.18) can be recast as

$$2\tilde{\nabla}_c P^c{}_{ab} + \frac{1}{2} g_{ab} \mathcal{Q} + \tilde{R}_{ab} + \tilde{\nabla}_c L_{ab}{}^c - \tilde{\nabla}_b L_{ca}{}^c = \Psi_{ab}. \tag{3.47}$$

Now, if we substitute Eq. (3.4) and we use the relation

$$2\tilde{\nabla}_c P^c{}_{ab} + \tilde{\nabla}_c L_{ab}{}^c - \tilde{\nabla}_b L_{ca}{}^c - \frac{1}{2} g_{ab} \tilde{\nabla}_c\left(q^c - Q^c\right) = -\frac{1}{2}\check{R}_{ab} = 0, \tag{3.48}$$

we obtain the Einstein equations,

$$\tilde{R}_{ab} - \frac{1}{2} g_{ab} \tilde{R} = \Psi_{ab}. \tag{3.49}$$

This result allows us to highlight the main difference between a generic STG theory and STEGR. STG uses the most general quadratic scalar we can build with the nonmetricity tensor, i.e.,

$$\mathbb{Q} = c_1 Q_{cab} Q^{cab} + c_2 Q_{cab} Q^{abc} + c_3 q_a q^a + c_4 Q_a Q^a + c_5 q_a Q^a, \tag{3.50}$$

with c_i $(i = 1, ..., 5)$ arbitrary coefficient, and not the scalar $\mathcal{Q}$ given in Eq. (3.3), which is recovered for

$$c_1 = -c_3 = \frac{1}{4}, \qquad c_2 = -c_5 = -\frac{1}{2}, \qquad c_4 = 0.$$

Therefore, in a generic STG, the metric field equations are not equivalent to GR ones.

3.2 Extensions of STEGR

In the previous section, we showed that STEGR and GR have equivalent metric field equations, so the two theories have both equal metric solutions and corresponding shortcomings. For this reason, generalizations and modifications of STEGR have been proposed as well.

Along the same line as the $f(R)$ theories, the most straightforward modification is the generalization of the STEGR action to an arbitrary function $f(\mathcal{Q})$ of the nonmetricity scalar [69, 70, 73],

$$A = \int \left[-\frac{1}{2}\sqrt{-g} f(\mathcal{Q}) + \lambda_a{}^{bcd} R^a{}_{bcd} + \lambda_c{}^{ab} T_{ab}{}^c + \sqrt{-g}\mathcal{L}_m(g_{ab}, \Gamma_{ab}{}^c, \psi)\right] \mathrm{d}^4 x. \tag{3.51}$$

By varying with respect to the Lagrange multipliers, we again obtain the constraints

$$R^a{}_{bcd} = 0 \qquad \text{and} \qquad T_{ab}{}^c = 0, \tag{3.52}$$

whereas variations with respect to the metric and the connection produce field equations of the form

$$\frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}f'P^c{}_{ab}\right) + \frac{1}{2}g_{ab}f + f'\left(P_{acd}Q_b{}^{cd} - 2Q^{cd}{}_a P_{cdb}\right) = \Psi_{ab}, \tag{3.53}$$

and

$$\nabla_d\lambda_c{}^{bad} + \lambda_c{}^{ab} - \sqrt{-g}f'P^{ab}{}_c = \Delta^{ab}{}_c, \tag{3.54}$$

respectively, with $f' = \mathrm{d}f/\mathrm{d}\mathcal{Q}$. The energy-momentum conservation is derived by a procedure similar to that used for the STEGR and the expression is the same,

$$\tilde{\nabla}_a\Psi^a{}_b + \frac{2}{\sqrt{-g}}\nabla_a\nabla_c\Delta^{ac}{}_b = 0. \tag{3.55}$$

The most significant difference is just the use in the derivation of (3.55) of the following relation,

$$2\nabla_b f' P^{bc}{}_d L^d{}_{ca} + \tilde{\nabla}_c f' P^{cdq} Q_{adq} = \nabla_b f' P^{bcd}\left(2L_{dca} + Q_{acd}\right) = 0. \tag{3.56}$$

Although $f(R)$ theories are a guideline for $f(\mathcal{Q})$ gravity, they are completely different theories, and in particular the last one has no issues related to projective invariance.

Another extension of STEGR is represented by theories in which nonmetricity is non-minimally coupled to the matter Lagrangian. As an example, in [104], the following action is considered

$$A = \int\left[\frac{1}{2}f_1(\mathcal{Q}) + \lambda_a{}^{bcd}R^a{}_{bcd} + \lambda_c{}^{ab}T_{ab}{}^c + f_2(\mathcal{Q})\mathcal{L}_m(g_{ab}, \Gamma_{ab}{}^c, \psi)\right]\sqrt{-g}\,\mathrm{d}^4x, \tag{3.57}$$

where f_1 and f_2 are generic functions of $\mathcal{Q}$. The field equations are equal to

$$\frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}FP^c{}_{ab}\right) + \frac{1}{2}g_{ab}f_1 + F\left(P_{acd}Q_b{}^{cd} - 2Q^{cd}{}_a P_{cdb}\right) = -f_2\Psi_{ij}, \tag{3.58}$$

and

$$\nabla_a\nabla_b\left(\sqrt{-g}FP^{ab}{}_c - f_2\Delta^{ab}{}_c\right) = 0, \tag{3.59}$$

where

$$F = f_1'(\mathcal{Q}) + 2f_2'(\mathcal{Q})\mathcal{L}_m, \tag{3.60}$$

has been introduced. The non-minimal coupling leads to the non-conservation of the energy-momentum tensor,

$$\begin{aligned}\tilde{\nabla}_a\Psi^a{}_b + \frac{2}{\sqrt{-g}}\nabla_a\nabla_c\Delta^{ac}{}_b = &-\frac{2}{\sqrt{-g}}\left[\delta^{ac}{}_b\nabla_a\nabla_c f_2 + 2\nabla_{(a}f_2\nabla_{c)}\delta^{ac}{}_b\right] + \\ &-\left(\Psi^a{}_b - \delta^a_b\mathcal{L}_m\right)\nabla_a\log f_2.\end{aligned} \tag{3.61}$$

In addition, it is worth highlighting the $f(\mathcal{Q},\mathcal{T})$ theory, which was first used in [105] and has been quite popular over time. Here, the nonmetricity is non-minimally coupled with the trace $\mathcal{T}$ of the matter energy-momentum tensor, and the gravitational Lagrangian density is given by a general function of both $\mathcal{Q}$ and $\mathcal{T}$ [2],

$$A = \int\left[\frac{1}{2}\sqrt{-g}f(\mathcal{Q},\mathcal{T}) + \lambda_a{}^{bcd}R^a{}_{bcd} + \lambda_c{}^{ab}T_{ab}{}^c + \sqrt{-g}\mathcal{L}_m(g_{ab}, \Gamma_{ab}{}^c, \psi)\right]\mathrm{d}^4x. \tag{3.62}$$

[2] To adopt the convention used in the literature, we used $\mathcal{T}$ instead of Ψ to refer to the energy-momentum tensor.

The variation of the action yields the following field equations

$$-\frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}f_{\mathcal{Q}}P^c{}_{ab}\right)-\frac{1}{2}g_{ab}f+f_{\mathcal{T}}\left(T_{ab}+\Theta_{ab}\right)-f_{\mathcal{Q}}\left(P_{acd}Q_b{}^{cd}-2Q^{cd}{}_aP_{cdb}\right)=T_{ab}, \tag{3.63}$$

$$\nabla_a\nabla_b\left(\sqrt{-g}f_{\mathcal{Q}}P^{ab}{}_c+\Delta^{ab}{}_c\right)=0, \tag{3.64}$$

with $f_{\mathcal{Q}}=\partial f/\partial\mathcal{Q}$, $f_{\mathcal{T}}=\partial f/\partial\mathcal{T}$, and

$$\Theta_{ab}=g^{cd}\frac{\delta T_{cd}}{\delta g^{ab}}. \tag{3.65}$$

Also in this theory, the energy-momentum tensor is not preserved.

II
Cosmology and Astrophysics

4
Homogeneous cosmology

The cosmological principle states that at large scales the universe is homogeneous and isotropic, that is, all points of the universe and all spatial directions of observation are equivalent. It stems from philosophical arguments based on a generalization of the Copernican idea that the Earth does not occupy a privileged position in the universe. This idea was generalized to the fact that no point in the universe occupies a privileged position but that they are all perfectly equivalent to each other. These assumptions were later supported (e.g., CMB [106]) showing that the structures are more and more homogeneous and isotropic as the scale increases.

In this chapter, we will initially examine the geometric description of homogeneity and isotropy. Next, we will introduce Friedmann-Lemaître-Robertson-Walker (FLRW) universes and see why introducing the concepts of cosmological constant and dark energy becomes necessary in GR [10, 11].

Although the present universe is believed to be essentially isotropic, it is possible that it may not have been so at its beginning and may not necessarily be so in the future. Moreover, homogeneity and isotropy are witnessed only at large scales. Therefore, the suitability of the FLRW metric in describing the universe at every epoch and scale is not certain. We will relax the isotropy assumption, and consider homogeneous but anisotropic cosmological models, introducing the Bianchi classification[1] for homogeneous spacetimes [108].

4.1 Homogeneity and isotropy

We say that a spacetime is homogeneous [78, 80] *if there exists a one-parameter family of spacelike hypersurfaces* Σ_t *foliating the spacetime*[2] *such that for each* t *and for any points* P, Q $\in \Sigma_t$, *there exists an isometry of the metric tensor* g_{ab} *which takes* P *into* Q.

On the other hand, a spacetime is isotropic at each point *if there exists a congruence*[3]

[1]Homogeneous and anisotropic models also include Kantowski–Sachs universes [107] which are not included in the Bianchi classification.

[2]This means that spacetime is decomposed into hypersurfaces of dimension $n = 3$.

[3]A congruence of curves is a set of curves that do not intersect and thus for each event there is only and only one curve of the congruence.

of timelike curves, which represent the observers, with 4-velocity u^a and satisfying the following property: given any point P and any two unit vectors ${s_1}^a$ and ${s_2}^a$, which are orthogonal to u^a, there exists an isometry of the metric tensor g_{ab} that leaves P and u^a at P fixed but rotates ${s_1}^a$ into ${s_2}^a$.

We should stress that at each event of spacetime there may be only one isotropic observer. To show this, let us represent the isotropic observer by a timelike curve, which we refer to as world line, with 4-velocity $u^a$and such that it is a rest frame for the matter filling the universe. If we consider another observer characterized by a 4-velocity v^a, which is in motion with respect to u^a, this observer will perceive matter flows in some direction, and thus the two observers see the universe in two different ways.

For the spacetime to be both homogeneous and isotropic, the congruence of timelike curves must be orthogonal to the spatial hypersurfaces[4]. If the 4-velocity of an observer were not perpendicular to the spatial hypersurfaces, we would have a projection of u^a onto Σ_t that selects preferred a direction, which is in contrast to the isotropy principle.

Let us now focus on a single hypersurface and introduce the metric h_{ab} induced by g_{ab} on it, whose signature is $(+,+,+)$. The corresponding Levi-Civita curvature tensor on Σ_t is denoted by ${}^3\tilde{R}^a{}_{bcd}$, where we emphasize that the hypersurface has dimension 3. On the other hand, we denote with $\tilde{D}_a$ the Levi-Civita covariant derivative on Σ_t with $\tilde{D}_c h_{ab} = 0$. Homogeneity and isotropy conditions must be satisfied even for isometries of the induced metric.

Considering the Killing vectors ξ^a of the isometries of h_{ab}, they satisfy the corresponding 3-dimensional equation of Eq. (1.97),

$$\tilde{D}_b\tilde{D}_c\xi_d = {}^3\tilde{R}^a{}_{bcd}\xi_a. \tag{4.1}$$

However, from the Ricci identity (1.48), we have

$$\tilde{D}_c\tilde{D}_d\tilde{D}_a\xi_b - \tilde{D}_d\tilde{D}_c\tilde{D}_a\xi_b = -{}^3\tilde{R}^e{}_{acd}\tilde{D}_e\xi_b - {}^3\tilde{R}^e{}_{bcd}\tilde{D}_a\xi_e, \tag{4.2}$$

which, using Eq. (4.1), can be recast as

$$\left(-{}^3\tilde{R}^e{}_{dab}h^p_c + {}^3\tilde{R}^e{}_{cab}h^p_d + {}^3\tilde{R}^e{}_{acd}h^p_b - {}^3\tilde{R}^e{}_{bcd}h^p_a\right)\tilde{D}_p\xi_e = \left(\tilde{D}_c\,{}^3\tilde{R}^e{}_{dab} - \tilde{D}_d\,{}^3\tilde{R}^e{}_{cab}\right)\xi_e, \tag{4.3}$$

where with h^b_a we refer to the 3-dimensional Kronecker delta, $h^a_a = 3$. From the concept of isotropy with respect to a given point P, we know that isotropy corresponds to an infinitesimal isometry that leaves the point P fixed, which implies, according to Eq. (1.92), that $\xi_e(P) = 0$. Moreover, the first derivative $\tilde{D}_p\xi_e$ assumes all possible values with the only constraint to be antisymmetric given by the Killing equation [78]. Hence, the bracketed term in the l.h.s. of Eq. (4.3) must be symmetric in the contravariant indices,

$$\begin{aligned}-{}^3\tilde{R}^e{}_{dab}h^p_c + {}^3\tilde{R}^e{}_{cab}h^p_d + {}^3\tilde{R}^e{}_{acd}h^p_b - {}^3\tilde{R}^e{}_{bcd}h^p_a = \\ = -{}^3\tilde{R}^p{}_{dab}h^e_c + {}^3\tilde{R}^p{}_{cab}h^e_d + {}^3\tilde{R}^p{}_{acd}h^e_b - {}^3\tilde{R}^p{}_{bcd}h^e_a.\end{aligned} \tag{4.4}$$

Contracting p with c,

$$-3\,{}^3\tilde{R}^e{}_{dab} + {}^3\tilde{R}^e{}_{dab} + {}^3\tilde{R}^e{}_{abd} - {}^3\tilde{R}^e{}_{bad} = -{}^3\tilde{R}^e{}_{dab} + {}^3\tilde{R}_{ad}h^e_b - {}^3\tilde{R}_{bd}h^e_a \tag{4.5}$$

and by using the Bianchi identity (2.18), we obtain

$$2\,{}^3\tilde{R}_{edab} = {}^3\tilde{R}_{bd}h_{ea} - {}^3\tilde{R}_{ad}h_{eb}. \tag{4.6}$$

[4]The requirements for u^a to be orthogonal to the spatial hypersurfaces are given in Appendix B.

However, ${}^3\tilde{R}_{edab}$ is antisymmetric in the first two indices, so

$$ {}^3\tilde{R}_{bd}h_{ea} - {}^3\tilde{R}_{ad}h_{eb} = -{}^3\tilde{R}_{be}h_{da} - {}^3\tilde{R}_{ae}h_{db}, \tag{4.7} $$

and the contraction of e with b yields

$$ {}^3\tilde{R}_{ad} = \frac{1}{3}\,{}^3\tilde{R}h_{ad}. \tag{4.8} $$

Thus,

$$ {}^3\tilde{R}_{edab} = \frac{1}{6}\,{}^3\tilde{R}\left(h_{bd}h_{ea} - h_{ad}h_{eb}\right). \tag{4.9} $$

From the contracted Bianchi identity (2.21), we have

$$ 0 = \tilde{D}_a\left({}^3\tilde{R}^a{}_b - \frac{1}{2}h^a_b\,{}^3\tilde{R}\right) = -\frac{1}{3}\partial_b\,{}^3\tilde{R}, \tag{4.10} $$

that is, the 3-dimensional Ricci scalar is independent of the spatial coordinates. It is useful to introduce the function $K(t)$ that depends only on the parameter t that identifies the hypersurfaces, such that

$$ {}^3\tilde{R} = 6K(t). \tag{4.11} $$

In this way we find

$$ {}^3\tilde{R}_{abcd} = K\left(h_{ac}h_{db} - h_{ad}h_{cb}\right). \tag{4.12} $$

A space endowed with a curvature given by Eq. (4.12) is called space of constant curvature.

It can be shown that for every real value of K, there exists one and only one 3-geometry with constant curvature. This means that if we have two metric tensors with the same signature, for which the values of K are equal, then it is always possible to find a diffeomorphism that carries one metric into the other [78]. It is then sufficient to construct a 3-dimensional metric for each value of K to classify all possible 3-dimensional geometries.

The simplest geometry is clearly the one represented by $K = 0$, that is, an Euclidean geometry with null curvature,

$$ {}^3\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{4.13} $$

For positive values of K, we have the geometry of a 3-sphere. To highlight this aspect, let us consider a 4-dimensional Euclidean space with metric

$$ \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2. \tag{4.14} $$

A 3-sphere is represented by the equation

$$ x^2 + y^2 + z^2 + w^2 = a^2, \tag{4.15} $$

with a the radius, and it is obviously a homogeneous and isotropic 3-geometry. To evaluate the induced metric on this 3-sphere, we start by introducing a polar coordinate system such that

$$ \begin{cases} w = a\cos\psi \\ x = a\sin\psi\sin\theta\cos\varphi \\ y = a\sin\psi\sin\theta\sin\varphi \\ z = a\sin\psi\cos\theta \end{cases}. \tag{4.16} $$

By differentiating Eq. (4.16), we derive the 3-dimensional metric

$$^{(3)}\mathrm{d}s^2 = a^2\left[\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right], \tag{4.17}$$

and from the evaluation of $^3\tilde{R}$, we obtain

$$K = \frac{1}{a^2}. \tag{4.18}$$

Therefore, as the radius of the 3-sphere varies, K assumes all the positive values.

In the case of negative values of K, we consider a 4-dimensional metric with Lorentzian signatures

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}w^2, \tag{4.19}$$

and a hyperboloid represented by the equation

$$w^2 - x^2 - y^2 - z^2 = a^2. \tag{4.20}$$

We can once again introduce polar coordinates,

$$\begin{cases} w = a\cosh\psi \\ x = a\sinh\psi\sin\theta\cos\varphi \\ y = a\sinh\psi\sin\theta\sin\varphi \\ z = a\sinh\psi\cos\theta \end{cases}, \tag{4.21}$$

leading to the following 3-dimensional metric

$$^{(3)}\mathrm{d}s^2 = a^2\left[\mathrm{d}\psi^2 + \sinh^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right]. \tag{4.22}$$

From the 3-dimensional Ricci scalar we have

$$K = -\frac{1}{a^2}. \tag{4.23}$$

This result proves that hyperboloids in the Lorentzian space (4.19) are spaces with constant negative curvature.

Eqs. (4.13), (4.17), and (4.22) are the only three constant curvature geometries possible unless diffeomorphism.

Given a congruence of timelike curves, the metric of spacetime has the following form

$$g_{ab} = -u_a u_b + h_{ab}, \tag{4.24}$$

where the 4-velocity u^a is normalized, and $c = 1$. We must now choose an appropriate coordinate system to write the 4-dimensional metric g_{ab}. In considering a spatial hypersurface, we know that the world lines are orthogonal to it. Therefore, we can consider spatial coordinates on the hypersurfaces that are carried along the world lines that represent the observers. In this way, by definition, isotropic observers have constant spatial coordinates that uniquely identify each of them. Moreover, we can take the proper time t measured by the observers as the time coordinate, which label each hypersurface. We know that this time has the same value for each observer, since if it did not, we would have a criterion to distinguish points on spatial hypersurfaces, going against the principle of homogeneity. The fixed spatial coordinates we have defined are called comoving coordinates[5], whereas the proper time is the cosmic time.

In this coordinate system, the 4-dimensional metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \begin{cases} \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ \mathrm{d}\psi^2 + \sin^2\psi \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \\ \mathrm{d}\psi^2 + \sinh^2\psi \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \end{cases} , \tag{4.25}$$

where t is the proper time, $\{x, y, z\}$ and $\{\psi, \theta, \varphi\}$ are the comoving coordinates, and $a(t)$ is called cosmic scale factor, which depends on the cosmic time due to Eqs. (4.18) and (4.23). Equation (4.25) can be written in a more compact form if we consider the following variable changes: for $K(t) > 0$, we can define

$$r = \sin\psi, \tag{4.26}$$

such that

$$^{(3)}\mathrm{d}s^2 = a^2(t) \left[\frac{\mathrm{d}r^2}{1-r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)\right]; \tag{4.27}$$

on the other hand, for $K(t) < 0$, if we define

$$r = \sinh\psi, \tag{4.28}$$

we have

$$^{(3)}\mathrm{d}s^2 = a^2(t) \left[\frac{\mathrm{d}r^2}{1+r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)\right]. \tag{4.29}$$

So, in general, we can write

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)\right], \tag{4.30}$$

which is the Friedmann-Lemaître-Robertson-Walker metric, and it is conformally flat, that is, its Weyl tensor $\tilde{C}^a{}_{bcd}$ is null. The coordinate r is dimensionless, and k is a constant parameter that assumes the following values

$$k = \begin{cases} 1 & \text{for} \quad K > 0 \\ 0 & \text{for} \quad K = 0 \\ -1 & \text{for} \quad K < 0 \end{cases}. \tag{4.31}$$

Once the geometry is fixed, the only aspect that cannot be derived from the cosmological principle is the evolution of the scale factor, which is obtained by solving the field equations. However, before that, we need to understand the consequences of homogeneity and isotropy on the energy-momentum tensor Ψ_{ab} describing ordinary matter. We consider energy-momentum tensors symmetric for exchanging of indices[6].

Let us consider the contraction of the energy-momentum tensor Ψ_{ab} with the 4-velocity u^a and call v^a the corresponding vector,

$$\Psi^a{}_b u^b = v^a. \tag{4.32}$$

[5]The term comoving refer to the fact that the isotropic observers are at rest with respect to the matter.

[6]The reasoning below is not valid in the case of a non-symmetric tensor.

So that v^a does not identify a preferred direction, it must be parallel to u^a. This means that the contraction of v^a with the hypersurface space vectors s^a is zero,

$$v^a s_a = 0 \quad \longrightarrow \quad T^a{}_b u^b s_a = 0, \tag{4.33}$$

so the energy-momentum tensor cannot have mixed spacetime components. On the other hand, the spatial components of Ψ_{ab} define a linear operator on the tangent spatial vector s^a of the hypersurfaces,

$$\hat{L}: \quad \hat{L}\, s^i = \Psi^j{}_i s^i \quad \in \Sigma_t \qquad i,j = 1,2,3. \tag{4.34}$$

Since Ψ_{ab} is symmetric, this operator is hermitian with respect to the scalar product induced by the metric h_{ab}, and therefore $\hat{L}$ have a complete set of eigenvectors. Isotropy tells us that the associated eigenvalues must all be equal to each other, otherwise, different eigenvalues would identify specific directions of the space. Hence, $\hat{L}$ is a multiple of the identity

$$\Psi^i{}_j = \lambda(t)\delta^i_j, \tag{4.35}$$

where λ is the eigenvalue which is constant on Σ_t. We obtained that the energy-momentum tensor is diagonal, then it corresponds to the Cauchy tensor of a perfect fluid,

$$\Psi_{ab} = \rho(t) u_a u_b + p(t) h_{ab}, \tag{4.36}$$

with $\rho(t)$ the matter density, and $p(t)$ the pressure.

The cosmological principle has the consequence that the energy-momentum tensor describing matter on a cosmological scale must have the form of the perfect fluid one which is at rest with respect to an isotropic observer.

4.2 FLRW metric in General Relativity

Now, we derive the dynamics of the scale factor $a(t)$ of the metric (4.30) in GR. From the definition (1.25), it follows that the only non-zero components of the Levi-Civita connection are

$$\begin{gathered}
\tilde{\Gamma}_{11}{}^0 = \frac{a\dot{a}}{1-kr^2}, \quad \tilde{\Gamma}_{22}{}^0 = a\dot{a}r^2, \quad \tilde{\Gamma}_{33}{}^0 = a\dot{a}r^2\sin^2\theta, \\
\tilde{\Gamma}_{01}{}^1 = \tilde{\Gamma}_{02}{}^2 = \tilde{\Gamma}_{03}{}^3 = \frac{\dot{a}}{a}, \\
\tilde{\Gamma}_{11}{}^1 = \frac{kr}{1-kr^2}, \quad \tilde{\Gamma}_{22}{}^1 = -r\left(1-kr^2\right), \quad \tilde{\Gamma}_{33}{}^1 = -r\left(1-kr^2\right)\sin^2\theta, \\
\tilde{\Gamma}_{12}{}^2 = \tilde{\Gamma}_{13}{}^3 = \frac{1}{r}, \\
\tilde{\Gamma}_{33}{}^2 = -\cos\theta\sin\theta, \quad \tilde{\Gamma}_{23}{}^3 = \cot\theta,
\end{gathered} \tag{4.37}$$

where the dot indicates the derivative with respect to time, $\dot{a} = \mathrm{d}a/\mathrm{d}t$. Hence, the 00-component and the 11-component of Einstein field equations (2.34) yield[7]

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2}, \tag{4.38}$$

[7] The 22-component and 33-component give the same field equation as the 11-one due to the isotropy of the FLRW metric.

where we have defined the Hubble parameter $H = \dot{a}/a$, and

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho + 3p\right), \tag{4.39}$$

respectively (we used the natural units $c = 8\pi G_N = 1$). Eq. (4.39) is usually called Raychaudhuri equation, and together Eq. (4.38) form the so-called Friedmann equations. On the other hand, the contracted Bianchi identity gives the energy-momentum conservation equation,

$$\dot{\rho} + 3H\left(\rho + p\right) = 0. \tag{4.40}$$

Henceforth, we will use the equation of state

$$p = w\rho, \tag{4.41}$$

with w a suitable constant. The range of validity of w for ordinary matter is imposed by the definition of the speed of sound c_S,

$$c_S^2 = c^2 \frac{\mathrm{d}p}{\mathrm{d}\rho}. \tag{4.42}$$

Since the speed of sound c_s must be less than the speed of light c, then $0 \leq w \leq 1$. Two characteristic values of w are $w = 0$, which represents dust, and $w = 1/3$, which corresponds to radiation. From Eqs. (4.40) and (4.41), we derive that for $w = 0$

$$\rho \sim \frac{1}{a^3}, \tag{4.43}$$

whereas for $w = 1/3$

$$\rho \sim \frac{1}{a^4}. \tag{4.44}$$

To analytically solve the Friedmann equations, we introduce the conformal time $\mathcal{T}$ [109],

$$\mathrm{d}\mathcal{T} = \frac{\mathrm{d}t}{a}. \tag{4.45}$$

It is called conformal time since the metric (4.30) assumes the form

$$\mathrm{d}s^2 = a^2(t)\left[-\mathrm{d}\mathcal{T}^2 + \frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right], \tag{4.46}$$

which is a conformal transformation (1.78) with scale factor a^2. Defining the quantity

$$\mathbb{H} = \frac{a'}{a}, \tag{4.47}$$

with $a' = \mathrm{d}a/\mathrm{d}\mathcal{T}$, and considering the matter to be dust, the Friedmann equations are recast as follows,

$$\mathbb{H}^2 = \frac{1}{3}\rho a^2 - k, \tag{4.48}$$

and

$$\mathbb{H}' = -\frac{1}{6}\rho a^2, \tag{4.49}$$

from which we derive the equation

$$2\mathbb{H}' + \mathbb{H}^2 + k = 0. \tag{4.50}$$

The solutions are given by

$$a(\mathcal{T}) = a_1 \begin{cases} \sin^2\left(\frac{\mathcal{T}}{2}\right) & k = 1 \\ \mathcal{T}^2 & k = 0 \\ \sinh^2\left(\frac{\mathcal{T}}{2}\right) & k = -1 \end{cases}, \tag{4.51}$$

where a_1 is a constant of integration. These solutions can be used to obtain the relation between the conformal time $\mathcal{T}$ and the cosmic one t,

$$t = \frac{a_1}{2} \begin{cases} \mathcal{T} - \sin\mathcal{T} & k = 1 \\ \frac{2}{3}\mathcal{T}^3 & k = 0 \\ \sinh\mathcal{T} - \mathcal{T} & k = -1 \end{cases}. \tag{4.52}$$

A plot of Eqs. (4.51) is given in Figure 4.1[8]. It is worth noticing that all solutions have a Big Bang singularity, that is, the scale factor vanishes at the origin of time, $a(0) = 0$. In addition, for $k = 0$ and $k = -1$, the universe always expands, while for $k = 1$ the universe begins to collapse after an expansion phase until the scale factor returns to zero. This phenomenon is called the Big Crunch.

As it is evident from Eq. (4.39) and Figure 4.1, for positive ρ and p, the universe must experience a decelerated expansion phase under our assumptions. However, in the late 1990s it was discovered through observation of distant Type Ia supernovae that the universe undergoes an accelerated expansion instead [110, 111]. The implication is that GR must be modified to include a new form of energy that compensates the ordinary matter. The currently most widely accepted model for cosmology that is in accordance with the accelerated expansion is the ΛCDM model, which assumes that the accelerated expansion is due to a positive cosmological constant Λ [10, 11, 112–115]. In the model, the cosmological equations are given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{4.53}$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3}. \tag{4.54}$$

From an analogy with the quantum field theory, we can identify the cosmological constant with the energy of the vacuum with energy-momentum tensor

$$\Psi^{\Lambda}_{ab} = -\rho_{\Lambda} g_{ab}, \tag{4.55}$$

that can be considered as the energy-momentum of a perfect fluid with the following equation of state

$$p_{\Lambda} = -\rho_{\Lambda} = -\frac{\Lambda}{3}. \tag{4.56}$$

[8] Eqs. (4.52) were solved numerically for $\mathcal{T}$ using the bisection method.

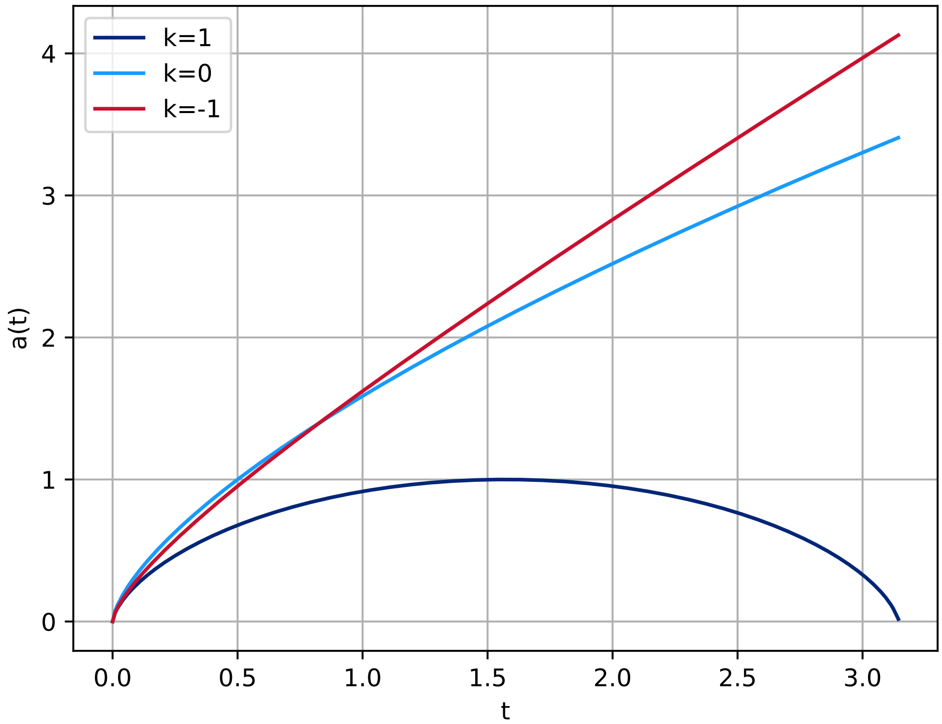


Figure 4.1: Evolution of the scale factor a in function of the cosmic time t with $a_1 = 1$.

For this reason, we can refer to Λ as cosmological constant or vacuum energy.

If the universe is dominated by the cosmological constant, from Eq. (4.53) we obtain that the Hubble parameter is a constant and then the scale factor is an exponential function of cosmic time,

$$a(t) = a_1 \exp\left(\sqrt{\frac{\Lambda}{3}}t\right). \tag{4.57}$$

Solution (4.57) is called de Sitter universe. This solution represents the late time evolution of the universe when both matter and radiation are considered. In fact, because of Eqs. (4.43) and (4.44), we have that with the expansion of the scale factor, the contributions in Eq. (4.53) of matter, radiation, and curvature k are negligible with respect to the cosmological constant. Thus, at a late epoch, the cosmological constant is the only quantity contributing to the dynamics of a to persist. What has just been described is known as the cosmological non-hair theorem.

Due to the principles of quantum field theory and the standard model, the theoretical value of the cosmological constant can be derived. However, this value differs from the observed value by 120 orders of magnitude [11, 116], leading to what is known as the cosmological constant problem. As a result, there is a question of whether ρ_Λ truly represents a cosmological constant or a more general fluid known as dark energy, which is characterized by negative pressure and an equation of state that evolves over time.

Dark energy, where the word "dark" is used because of the lack of clarity regarding its true nature, accounts for approximately 68% of the total energy in the universe [106]. The remaining 32% is attributed to matter, with 5% being baryonic matter and the remaining 27% belonging to another "dark sector" of our understanding of the universe, known as cold dark matter [15, 117, 118]. The term "cold" means that it is non-relativistic matter, while "dark" indicates its lack of interaction with the electromagnetic field. Therefore, the ΛCDM model is so called because it considers both the cosmological constant Λ and cold dark matter.

Among the open issues of standard cosmology, we can highlight the flatness and the horizon problem. The first is a consequence of observations showing that the universe is approximately flat. It represents a fine-tuning problem since any small amount of curvature present in the primordial universe should grow over time. The reason is that the curvature term in Eq. (4.38) scales as $1/a^2$, while both matter and radiation are washed out faster due to Eqs. (4.43) and (4.44), respectively. Hence, the value of curvature must be very small in the primordial universe to still be approximately zero in the present. The second derives from CMB observations that show homogeneity on scales involving regions that are not causally connected. One solution to both problems is an accelerated expansion phase at the beginning of the universe, called inflation. The flatness problem is solved by the fact that, intuitively, if we expand a curved region it will appear locally flat. On the other hand, the horizon problem is solved since the universe is assumed to be totally causally connected at the beginning of its life. Causality is lost as the light cones of the various points that make up the universe move apart further and furthe0,r until they are no longer connected to each other at the end of the inflation phase [119–123].

There are extended and alternative theories of gravity that incorporate the inflation phase of the universe and solutions to the “dark sector” [16–18]. Among the simplest extensions are certainly the theories that consider the addition of a scalar field as a mediator of gravity in addition to the metric, the aforementioned scalar-tensor theories (see. Sec. 2.2). Examples are the quintessence, a scalar field used as a substitute for the cosmological constant, or the inflaton, which instead is a scalar field that guides inflation as it unfolds [122–127]. We will see in Ch. 5 that nonmetricity can play both of these roles in $f(\mathcal{Q})$ gravity.

4.3 The Bianchi classification

In this section, we describe the Bianchi classification of 3-dimensional Lie algebras. Although the way of deriving the classification was introduced by Bianchi [128], we will follow the method invented by Schücking [129, 130] and later used in [131, 132].

Let us assume we have only homogeneity and consider a spatial hypersurface Σ_t [108, 133–135]. The homogeneity is represented by an isometry of the metric generated by infinitesimal translations of three independent Killing vector fields ξ_α, with $\alpha = 1, 2, 3$[9]. We showed in Sec. 1.5 that the Killing vectors form a Lie algebra. The commutator of the basis of the Lie algebra is given by

$$[\xi_\alpha, \xi_\beta] = C^\gamma{}_{\alpha\beta}\xi_\gamma, \tag{4.58}$$

where $C^\gamma{}_{\alpha\beta}$ are called structure constants and are antisymmetric in the two lower indices, $C^\gamma{}_{\alpha\beta} = -C^\gamma{}_{\beta\alpha}$. Moreover, the Jacobi identity (1.99) implies the following constraint on the structure constants,

$$C^\mu{}_{\alpha\beta}C^\gamma{}_{\mu\nu} + C^\mu{}_{\beta\nu}C^\gamma{}_{\mu\alpha} + C^\mu{}_{\nu\alpha}C^\gamma{}_{\mu\beta} = 0. \tag{4.59}$$

[9] In this section, the Greek letters are used as indices that range from 1 to 3.

With Eqs. (4.58) and (4.59), from the knowledge of structure constants, we can completely define the Lie algebra and therefore perform a classification.

It is useful to introduce the two indices quantities $C^{\alpha\beta}$ obtained by the dual transformation

$$C^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\gamma\mu}C^{\beta}{}_{\gamma\mu} \quad \Longleftrightarrow \quad C^{\gamma}{}_{\alpha\beta} = \epsilon_{\mu\alpha\beta}C^{\mu\gamma}, \tag{4.60}$$

where $\epsilon^{\alpha\beta\gamma}$ and $\epsilon_{\alpha\beta\gamma}$ are the Levi-Civita symbols. In this way, Eq. (4.59) can be recast as

$$C^{\alpha\beta}C^{\gamma\mu}\epsilon_{\gamma\alpha\beta} = 0. \tag{4.61}$$

The choice of the basis of generators is not unique; in fact, if ξ_α is a basis, so will be

$$\bar{\xi}_\alpha = A_\alpha{}^\beta \xi_\beta, \tag{4.62}$$

with $A_\alpha{}^\beta$ a non-singular matrix of constant elements. Under such a change of basis, the structure constants transform as follows

$$\bar{C}^{\alpha\beta} = A_\gamma{}^\alpha A_\mu{}^\beta C^{\gamma\mu}, \tag{4.63}$$

which means we have equivalent sets of structure constants related by the transformation (4.63). Hence, the classification of 3-dimensional Lie algebras, and thus homogeneous spaces, is related to the determination of all sets of structure constants that are not equivalent to each other.

We decompose $C^{\alpha\beta}$ in its symmetric and antisymmetric parts

$$C^{\alpha\beta} = n^{\alpha\beta} - \epsilon^{\alpha\beta\gamma}a_\gamma, \tag{4.64}$$

where $n^{\alpha\beta}$ is the symmetric part, whereas the antisymmetric one is expressed in terms of the vector a_γ. If we substitute the decomposition in Eq. (4.61), we obtain the relation

$$n^{\alpha\beta}a_\beta = 0. \tag{4.65}$$

Under the transformations (4.62), $n^{\alpha\beta}$ transforms as a contravariant tensor density

$$\bar{n}^{\gamma\mu} = (\det A)^{-1}(A^{-1})_\alpha{}^\gamma (A^{-1})_\beta{}^\mu n^{\alpha\beta}, \tag{4.66}$$

and a_β as a covariant vector

$$\bar{a}_\gamma = A_\gamma{}^\beta a_\beta. \tag{4.67}$$

With Eq. (4.66), being $n^{\alpha\beta}$ symmetric, we can diagonalize $n^{\alpha\beta}$,

$$n^{\alpha\beta} = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix}. \tag{4.68}$$

In addition, without loss of generality, we can set $a_\beta = (a, 0, 0)$. Therefore, if we assume that the basis we choose is precisely the one in which $n^{\alpha\beta}$ is diagonal, from Eq. (4.65) we find

$$n_1 a = 0, \tag{4.69}$$

Table 4.1: Bianchi classification of homogeneous spaces.

Class	Bianchi type	a	n_1	n_2	n_3
A	**I**	0	0	0	0
	II	0	1	0	0
	$\mathbf{VII}_0$	0	1	1	0
	$\mathbf{VI}_0$	0	1	-1	0
	IX	0	1	1	1
	VIII	0	1	1	-1
B	**V**	1	0	0	0
	IV	1	0	0	1
	$\mathbf{VII}_h$	a	0	1	1
	$\mathbf{VI}_h$	a	0	1	-1
	III	1	0	1	-1

and the commutators (4.58) become

$$\begin{aligned}
[\xi_1, \xi_2] =& a\xi_2 + n_3\xi_3, \\
[\xi_2, \xi_3] =& n_1\xi_1, \\
[\xi_3, \xi_1] =& n_2\xi_2 - a\xi_3.
\end{aligned} \tag{4.70}$$

Now, the last viable operation is a rescaling of the basis ξ_α without changing its direction (the direction has been fixed by the choice of a basis that diagonalizes $n^{\alpha\beta}$),

$$\xi_\alpha = K\bar{\xi}_\alpha, \tag{4.71}$$

with a consequent change in the commutation relations,

$$\begin{aligned}
[\bar{\xi}_1, \bar{\xi}_2] =& \frac{a}{K_1}\bar{\xi}_2 + \frac{K_3}{K_1K_2}n_3\bar{\xi}_3, \\
[\bar{\xi}_2, \bar{\xi}_3] =& \frac{K_1}{K_2K_3}n_1\bar{\xi}_1, \\
[\bar{\xi}_3, \bar{\xi}_1] =& \frac{K_2}{K_1K_3}n_2\bar{\xi}_2 - \frac{a}{K_1}\bar{\xi}_3.
\end{aligned} \tag{4.72}$$

With this rescaling, we can normalize each structure constant to ± 1. Finally, we can classify the possible types of homogeneous spaces according to which of a, n_1, n_2, and n_3 is zero. In particular, we distinguish two classes depending on whether a is zero or not, which are called class A and class B, respectively. The complete classification is given in Table 4.1

Throughout the thesis, we will consider only the Bianchi Type-I spacetimes, whose metric can be given in the form [136, 137]

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2, \tag{4.73}$$

where $a(t)$, $b(t)$ and $c(t)$ are the scale factors associated to each space direction. It is manifest that is a generalization of the spatially flat FLRW metric, since if we set $a = b = c$ we recover Eq. (4.30) with $k = 0$.

It can be easily verified that the metric (4.73) belongs to the BI case if we consider the Killing vectors

$$\xi_1^a = (0, 1, 0, 0), \quad \xi_2^a = (0, 0, 1, 0), \quad \xi_3^a = (0, 0, 0, 1), \tag{4.74}$$

which commute between them, and we use Eq. (1.85) to verify that the Lie derivative of the metric with respect to these vectors is zero.

4.4 Bianchi type-I in General Relativity

For the BI metric the non-zero components of the Levi-Civita connection are

$$\begin{aligned} \tilde{\Gamma}_{01}{}^{1} = \frac{\dot{a}}{a}, \quad \tilde{\Gamma}_{02}{}^{2} = \frac{\dot{b}}{b}, \quad \tilde{\Gamma}_{03}{}^{3} = \frac{\dot{c}}{c}, \\ \tilde{\Gamma}_{11}{}^{0} = a\dot{a}, \quad \tilde{\Gamma}_{22}{}^{0} = b\dot{b}, \quad \tilde{\Gamma}_{33}{}^{0} = c\dot{c}, \end{aligned} \tag{4.75}$$

so from the Einstein field equations, with the energy-momentum tensor of a perfect fluid, we obtain

$$\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{a}\dot{c}}{ac} = \rho, \tag{4.76}$$

$$-\frac{\ddot{b}}{b} - \frac{\ddot{c}}{c} - \frac{\dot{b}\dot{c}}{bc} = p, \tag{4.77}$$

$$-\frac{\ddot{a}}{a} - \frac{\ddot{c}}{c} - \frac{\dot{a}\dot{c}}{ac} = p, \tag{4.78}$$

$$-\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\dot{a}\dot{b}}{ab} = p. \tag{4.79}$$

On the other hand, the conservation of the energy-momentum tensor gives

$$\dot{\rho} + \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right) (\rho + p) = 0. \tag{4.80}$$

We can introduce the average scale factor for BI cosmologies,

$$\tau = abc, \tag{4.81}$$

and the average Hubble parameter $H = \dot{\tau}/\tau$.

As a first step, we analyze the vacuum case, $\rho = p = 0$ [133, 136, 138]. Due to Eq. (2.34), we know that in the vacuum scenario, the equations to solve are given by $\tilde{R}_{ab} = 0$, which components return the following equations,

$$\tilde{R}_{00} = -\dot{H} - H_a^2 - H_b^2 - H_c^2 = 0, \tag{4.82}$$

$$\tilde{R}_{11} = \dot{H}_a + H H_a = 0, \tag{4.83}$$

$$\tilde{R}_{22} = \dot{H}_b + H H_b = 0, \tag{4.84}$$

$$\tilde{R}_{33} = \dot{H}_c + H H_c = 0, \tag{4.85}$$

where we introduced the directional Hubble parameters

$$H_a = \frac{\dot{a}}{a}, \qquad H_b = \frac{\dot{b}}{b}, \qquad H_c = \frac{\dot{c}}{c}. \tag{4.86}$$

From Eq. (4.76) we have that

$$H_a H_b + H_b H_c + H_a H_c = 0, \tag{4.87}$$

thereby

$$H^2 = (H_a + H_b + H_c)^2 = H_a^2 + H_b^2 + H_c^2, \tag{4.88}$$

and Eq. (4.82) becomes a closed equation for H,

$$\dot{H} + H^2 = 0, \tag{4.89}$$

which solution is

$$H = \frac{1}{t - t_0}, \tag{4.90}$$

with t_0 a constant of integration that we set to zero, $t_0 = 0$. Equations (4.83), (4.84), and (4.85) are used to derive the directional Hubble parameters and the scale factors,

$$H_a = \frac{\mathbb{P}_a}{t} \quad \longrightarrow \quad a = a_0 t^{\mathbb{P}_a} \tag{4.91}$$

$$H_b = \frac{\mathbb{P}_b}{t} \quad \longrightarrow \quad b = b_0 t^{\mathbb{P}_b} \tag{4.92}$$

$$H_c = \frac{\mathbb{P}_c}{t} \quad \longrightarrow \quad c = c_0 t^{\mathbb{P}_c}, \tag{4.93}$$

where $\mathbb{P}_a$, $\mathbb{P}_b$, $\mathbb{P}_c$, a_0, b_0, and c_0 are constants of integration. Finally, we can write the so-called Kasner metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2\mathbb{P}_a}\mathrm{d}x^2 + t^{2\mathbb{P}_b}\mathrm{d}y^2 + t^{2\mathbb{P}_c}\mathrm{d}z^2, \tag{4.94}$$

where we set $a_0 = b_0 = c_0 = 1$. The definition of H, Eq. (4.88), and Eq. (4.90) imply the following constraints on the exponents,

$$\mathbb{P}_a + \mathbb{P}_b + \mathbb{P}_c = \mathbb{P}_a^2 + \mathbb{P}_b^2 + \mathbb{P}_c^2 = 1, \tag{4.95}$$

which guarantees that p_a, p_b and p_c cannot have all the same value and that if one of them is negative, the other two must be positive.

These kinds of metrics were initially studied to understand how anisotropy affected the cosmology. A clear example is how the point-like initial singularity typical of FLRW models changes. In Kasner metric, for t tending to zero, if two of the scale factors tend to zero, the other expands infinitely creating what is called a "cigar" singularity or to a finite value obtaining a "barrel" singularity. On the other hand, if it is only one of the scale factors that tends to zero, we will have the other two tend to a finite value obtaining a "pancake" singularity [133, 136, 139].

We now turn to consider the case with matter. From the sum of Eqs. (4.76), (4.77), (4.78), and (4.79), we obtain an equation for the average scale factor τ,

$$\frac{\ddot{\tau}}{\tau} = \frac{3}{2}(\rho - p) \tag{4.96}$$

from which we can derive its evolution. On the other hand, subtracting Eq. (4.78) and Eq. (4.79) from Eq. (4.77), we find, respectively,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tau\left(\frac{\dot{a}}{a}-\frac{\dot{b}}{b}\right)\right]=0 \quad \longrightarrow \quad \frac{\dot{a}}{a}-\frac{\dot{b}}{b}=\frac{k_{ab}}{\tau}, \tag{4.97}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tau\left(\frac{\dot{a}}{a}-\frac{\dot{c}}{c}\right)\right]=0 \quad \longrightarrow \quad \frac{\dot{a}}{a}-\frac{\dot{c}}{c}=\frac{k_{ac}}{\tau}, \tag{4.98}$$

with k_{ab} and k_{ac} constants of integration. A further integration yield

$$a = d_a \sqrt[3]{\tau}\, \exp\left[\frac{1}{3}(kab+kac)\int \frac{1}{\tau}\,dt\right], \tag{4.99}$$

$$b = d_b \sqrt[3]{\tau}\, \exp\left[\frac{1}{3}(-2kab+kac)\int \frac{1}{\tau}\,dt\right], \tag{4.100}$$

$$c = d_c \sqrt[3]{\tau}\, \exp\left[\frac{1}{3}(kab-2kac)\int \frac{1}{\tau}\,dt\right], \tag{4.101}$$

where d_a, d_b, and d_c are constants of integration as well, which satisfy the constraint

$$d_a d_b d_c = 1. \tag{4.102}$$

Equations (4.99), (4.100), and (4.101) are the general solutions for the scale factors of the system of equations (4.77), (4.78), and (4.79) with matter.

4.5 4-velocity with nonmetricity

So far we have introduced the FLRW spacetime and the Bianchi classification, but we never mention the nonmetricity tensor since the Killing equations are strictly related to the Levi-Civita connection, see Eq. (1.95). In defining the comoving coordinates we stated that the timelike 4-velocity of the isotropic and homogeneous observer is normalized and that the proper time is the curve parameter that we refer to as cosmic time. However, these are not conditions directly fulfilled by the full connection ${\Gamma_{ab}}^c$ since we know that nonmetricity represents the failure to covariantly preserve the norm of a parallel transported vector, whose consequence is also the non-coincidence of proper time t with the curve parameter λ. Indeed, from the definition of proper time, we know that

$$g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}t}\frac{\mathrm{d}x^b}{\mathrm{d}t}=-1, \tag{4.103}$$

while for the 4-velocity $u^a = \mathrm{d}x^a/\mathrm{d}\lambda$ we have

$$-l^2 = g_{ab}u^a u^b = g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}t}\frac{\mathrm{d}x^b}{\mathrm{d}t}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 = -\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2, \tag{4.104}$$

where l is a generic function which represent the norm of u^a. Therefore, the proper time and λ coincide if the 4-velocity is normalized. For this to happen, the following condition must be satisfied if we want the norm to be preserved along the curve with tangent vector u^a:

$$u^c\nabla_c\left(g_{ab}u^a u^b\right) = Q_{cab}u^c u^a u^b + 2u_b u^c \nabla_c u^b = 0. \tag{4.105}$$

If the curve is autoparallel, then $Q_{cab}u^c u^a u^b = 0$. In the $f(\mathcal{Q})$ gravity, since torsion is zero, by making the Levi-Civita covariant derivative explicit in Eq. (4.105), we have

$$u^c \nabla_c \left(g_{ab} u^a u^b \right) = Q_{cab} u^c u^a u^b + 2u_b u^c \tilde{\nabla}_c u^b + 2u_b u^c L_{cd}{}^b u^d = 2u_b u^c \tilde{\nabla}_c u^b = 0, \quad (4.106)$$

which is identically zero if the curve is autoparallel with respect to the Levi-Civita connection.

In the following chapters, we will consider different combinations of the above constraints that guarantee the 4-velocity normalization.

5

Reconstructing homogeneous $f(\mathcal{Q})$ cosmologies

As we mentioned in the previous chapters, alternative theories of gravity have been developed to overcome the shortcomings of GR in describing the desired behavior of the universe. Among the developed alternative theories there is $f(\mathcal{Q})$ gravity, the generalization of the STEGR introduced in Ch. 3. A first approach in this direction is given by the works [69, 70], in which it is shown that in a spatially flat FLRW metric using a function of the type

$$f(\mathcal{Q}) = \mathcal{Q} + \alpha \mathcal{Q}^n, \tag{5.1}$$

and the coincident gauge, we can obtain solutions in which the universe goes from a matter-dominated epoch to an asymptotically de Sitter universe without the introduction of any cosmological constant. However, perturbation analysis shows that these solutions present strong coupling problems due to the disappearance of the two scalar degrees of freedom that propagate when de Sitter backgrounds are considered. The results of these models were also tested by a comparison with experimental data showing good compatibility with the theoretical results [140–143]. Other models of spatially flat FLRW cosmology can be found in [144] and [145]; specifically, in the latter, it is shown that $f(\mathcal{Q})$ gravity can satisfy the constraints imposed by the Big Bang Nucleosynthesis. On the other hand, studies of non-flat FLRW metrics were conducted not using the coincident gauge but the symmetries of the metric, which impose constraints on the nonmetricity via Lie derivative [146, 147]. FLRW metric has been studied from a quantum point of view as well. For example, in [148], the form of the function $f(\mathcal{Q})$ that conducts cosmological bouncing solutions has been selected by order reduction; moreover, the Hamiltonian formalism is used to find the Wheeler-DeWitt equation and the wave function of the universe. On the other hand, in [149], by introducing Dirac brackets, it results that the cosmological quantum theory of $f(\mathcal{Q})$ gravity is highly dependent on the nature of matter.

There is no shortage of studies of BI metrics, although they are far fewer than FLRW metrics. In [150] a local rotationally symmetric BI metric is studied, where in Eq. (4.73) the assumption $b = c$ is set. It turns out that by choosing a second-order polynomial function for $f(\mathcal{Q})$, i.e., n=2 in Eq. (5.1), the universe approaches a phase of isotropy at late times. The same kind of BI metric is also considered in [151] and [152] but with

$f(\mathcal{Q}) = \alpha\mathcal{Q} + \beta$, where α and β are free parameter. Here, the solutions were compared with experimental data by obtaining constraints on the parameters of the theory.

In all previous works, the study procedure is to choose the form of $f(\mathcal{Q})$ a priori and then determine the dynamics of the cosmology. We decided to follow a different approach that is known as reconstruction method. Reconstruction methods were used for the first time by G. F. R. Ellis and M. S. Madsen [153] to find the potential functions needed in models of inflationary universes. The general idea consists of reversing the usual procedure of resolution: a given form for the spatial scale factor is assumed and, once substituted into the cosmological equations, further information is derived on the remaining unknown functions of the theory, for instance, the inflaton potential function in [153]. Subsequently, reconstruction was used in various frameworks, like in the cosmology of $f(R)$ gravity [154], where the generic function of the Ricci scalar is reconstructed starting from given scale factors, in scalar-tensor cosmologies [155] and in the study of static and spherically symmetric spacetimes [156–159].

In the following, we will consider such a method to study both BI and spatially flat FLRW universes in $f(\mathcal{Q})$ gravity.

The study performed in this chapter is based on the paper "Reconstructing isotropic and anisotropic $f(\mathcal{Q})$ cosmologies" [74].

5.1 $f(\mathcal{Q})$ cosmology

Let us now specialize the field equations (3.53) and (3.54) to the study of BI and FLRW universes. These equations will constitute the core of the reconstruction method we intend to apply.

In this chapter, we will consider an observer represented by an autoparallel curve with normalized 4-velocity $u^a = (1, 0, 0, 0)$ and the coincident gauge, so the condition (4.105) will reduce to $\partial_0 g_{00} = 0$, which is always satisfied by both the FLRW metric (4.30) and the BI one (4.73).

5.1.1 Bianchi Type-I metric

As we showed in Sec. 4.3, the BI metric represents spatially flat, homogeneous but not isotropic spacetimes,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2 + b^2(t)\mathrm{d}y^2 + c^2(t)\mathrm{d}z^2. \tag{5.2}$$

Because we adopt the coincident gauge[1], the nonmetricity scalar is

$$\mathcal{Q} = 2\left(\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc}\right), \tag{5.3}$$

where the dot represents the derivative with respect to time.

To derive the cosmological equations, we assume that, at cosmological level, the hypermomentum tensor ${\Delta_{ab}}^c$ is zero, and the matter is described by the energy-momentum tensor of a perfect fluid,

$$\Psi_{ab} = (\rho + p)\, u_a u_b + p g_{ab}, \tag{5.4}$$

[1]With this choice, we assert that the comoving coordinates and cosmic time used to write Eq. (5.2) are the same ones that ensure the vanishing of the total connection ${\Gamma_{ab}}^c$.

where pressure p and the energy density ρ are related by the equation of state $p = w\rho$, with w the barotropic factor, and ρ satisfies the continuity equation derived from Eq. (3.55),

$$\dot{\rho} + \frac{\dot{\tau}}{\tau}(1+w)\rho = 0, \tag{5.5}$$

whose solution in terms of the volume of the universe $\tau(t) = a(t)b(t)c(t)$ is

$$\rho = \rho_0 \tau^{-(1+w)}, \tag{5.6}$$

where ρ_0 is the density at a given initial time. We derive the cosmological equations from the temporal and spatial part of Eq. (3.53),

$$\frac{1}{2}f - 2f'\left(\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{a}\dot{c}}{ac}\right) = -\rho, \tag{5.7}$$

$$\dot{f}'\left(\frac{\dot{a}}{a} - \frac{\dot{\tau}}{\tau}\right) - f'\left(\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + 2\frac{\dot{b}\dot{c}}{bc}\right) + \frac{1}{2}f = p, \tag{5.8}$$

$$\dot{f}'\left(\frac{\dot{b}}{b} - \frac{\dot{\tau}}{\tau}\right) - f'\left(\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + 2\frac{\dot{a}\dot{c}}{ac}\right) + \frac{1}{2}f = p, \tag{5.9}$$

$$\dot{f}'\left(\frac{\dot{c}}{c} - \frac{\dot{\tau}}{\tau}\right) - f'\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} + 2\frac{\dot{a}\dot{b}}{ab}\right) + \frac{1}{2}f = p. \tag{5.10}$$

We can recast the above equations in a more useful form by performing some simple operations.

Let us consider the combination of Eqs. (5.7)-(5.10) given by Eq. (5.7) multiplied by $-\dot{\tau}/\tau$ and added to Eqs. (5.8), (5.9), and (5.10) multiplied by $3\dot{a}/a$, $3\dot{b}/b$, and $3\dot{c}/c$, respectively. The resulting expression is an equivalent of the Raychaudhuri equation (4.39),

$$f\frac{\dot{\tau}}{\tau} - f'\left(\frac{3}{2}\dot{\mathcal{Q}} + 2\mathcal{Q}\frac{\dot{\tau}}{\tau}\right) - 3f''\mathcal{Q}\dot{\mathcal{Q}} = \frac{\dot{\tau}}{\tau}(\rho + 3p), \tag{5.11}$$

where we used that

$$\dot{f}'(\mathcal{Q}) = f''(\mathcal{Q})\dot{\mathcal{Q}}, \tag{5.12}$$

and

$$\dot{\mathcal{Q}} = \frac{\ddot{a}\dot{b}}{ab} + \frac{\ddot{a}\dot{c}}{ac} + \frac{\dot{a}\ddot{b}}{ab} - \frac{\dot{a}^2\dot{b}}{a^2b} - \frac{\dot{a}\dot{b}^2}{ab^2} + \frac{\dot{a}\ddot{c}}{ac} - \frac{\dot{a}^2\dot{c}}{a^2c} - \frac{\dot{a}\dot{c}^2}{ac^2} + \frac{\ddot{b}\dot{c}}{bc} + \frac{\dot{b}\ddot{c}}{bc} - \frac{\dot{b}^2\dot{c}}{b^2c} - \frac{\dot{b}\dot{c}^2}{bc^2}. \tag{5.13}$$

However, we can also obtain Eq. (5.11) using only Eqs. (5.5) and (5.7) taking the time derivative of Eq. (5.7) multiplied by 3, adding Eq. (5.7) multiplied by $2\dot{\tau}/\tau$ and substituting (5.5). Thus, if the Eqs. (5.5) and (5.7) are satisfied, so is Eq. (5.11). This step will be crucial for developing the reconstruction algorithm as it allows us to remove one equation.

Instead, subtracting Eq. (5.9) and Eq. (5.10) from Eq. (5.8), we obtain, respectively:

$$\frac{\dot{a}}{a} - \frac{\dot{b}}{b} = \frac{k_{ab}}{f'\tau} \quad \rightarrow \quad \frac{a}{b} = e^{d_1}\exp\int\frac{k_{ab}}{f'\tau}dt \tag{5.14}$$

and

$$\frac{\dot{a}}{a} - \frac{\dot{c}}{c} = \frac{k_{ac}}{f'\tau} \quad \rightarrow \quad \frac{a}{c} = e^{d_2}\exp\int\frac{k_{ac}}{f'\tau}dt, \tag{5.15}$$

where k_{ab}, k_{ac}, d_1, and d_2 are constants of integration.

Therefore, we can consider the equivalent set of independent cosmological equations given by

$$\frac{1}{2}f - 2f'\left(\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{a}\dot{c}}{ac}\right) = -\rho, \tag{5.16}$$

$$\dot{\rho} + \frac{\dot{\tau}}{\tau}(1+w)\rho = 0, \tag{5.17}$$

$$\frac{\dot{a}}{a} - \frac{\dot{b}}{b} = \frac{k_{ab}}{f'\tau}, \tag{5.18}$$

$$\frac{\dot{a}}{a} - \frac{\dot{c}}{c} = \frac{k_{ac}}{f'\tau}. \tag{5.19}$$

One of the last two equations can be replaced, depending on the situation we are analyzing, by the relation that we derive with simple algebraic steps from Eqs. (5.18) and (5.19),

$$(k_{ab} - k_{ac})\frac{\dot{a}}{a} + k_{ac}\frac{\dot{b}}{b} - k_{ab}\frac{\dot{c}}{c} = 0. \tag{5.20}$$

Notice that the number of cosmological equations has decreased. This is possible because one of the Eqs. (5.8)-(5.10) can be replaced by Eq. (5.11), which is always satisfied given the solutions of the Eqs. (5.5) and (5.7).

5.1.2 FLRW metric

If we require isotropy in the BI metric, i.e., $a(t) = b(t) = c(t)$ in (5.2), we obtain the spatially flat FLRW metric. The nonmetricity scalar is related to the Hubble parameter $H = \dot{a}/a$ by

$$\mathcal{Q} = 6H^2, \tag{5.21}$$

and the Raychaudhuri equation (5.11) is equal to

$$\frac{1}{6}f - f'\left(\dot{H} + 2H^2\right) - 12f''H^2\dot{H} = \frac{1}{6}(\rho + 3p). \tag{5.22}$$

The cosmological equations (5.16)-(5.19) reduce to the following

$$\frac{1}{2}f - 6H^2 f' = -\rho, \tag{5.23}$$

$$\dot{\rho} + 3H(1+w)\rho = 0. \tag{5.24}$$

To facilitate the study of the examples that will be considered, it is useful to define the deceleration parameter,

$$q = -\frac{\ddot{a}a}{\dot{a}^2}, \tag{5.25}$$

and rewrite Eqs. (5.22) and (5.23) in a more expressive form,

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\hat{\rho}_M + 3\hat{p}_M) - \frac{1}{6}(\hat{\rho}_f + 3\hat{p}_f), \tag{5.26}$$

$$H^2 = \frac{1}{3}(\hat{\rho}_M + \hat{\rho}_f), \tag{5.27}$$

where

$$\hat{\rho}_M = \frac{1}{2}\frac{\rho}{f'} \qquad \text{and} \qquad \hat{p}_M = \frac{p}{f'} \tag{5.28}$$

represent the standard energy density and pressure, with an effective gravitational constant regulated by $f'(\mathcal{Q})$, while

$$\hat{\rho}_f = \frac{1}{4}\frac{f}{f'}, \quad \text{and} \quad \hat{p}_f = 2\left[\frac{\mathcal{Q}}{4} - \frac{1}{4}\frac{f}{f'} + \frac{f''}{f'}H\dot{\mathcal{Q}}\right] \tag{5.29}$$

represent the energy density and pressure of an effective fluid associated with the presence of nonmetricity.

5.2 Reconstruction method: Bianchi Type-I

In this section, we will apply the reconstruction algorithm to investigate some exact solutions in BI models. Given suitable scale factors, we will find the function $f(\mathcal{Q})$, which admits such scale factors as solutions of the corresponding cosmological equations.

5.2.1 Example 1: Power law scale factors

Let us start by assuming each scale factor as a power law as in the classical Kasner solution, but without the restrictions on the exponents (4.95),

$$a(t) = a_0 t^n, \qquad b(t) = b_0 t^m, \qquad c(t) = c_0 t^l, \tag{5.30}$$

$$\tau = abc = a_0 b_0 c_0 t^N \tag{5.31}$$

where $N = n + m + l$ and a_0, b_0 and c_0 are dimensional constants. In such a circumstance, the nonmetricity scalar takes the form

$$\mathcal{Q} = 2\,(nm + nl + ml)\,t^{-2} = \xi t^{-2}, \tag{5.32}$$

with $\xi = 2(nm + nl + ml)$. Making use of Eq. (5.32), from the definition of the spatial volume and the continuity equation we obtain the expressions of τ and ρ as functions of $\mathcal{Q}$,

$$\tau(\mathcal{Q}) = \tau_0 \left(\frac{\xi}{\mathcal{Q}}\right)^{\frac{N}{2}}, \tag{5.33}$$

$$\rho(\mathcal{Q}) = \rho_0 \tau^{-(1+\varpi)} = \rho_0 \tau_0^{-(1+\varpi)} \left(\frac{\mathcal{Q}}{\xi}\right)^{\frac{1}{2}(1+\varpi)N}, \tag{5.34}$$

where $\tau_0 = a_0 b_0 c_0$. In the above expression and the following, we will use ϖ instead of w to emphasize the fact that ϖ is just a parameter for the theory we will reconstruct, and it is not related to any matter source the final reconstructed theory might be coupled with. Inserting Eq. (5.34) into Eq. (5.16), we obtain the differential equation

$$\frac{1}{2}f - \mathcal{Q}f' = -\epsilon \mathcal{Q}^{\frac{1}{2}(1+\varpi)N} \tag{5.35}$$

with

$$\epsilon = \rho_0 \tau_0^{-(1+\varpi)} \xi^{-\frac{1}{2}(1+\varpi)N}. \tag{5.36}$$

Equation (5.35) admits the solution,

$$f(\mathcal{Q}) = f_0\sqrt{\mathcal{Q}} + 2\epsilon\frac{\mathcal{Q}^{\frac{1}{2}(1+\bar{w})N}}{N(1+\bar{w})-1}, \tag{5.37}$$

where f_0 is a constant of integration and will be so throughout the chapter[2].

Using Eqs. (5.30), (5.31) and (5.37), Eqs. (5.18), and (5.19) generate the following constraints on the integration constants:

$$f_0 = 0, \qquad n+m+l = \frac{1}{\bar{w}}, \tag{5.38}$$

$$k_{ab} = \frac{\bar{w}(1+\bar{w})\rho_0\tau_0^{-\bar{w}}(n-m)}{2\left[m+n-\bar{w}\left(m^2+mn+n^2\right)\right]}, \tag{5.39}$$

$$k_{ac} = \frac{(1+\bar{w})\rho_0\tau_0^{-\bar{w}}(\bar{w}m+2\bar{w}n-1)}{2\left[m+n-\bar{w}\left(m^2+mn+n^2\right)\right]}. \tag{5.40}$$

Notice that relation (5.38) implies that we are forced to exclude the case $\bar{w}=0$. If we set $\bar{w}=0$ from the beginning, then we would obtain:

$$m = n = l = \frac{1}{3}, \tag{5.41}$$

i.e., an isotropic solution.

5.2.2 Example 2: More complex scale factors.

In this second example, we choose the scale factors as follows:

$$\begin{gathered} a(t) = \alpha\mathtt{a}(t)\sqrt[3]{\tau(t)}, \qquad b(t) = \beta\mathtt{b}(t)\sqrt[3]{\tau(t)}, \\ c(t) = \gamma\mathtt{c}(t)\sqrt[3]{\tau(t)}, \end{gathered} \tag{5.42}$$

which is an interesting template for BI solutions (see, e.g. [136]). The quantities α, β, and γ are generic constants.

The first step is to derive $\mathtt{b}(t)$ from Eq. (5.20),

$$\mathtt{b} = b_0\mathtt{a}^{\frac{k_{ac}-k_{ab}}{k_{ac}}}\mathtt{c}^{\frac{k_{ab}}{k_{ac}}}, \tag{5.43}$$

where b_0 is a constant of integration. Then, using the definition of τ and $\mathcal{Q}$, we obtain the scale factor

$$\mathtt{a} = a_0\mathtt{c}^{\frac{k_{ab}+k_{ac}}{k_{ab}-2k_{ac}}}, \tag{5.44}$$

and the relation

$$\frac{\dot{\mathtt{c}}^2}{\mathtt{c}^2} = \frac{(k_{ab}-2k_{ac})^2}{18\Omega^2}\left(2\frac{\dot{\tau}^2}{\tau^2} - 3\mathcal{Q}\right), \tag{5.45}$$

[2] One might think that the result in Eq. (5.37) is only valid for the fluid chosen in the reconstruction process, however, such a conclusion would be incorrect. In fact, if we use Eq. (5.37) and fluids with $w \neq \bar{w}$ in Eqs. (5.7)-(5.10), then we obtain a different evolution for the scale factors from the one used for the reconstruction method. For the sake of simplicity, we will show this explicitly in Section 5.3 for the FRLW case.

with

$$a_0 = (b_0 \alpha \beta \gamma)^{\frac{k_{ac}}{k_{ab}-2k_{ac}}} \tag{5.46}$$

and

$$\Omega = \sqrt{k_{ab}^2 - k_{ab}k_{ac} + k_{ac}^2}. \tag{5.47}$$

If we now extrapolate $f'(\mathcal{Q})$ from Eq. (5.16),

$$f' = \frac{f + 2\rho_0 \tau^{-(1+\varpi)}}{2\mathcal{Q}}, \tag{5.48}$$

and replace all in Eq. (5.18), then we find that particular cosmological solutions can be found imposing the conditions,

$$f = K_f \tau^{-1-\varpi}, \tag{5.49}$$

$$\frac{\dot{\tau}^2}{\tau^2} = \frac{1}{4}\mathcal{Q}\left(K_\tau^2\, \mathcal{Q}\, \tau^{2\varpi} + 6\right), \tag{5.50}$$

$$K_\tau(K_f + 2\rho_0) - 4\Omega = 0. \tag{5.51}$$

Equations (5.49) and (5.50) can be resolved if we make explicit the dependence of $\mathcal{Q}$ on τ. A convenient choice is

$$\tau = \pm \left(\frac{\mathcal{Q}_0}{\mathcal{Q}}\right)^{\frac{1}{n}}, \tag{5.52}$$

with $\mathcal{Q}_0 > 0$ a dimensional constant. Relation (5.52), together with Eq. (5.49), allows us to write

$$f(\mathcal{Q}) = K_f \left[\pm \left(\frac{\mathcal{Q}_0}{\mathcal{Q}}\right)^{\frac{1}{n}}\right]^{-1-\varpi}, \tag{5.53}$$

which, when substituted into Eq. (5.16), provides the value of the constant K_f,

$$K_f = \frac{2n\rho_0}{2\varpi + 2 - n}, \tag{5.54}$$

and from Eq. (5.51), also the value of K_τ,

$$K_\tau = \frac{\Omega(2\varpi + 2 - n)}{\rho_0(1+\varpi)}. \tag{5.55}$$

To proceed further, we will analyze separately three subcases: $\varpi = 0$, $n = 2\varpi$, and $n = \varpi$.

Case $\varpi = 0$

If we set $\varpi = 0$ and

$$\tau = -\left(\frac{\mathcal{Q}_0}{\mathcal{Q}}\right)^{\frac{1}{n}}, \tag{5.56}$$

with n an odd integer, then the solution of Eq. (5.50) is

$$\tau(t) = a_1 b_1 c_1 \left[\frac{4K_\tau^2}{9n^2} - (t - t_0)^2\right]^{\frac{1}{n}}. \tag{5.57}$$

The quantities a_1, b_1 and c_1 are constants depending on α, β, γ, b_0, k_{ab}, k_{ac}, ρ_0 and $\mathcal{Q}_0$ and the parameter t_0 is the instant of time in which the initial data are assigned. This notation will also be used in the subsequent examples. The scale factors assume the form,

$$a(t) = a_1 \left[\frac{4K_\tau^2}{9n^2} - (t-t_0)^2\right]^{\frac{1}{3n}} \exp\left\{ - m_1 \tanh^{-1}\left[\frac{3n(t-t_0)}{2K_\tau}\right]\right\}, \tag{5.58}$$

$$b(t) = b_1 \left[\frac{4K_\tau^2}{9n^2} - (t-t_0)^2\right]^{\frac{1}{3n}} \exp\left\{ - m_2 \tanh^{-1}\left[\frac{3n(t-t_0)}{2K_\tau}\right]\right\}, \tag{5.59}$$

$$c(t) = c_1 \left[\frac{4K_\tau^2}{9n^2} - (t-t_0)^2\right]^{\frac{1}{3n}} \exp\left\{ - m_3 \tanh^{-1}\left[\frac{3n(t-t_0)}{2K_\tau}\right]\right\}, \tag{5.60}$$

where

$$m_1 = \frac{2(k_{ab}+k_{ac})}{3n\Omega}, \tag{5.61}$$

$$m_2 = \frac{2(k_{ac}-2k_{ab})}{3n\Omega}, \tag{5.62}$$

$$m_3 = \frac{2(k_{ab}-2k_{ac})}{3n\Omega}. \tag{5.63}$$

Notice that the sum of m_1, m_2, and m_3 is zero,

$$m_1 + m_2 + m_3 = 0, \tag{5.64}$$

and it will be the same in the other cases of BI metrics discussed in this section.

In Fig. 5.1 we show an example of the evolution of scale factors: at the beginning, the universe is spatially one dimensional (and therefore singular) and becomes spatially one dimensional again after a process of expansion and contraction, as described by the behavior of τ. Hence, the relative differences between the scale factors are greatest at the beginning and the end of cosmic history, with a time interval in which the values of the scale factors are close to each other.

On the other hand, if

$$\tau = \left(\frac{\mathcal{Q}_0}{\mathcal{Q}}\right)^{\frac{1}{n}}, \tag{5.65}$$

with n still an odd integer or a rational number with an odd denominator, the solution of Eq. (5.50) is

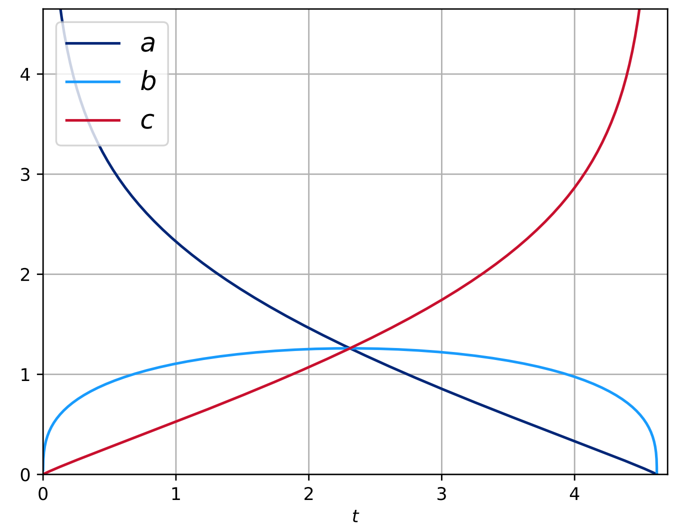
$$\tau(t) = a_1 b_1 c_1 \left[(t-t_0)^2 - \frac{4K_\tau^2}{9n^2}\right]^{\frac{1}{n}}, \tag{5.66}$$

Given Eq. (5.66), and choosing two scale factors proportional to each other, e.g. $a = b$, we have

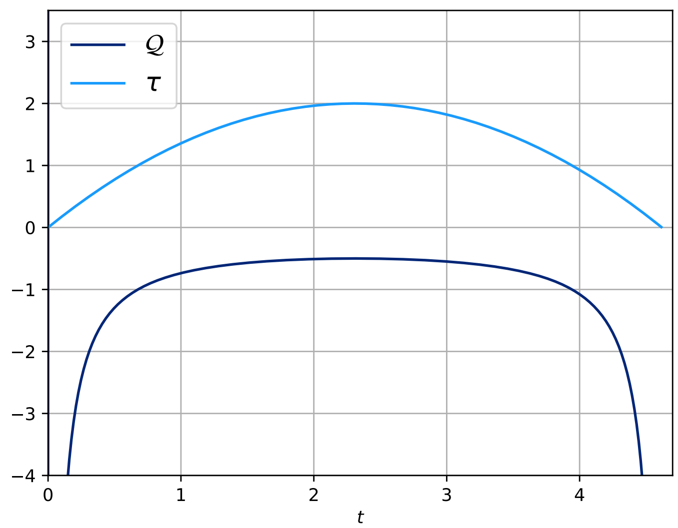
$$a(t) = a_1 \left[(t-t_0)^2 - \frac{4K_\tau^2}{9n^2}\right]^{\frac{1}{3n}} \left[\frac{3n\,(t-t_0) - 2K_\tau}{3n\,(t-t_0) + 2K_\tau}\right]^{m_1}, \tag{5.67}$$

and

$$c(t) = c_1 \left[(t-t_0)^2 - \frac{4K_\tau^2}{9n^2}\right]^{\frac{1}{3n}} \left[\frac{3n\,(t-t_0) - 2K_\tau}{3n\,(t-t_0) + 2K_\tau}\right]^{m_2}, \tag{5.68}$$



(a) Scale factors



(b) Nonmetricity scalar and τ.

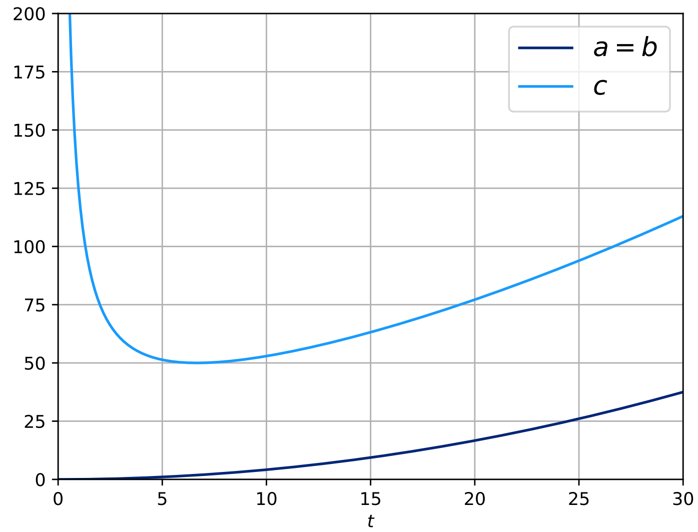
Figure 5.1: Evolution of (5.57)-(5.60) with values $n = 1$, $K_\tau = 2\sqrt{3}$, $\mathcal{Q}_0 = 1$, $a_1 = b_1 = c_1 = \frac{\sqrt[3]{3}}{2}$, $m_1 = \frac{2}{\sqrt{3}}$, $m_2 = 0$, $m_3 = -\frac{2}{\sqrt{3}}$, and $t_0 = \frac{4}{\sqrt{3}}$.

where

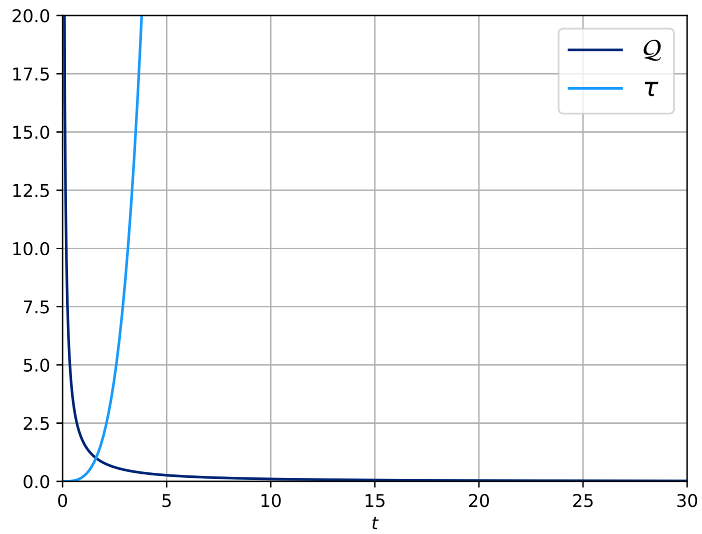
$$m_1 = \frac{1}{3n}, \tag{5.69}$$

$$m_2 = -\frac{2}{3n}. \tag{5.70}$$

A representation of the cosmological evolution is given in Fig. 5.2. After an initial singular phase, in which the universe is spatially one dimensional, the scale factors grow showing a similar behavior for large values of t. This trend can be immediately verified by taking the limit for $t \to \infty$ of Eq. (5.67) and of Eq. (5.68).



(a) Scale factors



(b) Nonmetricity scalar and τ.

Figure 5.2: Evolution of (5.66)-(5.68) with values $n = \frac{1}{3}$, $K_\tau = \frac{10}{3}$, $\mathcal{Q}_0 = 1$, $a_1 = c_1 = \frac{1}{24}$, $m_1 = 1$, $m_2 = 2$, and $t_0 = -\frac{20}{3}$.

Case $n = 2\varpi$

We now set $n = 2\varpi$, $\varpi \neq 0$, and

$$\tau = \left(\frac{\mathcal{Q}_0}{\mathcal{Q}}\right)^{\frac{1}{2\varpi}} . \tag{5.71}$$

In this case, the solution of Eq. (5.50) is

$$\tau(t) = a_1 b_1 c_1 \left(t + t_0\right)^{\frac{1}{\varpi}} . \tag{5.72}$$

Therefore, the scale factors are equal to

$$a(t) = a_1 \left(t + t_0\right)^{\frac{1}{3\varpi} + m_1} , \tag{5.73}$$

$$b(t) = b_1 \left(t+t_0\right)^{\frac{1}{3\varpi}+m_2}, \tag{5.74}$$

$$c(t) = c_1 \left(t+t_0\right)^{\frac{1}{3\varpi}+m_3}, \tag{5.75}$$

where

$$m_1 = \frac{K_\tau \sqrt{\mathcal{Q}_0}(k_{ab}+k_{ac})}{3\varpi\Omega\sqrt{\mathcal{Q}_0 K_\tau^2+6}}, \tag{5.76}$$

$$m_2 = \frac{K_\tau \sqrt{\mathcal{Q}_0}(k_{ac}-2k_{ab})}{3\varpi\Omega\sqrt{\mathcal{Q}_0 K_\tau^2+6}}, \tag{5.77}$$

$$m_3 = \frac{K_\tau \sqrt{\mathcal{Q}_0}(k_{ab}-2k_{ac})}{3\varpi\Omega\sqrt{\mathcal{Q}_0 K_\tau^2+6}}. \tag{5.78}$$

With $t_0 = 0$, we recover the results of Sec. 5.2.1,

$$a(t) = a_1 t^{\frac{1}{3\varpi}+m_1}, \quad b(t) = b_1 t^{\frac{1}{3\varpi}+m_2}, \quad c(t) = c_1 t^{\frac{1}{3\varpi}+m_3}. \tag{5.79}$$

We give a representation of the evolution of the system in Fig. 5.3. The scale factors are always growing, but their growth rate is different. We notice a particular instant of time where the scale factors coincide.

Case $n = \varpi$

If $n = \varpi$, $\varpi \neq 0$, and

$$\tau = \left(\frac{\mathcal{Q}_0}{\mathcal{Q}}\right)^{\frac{1}{\varpi}}, \tag{5.80}$$

then the solution of Eq. (5.50) is

$$\tau(t) = a_1 b_1 c_1 \left\{ \sinh\left[\frac{1}{4} K_\tau \mathcal{Q}_0 \varpi (t+t_0)\right] \right\}^{\frac{2}{\varpi}}. \tag{5.81}$$

The scale factors are

$$a(t) = a_1 \left\{ \sinh\left[\frac{1}{4} K_\tau \mathcal{Q}_0 \varpi (t+t_0)\right] \right\}^{\frac{2}{3\varpi}} e^{m_1 t}, \tag{5.82}$$

$$b(t) = b_1 \left\{ \sinh\left[\frac{1}{4} K_\tau \mathcal{Q}_0 \varpi (t+t_0)\right] \right\}^{\frac{2}{3\varpi}} e^{m_2 t}, \tag{5.83}$$

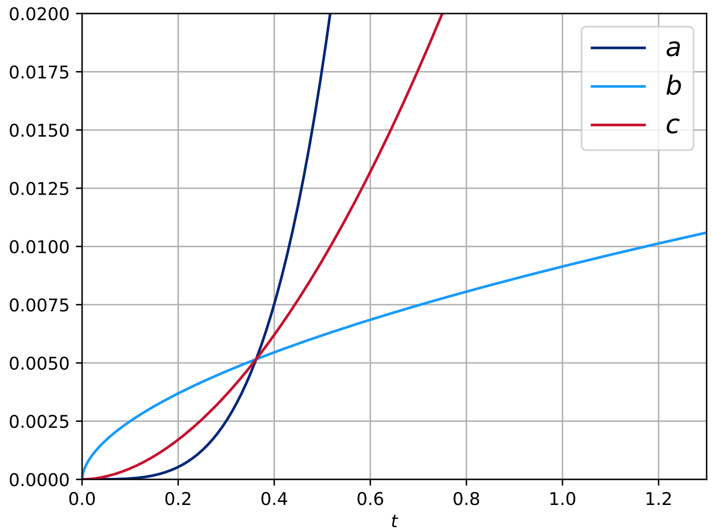
$$c(t) = c_1 \left\{ \sinh\left[\frac{1}{4} K_\tau \mathcal{Q}_0 \varpi (t+t_0)\right] \right\}^{\frac{2}{3\varpi}} e^{m_3 t}, \tag{5.84}$$

with

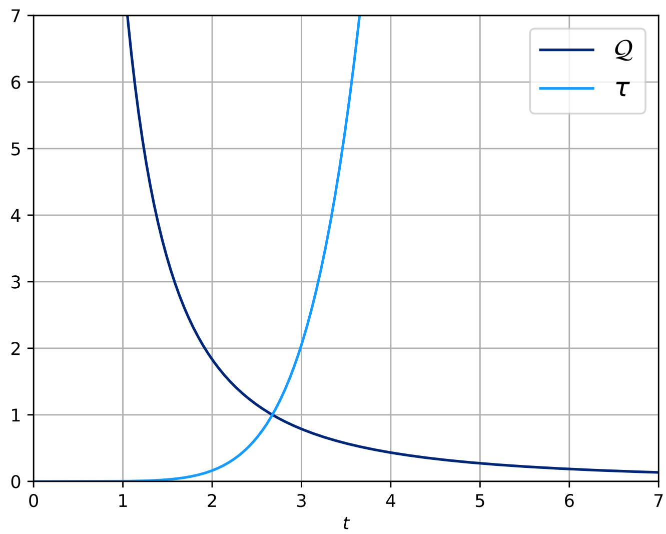
$$m_1 = \frac{K_\tau \mathcal{Q}_0 (k_{ab}+k_{ac})}{6\Omega}, \tag{5.85}$$

$$m_2 = \frac{K_\tau \mathcal{Q}_0 (k_{ac}-2k_{ab})}{6\Omega}, \tag{5.86}$$

$$m_3 = \frac{K_\tau \mathcal{Q}_0 (k_{ab}-2k_{ac})}{6\Omega}. \tag{5.87}$$



(a) Scale factors



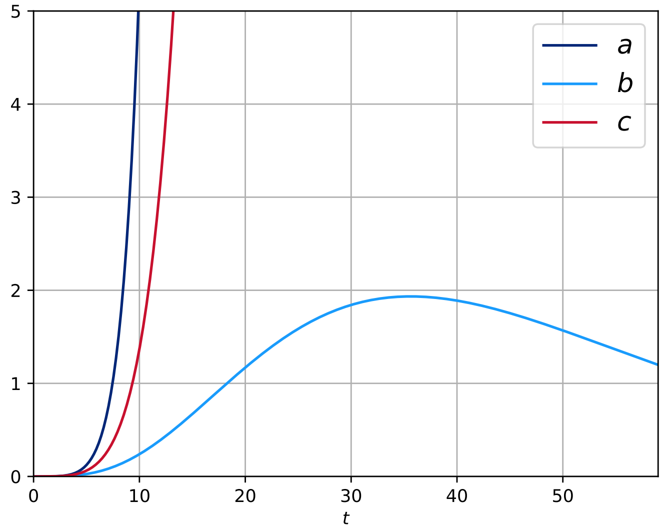
(b) Double logarithm plot of nonmetricity scalar and τ.

Figure 5.3: Evolution of (5.72)-(5.75) with values $w = 0.16$, $K_\tau = 1.25$, $\mathcal{Q}_0 = 1$, $a_1 = 0.25$, $b_1 = 9.14 \cdot 10^{-3}$, $c_1 = 3.42 \cdot 10^{-2}$, $m_1 = 1.74$, $m_2 = -1.52$, $m_3 = -0.22$, and $t_0 = 0$.

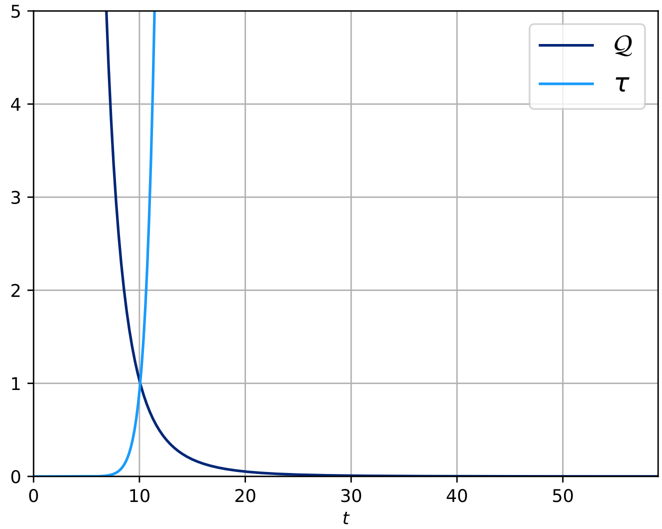
Let us consider the example shown in Fig. 5.4 where two scale factors grow up to infinity while the third one, which in the figure is represented by the scale factor b, once it has reached a maximum, tends to zero. Therefore, during its evolution, the universe undergoes a first phase where the scale factors are all growing and a second one where the spacetime tends to become spatially two-dimensional and therefore singular.

5.3 Reconstruction method: FLRW

In this section, we will consider spatially flat FLRW cosmologies. As it will be evident, the higher symmetry of the FLRW spacetime, compared to BI, will make it easier to apply



(a) Scale factors



(b) Nonmetricity scalar and τ.

Figure 5.4: Evolution of (5.81)-(5.84) with values $\varpi = 0.16$, $K_\tau = 0.55$, $\mathcal{Q}_0 = 1$, $a_1 = b_1 = c_1 = 515$, $m_1 = 0.17$, $m_2 = -0.14$, $m_3 = 3.45 \cdot 10^{-2}$, and $t_0 = 0$.

the reconstruction method.

5.3.1 Example 1: reconstruction from a time-dependent scale factor

To start with, we consider two scale factors $a = a(t)$: a power law and an exponential function of time.

Scale factor as a power law

Setting

$$a(t) = a_0 t^n, \tag{5.88}$$

with a_0 a dimensional constant, the nonmetricity scalar assumes the form

$$\mathcal{Q} = 6H^2 = 6n^2 t^{-2}. \tag{5.89}$$

By inverting the relation (5.89), we may express the scale factor and density ρ as function of $\mathcal{Q}$:

$$a(\mathcal{Q}) = a_0 \left(\frac{\alpha}{\mathcal{Q}}\right)^{\frac{n}{2}} \tag{5.90}$$

and

$$\rho = \rho_0 a^{-3(1+\varpi)} = \rho_0 a_0^{-3(1+\varpi)} \left(\frac{\mathcal{Q}}{\alpha}\right)^{\frac{3}{2}n(1+\varpi)}, \tag{5.91}$$

with $\alpha = 6n^2$. Replacing Eq. (5.91) in Eq. (5.23) and solving the resulting differential equation, we obtain the function

$$f(\mathcal{Q}) = f_0\sqrt{\mathcal{Q}} + f_1 \mathcal{Q}^{\frac{3}{2}(1+\varpi)n} \tag{5.92}$$

with

$$f_1 = \frac{2\rho_0}{3n(1+\varpi)-1} a_0^{-3(1+\varpi)} \alpha^{-\frac{3}{2}(1+\varpi)n}. \tag{5.93}$$

The function (5.92) is similar to the ones that have been mostly used in literature so far [69, 160][3].

Now we will show that, as anticipated in Sec. 5.2.1, the solution (5.92) is valid even if we choose to resolve the cosmological equations in the case of fluids with $w \neq \varpi$. For example, considering $\varpi = 0$, then

$$f(\mathcal{Q}) = f_0\sqrt{\mathcal{Q}} + \frac{2\rho_0}{3n-1} a_0^{-3} \left(\frac{\mathcal{Q}}{\alpha}\right)^{\frac{3}{2}n}, \tag{5.94}$$

and a fluid with $w = \frac{1}{3}$, i.e. radiation, for which

$$\rho_1 = \rho_{1,0} a^{-4}. \tag{5.95}$$

Substituting Eqs. (5.94) and (5.95) into Eq. (5.23), we obtain the following expression for $a(t)$,

$$a(t) = \left(\frac{2\sqrt[3]{\alpha}}{3}\right)^{\frac{9n}{8}} \sqrt[4]{\frac{a_0^3 \rho_{1,0}}{\rho_0}} \left(\frac{t+t_0}{n}\right)^{\frac{3n}{4}}, \tag{5.96}$$

which, for the same value of n, is clearly different from (5.88).

Scale factor as an exponential function

We set now

$$a(t) = a_0 e^{m(t-t_0)^{2n+1}}, \tag{5.97}$$

where $m > 0$, a_0 and t_0 are generic constants, and n is a natural number. The scale factor (5.97) describes a cosmic scenario in which all of the three main phases of the

[3]Notice that if we were to start with $n = \frac{2}{3(1+\varpi)}$, the solution that encompasses all the classical Friedmannian cosmological solutions, then the (5.92) would give $f(\mathcal{Q}) \propto \mathcal{Q}$. This result implies that $f(\mathcal{Q})$ gravity can have, at most, one cosmological solution in common with GR.

cosmological evolution (inflation, Friedmann phase, and dark phase) are represented (see Fig. 5.5a). The duration of the Friedmann phase is related to the value of the odd exponent $2n+1$, which therefore plays a crucial role. The nonmetricity scalar associated with the scale factor (5.97) has the form

$$\mathcal{Q} = 6(2n+1)^2 m^2 (t-t_0)^{4n}. \tag{5.98}$$

With the help of Eq. (5.98), we derive the expressions of the scale factor a and the density ρ as function of $\mathcal{Q}$:

$$a(\mathcal{Q}) = a_0 \exp\left[m\left(\frac{\mathcal{Q}}{\alpha}\right)^{\frac{2n+1}{4n}}\right] \tag{5.99}$$

and

$$\rho = \rho_0 a_0^{-3(1+\varpi)} \exp\left[-3m(1+\varpi)\left(\frac{\mathcal{Q}}{\alpha}\right)^{\frac{2n+1}{4n}}\right], \tag{5.100}$$

with $\alpha = 6(2n+1)^2 m^2$.

Inserting Eq. (5.100) into Eq. (5.23) and solving it, we obtain the function

$$f(\mathcal{Q}) = f_0\sqrt{\mathcal{Q}} - f_1 \mathcal{Q}^{\frac{1}{2}} \Gamma\left[-\frac{2n}{2n+1}, 3m(1+\varpi)\left(\frac{\mathcal{Q}}{\alpha}\right)^{\frac{2n+1}{4n}}\right] \tag{5.101}$$

where Γ is the incomplete gamma function, and

$$f_1 = \frac{4n}{2n+1}\frac{\rho_0}{\alpha^2} a_0^{-3(1+\varpi)} \left[3m(1+\varpi)\right]^{\frac{2n}{2n+1}}. \tag{5.102}$$

As it can be seen in Fig. 5.5b, here the effective gravitational constant $1/f'(\mathcal{Q})$ is positive, whereas the nonmetricity correction $\hat{\rho}_f$ is always negative. However, $\hat{\rho}_f$ grows slower than the matter term $\hat{\rho}_M$, so the scale factor will always tend to increase, and the cosmology expands.

5.3.2 Example 2: reconstruction from the time derivative of the scale factor

The previous examples relied explicitly on the inversion of the expression of the nonmetricity scalar, i.e., we always needed to obtain a relation of the form $t = t(\mathcal{Q})$. Such inversion is not always possible analytically. Another option is to give an implicit expression for the scale factor. In particular, we can consider the scale factor $a(t)$ as defined by a suitable differential equation,

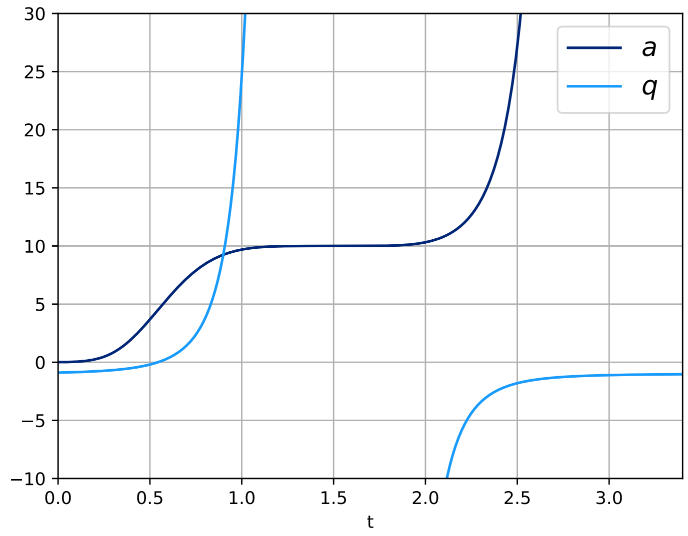
$$\dot{a} = h(a), \tag{5.103}$$

with $h(a)$ a generic function of the scale factor. Then, we can express the nonmetricity scalar in the form

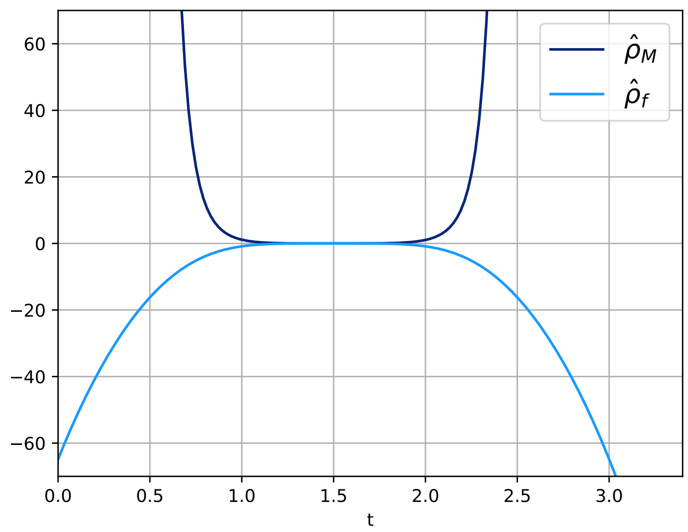
$$\mathcal{Q} = 6\left[\frac{h(a)}{a}\right]^2. \tag{5.104}$$

As an example, let us consider the relation

$$\dot{a} = \frac{2\Omega}{\sqrt{\Lambda}}\sqrt{a - \Lambda a^2}, \tag{5.105}$$



(a) Scale factor and deceleration parameter



(b) $\hat{\rho}_M$ and $\hat{\rho}_f$

Figure 5.5: Evolution of (5.97) with values $n = 2$, $m = 1$, $\rho_0 = 1$, $a_0 = 10$, $f_0 = 0$, $t_0 = \frac{3}{2}$, and $\varpi = 0$.

where Ω and Λ are generic constants. From Eq. (5.105) we derive the evolution of $a(t)$.

$$a(t) = \frac{1}{\Lambda}\sin^2(\Omega t). \tag{5.106}$$

Equation (5.104) allows us to obtain the scale factor and density ρ as a function of $\mathcal{Q}$,

$$\frac{1}{a} = \Lambda + \frac{\Lambda \mathcal{Q}}{24\Omega^2} \tag{5.107}$$

and

$$\rho = \rho_0 \left(\Lambda + \frac{\Lambda \mathcal{Q}}{24\Omega^2}\right)^{3(1+\varpi)}. \tag{5.108}$$

Replacing Eq. (5.108) in Eq. (5.23) and solving, we find the solution,

$$\begin{aligned} f(\mathcal{Q}) =& f_0\sqrt{\mathcal{Q}} + {}_2F_1\left[\frac{5}{2}, -3\bar{w}; \frac{7}{2}; -\frac{\mathcal{Q}}{24\Omega^2}\right] f_1\mathcal{Q}^3 + {}_2F_1\left[\frac{3}{2}, -3\bar{w}; \frac{5}{2}; -\frac{\mathcal{Q}}{24\Omega^2}\right] f_2\mathcal{Q}^2 + \\ &+ {}_2F_1\left[\frac{1}{2}, -3\bar{w}; \frac{3}{2}; -\frac{\mathcal{Q}}{24\Omega^2}\right] f_3\mathcal{Q} - {}_2F_1\left[-\frac{1}{2}, -3\bar{w}; \frac{1}{2}; -\frac{\mathcal{Q}}{24\Omega^2}\right] f_4, \end{aligned} \tag{5.109}$$

where ${}_2F_1$ denotes the hypergeometric function, and f_i $(i = 1, ..., 4)$ are constants depending on Λ, Ω, ρ_0, and $\bar{w}$.

As we can see in Fig. 5.6a, the scale factor (5.106) represents a cyclic universe in which every cycle is separated by a singularity. As in the previous example, the term $\hat{\rho}_M$ is always positive. However, $\hat{\rho}_f$ changes sign. When $\hat{\rho}_f < 0$, the expansion slows up to the point in which the cosmology reaches an equilibrium and then starts contracting. When $\hat{\rho}_f$ becomes positive, the contraction is slowed down up to the point in which the spacetime reaches the singularity with zero contraction rate but with positive acceleration. This fact suggests that the singularity might not be "stable" and, therefore, that one can use this solution for the analysis of pre-Big Bang scenarios in $f(\mathcal{Q})$ gravity.

5.3.3 Example 3: reconstruction from the deceleration parameter

Another way to avoid performing the inversion of the scale factor function is to express it indirectly in terms of a differential equation for the deceleration parameter. More specifically, we set

$$\dot{q} = h(q), \tag{5.110}$$

with $h(q)$ a generic function of the deceleration parameter. Equation (5.110) implies the following expression for the Hubble parameter and scale factor:

$$\frac{1}{H(q)} = \int \frac{1+q}{h(q)} dq, \tag{5.111}$$

$$a(q) = \exp\left[\int \frac{H(q)}{h(q)} dq\right]. \tag{5.112}$$

As an example, let us consider the equation

$$\dot{q} = q_0(1+q)\sqrt{q}, \tag{5.113}$$

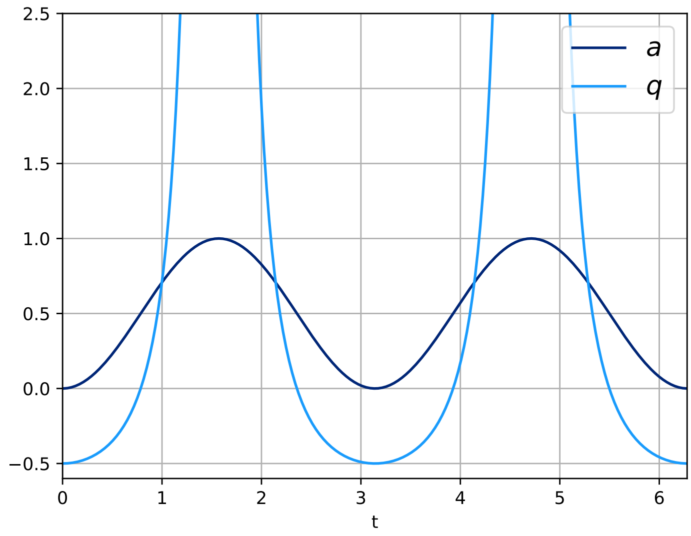
where q_0 is a generic constant. Using the above equations and remembering that $H^2 = \mathcal{Q}/6$, we can write q and a as functions of $\mathcal{Q}$:

$$q(\mathcal{Q}) = \frac{3}{2}\frac{q_0^2}{\mathcal{Q}}, \tag{5.114}$$

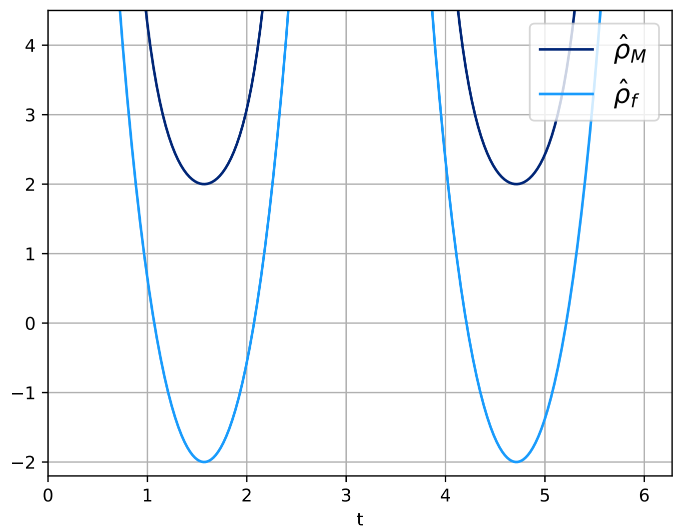
$$a(\mathcal{Q}) = \sqrt{\frac{3q_0^2}{3q_0^2 + 2\mathcal{Q}}}. \tag{5.115}$$

Replacing them in the Friedmann equation (5.23), we find

$$\begin{aligned} f(\mathcal{Q}) =& f_0\sqrt{\mathcal{Q}} + \frac{4}{3}\,{}_2F_1\left[\frac{1}{2}, \frac{1}{2}(-3\bar{w} - 1); \frac{3}{2}; -\frac{2\mathcal{Q}}{3q_0^2}\right] \frac{\rho_0}{q_0^2}\mathcal{Q} + \\ & - 2\,{}_2F_1\left[-\frac{1}{2}, \frac{1}{2}(-3\bar{w} - 1); \frac{1}{2}; -\frac{2\mathcal{Q}}{3q_0^2}\right] \rho_0. \end{aligned} \tag{5.116}$$



(a) Scale factor and deceleration parameter



(b) $\hat{\rho}_M$ and $\hat{\rho}_f$

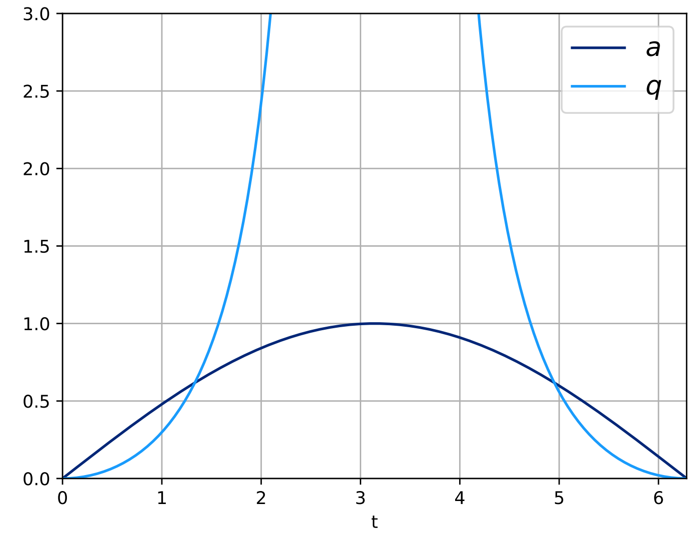
Figure 5.6: Evolution of (5.106) with values $\Lambda = 1$, $\Omega = 1$, $\rho_0 = 1$, $f_0 = 0$, and $\varpi = 0$.

The definition of $\mathcal{Q}$, combined with the definition of q, gives a differential equation for $a(t)$ from which we derive

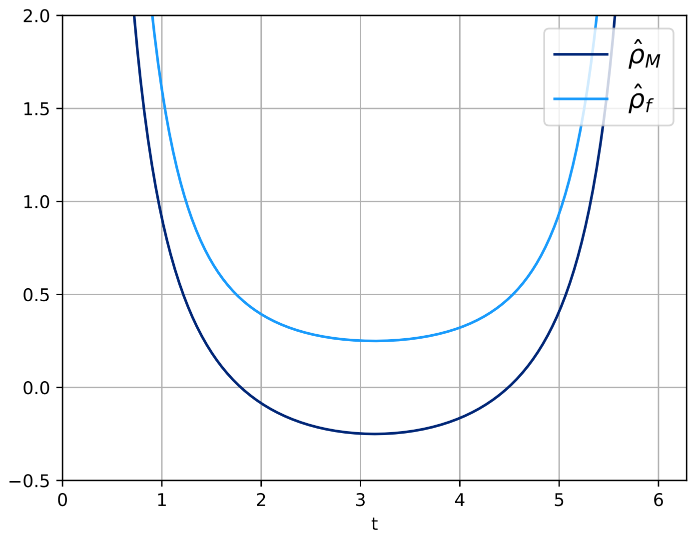
$$a(t) = a_0 \sin\left[\frac{q_0}{2}(t - t_0)\right] + a_1 \cos\left[\frac{q_0}{2}(t - t_0)\right], \tag{5.117}$$

with the condition $a_0^2 + a_1^2 = 1$.

As the above solution can be negative, we will limit ourselves to studying only the first half-period (Fig. 5.7). It is clear that this solution represents again a universe enclosed between two singularities as it happens in Sec. 5.3.2. The difference is that departure and approach to the initial and final singularities happen with an expansion/contraction velocity different from zero. The behavior of nonmetricity terms is similar to that in the previous subsection.



(a) Scale factor and deceleration parameter



(b) $\hat{\rho}_M$ and $\hat{\rho}_f$

Figure 5.7: Evolution of (5.117) with values $a_0 = a_1 = \frac{1}{\sqrt{2}}$, $q_0 = 1$, $\rho_0 = 1$, $f_0 = 0$, $t_0 = \frac{\pi}{2}$, and $\varpi = 0$.

5.3.4 Example 4: reconstruction from the time derivative of nonmetricity scalar

In this last example, we reconstruct the scale factor and the function $f(\mathcal{Q})$ by imposing a differential constraint on the nonmetricity scalar,

$$\dot{\mathcal{Q}}(t) = -\alpha \mathcal{Q}^n(t), \tag{5.118}$$

with α a dimensional constant. From Eq. (5.118), we obtain the solution

$$\mathcal{Q}(t) = \left[\alpha(n-1)\,(t-t_0)\right]^{\frac{1}{1-n}}. \tag{5.119}$$

Using the relation $\mathcal{Q} = 6H^2$, we first derive the scale factor as

$$a(\mathcal{Q}) = a_0 \exp\left[\sqrt{\frac{2}{3}}\frac{\mathcal{Q}^{\frac{3}{2}-n}}{\alpha(2n-3)}\right] \tag{5.120}$$

and then, from Eq. (5.119), we have

$$a(t) = a_0 \exp\left\{\sqrt{\frac{2}{3}}\frac{[\alpha(n-1)(t-t_0)]^{\frac{3-2n}{2-2n}}}{\alpha(2n-3)}\right\}. \tag{5.121}$$

The scale factor (5.121) has an increasing trend for $n > \frac{3}{2}$ and $\alpha > 0$ (see Fig. 5.8a). Equation (5.23) gives, for $\bar{w} = 0, 1/3, 1$

$$f(\mathcal{Q}) = f_0\sqrt{\mathcal{Q}} + f_1\sqrt{\mathcal{Q}}\,\Gamma\left[\frac{1}{2n-3}, \frac{(1+\bar{w})\sqrt{6}\mathcal{Q}^{\frac{3}{2}-n}}{(2n-3)\alpha}\right]. \tag{5.122}$$

where f_i $(i = 1, ..., 3)$ are constants depending on a_0, α, ρ_0, and n.

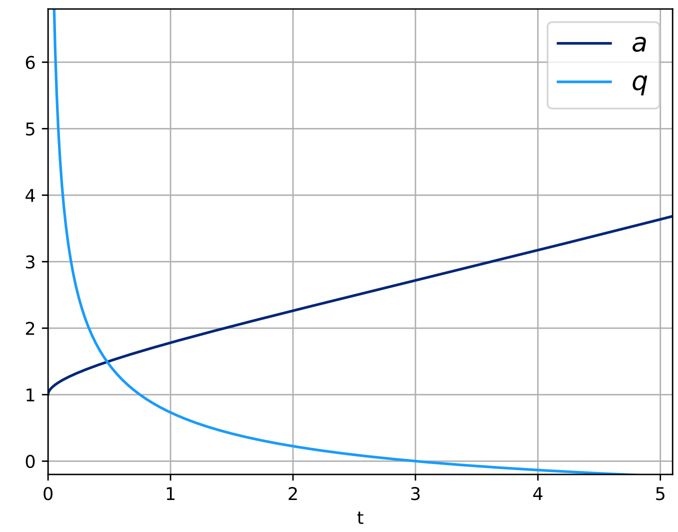
Looking at Fig. 5.8a, it is evident that the scale factor changes concavity. Therefore, after a decelerated phase, cosmology undergoes an accelerated expansion. The term $\hat{\rho}_M$ is always positive; thus, the effective gravitational constant $1/f'(\mathcal{Q})$ remains positive. The fact that the contribution of $\hat{\rho}_f$ is also positive and that we are considering only standard fluids suggests that nonmetric pressure term $\hat{p}_f$ in Eq. (5.26) must be responsible for the accelerated expansion.

5.4 Discussions

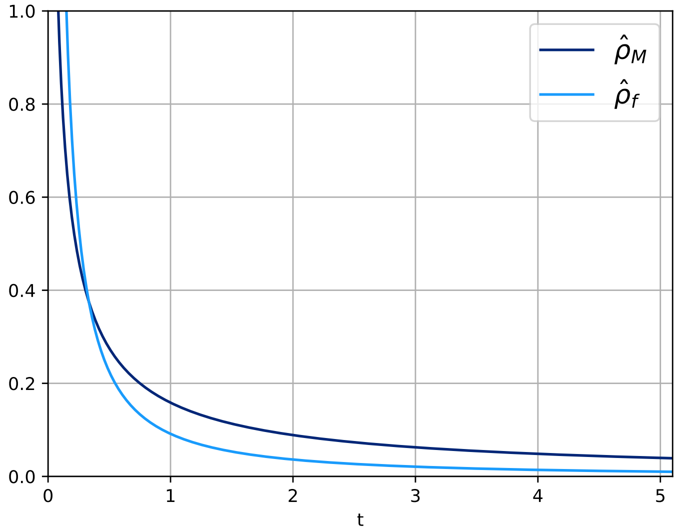
In this chapter, we derived and analyzed some exact cosmological solutions in the context of $f(\mathcal{Q})$ theory, with the aim of understanding the role the nonmetricity might play in the evolution of the universe.

The primary tools to perform this investigation were reconstruction techniques, in which the form of the scale factor(s) is assumed and the form of the function f is recovered *a posteriori*. An essential step for developing the reconstruction algorithm is to reduce the cosmological equation to the simplest set of independent equations. In this respect, the relation between the different components of the Einstein equations is pivotal. We were able to show that, as it is well known in the case of FLRW metrics, also in the case of BI metrics one of the Einstein equations is dependent on the other when the conservation laws are taken into account. This feature has allowed us to reduce the number of equations to be solved. In their original form, reconstruction techniques require some inversion of the scale factor or other related quantities. We have been able to go around this difficulty by assigning a differential relation rather than an exact expression for the scale factors. This approach led to the derivation of several nontrivial solutions.

We started by studying the case of an anisotropic universe endowed with a BI metric. In this context, we found several solutions, such as universes where initial and final states are singular configurations with only one or two spatial dimensions (Secs. 5.2.2 and 5.2.2), and more classical solutions where the scale factors are suitable power law functions (Secs. 5.2.1 and 5.2.2). In Sec. 5.2.2 we also obtained a universe that becomes more and more isotropic in the future.



(a) Scale factor and deceleration parameter



(b) $\hat{\rho}_M$ and $\hat{\rho}_f$

Figure 5.8: Evolution of (5.121) with values $n = 2$, $\alpha = 2$, $\rho_0 = 1$, $a_0 = 1$, $f_0 = 0$, $t_0 = 0$, and $\bar{w} = 0$. The scale factor a has an inflection point at $t = 3$.

In almost all of these solutions, we found that the nonmetricity scalar presents some special features when the difference between the behavior of the scale factors (and hence anisotropy) has a maximum or minimum. For example, in Fig. 5.1 we see that when the scale factors are all equal, the nonmetricity scalar has a maximum, whereas at the Big Bang/Big Crunch when we have the maximum anisotropy, the nonmetricity scalar has a(n infinite) minimum. Similar behaviors might be found in the other figures.

We then moved on to the study of spatially flat FLRW universes. In such a framework, we found different solutions: some of them represent Big Crunch models (Sec. 5.3.3) or oscillating models (Sec. 5.3.2) [161, 162], where the nonmetricity leads the universe to contract. The solution discussed in Sec. 5.3.1 shows that in the presence of nonmetricity, the scale factor can present all the principal phases of the universe history (inflation, decelerated expansion, and dark era). In this case, it turns out that the nonmetricity terms drive all three phases.

6
$1 + 3$ covariant formalism

In this chapter, we elaborate on the concept of spacetime foliation already introduced in Sec. 4.1. We endow the spacetime with a congruence of timelike curves, whose tangent vector field u determines, at each point, the local direction of the time flow. The existence of this vector field implies the existence of preferred rest frames at each point. Thus, the field u can be thought of as the 4-velocity field of a family of observers whose world lines coincide with the congruence. This assumption is the basis of the so-called $1 + 3$ formalism [89, 163, 164]. The reason behind the use of this formalism is that we do not need to introduce any kind of coordinates system, so we have a total covariant approach. Moreover, it gives us a direct insight into the physical relevance of nonmetricity in both geometric and dynamical aspects of spacetime. The approach was used in nonmetric theories of gravity for the first time in [165, 166], where both the Raychaudhuri equation was derived.

First, we will apply the formalism to nonmetric spacetimes and then to BI cosmologies in $f(\mathcal{Q})$ gravity to study how nonmetricity affects expansion and anisotropy.

The study performed in this chapter is based on the paper "Bianchi type-I cosmological dynamics in $f(\mathcal{Q})$ gravity: a covariant approach" [75].

6.1 $1 + 3$ formalism with nonmetricity

In this section, we apply the $1 + 3$ formalism to a spacetime endowed with a connection with nonmetricity but torsion-free, $T_{ab}{}^{c} = 0$. The aim is to analyze how nonmetricity affects the congruence of timelike curves, or world lines, which represent our preferred observers.

6.1.1 4-velocity

Given the congruence $x^a = x^a(\lambda)$, expressed in terms of a parameter λ, we define the 4-velocity as the timelike vector

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}, \qquad u_a = g_{ab}u^b. \tag{6.1}$$

As mentioned in Sec. 4.5, due to nonmetricity, in general, the proper time t is not an affine parameter. Therefore, we impose the condition

$$Q_{abc}u^a u^b u^c = 0, \tag{6.2}$$

together with the requirement that curves of the congruence are autoparallel, which ensures that proper time is the curve parameter. Assuming systematically the condition (6.2), we can arrange things to parameterize the curves using the proper time and define the 4-velocity as,

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}t}, \qquad u_i u^i = -1. \tag{6.3}$$

As we will deal exclusively with cosmological models of BI, we will assume in Sec. 6.3 that the condition (6.2) is satisfied. Indeed, we will show that, because of the gauge choice, such a condition is not restrictive for our purposes.

Once the 4-velocity has been defined, we may introduce the projection operator along u^i defined by means of the tensor

$$U^a{}_b = -u^a u_b, \tag{6.4}$$

which satisfies the properties,

$$U^a{}_b u^b = u^a, \qquad U^a{}_c U^c{}_b = U^a{}_b, \qquad U^a{}_a = 1. \tag{6.5}$$

Orthogonal projection

The choice of a preferred time direction allows us to single out a 3-dimensional subspace of the tangent bundle, at any point, orthogonal to the 4-velocity u^a. The restriction of the metric to this spatial subspace is the so-called transverse metric,

$$h_{ab} = g_{ab} + u_a u_b. \tag{6.6}$$

Associated with the transverse metric (6.6) there is the spatial projection operator,

$$h^a{}_b = \delta^a_b + u^a u_b, \tag{6.7}$$

satisfying the properties

$$h^a{}_c h^c{}_b = h^a{}_b, \qquad h^a{}_a = 3, \qquad h^a{}_b u^b = 0. \tag{6.8}$$

In the following discussion, we will also use the projected symmetric trace-free (PSTF) part of a tensor. In particular, for any 1-form V_b and covariant 2-tensor $\mathbb{T}_{ab}$, the PSTF is expressed as

$$V_{\langle a\rangle} = h_a{}^b V_b, \qquad \mathbb{T}_{\langle ab\rangle} = \left[h_{(a}{}^m h_{b)}{}^n - \frac{1}{3}h_{ab}h^{mn} \right] \mathbb{T}_{mn}. \tag{6.9}$$

6.1.2 Time and spatial derivative

The time derivative of a generic tensor $\mathbb{T}^{a\cdots}{}_{b\ldots}$ is defined as,

$$\begin{aligned}
\dot{\mathbb{T}}^{a\cdots}{}_{b\ldots} &= u^c \nabla_c \mathbb{T}^{a\cdots}{}_{b\ldots} = \\
&= u^c \tilde{\nabla}_c \mathbb{T}^{a\cdots}{}_{b\ldots} + u^c L_{cd}{}^a \mathbb{T}^{d\cdots}{}_{b\ldots} + \cdots - u^c L_{cb}{}^d \mathbb{T}^{a\cdots}{}_{d\ldots} - \cdots = \\
&= \mathring{\mathbb{T}}^{a\cdots}{}_{b\ldots} + u^c L_{cd}{}^a \mathbb{T}^{d\cdots}{}_{b\ldots} + \cdots - u^c L_{cb}{}^d \mathbb{T}^{a\cdots}{}_{d\ldots} - \cdots,
\end{aligned} \tag{6.10}$$

where

$$\mathring{\mathbb{T}}^{a\cdots}{}_{b\cdots} = u^c\tilde{\nabla}_c\mathbb{T}^{a\cdots}{}_{b\cdots} \tag{6.11}$$

is the time derivative with respect to the Levi-Civita connection.

On the other hand, the spatial derivative is defined as the spatial projection of the covariant derivative,

$$\begin{aligned} D_c\mathbb{T}^{a\cdots}{}_{b\cdots} =& h_c{}^p h^a{}_m\cdots h_b{}^n\cdots\nabla_p\mathbb{T}^{m\cdots}{}_{n\cdots} = \\ =& h_c{}^p h^a{}_m\cdots h_b{}^n\cdots\left(\tilde{\nabla}_p\mathbb{T}^{m\cdots}{}_{n\cdots} + L_{pq}{}^m\mathbb{T}^{q\cdots}{}_{n\cdots} + \cdots + \right. \\ & \left. -L_{pn}{}^q\mathbb{T}^{m\cdots}{}_{q\cdots} - \cdots\right) = \\ =& \tilde{D}_c\mathbb{T}^{a\cdots}{}_{b\cdots} + h_c{}^p h^a{}_m\cdots h_b{}^n\cdots L_{pq}{}^m\mathbb{T}^{q\cdots}{}_{n\cdots} + \\ & + \cdots - h_c{}^p h^a{}_m\cdots h_b{}^n\cdots L_{pn}{}^q\mathbb{T}^{m\cdots}{}_{q\cdots} - \cdots, \end{aligned} \tag{6.12}$$

with

$$\tilde{D}_c\mathbb{T}^{a\cdots}{}_{b\cdots} = h_c{}^p h^a{}_m\cdots h_b{}^n\cdots\tilde{\nabla}_p\mathbb{T}^{m\cdots}{}_{n\cdots} \tag{6.13}$$

the spatial derivative with respect to the Levi-Civita connection. It is worth noticing that the spatial derivative of the metric g_{ij} is equal to the spatial derivative of h_{ij}:

$$D_c g_{ab} = h_c{}^p h_a{}^m h_b{}^n\nabla_p g_{mn} = h_c{}^p h_a{}^m h_b{}^n\nabla_p\left(h_{mn} - u_m u_n\right) = D_c h_{ab}. \tag{6.14}$$

4-acceleration

Because of nonmetricity, scalar product, and covariant derivative do not commute in general. For this reason, we use the convention that the contravariant, or covariant, counterparts of objects related to the covariant derivative are obtained raising, or lowering, the indices by the metric. Accordingly, we define the 4-acceleration as

$$\dot{u}_a = u^b\nabla_b u_a = \mathring{u}_a - L_{ba}{}^c u_c u^b = \mathring{u}_a + \frac{1}{2}Q_{abc}u^b u^c, \tag{6.15}$$

where $\mathring{u}_i = u^h\tilde{\nabla}_h u_i$ is the 4-acceleration with respect to the Levi-Civita connection. After that, the contravariant counterpart of (6.15) is obtained as

$$\dot{u}^a = g^{ab}\dot{u}_b = g^{ab}u^c\nabla_c u_b = \mathring{u}^a + \frac{1}{2}g^{ab}Q_{bcd}u^c u^d. \tag{6.16}$$

If Eq. (6.2) holds, u^i and $\dot{u}_i$ are orthogonal to each other,

$$\dot{u}_a u^a = \dot{u}^a u_a = 0. \tag{6.17}$$

6.1.3 Kinematic quantities

The covariant derivative of the 4-velocity can be decomposed in its temporal and spatial projections,

$$\begin{aligned} \nabla_a u_b &= -u_a\dot{u}_b + D_a u_b - u_b h_a{}^c u^d\nabla_c u_d = \\ &= -u_a\dot{u}_b + \frac{1}{3}h_{ab}\Theta + \sigma_{ab} + \omega_{ab} - \frac{1}{2}u_b h_a{}^c Q_{cmn}u^m u^n, \end{aligned} \tag{6.18}$$

with

$$D_a u_b = \frac{1}{3}h_{ab}\Theta + \sigma_{ab} + \omega_{ab}, \tag{6.19}$$

and where:

- Θ is related to the rate of volume expansion,

$$\Theta = g^{ab} D_a u_b = g^{ab} h_a{}^p h_b{}^q \nabla_p u_q = h^{ab} \tilde{D}_a u_b - h^{ab} L_{ab}{}^c u_c = \tilde{\Theta} - h^{ab} L_{ab}{}^c u_c, \tag{6.20}$$

  with

$$\tilde{\Theta} = \tilde{D}_a u^a; \tag{6.21}$$

- σ_{ab} is the trace-free symmetric tensor called "shear tensor", describing the volume preserving distortion of the fluid flow,

$$\begin{aligned} \sigma_{ab} = D_{\langle a} u_{b\rangle} &= \left[h_{(a}{}^m h_{b)}{}^n - \frac{1}{3} h_{ab} h^{mn} \right] \left(\tilde{D}_m u_n - h_m{}^p h_n{}^q L_{pq}{}^c u_c \right) = \\ &= \tilde{\sigma}_{ab} - L_{\langle ab \rangle}{}^c u_c, \end{aligned} \tag{6.22}$$

$$\sigma_{ab} u^b = 0, \qquad \sigma_a{}^a = 0, \tag{6.23}$$

  with

$$\tilde{\sigma}_{ab} = \tilde{D}_{\langle a} u_{b\rangle}, \qquad \tilde{\sigma}_{ab} u^b = 0, \qquad \tilde{\sigma}_a{}^a = 0; \tag{6.24}$$

- ω_{ab} is the skew-symmetric tensor called "vorticity tensor" describing rotation of the fluid flow,

$$\omega_{ab} = D_{[a} u_{b]} = \tilde{D}_{[a} u_{b]} - h_{[a}{}^m h_{b]}{}^n L_{mn}{}^c u_c = \tilde{D}_{[a} u_{b]} = \tilde{\omega}_{ab}, \tag{6.25}$$

$$\tilde{\omega}_{ab} = \tilde{D}_{[a} u_{b]}, \qquad \omega_{ab} u^b = \tilde{\omega}_{ab} u^b = 0. \tag{6.26}$$

  So, the vorticity is not influenced by the nonmetricity tensor.

It is useful to introduce the magnitudes of shear and vorticity tensors,

$$\sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab}, \qquad \omega^2 = \frac{1}{2} \omega_{ab} \omega^{ab}, \tag{6.27}$$

$$\tilde{\sigma}^2 = \frac{1}{2} \tilde{\sigma}_{ab} \tilde{\sigma}^{ab}, \qquad \tilde{\omega}^2 = \frac{1}{2} \tilde{\omega}_{ab} \tilde{\omega}^{ab}, \tag{6.28}$$

which will be used in the cosmological equations. By substituting Eqs. (6.20), (6.22) and (6.25) in Eq. (6.18) and considering Eq. (6.2), we get the expression,

$$\begin{aligned} \nabla_a u_b &= \tilde{\nabla}_a u_b - \frac{1}{2} u_a h_b{}^c Q_{cmn} u^m u^n - h_a{}^m h_b{}^n L_{mn}{}^p u_p - \frac{1}{2} u_b h_a{}^c Q_{cmn} u^m u^n = \\ &= -u_a \mathring{u}_b + \frac{1}{3} \tilde{\Theta} h_{ab} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} - u_{(a} h_{b)}{}^c Q_{cmn} u^m u^n - h_a{}^m h_b{}^n L_{mn}{}^p u_p. \end{aligned} \tag{6.29}$$

6.1.4 Intrinsic and extrinsic curvature: the Gauss Relation

Given a spatial vector field v^b, we define the spatial Riemann tensor through the relation,

$${}^3R^a{}_{bcd} v^b = \left(D_c D_d - D_d D_c \right) v^a - 2\omega_{cd} u^p h_q{}^a \nabla_p v^q. \tag{6.30}$$

Besides the Riemann tensor, we can also define an extrinsic curvature tensor K_{ab} that measures the rate of change of the vector normal to a hypersurface. Therefore, in our framework, the extrinsic curvature is defined as the spatial derivative of the 4-velocity,

$$K_{ab} = D_a u_b = h_a{}^m h_b{}^n \nabla_m u_n = \tilde{K}_{ab} - h_a{}^m h_b{}^n L_{mn}{}^h u_h, \tag{6.31}$$

being

$$\tilde{K}_{ab} = \tilde{D}_a u_b \tag{6.32}$$

the extrinsic curvature induced by the Levi-Civita connection. Raising the second index, we obtain

$$\begin{aligned} K_c{}^a = g^{ab} K_{cb} =& \tilde{K}_c{}^a - g^{ab} h_c{}^i h_b{}^j L_{ij}{}^k u_k = \\ =& \tilde{K}_c{}^a + h_c{}^i h_k{}^a L_{ij}{}^k u^j + u^k h_c{}^i h^{aj} Q_{ijk}, \end{aligned} \tag{6.33}$$

which will be useful in the following.

The relation between the spatial Riemann tensor and the extrinsic curvature can be obtained by recasting Eq. (6.30). From the explicit evaluation of one of the terms in the bracket of Eq. (6.30), we have

$$\begin{aligned} D_c D_d v^a =& h_c{}^i h_d{}^j h_k{}^a \nabla_i \left(D_j v^k \right) = h_c{}^i h_d{}^j h_k{}^a \nabla_i \left(h_j{}^e h_g{}^k \nabla_e v^g \right) = \\ =& h_c{}^i h_d{}^j h_k{}^a \nabla_i \left(u^e \nabla_i u_j h_g{}^k \nabla_e v^g + h_j{}^e u_g \nabla_i u^k \nabla_e v^g + h_j{}^e h_g{}^k \nabla_i \nabla_e v^g \right) = \\ =& h_c{}^i h_d{}^j h_k{}^a h_g{}^k u^e \nabla_i u_j \nabla_e v^g - h_c{}^i h_d{}^j h_k{}^a h_j{}^e \nabla_i u^k \left(v^g \nabla_e u_g \right) + \\ & + h_c{}^i h_d{}^j h_k{}^a h_j{}^e h_g{}^k \nabla_i \nabla_e v^g = \\ =& K_{cd} h_g{}^a u^e \nabla_e v^g - K_c{}^a K_{dg} v^g + u^k h_c{}^i h^{aj} Q_{ijk} K_{dg} v^g + h_c{}^i h_d{}^e h_g{}^a \nabla_i \nabla_e v^g, \end{aligned} \tag{6.34}$$

where we used that u^a and v^a are orthogonal, so

$$u_a \nabla_b v^a = -v^a \nabla_b u_a, \tag{6.35}$$

and the relation

$$\nabla_a h_b{}^c = \nabla_a \left(g^{cd} h_{bd} \right) = -Q_a{}^{cd} h_{bd} + g^{cd} \nabla_a \left(g_{bd} + u_b u_d \right) = u^c \nabla_a u_b + u_b \nabla_a u^c. \tag{6.36}$$

Hence,

$$\begin{aligned} \left(D_c D_d - D_d D_c \right) v^a =& \left(K_d{}^a K_{cg} - K_c{}^a K_{dg} \right) v^g + h_c{}^i h_d{}^e h_g{}^a \left(\nabla_i \nabla_e - \nabla_e \nabla_i \right) v^g + \\ & + 2 u^k h_{[c}{}^i K_{d]b} h^{aj} Q_{ijk} v^b. \end{aligned} \tag{6.37}$$

Finally, using the Ricci identity (1.42), we obtain the so-called "Gauss relation",

$${}^3R^a{}_{bcd} = h_c{}^m h_d{}^n h^a{}_s h_b{}^r R^s{}_{rmn} + K_d{}^a K_{cb} - K_c{}^a K_{db} + 2 h_{[c}{}^m K_{d]b} h^{an} Q_{mns} u^s, \tag{6.38}$$

which proves that the 3-dimensional curvature is the projection of the 4-dimensional one corrected by terms of the extrinsic curvature.

Since the Riemann tensor is not antisymmetric in the first two indices, we have two "contracted Gauss relations." The first relation is obtained by contracting the first and third index of the Gauss relation,

$${}^3R_{bd} = {}^3R^i{}_{bid} = h_s{}^m h_d{}^n h_b{}^r R^s{}_{rmn} + K_d{}^i K_{ib} - K_i{}^i K_{db} + 2 h_{[i}{}^m K_{d]b} h^{in} b_{mns} u^s, \tag{6.39}$$

and the second one by contracting the second and third indices,

$${}^3\bar{R}_{bd} = {}^3R_b{}^i{}_{id} = h_r{}^m h_d{}^n h_b{}^s R_s{}^r{}_{mn} + K_{db} K_i{}^i - K_{ib} K_d{}^i + 2 h_{[i}{}^m K_{d]}{}^i h_b{}^n b_{mns} u^s. \tag{6.40}$$

The trace of both Eqs. (6.39) and Eq. (6.40) leads to the "scalar Gauss relation",

$$\begin{aligned}{}^3R &= g^{bd}\,{}^3R_{bd} = -g^{bd}\,{}^3\bar{R}_{bd} = \\ &= h_s{}^m h^{rn} R^s{}_{rmn} + K^{di}K_{id} - K_i{}^i K_d{}^d + 2h_{[i}{}^m K_{d]}{}^d h^{in} b_{mns} u^s,\end{aligned} \tag{6.41}$$

which generalizes the "Theorema Egregium" in the presence of nonmetricity.

It is also useful to rewrite Eq. (6.38) in a form in which the contributions due to Levi-Civita and nonmetricity terms are made evident:

$$\begin{aligned}{}^3R^a{}_{bcd} =& h_c{}^i h_d{}^e h_g{}^a h_b{}^m R^g{}_{mie} + \left(\tilde{K}_d{}^a + h_d{}^i h_j{}^a L_{ik}{}^j u^k + h_d{}^i h^{aj} Q_{ijk} u^k\right) K_{cd} + \\ &- \left(\tilde{K}_c{}^a + h_c{}^i h_j{}^a L_{ik}{}^j u^k + h_c{}^i h^{aj} Q_{ijk} u^k\right) K_{db} + 2u^k h_{[c}{}^i K_{d]b} h^{aj} Q_{ijk} = \\ =& h_c{}^i h_d{}^e h_g{}^a h_b{}^m R^g{}_{mie} + \left(\tilde{K}_d{}^a + h_d{}^i h_j{}^a L_{ik}{}^j u^k\right)\left(\tilde{K}_{cb} - h_c{}^m h_b{}^n L^p_{mn} u_p\right) + \\ &- \left(\tilde{K}_c{}^a + h_c{}^i h_j{}^a L_{ik}{}^j u^k\right)\left(\tilde{K}_{db} - h_d{}^m h_b{}^n L_{mn}{}^p u_p\right) = \\ =& h_c{}^i h_d{}^e h_g{}^a h_b{}^m R^g{}_{mie} - 2\tilde{K}_{[c}{}^a \tilde{K}_{d]b} + 2\tilde{K}_{[c}{}^a h_{d]}{}^m h_b{}^n L_{mn} p u_p + \\ &+ 2h_{[d}{}^i \tilde{K}_{c]b} h_j{}^a L_{ik}{}^j u^k + 2h_{[c}{}^i h_{d]}{}^m h_j{}^a h_b{}^n L_{ik}{}^j L_{mn}{}^p u^k u_p.\end{aligned} \tag{6.42}$$

Now, considering the decomposition of the Riemann tensor (1.56),

$$\begin{aligned}h_c{}^i h_d{}^e h_g{}^a h_b{}^m R^g{}_{mie} =& h_c{}^i h_d{}^e h_g{}^a h_b{}^m \tilde{R}^g{}_{mie} + h_c{}^i h_d{}^e h_g{}^a h_b{}^m \tilde{\nabla}_i L_{em}{}^g - h_c{}^i h_d{}^e h_g{}^a h_b{}^m \tilde{\nabla}_e L_{im}{}^g + \\ &+ h_c{}^i h_d{}^e h_g{}^a h_b{}^m L_{ij}{}^g Lem^j - h_c{}^i h_d{}^e h_g{}^a h_b{}^m L_{ej}{}^g L_{im}{}^j,\end{aligned} \tag{6.43}$$

and that

$$\begin{aligned}2h_{[c}{}^i h_{d]}{}^m h_j{}^a h_b{}^n L_{ik}{}^j L_{mn}{}^p u^k u_p + h_c{}^i h_d{}^e h_g{}^a h_b{}^m L_{ij}{}^g Lem^j + \\ - h_c{}^i h_d{}^e h_g{}^a h_b{}^m L_{ej}{}^g L_{im}{}^j = 2h_{[c}{}^i h_{d]}{}^m h_j{}^a h_b{}^n h_p{}^k L_{ik}{}^j L_{mn}{}^p,\end{aligned} \tag{6.44}$$

we find the desired relation

$$\begin{aligned}{}^3R^a{}_{bcd} =& {}^3\tilde{R}^a{}_{bcd} + 2\tilde{D}_{[c} L_{d]b}{}^a + 2\tilde{K}_{[c}{}^a h_{d]}{}^m h_b{}^n L_{mn}{}^k u_k + \\ &+ 2\tilde{K}_{[c|b} h_{|d]}{}^m h^a{}_n L_{mk}{}^n u^k + 2h_{[c}{}^r h_{d]}{}^m h_s{}^a h_b{}^n h_l{}^k L_{rk}{}^s L_{mn}{}^l,\end{aligned} \tag{6.45}$$

with

$${}^3\tilde{R}^a{}_{bcd} = h_c{}^m h_d{}^n h_s{}^a h_b{}^r \tilde{R}^s{}_{rmn} + \tilde{K}_d{}^a \tilde{K}_{cb} - \tilde{K}_c{}^a \tilde{K}_{db}. \tag{6.46}$$

6.1.5 Energy-momentum tensor

The energy-momentum tensor of the matter fluid can be decomposed in its irreducible parts as,

$$\Psi_{ab} = \rho u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab}, \tag{6.47}$$

where

$$\rho = \Psi_{ab} u^a u^b \tag{6.48}$$

is the relativistic energy density,

$$q_a = -h_a{}^c \Psi_{cb} u^b \tag{6.49}$$

the relativistic energy flux,

$$p = \frac{1}{3}h^{ab}\Psi_{ab} \tag{6.50}$$

the isotropic pressure, and

$$\pi_{ab} = \Psi_{\langle ab\rangle} \tag{6.51}$$

the trace-free anisotropic pressure. The trace of tensor (6.47) is equal to,

$$\Psi = \Psi_a{}^a = -\rho + 3p. \tag{6.52}$$

6.1.6 Nonmetricity decomposition

Similarly to what we have done for the energy-momentum tensor (6.47), we can decompose the nonmetricity tensor using u_a and h_{ab} as follows:

$$\begin{aligned} Q_{cab} =& - Q_0 u_c u_a u_b - \frac{1}{3}Q_1 u_c h_{ab} - \frac{2}{3}Q_2 u_{(a}h_{b)c} + Q^{(0)}{}_c u_a u_b + 2Q^{(1)}{}_{(a}u_{b)}u_c + \\ &+ \frac{1}{3}Q^{(2)}{}_c h_{ab} + \frac{2}{3}Q^{(3)}{}_{(a}h_{b)c} - Q^{(0)}{}_{ab}u_c - 2Q^{(1)}{}_{c(a}u_{b)} + {}^3Q_{cab}, \end{aligned} \tag{6.53}$$

where

$$Q_0 = Q_{cab}u^c u^a u^b, \quad Q_1 = Q_{cab}u^c h^{ab}, \quad Q_2 = Q_{cab}h^{ca}u^b \tag{6.54}$$

are scalar quantities,

$$Q^{(0)}{}_c = Q_{pab}h^p{}_c u^a u^b, \qquad Q^{(1)}{}_c = Q_{pab}u^p u^b h^a{}_c, \tag{6.55}$$

$$Q^{(2)}{}_c = Q_{pab}h^p{}_c h^{ab}, \qquad Q^{(3)}{}_c = Q_{pab}h^a{}_c h^{pb} \tag{6.56}$$

are covectors,

$$Q^{(0)}{}_{ab} = Q^{(0)}{}_{ba} = \left[h_{(a}{}^p h_{b)}{}^q - \frac{1}{3}h_{ab}h^{pq}\right] Q_{cpq}u^c, \tag{6.57}$$

$$Q^{(1)}{}_{ab} = \left(h_a{}^p h_b{}^q - \frac{1}{3}h_{ab}h^{pq}\right) Q_{pcq}u^c \tag{6.58}$$

are trace-free tensors and

$${}^3Q_{cab} = h_c{}^p h_a{}^q h_b{}^r Q_{pqr} - \frac{1}{3}h_{ab}h^{qr}h_c{}^p Q_{pqr} - \frac{1}{3}h_{ca}h^{pq}h_b{}^r Q_{pqr} - \frac{1}{3}h_{cb}h^{pr}h_a{}^q Q_{pqr} \tag{6.59}$$

is a fully spatial tensor, whose traces are given by

$${}^3Q_{ba}{}^b = -\frac{1}{3}Q^{(2)}{}_a - \frac{1}{3}Q^{(3)}_a \quad \text{and} \quad {}^3Q_{ab}{}^b = -\frac{2}{3}Q^{(3)}{}_a. \tag{6.60}$$

It is worth noting that, unlike the energy-momentum tensor, Eq. (6.53) is not an irreducible decomposition. Moreover, because of Eq. (6.2), $Q_0 = 0$.

We can now rewrite Eqs. (6.15), (6.20), and (6.22) in terms of different contributions of nonmetricity:

$$\dot{u}_a = \mathring{u}_a + \frac{1}{2}Q^{(0)}{}_a, \tag{6.61}$$

$$\Theta = \tilde{\Theta} - \frac{1}{2}Q_1 + Q_2, \tag{6.62}$$

$$\sigma_{ab} = \tilde{\sigma}_{ab} - \frac{1}{2}Q^{(0)}{}_{ab} + Q^{(1)}{}_{(ab)}. \tag{6.63}$$

Eqs. (6.61), (6.62) and (6.63) show how nonmetricity affects the kinematic quantities associated with the given congruence.

6.2 Covariant cosmological equations in $f(\mathcal{Q})$ gravity

Before writing the cosmological equation in the $1+3$ formalism, we need to recast the field equations (3.53) in a form more suitable for the present formalism. The process is very similar to the one used in Sec. 3.1 to show that in STEGR the field equations are equivalent to those in GR.

Let us consider the field equation

$$\frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}f'P^c{}_{ab}\right)+\frac{1}{2}g_{ab}f+f'\left(P_{acd}Q_b{}^{cd}-2Q^{cd}{}_aP_{cdb}\right)=\Psi_{ab}. \tag{6.64}$$

From the condition $R^a{}_{bcd}=0$, we know that the following relation holds for the Levi-Civita Riemann tensor

$$\tilde{R}_{ab}=-\tilde{\nabla}_cL_{ab}{}^c+\tilde{\nabla}_bL_{ca}{}^c-L_{cd}{}^cL_{ab}{}^d+L_{bd}{}^cL_{ca}{}^d. \tag{6.65}$$

Moreover, by inserting the identities

$$\frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}f'P^c{}_{ab}\right)=2f'\tilde{\nabla}_cP^c{}_{ab}-2f'L_{ca}{}^dP^c{}_{db}-2f'L_{cb}{}^dP^c{}_{da}+2f''\partial_c\mathcal{Q}P^c{}_{ab}, \tag{6.66}$$

and

$$L_{bd}{}^cL_{ca}{}^d-L_{cd}{}^cL_{ab}{}^d=-2L_{ca}{}^dP^c{}_{db}-2L_{cb}{}^dP^c{}_{da}+P_{acd}Q_b{}^{cd}-2Q^{cd}{}_aP_{cdb}, \tag{6.67}$$

into Eq. (6.64), we obtain

$$2f'\tilde{\nabla}_cP^c{}_{ab}+\frac{1}{2}g_{ab}f+f'\tilde{R}_{ab}+2f''\partial_c\mathcal{Q}P^c{}_{ab}+f'\left(\tilde{\nabla}_cL_{ab}{}^c-\tilde{\nabla}_bL_{ca}{}^c\right)=\Psi_{ab}. \tag{6.68}$$

From the trace of Eq. (6.68), we derive the Ricci scalar,

$$\tilde{R}=\frac{1}{f'}\Psi-2\frac{f}{f'}-2\tilde{\nabla}_cP^{ca}{}_a-2\tilde{\nabla}_{[c}L_{a]}{}^{ac}-2\frac{f''}{f'}P^cab\partial_c\mathcal{Q}. \tag{6.69}$$

and subtracting $\frac{1}{2}g_{ij}\tilde{R}$ from (6.68), we find

$$\begin{aligned}\tilde{R}_{ab}-\frac{1}{2}g_{ab}\tilde{R}=&\frac{1}{f'}\left(\Psi_{ab}-\frac{1}{2}g_{ab}\Psi\right)+\frac{1}{2}g_{ab}\frac{f}{f'}-g_{ab}\tilde{\nabla}_cP^{cd}{}_d-\tilde{\nabla}_cL_{ab}{}^c+\tilde{\nabla}_bL_{ca}{}^c+\\&-2\frac{f''}{f'}\left(P^c{}_{ab}-\frac{1}{2}g_{ab}P^{cd}{}_d\right)\partial_c\mathcal{Q}-2\tilde{\nabla}_cP^c{}_{ab}+\frac{1}{2}g_{ab}\tilde{\nabla}_c\left(q^c-Q^c\right).\end{aligned} \tag{6.70}$$

Using again the relation (3.48)

$$2\tilde{\nabla}_cP^c{}_{ab}+\tilde{\nabla}_cL_{ab}{}^c-\tilde{\nabla}_bL_{ca}{}^c-\frac{1}{2}g_{ab}\tilde{\nabla}_c\left(q^c-Q^c\right)=-\frac{1}{2}\check{R}_{ab}=0, \tag{6.71}$$

and the Ricci scalar decomposition

$$\tilde{R}=-\mathcal{Q}-\tilde{\nabla}_a\left(q^a-Q^a\right), \tag{6.72}$$

we obtain the final equation,

$$\tilde{R}_{ab}=\frac{1}{f'}\left(\Psi_{ab}-\frac{1}{2}g_{ab}\Psi\right)+\frac{1}{2}g_{ab}\left(\frac{f}{f'}-\mathcal{Q}\right)-2\frac{f''}{f'}\left(P^c{}_{ab}-\frac{1}{2}g_{ab}P^{cd}{}_d\right)\partial_c\mathcal{Q}, \tag{6.73}$$

or in the Einstein-like form

$$\tilde{R}_{ab} - \frac{1}{2} g_{ab} \tilde{R} = \frac{1}{f'} \left(\Psi_{ab} + \Psi^{eff}_{ab} \right), \tag{6.74}$$

with

$$\Psi^{eff}_{ab} = -\frac{1}{2} g_{ab} \left(f - f' \mathcal{Q} \right) - 2 f'' P^{c}{}_{ab} \partial_c \mathcal{Q} \tag{6.75}$$

the effective energy-momentum tensor due to nonmetricity. By replacing $f(\mathcal{Q}) = \mathcal{Q}$ into Eq. (6.74), we recover the Einstein field equations,

$$\tilde{R}_{ab} - \frac{1}{2} g_{ab} \tilde{R} = \Psi_{ab}. \tag{6.76}$$

It is possible to prove that, on shell, the conditions $\partial_c \mathcal{Q} = 0$ and $f''(\mathcal{Q}) = 0$ always yield the GR regime.

For $\mathcal{Q}_{sol} = const.$ we have that the function $f(\mathcal{Q}_{sol})$ and its derivatives are constant, and Eq. (6.74) can be written as

$$\tilde{R}_{ab} - \frac{1}{2} g_{ab} \tilde{R} + g_{ab} \Lambda_{eff} = \frac{1}{f'(\mathcal{Q}_{sol})} \Psi_{ab}, \tag{6.77}$$

with

$$\Lambda_{eff} = \frac{1}{2} \left[\frac{f(\mathcal{Q}_{sol})}{f'(\mathcal{Q}_{sol})} - \mathcal{Q}_{sol} \right], \tag{6.78}$$

an effective cosmological constant. Therefore, we obtain the GR field equations with a cosmological constant. On the other hand, by a reductio ad absurdum, we can prove that for $f''(\mathcal{Q}_{sol}) = 0$ and $\mathcal{Q}_{sol} \neq const.$, the function is necessarily linear, $f(\mathcal{Q}) = a\mathcal{Q} + b$, with a and b arbitrary real constant. If we assume that $f(\mathcal{Q})$ is a generic function, its second derivative equal to zero imposes that Q_{sol} is constant. Hence, we return to the previous case of $\partial_c \mathcal{Q} = 0$ which we have already shown to produce Einstein equations. Therefore, the function must be linear. However, a linear $f(\mathcal{Q})$ yields Eq. (6.74). We have thus demonstrated that both conditions $\partial_c \mathcal{Q} = 0$ and $f''(\mathcal{Q}) = 0$ lead back to GR with a cosmological constant.

6.2.1 Cosmological equations

To write the cosmological equations we first need to derive the contraction of $\tilde{R}_{ab}$ with u^a and u^b from the Ricci identity for the Levi-Civita Ricci tensor,

$$\begin{aligned} \tilde{R}_{ab} u^a u^b =& u^b \tilde{\nabla}_a \tilde{\nabla}_b u^a - \left(\tilde{\nabla}_a u^a \right)^{\cdot} = \tilde{\nabla}_a \mathring{u}^a - \tilde{\nabla}_a u^b \tilde{\nabla}_b u^a - \mathring{\tilde{\Theta}} = \\ =& -\mathring{\tilde{\Theta}} - \frac{1}{3} \tilde{\Theta}^2 - 2 \left(\tilde{\sigma}^2 - \tilde{\omega}^2 \right) + \tilde{D}_h \mathring{u}^h + \mathring{u}^h \mathring{u}_h, \end{aligned} \tag{6.79}$$

where we used that

$$\tilde{\nabla}_a \mathring{u}^a = g_b{}^a \tilde{\nabla}_a \mathring{u}^b = h_b{}^a \tilde{\nabla}_a \mathring{u}^b - u_b u^a \tilde{\nabla}_a \mathring{u}^b = \tilde{D}_a \mathring{u}^a + \mathring{u}^a \mathring{u}_a. \tag{6.80}$$

In addition, we need the Levi-Civita contracted Gauss relation

$$\begin{aligned} {}^3\tilde{R}_{bd} =& h_b{}^r h_d{}^n h_s{}^m \tilde{R}^s{}_{rmn} + \tilde{K}_d{}^a \tilde{K}_{ab} - \tilde{K}_a{}^a \tilde{K}_{db} = \\ =& h_b{}^r h_d{}^n \tilde{R}_{rn} + h_d{}^n h_{br} u^m u^s \tilde{R}^r{}_{mns} + \tilde{K}_d{}^a \tilde{K}_{ab} - \tilde{K}_a{}^a \tilde{K}_{db}, \end{aligned} \tag{6.81}$$

which because of the following relation

$$\begin{aligned} h_{ia}h_j{}^c u^b u^d \tilde{R}^a{}_{bcd} =& h_{ia}h_j{}^c u^d \left(\tilde{\nabla}_c\tilde{\nabla}_d - \tilde{\nabla}_d\tilde{\nabla}_c\right) u^a = \\ =& h_{ia}h_j{}^c u^d \left[\tilde{\nabla}_c \left(\tilde{K}_d{}^a - u_d\mathring{u}^a\right) - \tilde{\nabla}_d \left(\tilde{K}_c{}^a - u_c\mathring{u}^a\right)\right] = \\ =& -\tilde{K}_{di}\tilde{K}_j{}^d - h_{ia}h_j{}^c u^d \tilde{\nabla}_d \tilde{K}_c{}^a + \tilde{D}_j\mathring{u}_i + \mathring{u}_i\mathring{u}_j \end{aligned} \tag{6.82}$$

can be written as

$$ {}^3\tilde{R}_{bd} = h_b{}^p h_d{}^q \tilde{R}_{pq} - \tilde{K}_p{}^p\tilde{K}_{db} - h_d{}^q h_b{}^p u^m \tilde{\nabla}_m \tilde{K}_{qp} + \tilde{D}_d\mathring{u}_b + \mathring{u}_b\mathring{u}_d. \tag{6.83}$$

Therefore, the $1+3$ cosmological equations for a generic $f(\mathcal{Q})$ theory are given by:

- Raychaudhuri equation, obtained from Eq. (6.79),

$$\begin{aligned} \mathring{\tilde{\Theta}} + \frac{1}{3}\tilde{\Theta}^2 + 2\left(\tilde{\sigma}^2 - \tilde{\omega}^2\right) - \tilde{D}_i\mathring{u}^i - \mathring{u}^i\mathring{u}_i + \frac{1}{2f'}\left(\rho + 3p\right) + \\ -\frac{1}{2}\left(\frac{f}{f'} - \mathcal{Q}\right) - 2\frac{f''}{f'}\left(P^h{}_{ij}u^i u^j + \frac{1}{2}P^{hk}{}_k\right)\partial_h\mathcal{Q} = 0; \end{aligned} \tag{6.84}$$

- Spatial equations, derived from Eqs. (6.83) and (6.84), given by the 3-dimensional Ricci scalar, i.e. the Friedmann equation,

$$ {}^3\tilde{R} = \frac{2}{f'}\rho + \frac{f}{f'} - \mathcal{Q} - \frac{2}{3}\tilde{\Theta}^2 + 2\left(\tilde{\sigma}^2 - \tilde{\omega}^2\right) + 2\frac{f''}{f'}\partial_h\mathcal{Q}\left(P^{hi}{}_i - h^{ij}P^h{}_{ij} - P^h{}_{ij}u^i u^j\right), \tag{6.85}$$

  and the projected traceless 3-dimensional Ricci tensor,

$$\begin{aligned} \left(h_a{}^p h_b{}^q - \frac{1}{3}h_{ab}h^{pq}\right){}^3\tilde{R}_{pq} =& \frac{1}{f'}\left[\pi_{ab} - 2f''\partial_h\mathcal{Q}\left(h_a{}^p h_b{}^q - \frac{1}{3}h_{ab}h^{pq}\right)P^h{}_{pq}\right] + \\ & -\tilde{\Theta}\tilde{\sigma}_{ab} + \tilde{\Theta}\tilde{\omega}_{ab} + \tilde{D}_{\langle a}\mathring{u}_{b\rangle} - \tilde{D}_{[a}\mathring{u}_{b]} + \mathring{u}_{\langle a}\mathring{u}_{b\rangle} + \\ & -\mathring{\tilde{\sigma}}_{ab} + \mathring{\tilde{\omega}}_{ab}. \end{aligned} \tag{6.86}$$

As we will see in the following sections, Eqs. (6.84)-(6.86), together with the energy-momentum conservation law

$$\tilde{\nabla}_a\Psi^a{}_b = 0. \tag{6.87}$$

and Eq. (6.53), form a closed system able to describe the evolution of BI universes.

If we impose $f(\mathcal{Q}) = \mathcal{Q}$, and the conditions that characterize FLRW universes, i.e., absence of acceleration, distortion, and vorticity of the fluid flow, $\mathring{u}^a = \tilde{\sigma}_{ab} = \tilde{\omega}_{ab} = 0$, and if we consider that the expansion rate $\tilde{\Theta}$ can be defined as a function of the scale factor a as $\tilde{\Theta} = 3\mathring{a}/a$, we obtain that Eqs. (6.84) and (6.85) are equal to Eqs. (4.39) and (4.38), respectively.

6.3 Bianchi type-I model in $1+3$ approach

Bianchi type-I models describe anisotropic and homogeneous universes characterized by a timelike congruence that is orthogonal to the spatial hypersurfaces foliating the spacetime. Because of Frobenius' theorem, the orthogonality condition is guaranteed by $\omega_{ab} = 0$ (see Appendix B). Furthermore, in BI models the spatial hypersurfaces are assumed flat, i.e., ${}^3R^a{}_{bcd} = 0$.

The BI models' symmetries also impact the form of the nonmetricity tensor. Remembering that we have chosen $Q_0 = 0$, we can assume without loss of generality that the only non-zero projections of the nonmetricity tensor are Q_1 and $Q^{(0)}{}_{ab}$, so the nonmetricity tensor results to be of the particular form

$$Q_{cab} = -\frac{1}{3}Q_1 u_c h_{ab} - Q^{(0)}{}_{ab} u_c. \tag{6.88}$$

In local coordinates, expression (6.88), and in particular the condition $Q_0 = 0$, can be justified by adopting the coincidence gauge $\Gamma_{ab}{}^c = 0$. Since the projections of the nonmetricity tensor are tensor quantities, once the identity (6.88) has been proved in the coincidence gauge, it remains valid in any other gauge. A straightforward calculation shows that in the coincident gauge, using the metric (5.2), the only non-zero projections of the nonmetricity tensor Q_{cab} are,

$$Q_1 = 2\frac{\dot{\tau}}{\tau}, \tag{6.89}$$

$$Q^{(0)}{}_{11} = \left(2a\dot{a} - \frac{2}{3}a^2\,\frac{\dot{\tau}}{\tau}\right), \tag{6.90}$$

$$Q^{(0)}{}_{22} = \left(2b\dot{b} - \frac{2}{3}b^2\,\frac{\dot{\tau}}{\tau}\right), \tag{6.91}$$

$$Q^{(0)}{}_{33} = \left(2c\dot{c} - \frac{2}{3}c^2\,\frac{\dot{\tau}}{\tau}\right), \tag{6.92}$$

thus Eq. (6.88) is proved. With these assumptions, and separating the Levi-Civita contributions from the nonmetricity ones, we can write:

$$\Theta = \tilde{\Theta} - \frac{1}{2}Q_1, \tag{6.93}$$

$$\sigma_{ab} = \tilde{\sigma}_{ab} - \frac{1}{2}Q^{(0)}{}_{ab}, \tag{6.94}$$

and

$$u^a\nabla_a u^b = u^a\tilde{\nabla}_a u^b + L_{ac}{}^b u^a u^c. \tag{6.95}$$

However, in the formulation of $f(\mathcal{Q})$ gravity that we are considering, the curvature tensor is identically zero, which corresponds to flat spacetime. Therefore, from Eq. (6.38), for $R^a{}_{bcd} = 0$, ${}^3R^a{}_{bcd} = 0$, and $u^a\nabla_a u^b = 0$, we can set $\Theta = 0$ and $\sigma_{ab} = 0$, in such a way that

$$\tilde{\Theta} = \frac{1}{2}Q_1, \tag{6.96}$$

$$\tilde{\sigma}_{ab} = \frac{1}{2}Q^{(0)}{}_{ab}, \tag{6.97}$$

$$\mathcal{Q} = -\frac{1}{4}Q^{(0)}{}_{ab}Q^{(0)ab} + \frac{1}{6}{Q_1}^2 = -2\tilde{\sigma}^2 + \frac{2}{3}\tilde{\Theta}^2, \tag{6.98}$$

and, by inserting Eq. (6.88) into (6.95) we have

$$\mathring{u}^b = 0. \tag{6.99}$$

In addition, Eq. (6.45) leads to

$$^3\tilde{R}^a{}_{bcd} = 0. \tag{6.100}$$

In the subsequent discussion, we consider a matter source described by the energy-momentum tensor

$$\Psi_{ab} = \rho u_a u_b + p h_{ab} + \pi_{ab}, \tag{6.101}$$

where p and ρ satisfy the barotropic linear equation of state,

$$p = w\rho, \qquad w = const. \tag{6.102}$$

Using all the above results in Eqs. (6.84), (6.85) and (6.86), we can write the $1+3$ cosmological equations for Bianchi type-I universes:

- Raychaudhuri equation,

$$\mathring{\tilde{\Theta}} + \frac{1}{3}\tilde{\Theta}^2 + 2\tilde{\sigma}^2 + \frac{1}{2f'}(\rho + 3p) - \frac{1}{2}\left(\frac{f}{f'} - \mathcal{Q}\right) + \frac{f''}{f'}\tilde{\Theta}\mathring{\mathcal{Q}} = 0; \tag{6.103}$$

- Spatial equations,

$$2\tilde{\sigma}^2 - \frac{2}{3}\tilde{\Theta}^2 + \frac{2}{f'}\rho + \frac{f}{f'} - \mathcal{Q} = 0, \tag{6.104}$$

$$\mathring{\tilde{\sigma}} + \tilde{\Theta}\tilde{\sigma} + \frac{f''}{f'}\tilde{\sigma}\mathring{\mathcal{Q}} - \frac{1}{2f'}\frac{\pi^{ab}\tilde{\sigma}_{ab}}{\tilde{\sigma}} = 0; \tag{6.105}$$

- Energy-momentum conservation,

$$\mathring{\rho} + \tilde{\Theta}(\rho + p) + \pi^{ab}\tilde{\sigma}_{ab} = 0. \tag{6.106}$$

Equation (6.105) is obtained by multiplying Eq. (6.86) by $\tilde{\sigma}^{ij}/(2\tilde{\sigma})$, whereas Eq. (6.106) is derived by the temporal projection of Eq. (6.87)[1].

[1] We should remark here that the derivatives are with respect to the proper time.

7 Dynamics of Bianchi type-I cosmology in $f(\mathcal{Q})$ gravity

In general, cosmological equations can be difficult to solve because they form a system of non-linear (partial) differential equations, so we attempt to obtain a global picture of cosmic evolution by means of the so-called Dynamical Systems Approach (DSA), see e.g., [167, 168]. This technique allows us to study a cosmological model by analyzing the behavior of the orbits in a phase space connected with the geometrical features and matter sources of spacetime. Making use of DSA, it is possible to achieve a semi-quantitative analysis of the solutions of the dynamical equations and their stability. DSA has been widely used in gravitational theories [63, 169–174], including the $f(\mathcal{Q})$ theory [175–179], where it is shown that in FLRW cosmologies, using different functions for $f(\mathcal{Q})$, the evolution of the background is the same as in the ΛCDM model; however, this behavior is due only to the nonmetric structure of spacetime, without the need to introduce any form of exotic energy. The $1+3$ approach introduced above provides an ideal framework to employ the DSA, as it makes available convenient variables for the description of the phase space of the dynamical system. As we will see, in the context of the BI metric, the application of the $1+3$ approach leads to a remarkable simplification of the involved equations. For example, we will be able to deal with complex matter sources as well as non-trivial forms of the function $f(\mathcal{Q})$.

The study performed in this chapter is based on the paper "Bianchi type-I cosmological dynamics in $f(\mathcal{Q})$ gravity: a covariant approach" [75].

7.1 Dynamical Systems Approach

Let us consider a phase space, that is, the space where each point $\mathbf{x} = (x_1, x_2, ..., x_n)$ corresponds to a possible state of our system. The dynamical system is defined as the system of equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{7.1}$$

where the function $\mathbf{f}$ is a vector field such that

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))\,, \tag{7.2}$$

and the dot represents a derivative with respect to a generic time parameter. In cosmology, we are used to dealing with finite and continuous dynamical systems, so we will restrict ourselves to dealing only with these cases. We define critical points as points $\mathbf{x} = \mathbf{x}_0$ in the phase space for which $\mathbf{f}(\mathbf{x}_0) = 0$. These points can be stable or asymptotically stable:

- Stable critical point. *Let $\mathbf{x}_0$ be a critical point of the system (7.1). It is called stable if for every $\epsilon > 0$ we can fund a δ such that if $\psi(t)$ is any solution of (7.1) satisfying $\|\psi(t_0) - \mathbf{x}_0\| < \delta$, then the solution $\psi(t)$ exists for all $t \geq t_0$ and it will satisfy $\|\psi(t_0) - \mathbf{x}_0\| < \varepsilon$ for all $t \geq t_0$;*
- Asymptotically stable critical point. *Let $\mathbf{x}_0$ be a critical point of the system (7.1). It is called asymptotically stable if there exists a number δ such that if $\psi(t)$ is any solution of (7.1) satisfy $\|\psi(t_0) - \mathbf{x}_0\| < \delta$, then $\lim_{t->\infty} \psi(t) = \mathbf{x}_0$.*

Thus, for asymptotically stable points, the trajectories that represent a solution of the system, also called orbit, will sooner or later reach the critical point; while for stable points, it is not certain that trajectories will reach the critical point, however they will remain in a region close to it. On the other hand, a point is said to be unstable if the trajectories move away from the critical point.

There are various techniques for studying the stability of these points; we will use the so-called linear stability theory, based on the idea of linearizing the system near a fixed point. Since we consider smooth functions, we can use the Taylor expansion near $\mathbf{x}_0$ up to the first order,

$$f_i(\mathbf{x}) = f_i(\mathbf{x}_0) + \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)(x_j - x_{0j}). \tag{7.3}$$

The study of the eigenvalues of the Jacobian at the critical point is what gives us information about the stability of the point. A critical point is said hyperbolic if none of the eigenvalues have zero real part. Otherwise, it is called non-hyperbolic. If the critical point is not a point but is represented by a submanifold of dimension greater than or equal to 1, we will have that hyperbolic submanifold has a number of zero eigenvalues equal to the dimension of the submanifold.

Now, from the Hartman-Grobman theorem we know that near a hyperbolic critical point, the nonlinear system (7.1) has the same qualitative structure as the linear system given by the combination of Eqs. (7.1) and (7.3) [167]. Therefore, the study of the stability of the linearized system gives the stability of the nonlinear one. In case the points are non-hyperbolic we have to refer to other analysis techniques such as the Lyapunov's method or the center manifold theorem [167, 168]. A critical point is called an attractor (sink) if all the eigenvalues of the Jacobian have a negative real part; it is called a repeller (source) if all the eigenvalues of the Jacobian have a positive real part, and it is called a saddle if at least two eigenvalues of the Jacobian have real parts with opposite signs. Due to the definition given above, attractors are stable points, whereas repellers are unstable ones. The saddle points instead attract trajectories in some directions but repel them along others.

7.1.1 Dynamical systems in the ΛCDM model

Before studying the BI model in $f(\mathcal{Q})$ gravity, we give an example of how to use the DSA by analyzing spatially flat ΛCDM model[1].

Let us consider Eqs. (4.53), (4.54), and (4.40) with $k = 0$ and with both matter and radiation not interacting with each other,

$$3H^2 = \rho_m + \rho_r + \Lambda, \tag{7.4}$$

$$\dot{H} + H^2 = -\frac{1}{6}\left(\rho_m + 2\rho_r\right) + \frac{\Lambda}{3}, \tag{7.5}$$

$$\dot{\rho}_m + 3H\rho_m = 0, \tag{7.6}$$

$$\dot{\rho}_r + 4H\rho_r = 0, \tag{7.7}$$

where ρ_m and ρ_r are the energy density of matter and radiation, respectively. Now, we introduce the dimensionless variables

$$\Omega_m = \frac{\rho_m}{3H^2}, \qquad \Omega_r = \frac{\rho_r}{3H^2}, \qquad \Omega_\Lambda = \frac{\Lambda}{3H^2}, \tag{7.8}$$

where we assume that $H \geq 0$, so we have an expanding universe. This dimensionless representation minimizes the number of parameters needed to describe the system and can provide additional insight into the influence of parameters on the dynamic response. Also, we normalized with respect to the rate of expansion to have comoving quantities.

The Friedmann equation (7.4) can be rewritten as

$$1 = \Omega_m + \Omega_r + \Omega_\Lambda. \tag{7.9}$$

We can use this equation as a constraint to derive Ω_Λ, so the dynamical system reduces to a system of only two dimensions. Since we consider positive energy densities and positive cosmological constant, we also have the constraint $\Omega_m + \Omega_r \leq 1$, which bounds the physically significant region of the phase space. To obtain the final dynamical system, we have to introduce a dimensionless time variable,

$$\mathcal{T} = \ln a, \tag{7.10}$$

and perform the derivative with respect to it of the dynamical variables,

$$\frac{\mathrm{d}\Omega_m}{\mathrm{d}\mathcal{T}} = \frac{1}{H}\frac{\mathrm{d}\Omega_m}{\mathrm{d}t} = \frac{\dot{\rho}_m}{3H^3} - \frac{2}{3}\frac{\dot{H}}{H^3}\rho_m, \tag{7.11}$$

$$\frac{\mathrm{d}\Omega_r}{\mathrm{d}\mathcal{T}} = \frac{1}{H}\frac{\mathrm{d}\Omega_r}{\mathrm{d}t} = \frac{\dot{\rho}_r}{3H^3} - \frac{2}{3}\frac{\dot{H}}{H^3}\rho_r. \tag{7.12}$$

Finally, using Eqs. (7.4), (7.5), (7.6), and (7.7) we find

$$\Omega_\Lambda = 1 - \Omega_m - \Omega_r \tag{7.13}$$

$$\frac{\mathrm{d}\Omega_m}{\mathrm{d}\mathcal{T}} = \Omega_m\left(3\Omega_m + 4\Omega_r - 3\right) \tag{7.14}$$

$$\frac{\mathrm{d}\Omega_r}{\mathrm{d}\mathcal{T}} = \Omega_r\left(3\Omega_m + 4\Omega_r - 4\right) \tag{7.15}$$

[1] In this example we do not use the $1+3$ formalism, because given the simplicity of the field equations the only difference would be the use of $\tilde{\Theta}$ instead of H.

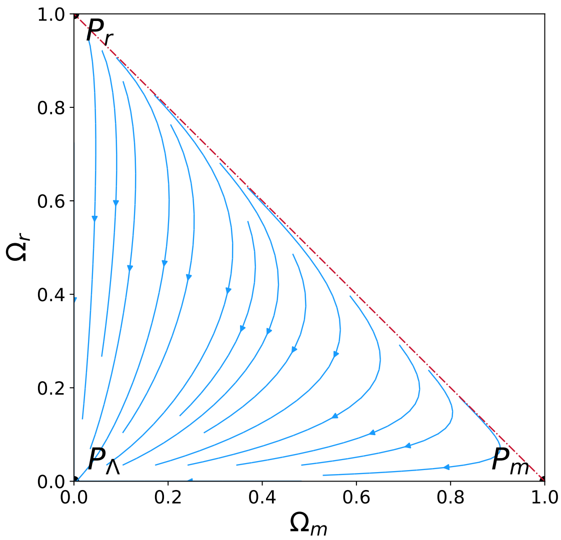


Figure 7.1: Phase space portrait of the system (7.52)-(7.53) for $w = 0$ and $n = 3$.

The system has three critical points

$$P_m = \{\Omega_m = 1, \Omega_r = 0, \Omega_\Lambda = 0\}, \qquad P_r = \{\Omega_m = 0, \Omega_r = 1, \Omega_\Lambda = 0\},$$
$$P_\Lambda = \{\Omega_m = 0, \Omega_r = 0, \Omega_\Lambda = 1\}, \tag{7.16}$$

with eigenvalues of the Jacobian equal to $\{-1, 3\}$, $\{1, 4\}$, and $\{-4, -3\}$, respectively. Therefore, the point P_m is a saddle, P_r is a repeller, whereas the point P_Λ is an attractor. Hence, in its evolution, the universe starts from a situation of radiation dominance, and then reaches a situation of dark energy dominance, with an intermediate phase dominated by matter. This description is in agreement with what was said in Sec. 4.2 where it was shown that the universe tends to be de Sitter type at a late epoch. A graphic representation of the phase space is given in Figure 7.1.

In Eqs. (7.14) and (7.15), the conditions $\Omega_m = 0$ and $\Omega_r = 0$ identify what are called invariant submanifold. The name is due to the fact that they correspond to a null variation of the dynamical variables with respect to the dimensionless time. So, an orbit belonging to an invariant submanifold at a certain instant will always belong to that invariant submanifold. Invariant submanifolds split the phase space into separate regions that are not connected by orbits, because if they were, the solutions would have a discontinuity in their first derivative at the invariant submanifold, but we assumed smooth functions. A consequence is that the only global critical points are given by the intersections of the invariant submanifolds. In FLRW cosmology the point P_Λ is a global attractor since it is the intersection between $\Omega_m = 0$ and $\Omega_r = 0$.

7.2 Dynamical Systems in $f(\mathcal{Q})$ gravity

In this section, we will apply the DSA to analyze the dynamics of BI universes in the framework of $f(\mathcal{Q})$ gravity. We will deal with some specific models associated with particular functions $f(\mathcal{Q})$, all widely used in the literature. In one of the examples, we will also consider the presence of anisotropic pressure.

In our analysis, we will always consider an expanding universe. This means that if we define the "average length scale" l by using the expansion rate $\tilde{\Theta}$ as follows

$$\frac{\mathring{l}}{l} = \frac{1}{3}\tilde{\Theta}, \tag{7.17}$$

then we have to impose $\tilde{\Theta} > 0$.

7.2.1 $f(\mathcal{Q})$ as a power law without anisotropic pressure

As a first example, we consider the function[2]

$$f(\mathcal{Q}) = \alpha \mathcal{Q}^n, \tag{7.18}$$

with α a dimensional constant, and a null anisotropic pressure $\pi_{ij} = 0$. In this case, Eqs. (6.103)-(6.106) assume the form,

$$\mathring{\tilde{\Theta}} + \frac{1}{3}\tilde{\Theta}^2 + 2\tilde{\sigma}^2 + \frac{n-1}{2n}\mathcal{Q} + (n-1)\,\tilde{\Theta}\frac{\mathring{\mathcal{Q}}}{\mathcal{Q}} + \frac{1}{2\alpha n}\,(1+3w)\,\mathcal{Q}^{1-n}\rho = 0, \tag{7.19}$$

$$2\tilde{\sigma}^2 - \frac{2}{3}\tilde{\Theta}^2 + \frac{1-n}{n}\mathcal{Q} + \frac{2}{\alpha n}\mathcal{Q}^{1-n}\rho = 0, \tag{7.20}$$

$$\mathring{\tilde{\sigma}} + \tilde{\Theta}\tilde{\sigma} + (n-1)\,\tilde{\sigma}\frac{\mathring{\mathcal{Q}}}{\mathcal{Q}} = 0, \tag{7.21}$$

$$\mathring{\rho} + \tilde{\Theta}\,(1+w)\,\rho = 0. \tag{7.22}$$

In order to recast these equations in a form more suitable for dynamical system analysis, we define the following dimensionless variables [3],

$$\Sigma^2 = 3\frac{\tilde{\sigma}^2}{\tilde{\Theta}^2}, \qquad \Omega^2 = 3\frac{1}{\alpha}\frac{1}{\tilde{\Theta}^{2n}}\rho. \tag{7.23}$$

Notice that the dynamical variables related to the shear and the matter sources have been chosen non-negative, offering the advantage of a partial compactification of the phase space. The choice of the matter variable should also be discussed. In general, one chooses as the variable associated with ρ, simply $3\rho^2/\tilde{\Theta}^2$ or $3\rho^2/(f'\tilde{\Theta}^2)$, which directly relates to the cosmic matter parameters, and therefore, it is easier to compare with observational results. These parameters appear via Eq. (6.104) in many observable quantities, such as the luminosity distance relation of the look back time, etc. However, here and in the following examples, except for Sec. 7.2.4, we choose a different form for Ω. The reason is that such a form allows us to introduce fewer dynamical variables. In addition, our choice allows us to obtain a variable that involves only the expansion rate and energy density, which can be measured independently, leading to an equally good variable in terms of comparison with observations.

Introducing the dimensionless time variable,

$$\mathcal{T} = \ln l, \tag{7.24}$$

[2]In this example, the case $n = 1$ and $\alpha = 1$ correspond to GR solutions.

[3]In the natural units we are using (i.e., $c = 8\pi G_N = 1$), the quantities Θ^2, σ^2, ρ, and $\mathcal{Q}$ have the dimension of a length to the power of -2.

Table 7.1: The stability of the fixed points and evolution of l, $\tilde{\sigma}$, and ρ for $f(\mathcal{Q}) = \alpha \mathcal{Q}^n$ and $\pi_{ij} = 0$. The parameters t_0, l_0, σ_0, σ_1, ρ_0, and ρ_1 are constants of integration.

	$w=0$			$0<w\leq 1$		
Point	Attractor	Repeller	Saddle	Attractor	Repeller	Saddle
P_1	$n \geq \frac{1}{2}$			$\frac{1}{2} \leq n < \frac{1+w}{2w}$	$n > \frac{1+w}{2w}$	
	Average length	Shear		Energy density		
P_1	$l = l_0\,(t-t_0)^{\frac{2n}{3(1+w)}}$	$\tilde{\sigma} = \sigma_0 = 0$		$\rho = \rho_0 + \frac{\rho_1}{(t-t_0)^{2n}}$		

making use of the above variables and recalling the identity (6.98), we can rewrite the system of cosmological equations in the new form,

$$(1-2n)\left(1-\Sigma^2\right) + \left(\frac{3}{2}\right)^{n-1} \Omega^2 \left(1-\Sigma^2\right)^{1-n} = 0, \tag{7.25}$$

$$\frac{d\Sigma}{d\mathcal{T}} = \frac{1}{3n}\Sigma\left(\Sigma^2-1\right)\left[3(n+1) - 3^n(3w+1)\left(2-2\Sigma^2\right)^{-n}\Omega^2\right]. \tag{7.26}$$

From Eq. (7.25) we derive Ω as a function of Σ,

$$\Omega = \sqrt{2n-1}\left(\frac{3}{2}\right)^{\frac{1-n}{2}}\left(1-\Sigma^2\right)^{\frac{n}{2}}, \tag{7.27}$$

which makes Eq. (7.26) a differential equation for Σ,

$$\frac{d\Sigma}{d\mathcal{T}} = \frac{3}{2n}\Sigma\left(1-\Sigma^2\right)\left[(2n-1)w-1\right]. \tag{7.28}$$

A first consideration about the above equations is that $\Sigma = 1$ is not an acceptable value, since we derive Eq. (7.27) from Eq. (7.25), assuming that $\Sigma \neq 1$. The same problem will occur in Sec. 7.2.2, in which the function $f(\mathcal{Q})$ is again (7.18).

Furthermore, being Ω and Σ real, Eq. (7.27) is only meaningful if $n \geq 1/2$ and $0 \leq \Sigma < 1$, and for some values of n in the intervals $n \geq 1/2$ and $\Sigma > 1$, or $n \leq 1/2$ and $\Sigma > 1$. However, we consider only the condition $0 \leq \Sigma < 1$. This choice has two motivations. The first is that, from a physical point of view, we are interested in the states of phase space describing an isotropic universe, i.e. $\Sigma = 0$. This state cannot be reached by any orbit starting at $\Sigma > 1$. A second motivation is that in Eq. (7.27) there is the term $(1-\Sigma^2)^{\frac{n}{2}}$, the value of which depends strictly on the choice of n (e.g. even, odd, or a rational number) when $\Sigma > 1$. The case $n \geq 1/2$ and $0 \leq \Sigma \leq 1$, on the other hand, being a continuous interval for n, offers a wider setting for a parameter analysis aimed at comparison with observations. Similar constraints will be necessary also in the models we will consider in the following sections.

The system (7.27) and (7.28) presents only one critical point,

$$P_1 = \left\{\Sigma = 0,\ \Omega = \sqrt{2n-1}\left(\frac{3}{2}\right)^{\frac{1-n}{2}}\right\}, \tag{7.29}$$

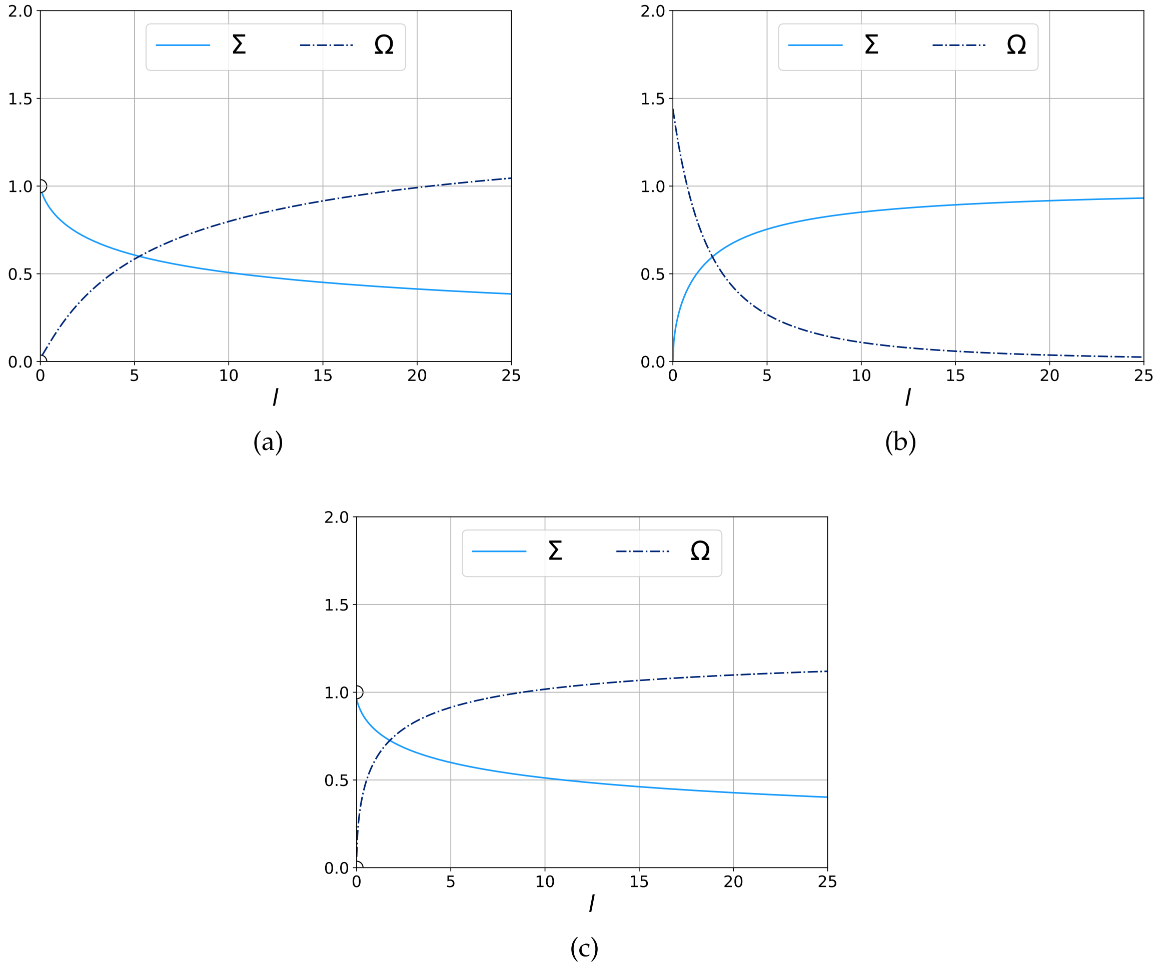


Figure 7.2: Evolution of $\Sigma(l)$ and $\Omega(l)$ with: (a) $w = 0$, $n = 3$, and $\Sigma_1 = -1/3$; (b) $w = 1/3$, $n = 3$, and $\Sigma_1 = 2/3$; (c) $w = 1/3$, $n = 3/2$, and $\Sigma_1 = -1/4$. The empty (half-)circles represent the conditions $\Sigma \neq 1$ and $\Omega \neq 0$.

which represents a universe where the shear is negligible with respect to the matter.

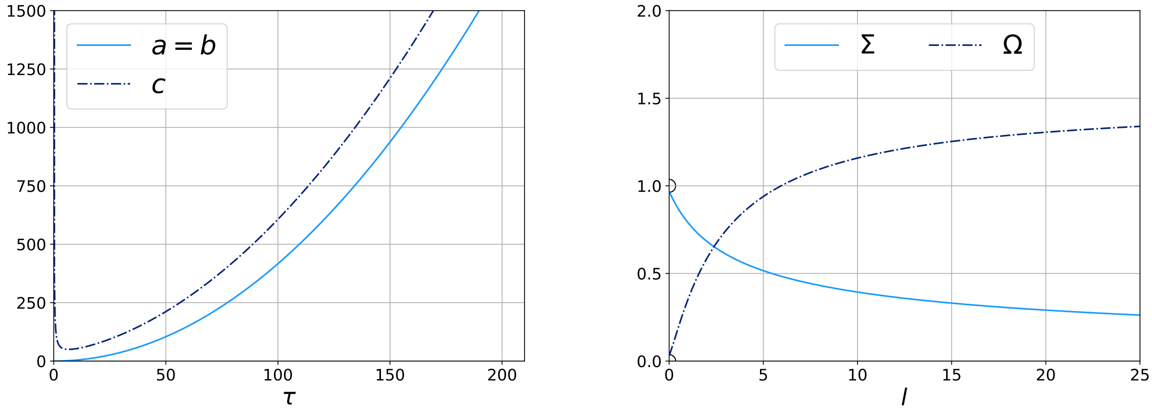
The derivative of Eq. (7.28) with respect to Σ allows us to discuss the stability of the solutions near the critical point, which depends on the values of w and n. The results are shown in Table 7.1.

We can obtain an "approximation" for the time dependence of l, $\tilde{\sigma}$, and ρ near a critical point by substituting Eq. (7.23) into Eqs. (7.19), (7.21), and (7.22). The results are again reported in Table 7.1.

Equation (7.28) can be solved analytically, so we can also obtain exact solutions for Σ and Ω as a function of average length scale l,

$$\begin{aligned} \Sigma(l) &= \frac{1}{\sqrt{1 + e^{2\Sigma_1} l^{\frac{3(1+w-2nw)}{n}}}}, \\ \Omega(l) &= \sqrt{2n-1}\left(\frac{3}{2}\right)^{\frac{1-n}{2}} \left(\frac{e^{2\Sigma_1} l^{\frac{3(1+w-2nw)}{n}}}{1 + e^{2\Sigma_1} l^{\frac{3(1+w-2nw)}{n}}}\right)^{n/2}, \end{aligned} \tag{7.30}$$

where Σ_1 is a constant of integration. Using Eq. (7.30), we can compare the evolution of Σ and Ω with the results coming from the stability analysis in Table 7.1. As it can be seen in



(a)

(b)

Figure 7.3: (a) Evolution of the scale factor a, b, and c in function of the proper time t, with $w = 0$, $n = 3$, $K_\tau = \frac{10}{3}$, $a_1 = c_1 = \frac{1}{24}$, and $t_0 = -\frac{20}{3}$. (b) Evolution of Σ, and Ω in function of the average length scale l, with $w = 0$, $n = 3$, $K_\tau = \frac{10}{3}$, and $a_1 = c_1 = \frac{1}{24}$. The empty (half-)circles represent the conditions $\Sigma \neq 1$ and $\Omega \neq 0$.

Figure 7.2, once the appropriate parameters have been chosen, the results are consistent with Table 7.1. In Figure 7.2a, where $w = 0$ and $n \geq 1/2$, we have that P_1 is an attractor, whereas in Figure 7.2b, and Figure 7.2c, with $0 < w \leq 1$, P_1 is a repeller or an attractor, respectively, depending on the value of n.

In Sec. 5.2.2, the reconstruction method was used to find exact Bianchi type-I cosmologies in $f(\mathcal{Q})$ gravity. It is interesting to compare these results with the more general description we have obtained from the above phase space analysis.

For example, we found, for $f(\mathcal{Q}) = \beta \mathcal{Q}^n$, $w = 0$ and n an odd integer, the following solution for the scale factors[4],

$$\begin{aligned} a(t) =& a_1 \left[(t - t_0)^2 - \frac{4n^2 K_\tau^2}{9} \right]^{\frac{n}{3}} \left[\frac{3\frac{(t-t_0)}{n} - 2K_\tau}{3\frac{(t-t_0)}{n} + 2K_\tau} \right]^{\frac{n}{3}}, \\ c(t) =& c_1 \left[(t - t_0)^2 - \frac{4n^2 K_\tau^2}{9} \right]^{\frac{n}{3}} \left[\frac{3\frac{(t-t_0)}{n} - 2K_\tau}{3\frac{(t-t_0)}{n} + 2K_\tau} \right]^{-\frac{2n}{3}}, \end{aligned} \tag{7.31}$$

represented by Figure 7.3a. As has been pointed out previously, it is clear that the scale factors tend to have the same expansion rate as the time increases, thus describing a universe that tends to isotropize. To perform the comparison we need the dynamical variables Σ and Ω calculated for the (7.31). Since we can impose without loss of generally $\tau = l^3$, from Eq. (5.66) we derive the proper time t as a function of l. Therefore, from Eqs.

[4]In Sec. 5.2.2 we have $f(\mathcal{Q}) = \beta \mathcal{Q}^{\frac{1}{n}}$, thus here we perform the transformation $\frac{1}{n} \to n$.

(6.89), (6.90), (6.91), (6.92), (6.96), and (6.97), we obtain

$$\begin{aligned}
\Sigma^2 =& \frac{4K_\tau^2}{\frac{9}{n^2}\left(\frac{l^3}{a_1 b_1 c_1}\right)^{\frac{1}{n}} + 4K_\tau^2}, \\
\Omega^2 =& \frac{1}{\alpha}\frac{3^{1+2n}}{4^n a_1^2 b_1^2 c_1^2}\left(\frac{1}{n}\right)^{4n}\rho_0\left[\frac{9}{n^2}\left(\frac{l^3}{a_1 b_1 c_1}\right)^{\frac{1}{n}} + 4K_\tau^2\right]^{-n} l^3.
\end{aligned} \tag{7.32}$$

The isotropization is evident in Figure 7.3b, which shows the behavior of Σ and Ω. As expected, this behavior matches exactly the one of Figure 7.2a once the parameters are chosen consistently.

7.2.2 $f(\mathcal{Q})$ as a power law with anisotropic pressure

We consider again the function $f(\mathcal{Q}) = \alpha\mathcal{Q}^n$, but now we add an anisotropic pressure of the form (see e.g. [163, 180, 181]),

$$\pi_{ij} = -\mu\tilde{\sigma}_{ij}, \tag{7.33}$$

being μ a suitable dimensional constant.

Under these assumptions, the dynamical equations are:

$$\mathring{\tilde{\Theta}} + \frac{1}{3}\tilde{\Theta}^2 + 2\tilde{\sigma}^2 + \frac{n-1}{2n}\mathcal{Q} + (n-1)\,\tilde{\Theta}\frac{\mathring{\mathcal{Q}}}{Q} + \frac{1}{2\alpha n}\left(1+3w\right)Q^{1-n}\rho = 0, \tag{7.34}$$

$$2\tilde{\sigma}^2 - \frac{2}{3}\tilde{\Theta}^2 + \frac{1-n}{n}\mathcal{Q} + \frac{2}{\alpha n}\mathcal{Q}^{1-n}\rho = 0, \tag{7.35}$$

$$\mathring{\tilde{\sigma}} + \tilde{\Theta}\tilde{\sigma} + (n-1)\,\tilde{\sigma}\frac{\mathring{\mathcal{Q}}}{Q} + \frac{\mu}{\alpha n}\mathcal{Q}^{1-n}\tilde{\sigma} = 0, \tag{7.36}$$

$$\mathring{\rho} + \tilde{\Theta}\left(1+w\right)\rho - 2\mu\sigma^2 = 0. \tag{7.37}$$

In this case, we have an additional variable related to the anisotropic pressure,

$$\mathcal{M} = \frac{\mu}{\alpha}\tilde{\Theta}^{1-2n}, \tag{7.38}$$

together with

$$\Sigma^2 = 3\frac{\tilde{\sigma}^2}{\tilde{\Theta}^2}, \quad \Omega^2 = 3\frac{1}{\alpha}\frac{1}{\tilde{\Theta}^{2n}}\rho. \tag{7.39}$$

By following a similar procedure as in the previous example, we obtain the final system of dynamical equations:

$$\Omega = \sqrt{2n-1}\left(\frac{3}{2}\right)^{\frac{1-n}{2}}\left(1-\Sigma^2\right)^{\frac{n}{2}}, \tag{7.40}$$

$$\frac{d\Sigma}{d\mathcal{T}} = -\frac{1}{2n}\left(\frac{3}{2}\right)^n \Sigma\left(1-\Sigma^2\right)^{1-n}\left[4\mathcal{M} + 3^{1-n}\left(2-2\Sigma^2\right)^n\left(1+w-2nw\right)\right], \tag{7.41}$$

$$\begin{aligned}
\frac{d\mathcal{M}}{d\mathcal{T}} =& \frac{3^n}{2n}\mathcal{M}\Big\{8\left(n-1\right)\mathcal{M}\Sigma^2\left(2-2\Sigma^2\right)^{-n} + \\
& - 3^{1-n}(2n-1)\left[\Sigma^2\left(2nw-w-1\right) - w - 1\right]\Big\}.
\end{aligned} \tag{7.42}$$

Table 7.2: The stability of the fixed points and evolution of l, $\tilde{\sigma}$, and ρ for $f(\mathcal{Q}) = \alpha\mathcal{Q}^n$ and $\pi_{ij} = -\mu\sigma_{ij}$. The parameters t_0, l_0, σ_0, ρ_0, and ρ_1 are constants of integration.

	$w = 0$			$0 < w \leq 1$		
Point	Attractor	Repeller	Saddle	Attractor	Repeller	Saddle
P_1			$n \geq \frac{1}{2}$		$n > \frac{w+1}{2w}$	$\frac{1}{2} \leq n < \frac{w+1}{2w}$
	Average length		Shear	Energy density		
P_1	$l = l_0 \left(t - t_0\right)^{\frac{2n}{3(1+w)}}$		$\tilde{\sigma} = \sigma_0 = 0$	$\rho = \rho_0 + \frac{\rho_1}{(t-t_0)^{2n}}$		

As anticipated in Sec. 7.2.1, in the above system of equations, $\Sigma = 1$ is not an acceptable value. In addition, the conditions to have Ω and Σ real are $n \geq 1/2$ and $0 \leq \Sigma < 1$.

The invariant submanifolds of the system include $\Sigma = 0$ and $\mathcal{M} = 0$. Notice that the invariant submanifold $\Sigma = 0$ represents isotropic universes, whereas $\mathcal{M} = 0$ implies that either we are in a situation in which the terms associated with the coupling μ are negligible (and therefore the universe described in Sec. 7.2.1) or that the expansion is going to infinity. However, it is not immediate in this framework to distinguish these two cases. Only a more detailed analysis of the equations or a different choice of variables might shed clarity on this point. We will not attempt such an analysis here.

In the parameter range we consider, there is one critical point,

$$P_1 = \left\{\Sigma = 0,\, \mathcal{M} = 0,\, \Omega = \sqrt{2n-1}\left(\frac{3}{2}\right)^{\frac{1-n}{2}}\right\}, \tag{7.43}$$

where matter dominates over the shear and the anisotropic pressure. The analysis of the stability and the approximate evolution of l, $\tilde{\sigma}$ and ρ are summarized in Table 7.2.

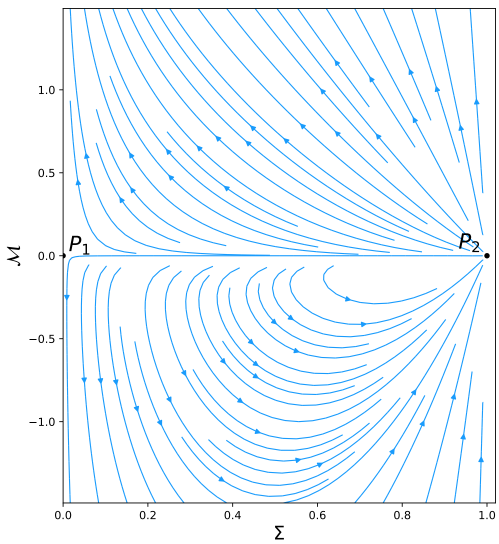
The phase space is described in Figures 7.4a, 7.4b and 7.4c, for different values of w and n. To proceed in the analysis, we define

$$P_2 := \left\{\Sigma = 1,\, \mathcal{M} = 0,\, \Omega = 0\right\}, \tag{7.44}$$

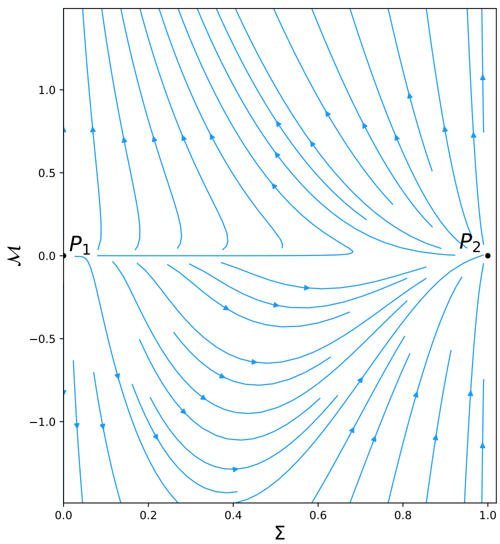
which is *not* a critical point, but it will be useful to describe the orbits of the phase space.

The phase space we obtained shows several types of cosmic evolutions. For example in Figure 7.4a, close to P_2, with $\mathcal{M}$ positive, we are in a universe where matter is negligible compared to shear. As the time progresses, the universe isotropizes with a decreasing expansion rate. In contrast, in the negative half-plane for M, after a phase of isotropization and approach to P_1, the orbits return to their starting point, i.e. to an anisotropic state. A similar behavior is found in Figure 7.4c. On the other hand, in Figure 7.4b we have that orbits move away from an isotropic universe, represented by the points of the phase space near P_1. In the positive half-plane, the region near P_2 is a transition phase for the system leading to a decelerated isotropization, whereas in the negative half-plane, there are decelerated and accelerated expansion phases that lead the universe to anisotropy.

As expected, the invariant submanifold $\mathcal{M} = 0$ mirrors exactly the phase space of the case of Section 7.2.1.



(a)



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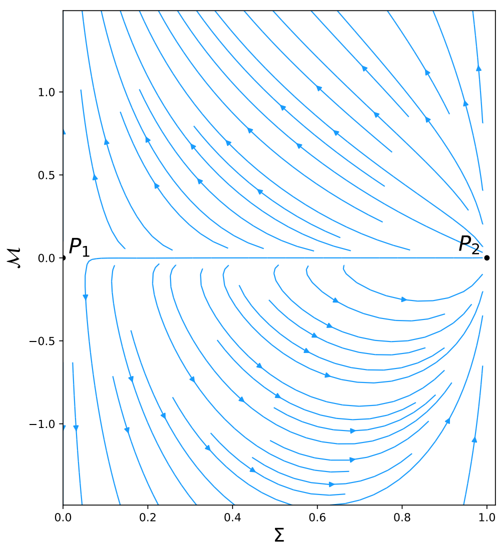

(c)

Figure 7.4: Phase space portrait of the system (7.41)-(7.42) with (a) $w = 0$ and $n = 3$, (b) $w = \frac{1}{3}$ and $n = 3$, (c) $w = \frac{1}{3}$ and $n = 3/2$.

7.2.3 The case $f(\mathcal{Q}) = \alpha\left(\sqrt{\mathcal{Q}} + \beta\mathcal{Q}^n\right)$

We now consider the following function,

$$f\left(\mathcal{Q}\right) = \alpha\left(\sqrt{\mathcal{Q}} + \beta\mathcal{Q}^n\right), \tag{7.45}$$

Table 7.3: The stability of the fixed points and evolution of l, $\tilde{\sigma}$, and ρ for $f(\mathcal{Q}) = \alpha\left(\sqrt{\mathcal{Q}} + \beta\mathcal{Q}^n\right)$ and $\pi_{ij} = 0$. The parameters t_0, l_0, σ_0, σ_1, and ρ_0 are constants of integration.

	$0 \leq w \leq 1$		
Point	Attractor	Repeller	Saddle
P_1	$n > \frac{1}{2}$		
P_2			$n > \frac{1}{2}$
	Average length	Shear	Energy density
P_1	$l = l_0\,(t - t_0)^{\frac{2n}{3(1+w)}}$	$\tilde{\sigma} = \sigma_0 = 0$	$\rho = \rho_0 = 0$
P_2	$l = l_0\,(t - t_0)^{\frac{2n}{3(1+2n+w)}}$	$\sigma = \sigma_0 + \sigma_1\,(t - t_0)^{-1}$	$\rho = \rho_0 = 0$

where α and β are dimensional constants, and set the anisotropic pressure π_{ij} equal to zero. The resulting cosmological equations are,

$$\begin{aligned}
&\mathring{\tilde{\Theta}} + \frac{1}{3}\tilde{\Theta}^2 + 2\tilde{\sigma}^2 + \frac{\mathcal{Q}}{2} - \frac{\mathcal{Q} + \beta\mathcal{Q}^{n+\frac{1}{2}}}{1 + 2\beta n\mathcal{Q}^{n-\frac{1}{2}}} + \\
&\qquad - \frac{1}{2}\frac{\mathring{\mathcal{Q}}}{\mathcal{Q}}\frac{1 - 4\beta(n-1)n\mathcal{Q}^{n-\frac{1}{2}}}{1 + 2\beta n\mathcal{Q}^{n-\frac{1}{2}}}\tilde{\Theta} + \frac{\sqrt{\mathcal{Q}}}{\alpha\left(1 + 2\beta n\mathcal{Q}^{n-\frac{1}{2}}\right)}\left(1 + 3w\right)\rho = 0,
\end{aligned} \tag{7.46}$$

$$2\tilde{\sigma}^2 - \frac{2}{3}\tilde{\Theta}^2 - \mathcal{Q} + 2\frac{\mathcal{Q} + \beta\mathcal{Q}^{n+\frac{1}{2}}}{1 + 2\beta n\mathcal{Q}^{n-\frac{1}{2}}} + \frac{4\sqrt{\mathcal{Q}}}{\alpha\left(1 + 2\beta n\mathcal{Q}^{n-\frac{1}{2}}\right)}\rho = 0, \tag{7.47}$$

$$\mathring{\tilde{\sigma}} + \tilde{\Theta}\tilde{\sigma} - \frac{1}{2}\frac{\mathring{\mathcal{Q}}}{\mathcal{Q}}\frac{1 - 4\beta(n-1)n\mathcal{Q}^{n-\frac{1}{2}}}{1 + 2\beta n\mathcal{Q}^{n-\frac{1}{2}}}\tilde{\sigma} = 0, \tag{7.48}$$

$$\mathring{\rho} + \tilde{\Theta}\left(1 + w\right)\rho = 0. \tag{7.49}$$

By defining the dynamical variables,

$$\Sigma^2 = 3\frac{\tilde{\sigma}^2}{\tilde{\Theta}^2}, \qquad \mathcal{B} = \beta\tilde{\Theta}^{2n-1}, \qquad \Omega^2 = 3\frac{1}{\alpha}\frac{1}{\tilde{\Theta}}\rho, \tag{7.50}$$

the reduced system of dynamical equations is

$$\Omega = \left(\frac{3}{2}\right)^{\frac{1-n}{2}}\sqrt{\left(2n-1\right)\mathcal{B}}\left(1 - \Sigma^2\right)^{\frac{n}{2}}, \tag{7.51}$$

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\mathcal{T}} = \frac{3\Sigma\left(\Sigma^2 - 1\right)}{3^n\sqrt{2} + 2^{n+1}\sqrt{3}n\mathcal{B}\left(1 - \Sigma^2\right)^{n-\frac{1}{2}}}\left[3^n\sqrt{2} + 2^n\sqrt{3}\mathcal{B}(1 + w - 2nw)\left(1 - \Sigma^2\right)^{n-\frac{1}{2}}\right], \tag{7.52}$$

$$\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\mathcal{T}} = \frac{3\left(1 - 2n\right)\mathcal{B}}{n\left[3^n\sqrt{2} + 2^{n+1}\sqrt{3}n\mathcal{B}\left(1 - \Sigma^2\right)^{n-\frac{1}{2}}\right]}\left\{3^n\sqrt{2}n\Sigma^2 + \right.$$

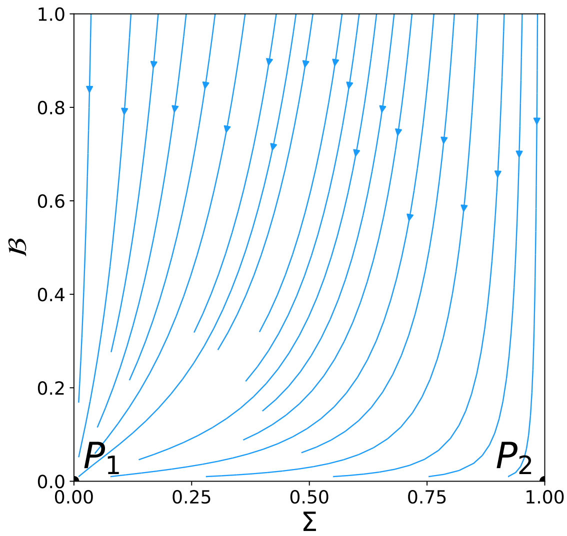

Figure 7.5: Phase space portrait of the system (7.52)-(7.53) for $w = 0$ and $n = 3$.

$$+\frac{3^n}{\sqrt{2}}(1+w) + 2^n\sqrt{3}n\mathcal{B}\left[1+w+(1+w-2nw)\Sigma^2\right]\left(1-\Sigma^2\right)^{n-\frac{1}{2}}\Big\}. \tag{7.53}$$

We assume $\mathcal{B} \geq 0$, $n \geq 1/2$, and $0 \leq \Sigma \leq 1$, so that Ω is real.

The invariant submanifold $\Sigma = 0$ represents isotropic universes, whereas $\Sigma = 1$ anisotropic ones, and $\mathcal{B} = 0$, similarly to the previous section, a surface where the Lagrangian function is $f(\mathcal{Q}) = \alpha\sqrt{\mathcal{Q}}$, when β is negligible, or the expansion rate $\tilde{\Theta}$ is zero.

The critical points are

$$\begin{aligned} P_1 &= \{\Sigma = 0,\ \mathcal{B} = 0,\ \Omega = 0\}, & (7.54)\\ P_2 &= \{\Sigma = 1,\ \mathcal{B} = 0,\ \Omega = 0\}. & (7.55)\end{aligned}$$

Both critical points have $\mathcal{B}$ and Ω equal to zero, and they are distinguished by the presence or absence of the shear Σ. The stability of the system and the approximate solutions are summarized in Table 7.3.

A representation of the stability is given in Figure 7.5. We notice that all the orbits converge in P_1, which is a global attractor. Hence, in this theory the universe always becomes isotropic.

7.2.4 $f(\mathcal{Q})$ as Lambert function

For this last example, we consider the function,

$$f(\mathcal{Q}) = \mathcal{Q}\, e^{\alpha\mathcal{Q}}, \tag{7.56}$$

where α is a dimensional constant, and the anisotropic pressure π_{ij} is zero.

The cosmological equations are,

$$\mathring{\tilde{\Theta}} + \frac{1}{3}\tilde{\Theta}^2 + 2\tilde{\sigma}^2 + \frac{\alpha\mathcal{Q}^2}{2(1+\alpha\mathcal{Q})} + \frac{\alpha(2+\alpha\mathcal{Q})}{1+\alpha\mathcal{Q}}\mathring{\mathcal{Q}}\tilde{\Theta} + \frac{e^{-\alpha\mathcal{Q}}}{2(1+\alpha\mathcal{Q})}(1+3w)\rho = 0, \tag{7.57}$$

$$2\tilde{\sigma}^2 - \frac{2}{3}\tilde{\Theta}^2 - \mathcal{Q} + \frac{\mathcal{Q}}{1+\alpha\mathcal{Q}} + \frac{2\,e^{-\alpha\mathcal{Q}}}{1+\alpha\mathcal{Q}}\rho = 0, \tag{7.58}$$

$$\mathring{\tilde{\sigma}} + \tilde{\Theta}\tilde{\sigma} + \frac{\alpha\,(2+\alpha\mathcal{Q})}{1+\alpha\mathcal{Q}}\mathring{\mathcal{Q}}\sigma = 0, \tag{7.59}$$

$$\mathring{\rho} + \tilde{\Theta}\,(1+w)\,\rho = 0. \tag{7.60}$$

The introduction of the following dynamical variables,

$$\Sigma^2 = 3\frac{\tilde{\sigma}^2}{\tilde{\Theta}^2}, \qquad \mathcal{A} = \alpha\tilde{\Theta}^2, \qquad \Omega^2 = 3\frac{1}{\tilde{\Theta}^2}\rho, \tag{7.61}$$

leads to the equation for Ω,

$$\Omega = \sqrt{(1-\Sigma^2)\left[1+\frac{4}{3}\mathcal{A}\,(1-\Sigma^2)\right]e^{\frac{1}{3}\mathcal{A}(1-\Sigma^2)}} \tag{7.62}$$

and the system of two differential equations,

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\mathcal{T}} = -\frac{3\Sigma\left(1-\Sigma^2\right)\left\{3-w\left[3+4\mathcal{A}\left(1-\Sigma^2\right)\right]\right\}}{2\left[3+2\mathcal{A}\,(1-\Sigma^2)\right]}, \tag{7.63}$$

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\mathcal{T}} = 6w\mathcal{A}\,\Sigma^2 - \frac{9}{2}\,(1+w)\,\mathcal{A}\left\{\frac{2\Sigma^2}{3+2\mathcal{A}\,(1-\Sigma^2)} + \frac{2\left[3+4\mathcal{A}\left(1-\Sigma^2\right)\right]}{9+2\mathcal{A}\,(1-\Sigma^2)\,[15+4\mathcal{A}\,(1-\Sigma^2)]}\right\}. \tag{7.64}$$

To guarantee that $\rho > 0$, we need to impose that Ω is real, which in turn implies the conditions,

$$\mathcal{A} \leq -\frac{3}{4} \quad \text{and} \quad \frac{1}{2}\sqrt{\frac{3+4\mathcal{A}}{\mathcal{A}}} \leq \Sigma \leq 1 \tag{7.65}$$

or

$$\mathcal{A} > -\frac{3}{4} \quad \text{and} \quad 0 \leq \Sigma \leq 1. \tag{7.66}$$

We identify the invariant submanifolds $\Sigma = 0$, $\Sigma = 1$, and $\mathcal{A} = 0$. The first two outline isotropic and anisotropic universes, respectively; $\mathcal{A} = 0$ is the surface where the theory reduces to $f(\mathcal{Q}) = \mathcal{Q}$, or to a cosmology where $\tilde{\Theta} = 0$.

In the range given by Eqs. (7.65) and (7.66), the critical points are,

$$P_1 = \{\Sigma = 0,\ \mathcal{A} = 0,\ \Omega = 1\}, \tag{7.67}$$

$$P_2 = \{\Sigma = 1,\ \mathcal{A} = 0,\ \Omega = 0\}, \tag{7.68}$$

$$P_3 = \left\{\Sigma = 0,\ \mathcal{A} = -\frac{3}{4},\ \Omega = 0\right\}. \tag{7.69}$$

Moreover, for $w = 1$ and $\mathcal{A} = 0$, the system of Eqs. (7.63) and (7.64) admits the solution,

$$P_4 = \left\{\Sigma = \Sigma^*,\ \mathcal{A} = 0,\ \Omega = \sqrt{1-(\Sigma^*)^2}\right\}, \tag{7.70}$$

where Σ^* is an arbitrary constant.

Table 7.4: The stability of the fixed points and evolution of l, $\tilde{\sigma}$, and ρ for $f(\mathcal{Q}) = \mathcal{Q}e^{\alpha \mathcal{Q}}$ and $\pi_{ij} = 0$. The parameters t_0, l_0, σ_0, ρ_0 and ρ_1 are constants of integration.

Point	Attractor	Repeller	Saddle
P_1	$0 \leq w < 1$		
P_2			$0 \leq w < 1$
P_3	$0 \leq w \leq 1$		
P_4	$w = 1$		
	Average length	**Shear**	**Energy density**
P_1	$l(t) = l_0 (t - t_0)^{\frac{2}{3(1+w)}}$	$\tilde{\sigma} = \sigma_0 = 0$	$\rho(t) = \rho_0 + \frac{\rho_1}{(t-t_0)^2}$
P_2	$l(t) = l_0 \sqrt[3]{3(t - t_0)}$	$\tilde{\sigma}(t) = \sigma_0 + \frac{1}{\sqrt{3}(t-t_0)}$	$\rho(t) = \rho_0 = 0$
P_3	$l(t) = l_0 e^{\frac{t}{t_0}}$	$\tilde{\sigma} = \sigma_0 = 0$	$\rho(t) = \rho_0 = 0$
P_4	$l(t) = l_0 \sqrt[3]{3(t - t_0)}$	$\tilde{\sigma}(t) = \sigma_0 + \frac{\Sigma^*}{\sqrt{3}(t-t_0)}$	$\rho(t) = \rho_0 + \frac{1-\Sigma^{*2}}{3(t-t_0)^2}$

The results of the stability analysis near the critical points and the approximate solutions are outlined in Table 7.4.

The phase space of the Eqs. (7.63), and (7.64) is represented in Figure 7.6. In Figure 7.6a, P_1 and P_3 are attractors, and P_2 is a saddle point. In Figure 7.6b, in addition to the point P_3, all the space identified by $\mathcal{A} = 0$, i.e. the central heavy line in the figure, is an attractor. In both figures, the phase space is divided into three regions by two curves. The dash-dotted line indicates the curve

$$3 + 4\mathcal{A}\left(1 - \Sigma^2\right) = 0, \tag{7.71}$$

determining the lower boundary for which, by Eq. (7.65), Ω is real. Therefore, the phase space is not physical below this line, and in the figures, this area corresponds to the shaded one. Instead, the dashed curve represents one of the denominators of Eq. (7.64),

$$9 + 2\mathcal{A}\left(1 - \Sigma^2\right)\left[15 + 4\mathcal{A}\left(1 - \Sigma^2\right)\right] = 0, \tag{7.72}$$

the other denominator of Eq. (7.64) is irrelevant as it lies below the dash-dotted curve.

The presence of the sectors delimited by the dashed and dash-dotted curves is an essential difference from the other examples discussed above. In Sec. 7.2.2 we have analyzed different behaviors of the orbits according to the positivity or negativity of the constants related to the dynamical parameter. Here, however, for $\alpha < 0$, there are different attractors, depending on whether an orbit is above or below the divergence line. Therefore, the final state of cosmology depends crucially on the initial conditions. For example, in the case $w = 1$, the orbits below the divergence line describe universes that tend toward isotropy, whereas orbits above it tend to a finite value of Σ, i.e. the universe approaches an anisotropic state.

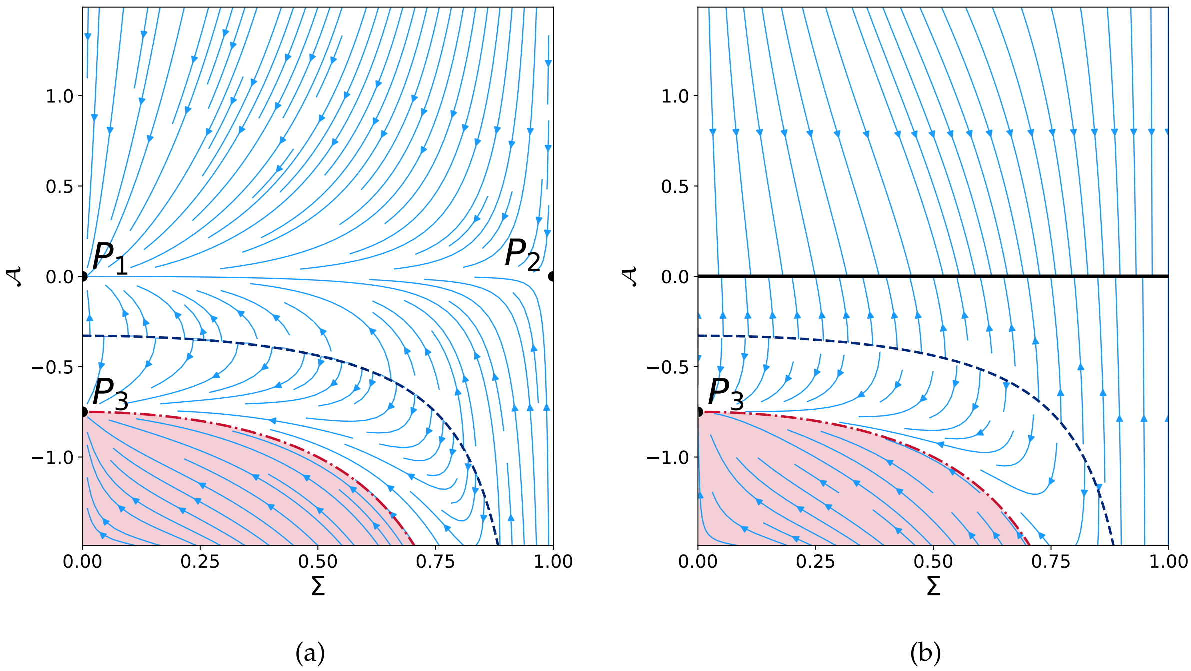

Figure 7.6: Phase space portrait of the system (7.63)-(7.64) for (a) $w = 0$, and (b) $w = 1$. Shaded areas are non-physical regions for the phase space.

7.3 Discussions

We investigated the dynamics of Bianchi type-I cosmologies within the framework of $f(\mathcal{Q})$ gravity using a combination of the $1+3$ covariant formalism and the dynamical systems approach.

The $1+3$ formalism allowed us to obtain a very clear and detailed description of the geometric and dynamic properties of $f(\mathcal{Q})$ cosmologies. In particular, we were able to characterize the effect of nonmetricity on the autoparallel motion of the observers and to obtain cosmological equations that are independent of any specific coordinate system. In addition, the $1+3$ decomposition made it possible to single out the different contributions of the nonmetricity tensor Q_{abc}, making more explicit the effect of nonmetricity on the kinematic quantities. We proved that in Bianchi type-I metric the decomposition of the tensor Q_{abc} involves only the scalar and traceless symmetric tensors which affect the expansion rate Θ and shear σ.

One of the main difficulties of applying the $1+3$ formalism to nonmetric theories of gravity is that in general one cannot assume proper time to be an affine parameter along the timelike congruence. However, in the case of Bianchi type-I cosmologies, this problem can be overcome, thus obtaining complete equivalence between the affine parameter of the world lines and the proper time of the observers associated with the congruence. This aspect is crucial, allowing the introduction of an unambiguous cosmic time and then the definition of a cosmic history.

After writing the cosmological equations in the $1+3$ framework, we separated the contributions due to Levi-Civita from the nonmetricity terms, to better understand the differences between GR and $f(\mathcal{Q})$ gravity. As it happens in many other extensions of GR, we were able to describe in a complete way the additional terms that nonmetricity induces in the gravitational field equations as contributions due to an effective energy-momentum tensor. This formulation allowed an immediate application of the DSA. We considered here four applications, involving different functions $f(\mathcal{Q})$ and thermodynamical properties of the sources.

In the first application, the function $f(\mathcal{Q})$ was a power law (Sec. 7.2.1), which was chosen because of its simplicity and because it is commonly used in literature. We obtained a one-dimensional dynamical system that was solvable analytically. We compared the results with those of Sec. 5.2.2, exhibiting a perfect match when the universe, filled with dust, is initially anisotropic and then isotropizes. This is not surprising as the phase space contains all cosmological solutions, and thus it must include the one reconstructed in Sec. 5.2.2.

We also analyzed a cosmology with the same power law action but in the presence of an anisotropic pressure which we assumed proportional to the shear (Sec. 7.2.2). In this scenario, an isotropic universe is seen to have a transition phase associated with a saddle point, from which the orbits either diverge completely from it or return to the anisotropic state from which they started. This behavior suggests a universe with a "cyclic" evolution, in which after a phase of isotropy, anisotropies start to grow again.

In Sec. 5.2.2 it was found that the reconstructed forms of $f(\mathcal{Q})$ always have a $\sqrt{\mathcal{Q}}$ term which plays a role similar to an integration constant. As another application (Sec. 7.2.3), we investigated the effect of this term when it is added in the functions $f(\mathcal{Q})$ used in the previous two examples. Our analysis showed that the main effect of this additional term is, as expected, constraining the sign of the nonmetricity scalar $\mathcal{Q}$, which in turn excludes some possible cosmic histories (the ones for $\Sigma > 1$). In the cases we considered, the additional term forces all cosmologies to become isotropic in the future.

As a final example, we attempted the evaluation of the effects due to a gravitational action consisting of an infinite series of power-law terms. Such effects can be evaluated considering a function $f(\mathcal{Q})$ as the Lambert function (Sec. 7.2.4). In this case, the phase space differs considerably from the ones in the previous examples. The most important difference turned out to be the appearance of separate regions of the phase space. The presence of these regions shows that the cosmology will have different behaviors and different final attractors depending on the initial conditions.

In all the examples we considered, some areas of the phase space needed to be excluded. We saw that these forbidden regions can appear for different reasons. For instance, in Secs 7.2.1 and 7.2.2, the chosen dynamical variables and the request to have a matter with physical thermodynamical quantities implied the exclusion of the line $\Sigma = 1$. In other cases, like the ones given in Sec. 7.2.3 and 7.2.4, the limitations were related to the nature of the function $f(\mathcal{Q})$. For example, the condition $\Sigma \neq 1$ is connected to the fact that the function $f(\mathcal{Q})$ might take, along an orbit, values that change dramatically the structure of the gravitational field equations, giving rise to singularities or degeneracies.

8

$1+1+2$ covariant formalism

In this chapter, we aim to provide a new framework to study static and spherically symmetric solutions in $f(\mathcal{Q})$ gravity. The usual "coincident gauge", which is characterized by a null connection, fails to describe spherically symmetric spacetimes in vacuum unless the function $f(\mathcal{Q})$ is linear. Because of this limitation, various approaches have been proposed. For example, the total connection is made to coincide with the Levi-Civita one in the gravity-free case [182–184], or alternatively, via the imposition of connection invariance under $SO(3)$ group transformations, restrictions are found on the connection itself [185–189].

We will analyze the class of spacetimes which are static and Locally Rotationally Symmetric (LRS) which contain, as a particular case, static and spherically symmetric spacetimes. Our analysis will be performed using the $1+1+2$ covariant formalism, a natural extension of $1+3$ formalism [190–194] that we have used in Ch. 6. In this approach, in addition to splitting spacetime in a preferred time direction and spatial hypersurfaces, the space itself is split into a preferred spatial direction and surfaces orthogonal to it. Therefore, it is well adapted to describe LRS spacetimes, which have a local preferred spatial direction corresponding to the local axis of symmetry. We will see that the combination of the formalism and the symmetries of LRS spacetimes will allow us to build a complete set of scalar quantities able to describe the structure of static and spherically symmetric spacetimes and to deduce the physical consequences of nonmetricity in this framework.

The study performed in this chapter is based on the paper "Static and LRS spacetimes of type II in $f(\mathcal{Q})$ gravity" [76].

8.1 Static and Locally Rotationally Symmetric spacetime

Considering the splitting $1+3$ introduced in Ch. 6, a spacetime is said to be locally rotationally symmetric in a neighborhood $N(P^*)$ of a point P^*, *if at each point P in $N(P^*)$ there exists a non-discrete subgroup G of the Lorentz group in the tangent space T_P which leaves invariant u^a, the curvature tensor and their derivatives up to third order* [195]. If G is one-dimensional, there is a single preferred spatial direction that constitutes a local axis of symmetry. Therefore, the LRS definition implies that all the spacelike projections of tensors in spatial directions different from the axis of symmetry must be zero. Three different classes of LRS spacetime can be distinguished:

- LRS spacetime of class I, the preferred spatial direction is hypersurface orthogonal;
- LRS spacetime of class II, both the preferred spatial direction and timelike congruence are hypersurface orthogonal;
- LRS spacetime of class III, the timelike congruence is hypersurface orthogonal.

The spacetime is said stationary if it admits a timelike Killing vector field $\mathbf{X}$ such that

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 0. \tag{8.1}$$

In our geometrical setting in which torsion is zero, we also require the further condition

$$\mathcal{L}_{\mathbf{X}} Q_{abc} = 0, \tag{8.2}$$

i.e., both metric and nonmetricity tensors are invariant under the action of the 1-parameter group of local diffeomorphisms generated by $\mathbf{X}$.

If the congruence associated with $\mathbf{X}$ is hypersurface orthogonal, the spacetime is said static.

Before moving on to the description of the $1+1+2$ formalism and LRS spacetimes in $f(\mathcal{Q})$ gravity, we will derive the simplest static and LRS solution in GR, the Schwarzschild metric.

8.1.1 Schwarzschild solution in General Relativity

In deriving the Schwarzschild metric we will also prove Birkhoff's theorem, which states that all spherically symmetric solutions of the Einstein field equations in vacuum must be static and asymptotically flat.

A metric is spherically symmetric if the metric tensor g_{ab} preserves its form under coordinate change of kind

$$\begin{cases} t' = t \\ x'^a = A^a{}_b x^b \end{cases}, \tag{8.3}$$

with $A^a{}_b \in SO(3)$. Therefore, to construct the most general metric such that the above condition is satisfied, we must find all the possible invariants under rotation. First, surely there are the time t and the radius $r = \sqrt{x^2+y^2+z^2}$ of the 2-sphere. The only other two are given by the quantities $\mathbf{x} \cdot \mathrm{d}\mathbf{x} = x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z$ and $\mathrm{d}\mathbf{x}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. Hence, the most general metric constructed with the rotational invariants is equal to ($c = 1$)

$$\mathrm{d}s^2 = -C(t,r)\mathrm{d}t^2 + 2A(t,r)\,(\mathbf{x}\cdot\mathrm{d}\mathbf{x})\,\mathrm{d}t + B(t,r)\,(\mathbf{x}\cdot\mathrm{d}\mathbf{x})^2 + H(t,r)\mathrm{d}\mathbf{x}^2, \tag{8.4}$$

where A, B, C, H are arbitrary functions. Introducing the polar coordinate system,

$$\begin{cases} x = r\sin\Theta\cos\varphi \\ y = r\sin\Theta\cos\varphi \\ z = r\cos\Theta \end{cases}, \tag{8.5}$$

we have

$$\mathbf{x}\cdot\mathrm{d}\mathbf{x} = r\mathrm{d}r, \tag{8.6}$$

and

$$\mathrm{d}\mathbf{x}^2 = \mathrm{d}r^2 + r^2\left(\mathrm{d}\Theta^2 + \sin\Theta^2\mathrm{d}\varphi^2\right), \tag{8.7}$$

thus the metric takes the form

$$\mathrm{d}s^2 = -C(t,r)\mathrm{d}t^2 + 2E(t,r)\mathrm{d}t\mathrm{d}r + D(t,r)\mathrm{d}r^2 + H(t,r)r^2\left(\mathrm{d}\Theta^2 + \sin^2\Theta\mathrm{d}\varphi^2\right), \tag{8.8}$$

with

$$E(t,r) = rA(t,r), \quad \text{and} \quad D(t,r) = r^2B(t,r) + H(t,r), \tag{8.9}$$

Because $H(t,r)$ must be positive to preserve the signature, we can redefine the radial coordinate as follows,

$$r' = \sqrt{H(t,r)}r, \tag{8.10}$$

thereby we obtain

$$\mathrm{d}s^2 = -C(t,r')\mathrm{d}t^2 + 2E(t,r')\mathrm{d}t\mathrm{d}r' + D(t,r')\mathrm{d}r'^2 + r'^2\left(\mathrm{d}\Theta^2 + \sin^2\Theta\mathrm{d}\varphi^2\right). \tag{8.11}$$

Now, we can rewrite the first two terms in such a way as to complete the square, namely,

$$-C(t,r')\mathrm{d}t^2 + 2E(t,r')\mathrm{d}t\mathrm{d}r' = -\left(\sqrt{C(t,r')}\mathrm{d}t - \frac{E(t,r')}{\sqrt{C(t,r')}}\mathrm{d}r'\right)^2 + \frac{E^2(t,r')}{C(t,r')}\mathrm{d}r'^2, \tag{8.12}$$

then the line element is equal to

$$\mathrm{d}s^2 = -\left(\sqrt{C(t,r')}\mathrm{d}t - \frac{E(t,r')}{\sqrt{C(t,r')}}\mathrm{d}r'\right)^2 + \tilde{D}(t,r')\mathrm{d}r'^2 + r'^2\left(\mathrm{d}\Theta^2 + \sin^2\Theta\mathrm{d}\varphi^2\right), \tag{8.13}$$

where

$$\tilde{D}(t,r') = D(t,r') + \frac{E^2(t,r')}{C(t,r')}. \tag{8.14}$$

We want to redefine the terms in parentheses so that

$$\sqrt{C(t,r')}\mathrm{d}t - \frac{E(t,r')}{\sqrt{C(t,r')}}\mathrm{d}r' = \mathrm{d}t', \tag{8.15}$$

with $t' = t'(t,r')$. The redefinition is possible if the r.h.s. of Eq. (8.15) is an exact differential form. However, in the plane identified by t and r', the differential forms are exact minus an integrating factor α,

$$\sqrt{C(t,r')}\mathrm{d}t - \frac{E(t,r')}{\sqrt{C(t,r')}}\mathrm{d}r' = \alpha(t,r')\mathrm{d}t', \tag{8.16}$$

and since the domain we are considering is simply connected, we just need to prove that the differential form is closed. The condition that guarantees that a form is closed gives the following

$$\frac{\partial}{\partial r'}\left(\frac{1}{\alpha(t,r')}\sqrt{C(t,r')}\right) + \frac{\partial}{\partial t}\left(\frac{E(t,r')}{\alpha(t,r')\sqrt{C(t,r')}}\right) = 0, \tag{8.17}$$

which is a first-order partial differential equation whose solution always exists given the initial conditions. Thus, α can always be derived and defined in such a way that Eq. (8.16) holds.

So, the metric can be recast as follows

$$ds^2 = -\mathbb{B}(t,r)\mathrm{d}t^2 + \mathbb{A}(t,r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\Theta^2 + \sin^2\Theta\mathrm{d}\varphi^2\right), \tag{8.18}$$

where we dropped the $'$ from t and r and renamed the arbitrary functions.

Now, we have to apply the Einstein equations $\tilde{R}_{ab} = 0$. The components of the Ricci tensors that are not identically null are given by

$$\tilde{R}_{00} = \frac{\mathbb{B}''}{2\mathbb{A}} - \frac{1}{4}\frac{\mathbb{A}'\mathbb{B}}{\mathbb{A}^2} + \frac{\mathbb{B}'}{r\mathbb{A}} - \frac{1}{4}\frac{\mathbb{B}'^2}{\mathbb{A}\mathbb{B}}\ , \tag{8.19}$$

$$\tilde{R}_{11} = -\frac{\mathbb{B}''}{2\mathbb{B}} + \frac{1}{4}\frac{\mathbb{B}'^2}{\mathbb{B}^2} + \frac{1}{4}\frac{\mathbb{A}'\mathbb{B}'}{\mathbb{A}\mathbb{B}} + \frac{\mathbb{A}'}{r\mathbb{A}}\ , \tag{8.20}$$

$$\tilde{R}_{01} = \frac{\dot{\mathbb{A}}}{r\mathbb{A}}\ , \tag{8.21}$$

$$\tilde{R}_{22} = 1 - \frac{1}{\mathbb{A}} + \frac{1}{2}\frac{r\mathbb{A}'}{\mathbb{A}^2} - \frac{1}{2}\frac{r\mathbb{B}'}{\mathbb{A}\mathbb{B}}\ , \tag{8.22}$$

$$\tilde{R}_{33} = \tilde{R}_{22}\sin^2\Theta\ , \tag{8.23}$$

where $\dot{}$ and $'$ represent the derivative with respect to the time and radius, respectively. From the component $\tilde{R}_{01}$ we obtain that the function $\mathbb{A}$ is independent of t, whereas from the sum

$$\frac{1}{\mathbb{B}}\tilde{R}_{00} + \frac{1}{\mathbb{A}}\tilde{R}_{11} = \frac{\mathbb{B}'}{r\mathbb{A}^2} + \frac{\mathbb{A}'}{r\mathbb{A}^2} = \frac{1}{r\mathbb{A}}\left(\frac{\mathbb{A}'}{\mathbb{A}} + \frac{\mathbb{B}'}{\mathbb{B}}\right) = 0 \quad \longrightarrow \quad \frac{\partial}{\partial r}\ln\left(\mathbb{A}\mathbb{B}\right) = 0, \tag{8.24}$$

we have that the product $\mathbb{A}\mathbb{B}$ depends on time only. This implies

$$\mathbb{A}(r)\mathbb{B}(t,r) = k_1 f(t), \tag{8.25}$$

with k_1 an arbitrary constant. Redefining the time as

$$\mathrm{d}t \quad \longrightarrow \quad \sqrt{f(t)}\mathrm{d}t, \tag{8.26}$$

the metric assumes the form

$$ds^2 = -\frac{k_1}{\mathbb{A}(r)}\mathrm{d}t^2 + \mathbb{A}(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\Theta^2 + \sin^2\Theta\mathrm{d}\varphi^2\right). \tag{8.27}$$

This metric is static since it is stationary and the scalar products of the vectors tangent to the hypersurfaces with the vectors tangent to the timelike congruence are zero, being the metric diagonal. Therefore, the congruence is hypersurface orthogonal, and we have proved our previous statement that all spherically symmetric solutions of the Einstein field equations in vacuum must be static. If we substitute

$$\mathbb{A}(r) = \frac{k_1}{\mathbb{B}(r)} \tag{8.28}$$

in $\tilde{R}_{22}$ and $\tilde{R}_{11}$, we have

$$\tilde{R}_{22} = 1 - \frac{\mathbb{B}}{k_1} - r\frac{\mathbb{B}'}{k_1}, \tag{8.29}$$

$$\tilde{R}_{11} = -\frac{1}{2}\frac{\mathbb{B}''}{\mathbb{B}} - \frac{1}{r}\frac{\mathbb{B}'}{\mathbb{B}} = \frac{k_1}{2r\mathbb{B}}\tilde{R}'_{22}. \tag{8.30}$$

Therefore, we just need to solve $\tilde{R}_{22} = 0$. From this it follows

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\mathbb{B}) = k_1 \quad \longrightarrow \quad \mathbb{B} = k_1\left(1 + \frac{k_2}{k_1}\frac{1}{r}\right). \tag{8.31}$$

Imposing that the metric is asymptotically flat,

$$\lim_{r\to\infty} \mathbb{A} = \lim_{r\to\infty} \mathbb{B} = 1, \tag{8.32}$$

and that we must recover the Newtonian limit, we set

$$k_1 = 1 \qquad \text{and} \qquad k_2 = -2G_N M, \tag{8.33}$$

where M is the mass that generates the gravitational field. Finally, we can write the so-called Schwarzschild metric

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_S}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\varphi^2\right), \tag{8.34}$$

where $r_S = 2G_N M$ is the Schwarzschild radius.

In the case the cosmological constant Λ is considered, the metric becomes

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r} - \frac{1}{3}\Lambda r^2\right)\mathrm{d}t^2 + \left(1 - \frac{r_S}{r} - \frac{1}{3}\Lambda r^2\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\varphi^2\right), \tag{8.35}$$

which is called Schwarzschild-de Sitter metric. It is worth noticing that Eq. (8.35) is no longer asymptotically flat.

8.2 $1+1+2$ decomposition

As for the $1+3$ formalism, here we consider a spacetime endowed with a torsion-free connection.

Let us introduce a congruence of timelike curves $x^a = x^a(\lambda)$, expressed in terms of an affine parameter λ and filling all spacetime. The vector field tangent to the congruence is

$$u^a = \frac{dx^a}{d\lambda}, \qquad u_a = g_{ab}u^b. \tag{8.36}$$

At any point, we can identify a 3-dimensional subspace of the tangent bundle orthogonal to u^a. The projector onto the spatial subspace is defined as

$$h_{ab} = g_{ab} + u_a u_b, \tag{8.37}$$

and satisfies the properties

$$h^a{}_c h^c{}_b = h^a{}_b, \qquad h_{ab}u^b = 0. \tag{8.38}$$

The $1+1+2$ decomposition is implemented by introducing a congruence of spacelike curves with tangent vector field e everywhere orthogonal to u: $g_{ab}u^a e^b = 0$. This allows us to split the spatial subspace into a preferred direction parallel to e and a 2-dimensional subspace, called "sheet", orthogonal to e. The projection tensor onto the sheet is given by

$$N_{ab} = h_{ab} - e_a e_b, \tag{8.39}$$

with

$$N^a{}_c N^c{}_b = N^a{}_b, \qquad N_{ab}u^b = N_{ab}e^b = 0. \tag{8.40}$$

Norm conservation of vectors u and e along both the congruences, and with respect to the covariant derivative induced by the full connection $\Gamma_{ab}{}^c$, as well as the preservation of their orthogonality impose restrictions on the nonmetricity tensor Q_{cab}:

$$\begin{aligned} u^c\nabla_c\left(g_{ab}u^a u^b\right) &= Q_{cab}u^c u^a u^b + 2u_b u^c \nabla_c u^b = 0 \\ \longrightarrow \quad Q_{cab}u^c u^a u^b &= 2u^b u^c \nabla_c u_b, \end{aligned} \tag{8.41}$$

$$\begin{aligned} e^c\nabla_c\left(g_{ab}e^a e^b\right) &= Q_{cab}e^c e^a e^b + 2e_b e^c \nabla_c e^b = 0 \\ \longrightarrow \quad Q_{cab}e^c e^a e^b &= 2e^b e^c \nabla_c e_b, \end{aligned} \tag{8.42}$$

$$\begin{aligned} e^c\nabla_c\left(g_{ab}u^a u^b\right) &= Q_{cab}e^c u^a u^b + g_{ab}e^c\nabla_c\left(u^a u^b\right) = 0 \\ \longrightarrow \quad Q_{cab}e^c u^a u^b &= 2u^b e^c \nabla_c u_b, \end{aligned} \tag{8.43}$$

$$\begin{aligned} u^c\nabla_c\left(g_{ab}e^a e^b\right) &= Q_{cab}u^c e^a e^b + g_{ab}u^c\nabla_c\left(e^a e^b\right) = 0 \\ \longrightarrow \quad Q_{cab}u^c e^a e^b &= 2e^b u^c \nabla_c e_b, \end{aligned} \tag{8.44}$$

$$\begin{aligned} u^c\nabla_c\left(g_{ab}u^a e^b\right) &= Q_{cab}u^c u^a e^b + g_{ab}u^c\nabla_c\left(u^a e^b\right) = 0 \\ \longrightarrow \quad Q_{cab}u^c u^a e^b &= u^b u^c \nabla_c e_b + e^b u^c \nabla_c u_b, \end{aligned} \tag{8.45}$$

$$\begin{aligned} e^c\nabla_c\left(g_{ab}u^a e^b\right) &= Q_{cab}e^c u^a e^b + g_{ab}e^c\nabla_c\left(u^a e^b\right) = 0 \\ \longrightarrow \quad Q_{cab}e^c u^a e^b &= u^b e^c \nabla_c e_b + e^b e^c \nabla_c u_b. \end{aligned} \tag{8.46}$$

The normalization of u^a ($g_{ab}u^a u^b = -1$) and e^a ($g_{ab}e^a e^b = 1$) implies the following properties for the projector tensors:

$$h_a{}^a = 3, \qquad \text{and} \qquad N_a{}^a = 2. \tag{8.47}$$

Making use of h_{ab} and N_{ab}, we can define the projected symmetric trace-free (PSTF) part of a tensor with respect to u^a and e^a, respectively: given a second rank covariant tensor $\mathbb{T}_{ab}$ we have,

$$\mathbb{T}_{\langle ab\rangle} = \left[h_{(a}{}^c h_{b)}{}^d - \frac{1}{3}h_{ab}h^{cd}\right]\mathbb{T}_{cd}, \tag{8.48}$$

and

$$\mathbb{T}_{\{ab\}} = \left[N_{(a}{}^c N_{b)}{}^d - \frac{1}{2}N_{ab}N^{cd}\right]\mathbb{T}_{cd}. \tag{8.49}$$

It is also useful to introduce the volume elements derived from the Levi-Civita tensor ε_{abcd},

$$\varepsilon_{abc} = \varepsilon_{abcd}u^d \qquad\qquad \varepsilon_{ab} = \varepsilon_{abc}e^c, \tag{8.50}$$

which are characterized by the following properties:

$$\varepsilon_{abc} = \varepsilon_{[abc]}, \qquad \varepsilon_{ab} = \varepsilon_{[ab]}, \tag{8.51}$$

$$\varepsilon_{abc}u^c = 0, \qquad \varepsilon_{abc}\varepsilon^{dec} = 2h_{[a}{}^d h_{b]}{}^e, \qquad \varepsilon_{abc}\varepsilon^{abd} = 2h_c{}^d \tag{8.52}$$

$$\begin{gathered}\varepsilon_{ab}e^b = \varepsilon_{ab}u^b = 0, \qquad \varepsilon_a{}^c\varepsilon_{cb} = N_{ab},\\ \varepsilon_{abc} = e_a\varepsilon_{bc} - e_b\varepsilon_{ac} + e_c\varepsilon_{ab}.\end{gathered} \tag{8.53}$$

Given a generic tensor $\mathbb{T}^{a\cdots}{}_{b\ldots}$, there are several kinds of covariant derivative we can define in the $1+1+2$ formalism. In particular:

- the time derivative, the covariant derivative along u^a,

$$\dot{\mathbb{T}}^{a\cdots}{}_{b\ldots} = u^c\nabla_c\mathbb{T}^{a\cdots}{}_{b\ldots}; \tag{8.54}$$

- the spatial derivative, the covariant derivative projected onto the 3-dimensional subspace orthogonal to u^a,

$$D_c\mathbb{T}^{a\cdots}{}_{b\ldots} = h_c{}^d h^a{}_e \cdots h_b{}^f \cdots \nabla_d\mathbb{T}^{e\cdots}{}_{f\ldots}; \tag{8.55}$$

- the hat derivative, the covariant spatial derivative along e^a,

$$\hat{\mathbb{T}}^{a\cdots}{}_{b\ldots} = e^c D_c\mathbb{T}^{a\cdots}{}_{b\ldots}; \tag{8.56}$$

- the δ-derivative, the covariant spatial derivative projected onto the sheet orthogonal to e^a,

$$\delta_c\mathbb{T}^{a\cdots}{}_{b\ldots} = N_c{}^d N^a{}_e \cdots N_b{}^f \cdots D_d\mathbb{T}^{e\cdots}{}_{f\ldots}. \tag{8.57}$$

As we have done in Ch. 6, we now decompose the covariant derivative of the 4-velocity,

$$\begin{aligned}\nabla_a u_b =& D_a u_b - u_a\left(\mathcal{A}_b + e_b\mathcal{A}^{(u)}\right) + \frac{1}{2}u_a u_b Q_{cde}u^c u^d u^e +\\ &- \frac{1}{2}u_b e_a Q_{cde}e^c u^d u^e - \frac{1}{2}u_b N_a{}^c Q_{cde}u^d u^e.\end{aligned} \tag{8.58}$$

where

$$\mathcal{A}^{(u)} = e^a u^b\nabla_b u_a, \qquad \text{and} \qquad \mathcal{A}_b = N_b{}^a u^c\nabla_c u_a. \tag{8.59}$$

Moreover, the spatial derivative of u_a can be expressed as

$$D_a u_b = \frac{1}{3}h_{ab}\Theta + \sigma_{ab} + \omega_{ab}, \tag{8.60}$$

where

$$\Theta = h^{ab}D_a u_b \tag{8.61}$$

represents the expansion of the time congruence,

$$\sigma_{ab} = \left[h_{(a}{}^c h_{b)}{}^d - \frac{1}{3}h_{ab}h^{cd}\right]D_c u_d = \Sigma_{ab} + 2\Sigma_{(a}e_{b)} + \Sigma\left(e_a e_b - \frac{1}{2}N_{ab}\right) \tag{8.62}$$

is the shear tensor, with

$$\Sigma_{ab} = \sigma_{\{ab\}}, \qquad \Sigma_a = N_a{}^c e^b\sigma_{cb}, \qquad \Sigma = e^a e^b\sigma_{ab} = -N^{ab}\sigma_{ab}, \tag{8.63}$$

the tensor, vector, and scalar parts of the shear, respectively. Finally, the quantity

$$\omega_{ab} = D_{[a}u_{b]} = \varepsilon_{ab}\Omega + \varepsilon_{abc}\Omega^c \tag{8.64}$$

is the vorticity tensor with

$$\Omega_c = \frac{1}{2}N_{cd}\varepsilon^{abd}D_a u_b, \qquad \Omega = \frac{1}{2}\varepsilon^{ab}N_a{}^c N_b{}^d D_c u_d. \tag{8.65}$$

In an analogous way, the covariant derivative of e_a can be expressed as,

$$\begin{aligned}\nabla_a e_b =& D_a e_b - u_a \alpha_b - \mathcal{A}^{(e)}u_a u_b + \left(\frac{1}{3}\Theta + \Sigma\right) e_a u_b + \Sigma_a u_b - \varepsilon_{ab}\Omega^d u_b + \\ & - \frac{1}{2}u_a e_b Q_{cdf}u^c e^d e^f - N_a{}^c u_b Q_{cdf}u^d e^f - e_a u_b Q_{cdf}e^c u^d e^f.\end{aligned} \tag{8.66}$$

The spatial derivative of e_a is given by

$$D_a e_b = \frac{1}{2}N_{ab}\phi + \zeta_{ab} + \varepsilon_{ab}\xi + e_a \mathrm{a}_b - N_a{}^c e_b L^f_{cd}e^d e_f + \frac{1}{2}e_a e_b Q_{cdf}e^c e^d e^f \tag{8.67}$$

where

$$\phi = N^{ab}\delta_a e_b \qquad \zeta_{ab} = \delta_{\{a}e_{b\}} \qquad \xi = \frac{1}{2}\varepsilon^{ab}\delta_a e_b \tag{8.68}$$

are the analogous of expansion, shear, and vorticity for the spatial vector e_a. The scalar

$$\mathcal{A}^{(e)} = -u^b u^a \nabla_a e_b \tag{8.69}$$

is the projection of the temporal acceleration of e^a along u^a, the vector

$$\alpha_b = u^c N_b{}^d \nabla_c e_d \tag{8.70}$$

is the projection of the temporal acceleration of e^a onto the sheet, and

$$\mathrm{a}_b = e^c N_b{}^d \nabla_c e_d \tag{8.71}$$

is the projection of the spatial acceleration onto the sheet.

In addition to this, making use of the Weyl tensor (1.76), we introduce six new tensors

$$\begin{aligned} H_{cd} &= \frac{1}{2}\varepsilon_c{}^{ab}u^e C_{abde}, \\ \bar{H}_{ab} &= \frac{1}{2}\varepsilon_a{}^{cd}u^e C_{becd} = H_{ab} + \frac{1}{2}\varepsilon_a{}^{cd}u^e Z^{(1)}{}_{becd}, \\ \check{H}_{ab} &= \frac{1}{2}\varepsilon_a{}^{cd}u^e C_{ebcd} = -H_{ab} + \frac{1}{2}\varepsilon_a{}^{cd}u^e Z^{(1)}{}_{ebcd}, \end{aligned} \tag{8.72}$$

$$E_{ac} = C_{abcd}u^b u^d, \qquad \bar{E}_{bc} = C_{abcd}u^a u^d, \qquad \check{E}_{cd} = C_{abcd}u^a u^b, \tag{8.73}$$

which are the “magnetic” and “electric” parts of the Weyl tensor, respectively. These tensors satisfy the following properties:

The magnetic and electric parts can be decomposed as well:

$$H_{ab} = \mathcal{H}_{ab} + \mathcal{H}_a e_b + e_a \mathcal{H}_b + \left(e_a e_b - \frac{1}{2}N_{ab}\right)\mathcal{H}, \tag{8.74}$$

$$\bar{H}_{ab} = \bar{\mathcal{H}}_{ab} + \bar{\mathcal{H}}_a e_b + e_a \mathbf{H}_b - \check{\bar{\mathcal{H}}}_a u_b + \varepsilon_{ab} \mathbb{H} - e_a u_b \check{\bar{\mathcal{H}}} + \left(e_a e_b - \frac{1}{2} N_{ab} \right) \bar{\mathcal{H}}, \tag{8.75}$$

$$\check{H}_{ab} = \check{\mathcal{H}}_{ab} + \check{\mathcal{H}}_a e_b + e_a \check{\mathbf{H}}_b - \mathfrak{H}_a u_b + \varepsilon_{ab} \check{\mathbb{H}} - e_a u_b \mathfrak{H} + \left(e_a e_b - \frac{1}{2} N_{ab} \right) \check{\mathcal{H}}, \tag{8.76}$$

$$E_{ab} = \mathcal{E}_{ab} + \mathcal{E}_a e_b + e_a \mathbf{E}_b - u_a \check{\mathcal{E}}_b + \varepsilon_{ab} \mathbb{E} - u_a e_b \check{\mathcal{E}} + \left(e_a e_b - \frac{1}{2} N_{ab} \right) \mathcal{E}, \tag{8.77}$$

$$\bar{E}_{ab} = \bar{\mathcal{E}}_{ab} + \bar{\mathcal{E}}_a e_b + e_a \bar{\mathbf{E}}_b - u_a \check{\bar{\mathcal{E}}}_b + \varepsilon_{ab} \bar{\mathbb{E}} - u_a e_b \check{\bar{\mathcal{E}}} + \left(e_a e_b - \frac{1}{2} N_{ab} \right) \bar{\mathcal{E}}, \tag{8.78}$$

$$\check{E}_{ab} = \mathfrak{E}_{ab} + 2\mathfrak{E}_{[a} e_{b]} - 2\check{\mathfrak{E}}_{[a} u_{b]}. \tag{8.79}$$

Eqs. (8.74)-(8.79) involve the following tensor, vector, and scalar quantities:

$$\mathcal{H}_{ab} = \left(N_a{}^c N_b{}^d - \frac{1}{2} N_{ab} N^{cd} \right) H_{cd}, \tag{8.80}$$

$$\mathcal{H}_a = N_a{}^c e^d H_{cd}, \qquad \mathcal{H} = e^a e^b H_{ab} = -N^{ab} H_{ab}, \tag{8.81}$$

$$\bar{\mathcal{H}}_{ab} = \left(N_{(a}{}^c N_{b)}{}^d - \frac{1}{2} N_{ab} N^{cd} \right) \bar{H}_{cd}, \quad \bar{\mathcal{H}}_a = N_a{}^c e^d \bar{H}_{cd}, \quad \mathbf{H}_b = e^c N_b{}^d \bar{H}_{cd}, \tag{8.82}$$

$$\check{\bar{\mathcal{H}}}_a = N_a{}^c u^d \bar{H}_{cd}, \quad \mathbb{H} = \frac{1}{2} \varepsilon^{ab} \bar{H}_{ab}, \quad \check{\bar{\mathcal{H}}} = e^a u^b \bar{H}_{ab}, \quad \bar{\mathcal{H}} = e^a e^b \bar{H}_{ab} = -N^{ab} \bar{H}_{ab}, \tag{8.83}$$

$$\check{\mathcal{H}}_{ab} = \left(N_{(a}{}^c N_{b)}{}^d - \frac{1}{2} N_{ab} N^{cd} \right) \check{H}_{cd}, \quad \check{\mathcal{H}}_a = N_a{}^c e^d \check{H}_{cd}, \quad \check{\mathbf{H}}_b = e^c N_b{}^d \check{H}_{cd}, \tag{8.84}$$

$$\mathfrak{H}_a = N_a{}^c u^d \check{H}_{cd}, \quad \check{\mathbb{H}} = \frac{1}{2} \varepsilon^{ab} \check{H}_{ab}, \quad \mathfrak{H} = e^a u^b \check{H}_{ab}, \quad \check{\mathcal{H}} = e^a e^b \check{H}_{ab} = -N^{ab} \check{H}_{ab}, \tag{8.85}$$

$$\mathcal{E}_{ab} = \left(N_{(a}{}^c N_{b)}{}^d - \frac{1}{2} N_{ab} N^{cd} \right) E_{cd}, \tag{8.86}$$

$$\mathcal{E}_a = N_a{}^c e^d E_{cd}, \qquad \mathbf{E}_b = e^c N_b{}^d E_{cd}, \qquad \check{\mathcal{E}}_b = u^c N_b{}^d E_{cd}, \tag{8.87}$$

$$\mathbb{E} = \frac{1}{2} \varepsilon^{ab} E_{ab}, \qquad \check{\mathcal{E}} = u^a e^b E_{ab}, \qquad \mathcal{E} = e^a e^b E_{ab} = -N^{ab} E_{ab}, \tag{8.88}$$

$$\bar{\mathcal{E}}_{ab} = \left(N_{(a}{}^c N_{b)}{}^d - \frac{1}{2} N_{ab} N^{cd} \right) E_{cd}, \tag{8.89}$$

$$\bar{\mathcal{E}}_a = N_a{}^c e^d E_{cd}, \qquad \bar{\mathbf{E}}_b = e^c N_b{}^d E_{cd}, \qquad \check{\bar{\mathcal{E}}}_b = u^c N_b{}^d E_{cd}, \tag{8.90}$$

$$\bar{\mathbb{E}} = \frac{1}{2} \varepsilon^{ab} \bar{E}_{ab}, \qquad \check{\bar{\mathcal{E}}} = u^a e^b \bar{E}_{ab} = \check{\mathcal{E}}, \qquad \bar{\mathcal{E}} = e^a e^b \bar{E}_{ab} = -N^{ab} \bar{E}_{ab}, \tag{8.91}$$

$$\mathfrak{E}_{ab} = N_a{}^c N_b{}^d, \qquad \mathfrak{E}_a = N_a{}^c e^d \check{E}_{cd}, \qquad \check{\mathfrak{E}}_a = N_a{}^c u^d \check{E}_{cd}. \tag{8.92}$$

Finally, we perform the decomposition of the (symmetric) energy-momentum tensor Ψ_{ab} with respect to u^a, e^a and N_{ab}:

$$\begin{aligned} \Psi_{ab} =& \rho u_a u_b + 2q e_{(a} u_{b)} + \left(p - \frac{1}{2} \pi \right) N_{ab} + \\ &+ (p + \pi) e_a e_b + 2\bar{q}_{(a} u_{b)} + 2\pi_{(a} e_{b)} + \pi_{ab}, \end{aligned} \tag{8.93}$$

with

$$\begin{aligned} &\rho = u^a u^b \Psi_{ab}, \quad q = -e^a u^b \Psi_{ab}, \quad p = \frac{1}{3} h^{ab} \Psi_{ab}, \quad \pi = \frac{1}{3}\left(2e^a e^b - N^{ab}\right) \Psi_{ab}, \\ &q_a = -N_a{}^b u^c \Psi_{bc}, \qquad \pi_a = N_a{}^b e^c \Psi_{bc}, \qquad \pi_{ab} = \left(N_a{}^c N_b{}^d - \frac{1}{2} N_{ab} N^{cd}\right) \Psi_{cd}. \end{aligned} \tag{8.94}$$

The quantities $p_\perp = \left(p - \frac{1}{2}\pi\right)$ and $p_r = (p + \pi)$ represent the transverse and radial pressure of the fluid, respectively.

8.3 Static and LRS spacetime in $1+1+2$ formalism

In the $1+1+2$ formalism, the natural choice for the local axis of symmetry is the above-defined spatial vector e^a. The non-zero kinematic and thermodynamic quantities are the scalars

$$\{\Theta, \Sigma, \Omega, \phi, \xi, \mathcal{A}^{(u)}, \mathcal{A}^{(e)}, \mu, p, q, \pi, \mathcal{H}, \bar{\mathcal{H}}, \bar{\mathbb{H}}, \check{\bar{\mathcal{H}}}, \mathcal{E}, \mathbb{E}, \check{\mathcal{E}}, \bar{\mathcal{E}}, \bar{\mathbb{E}}, \check{\bar{\mathcal{E}}}\}. \tag{8.95}$$

As we said above, in our framework the torsion is zero. On the other hand, torsion is strictly related to vorticity (see [164] for details) and therefore we assume the condition

$$\Omega = \xi = 0. \tag{8.96}$$

Under this assumption, because of Frobenius's theorem (see Appendix B), both timelike and spacelike congruences are hypersurface orthogonal, i.e., we are working within an LRS spacetime of class II.

8.3.1 Decomposition of nonmetricity tensor

In an LRS spacetime, the decomposition of the nonmetricity tensor involves only scalar quantities, according to[1]

$$\begin{aligned} Q_{abc} =& - Q_0 u_a u_b u_c + Q_1 e_a e_b e_c - Q_2 u_a e_b e_c - 2Q_3 e_a u_{(b} e_{c)} + \\ &+ Q_4 e_a u_b u_c + 2Q_5 u_a e_{(b} u_{c)} - \frac{1}{2} Q_6 u_a N_{bc} - Q_7 u_{(b} N_{c)a} + \\ &+ \frac{1}{2} Q_8 e_a N_{bc} + Q_9 e_{(b} N_{c)a} - Q_{10} \varepsilon_{a(b} u_{c)} + Q_{11} \varepsilon_{a(b} e_{c)}, \end{aligned} \tag{8.97}$$

with

$$\begin{aligned} &Q_0 = Q_{abc} u^a u^b u^c, \quad Q_1 = Q_{abc} e^a e^b e^c, \quad Q_2 = Q_{abc} u^a e^b e^c, \quad Q_3 = Q_{abc} e^a e^b u^c, \\ &Q_4 = Q_{abc} e^a u^b u^c, \quad Q_5 = Q_{abc} u^a u^b e^c, \quad Q_6 = Q_{abc} u^a N^{bc}, \quad Q_7 = Q_{abc} N^{ab} u^c, \\ &Q_8 = Q_{abc} e^a N^{bc}, \quad Q_9 = Q_{abc} N^{ab} e^c, \quad Q_{10} = Q_{abc} \varepsilon^{ab} u^c, \quad Q_{11} = Q_{abc} \varepsilon^{ab} e^c. \end{aligned} \tag{8.98}$$

As a consequence, the nonmetricity scalar $\mathcal{Q}$ is seen to assume the form

$$\begin{aligned} \mathcal{Q} =& \frac{1}{2} Q_0 (Q_3 + Q_7) + \frac{1}{2} Q_1 (-Q_5 + Q_9) + \frac{1}{2} Q_2 (Q_3 + Q_6 - Q_7) + \\ &- \frac{1}{2} Q_4 (Q_5 - Q_8 + Q_9) + \frac{1}{8} Q_6 (-4Q_3 + Q_6) - \frac{1}{8} Q_8 (4Q_5 + Q_8) + \\ &- \frac{1}{2} {Q_{10}}^2 + \frac{1}{2} {Q_{11}}^2. \end{aligned} \tag{8.99}$$

[1] In LRS spacetime, both the vector and tensor components are zero.

In view of Eq. (8.97), Eqs. (8.58) and (8.66) are now written as

$$\nabla_a u_b = \frac{1}{3} h_{ab}\Theta + \left(e_a e_b - \frac{1}{2} N_{ab} \right) \Sigma - u_a e_b \mathcal{A}^{(u)} + \varepsilon_{ab}\Omega + \frac{1}{2} u_a u_b Q_0 - \frac{1}{2} e_a u_b Q_4, \tag{8.100}$$

and

$$\nabla_a e_b = \frac{1}{2} N_{ab}\phi + \left(\frac{1}{3}\Theta + \Sigma \right) e_a u_b - u_a u_b \mathcal{A}^{(e)} + \varepsilon_{ab}\xi + \frac{1}{2} e_a e_b Q_1 - \frac{1}{2} u_a e_b Q_2 - e_a u_b Q_3. \tag{8.101}$$

For later use, it is useful to distinguish the contributions due to the Levi-Civita connection from the ones due to nonmetricity in the following scalars,

$$\begin{aligned} \Theta &= \tilde{\Theta} - \frac{1}{2}Q_2 + Q_3 - \frac{1}{2}Q_6 + Q_7, \qquad \Sigma = \tilde{\Sigma} - \frac{1}{3}Q_2 + \frac{2}{3}Q_3 + \frac{1}{6}Q_6 - \frac{1}{3}Q_7, \\ \phi &= \tilde{\phi} - \frac{1}{2}Q_8 + Q_9, \qquad \mathcal{A}^{(u)} = \tilde{\mathcal{A}} + \frac{1}{2}Q_4, \qquad \mathcal{A}^{(e)} = \tilde{\mathcal{A}} + \frac{1}{2}Q_4 - Q_5. \end{aligned} \tag{8.102}$$

8.3.2 Static spacetime

With nonmetricity, the Killing equations (8.1) assume the form

$$\begin{aligned} 0 = \mathcal{L}_{\mathbf{X}} g_{ab} &= 2\nabla_{(a}X_{b)} + 2X^c L_{abc} = \\ &= 2\tilde{\nabla}_{(a}X_{b)}. \end{aligned} \tag{8.103}$$

In the following analysis, the form involving the total covariant derivative will be used.

We choose the Killing vector expressed as

$$\mathbf{X} = C(x)u, \tag{8.104}$$

where $C(x)$ is a generic smooth function of the coordinates. Contracting Eq. (8.103) twice with u^a, we obtain

$$\dot{C} = 0, \tag{8.105}$$

that is, the function C is independent of time. On the other hand, the contraction with u^a and e^b gives the hat derivative of the function C,

$$\hat{C} = \tilde{\mathcal{A}} C. \tag{8.106}$$

Finally, from the contraction with $h_c{}^a h_d{}^b$ we find

$$C \left[N_{ab} \left(\frac{2}{3}\Theta - \Sigma + \frac{1}{2}Q_6 - Q_7 \right) + e_a e_b \left(\frac{2}{3}\Theta + 2\Sigma + Q_2 - 2Q_3 \right) \right] = 0, \tag{8.107}$$

from which we derive

$$\begin{aligned} \Theta &= -\frac{1}{2}Q_2 + Q_3 - \frac{1}{2}Q_6 + Q_7, \\ \Sigma &= -\frac{1}{3}Q_2 + \frac{2}{3}Q_3 + \frac{1}{6}Q_6 - \frac{1}{3}Q_7. \end{aligned} \tag{8.108}$$

The condition (8.2) and Eq. (8.108) ensure that the time derivative of the scalar components of nonmetricity tensor is zero. Hence, because of the LRS symmetries and the stationary conditions, the covariant derivatives of the scalars (8.95) are zero except those along the radial direction e^a.

Because of condition (8.96), we have that $\mathbf{X}$ is hypersurface orthogonal since it is proportional to the 4-velocity. Thus, our spacetime is static.

By a direct comparison of Eq. (8.102) with Eq. (8.108), we obtain the identities

$$\tilde{\Theta} = 0, \qquad \text{and} \qquad \tilde{\Sigma} = 0. \tag{8.109}$$

This is not a surprising result, since $\tilde{\Theta}$ and $\tilde{\Sigma}$ are respectively the expansion and shear we would have in GR, where both quantities are known to be null in a stationary LRS spacetime of class II. Moreover, it is a direct consequence of the Killing equations (8.103) when using the form with the Levi-Civita covariant derivative.

Equation (8.108) means that in the presence of a nonmetric connection, expansion, and shear can manifest exclusively due to the nonmetricity tensor.

8.4 $f(\mathcal{Q})$ gravity

Here, we use the same metric field equations derived for the $1+3$ formalism in Sec. 6.2

$$\tilde{R}_{ab} = \frac{1}{f'}\left(\Psi_{ab} - \frac{1}{2}g_{ab}\Psi\right) + \frac{1}{2}g_{ab}\left(\frac{f}{f'} - \mathcal{Q}\right) - 2\frac{f''}{f'}\left(P^c{}_{ab} - \frac{1}{2}g_{ab}P^{cd}{}_d\right)\partial_c\mathcal{Q}, \tag{8.110}$$

We will perform projections along both the timelike and spacelike curves as well as onto the sheets.

8.4.1 $1+1+2$ field equations in LRS spacetimes of class II

To make explicit the field equations in the $1+1+2$ formalism, we need the Ricci identity for the Levi-Civita Riemann tensor, written for both u_a and e_a,

$$\left[\tilde{\nabla}_c\tilde{\nabla}_d - \tilde{\nabla}_d\tilde{\nabla}_c\right]u_b = -\tilde{R}^a{}_{bcd}u_a, \qquad \left[\tilde{\nabla}_c\tilde{\nabla}_d - \tilde{\nabla}_d\tilde{\nabla}_c\right]e_b = -\tilde{R}^a{}_{bcd}e_a. \tag{8.111}$$

From Eqs. (6.73) and (8.111), we obtain the following relations[2]:

- $\tilde{R}_{ad}u^a u^d = g^{bc}u^d\left[\tilde{\nabla}_c\tilde{\nabla}_d - \tilde{\nabla}_d\tilde{\nabla}_c\right]u_b$,

$$\hat{\tilde{\mathcal{A}}} + \tilde{\mathcal{A}}^2 + \tilde{\mathcal{A}}\tilde{\phi} - \frac{1}{2}\frac{1}{f'}(\rho + 3p) + \frac{1}{2}\frac{f}{f'} - \frac{1}{2}\mathcal{Q} - \frac{1}{2}\frac{f''}{f'}(Q_4 - 2Q_5)\,\hat{\mathcal{Q}} = 0; \tag{8.112}$$

- $\tilde{R}_{ad}e^a e^d = g^{bc}e^d\left[\tilde{\nabla}_c\tilde{\nabla}_d - \tilde{\nabla}_d\tilde{\nabla}_c\right]e_b$,

$$\hat{\tilde{\phi}} + \frac{1}{2}\tilde{\phi}^2 + \hat{\tilde{\mathcal{A}}} + \tilde{\mathcal{A}}^2 + \frac{1}{2}\frac{1}{f'}(\rho - p + 2\pi) + \frac{1}{2}\frac{f}{f'} - \frac{1}{2}\mathcal{Q} - \frac{1}{2}\frac{f''}{f'}(Q_4 - Q_8)\,\hat{\mathcal{Q}} = 0; \tag{8.113}$$

- $\tilde{R}_{ad}u^a e^d = g^{bc}e^d\left[\tilde{\nabla}_c\tilde{\nabla}_d - \tilde{\nabla}_d\tilde{\nabla}_c\right]u_b$,

$$f''(Q_0 + Q_2 - Q_6)\,\hat{\mathcal{Q}} = q; \tag{8.114}$$

[2]We report only contractions that do not return trivial results.

- $-\tilde{R}^a{}_{bcd}e_a u^b u^c e^d = u^b u^c e^d \left[\tilde{\nabla}_c\tilde{\nabla}_d - \tilde{\nabla}_d\tilde{\nabla}_c\right] e_b$,

$$\begin{aligned}&\hat{\tilde{A}} + \tilde{A}^2 - \tilde{\mathcal{E}} - \frac{1}{6}\frac{1}{f'}\left(\rho + 3p - 3\pi\right) + \frac{1}{6}\frac{f}{f'} - \frac{1}{6}\mathcal{Q} + \\ &- \frac{1}{12}\frac{f''}{f'}\left(4Q_4 - 4Q_5 - Q_8 - 2Q_9\right)\hat{\mathcal{Q}} = 0;\end{aligned} \tag{8.115}$$

- $-\tilde{R}^a{}_{bcd}e_a u^b \varepsilon^{cd} = \varepsilon^{cd}u^b \left[\tilde{\nabla}_c\tilde{\nabla}_d - \tilde{\nabla}_d\tilde{\nabla}_c\right] e_b$,

$$\tilde{\mathcal{H}} = 0. \tag{8.116}$$

In vacuum, for Eq. (8.114) to be satisfied, there are three possible solutions: $f'' = 0$, and $\hat{\mathcal{Q}} = 0$, from which only Schwarzschild-de Sitter spacetime can be obtained (in Sec. 6.2 we showed that these two conditions always give the GR field equations), or $Q_0 + Q_2 - Q_6 = 0$, which instead imposes a restriction on the connection. The latter condition results in being automatically satisfied under the assumptions we will use to solve the field equations.

8.4.2 Weyl tensor: magnetic and electric parts

Keeping in mind that the Weyl tensor is zero because the Riemann tensor is null, $R^a{}_{bcd} = 0$, we can make explicit the contributions due to Levi-Civita connection and nonmetricity in the magnetic and electric parts of the Weyl tensor to derive constraints on nonmetricity. Equations $\check{\mathcal{E}} = \check{\check{\mathcal{E}}} = 0$ are automatically satisfied, so they do not provide any constraint. On the other hand, from the scalar part of (8.74), we have

$$\mathcal{H} = 0 = \tilde{\mathcal{H}} - \frac{1}{8}\tilde{\phi}Q_{10} + \frac{1}{4}\tilde{A}Q_{10} + \frac{1}{4}\hat{Q}_{10} + \frac{1}{16}\left(Q_8 - 2Q_9\right)Q_{10} - \frac{1}{8}\left(Q_6 - 2Q_7\right)Q_{11}, \tag{8.117}$$

from which, considering Eq. (8.116), we derive

$$Q_{10} = Q_{11} = 0. \tag{8.118}$$

The condition (8.118) ensures $\bar{\mathcal{H}} = \mathbb{H} = \check{\mathcal{H}} = \check{\mathcal{H}} = \check{\mathbb{H}} = \mathfrak{H} = \mathbb{E} = \bar{\mathbb{E}} = 0$ as well. Now, we just have to write the scalars $\mathcal{E}$ and $\bar{\mathcal{E}}$, which are derived by Eqs. (8.88) and (8.91):

$$\begin{aligned}\mathcal{E} = 0 = &\tilde{\mathcal{E}} - \frac{1}{24}\tilde{A}\left(6Q_1 + 2Q_4 + 10Q_5 - 2Q_8 - Q_9\right) + \\ &+ \frac{1}{48}\tilde{\phi}\left(6Q_1 - 4Q_4 + 10Q_5 - 5Q_8 - Q_9\right) + \\ &- \frac{1}{48}Q_0\left(6Q_2 - 2Q_3 - 4Q_6 + Q_7\right) + \\ &+ \frac{1}{48}Q_1\left(-6Q_4 + 10Q_5 - 3Q_8 + 5Q_9\right) + \\ &- \frac{1}{48}Q_2\left(-2Q_3 - 2Q_6 - 5Q_7 + 6Q_2\right) + \frac{1}{48}Q_3Q_6 + \\ &+ \frac{1}{48}Q_4\left(10Q_5 + 2Q_8 + Q_9 - 6Q_4\right) + \\ &- \frac{5}{48}Q_5Q_8 + \frac{1}{32}Q_6Q_7 + \frac{Q_6^2}{96} - \frac{3}{32}Q_8Q_9 + \\ &+ \frac{5Q_8^2}{96} + \frac{1}{6}\hat{Q}_4 - \frac{5}{12}\hat{Q}_5 + \frac{1}{12}\hat{Q}_8 - \frac{5}{24}\hat{Q}_9,\end{aligned} \tag{8.119}$$

and

$$\begin{aligned}\bar{\mathcal{E}} = 0 = & -\tilde{\mathcal{E}} - \frac{1}{24}\tilde{A}\left(6Q_1 + 10Q_4 + 2Q_5 + 2Q_8 - 5Q_9\right) + \\ & + \frac{1}{48}\tilde{\phi}\left(6Q_1 + 4Q_4 + 2Q_5 - Q_8 - 5Q_9\right) + \\ & - \frac{1}{48}Q_0\left(6Q_2 - 10Q_3 - 4Q_6 + 5Q_7\right) + \\ & + \frac{1}{48}Q_1\left(-6Q_4 + 2Q_5 - 3Q_8 + Q_9\right) + \\ & - \frac{1}{48}Q_2\left(-10Q_3 + 2Q_2Q_6 + Q_7 + 6Q_2\right) + \\ & + \frac{5}{48}Q_3Q_6 + \frac{1}{48}Q_4\left(5Q_5 - 2Q_8 + 5Q_9 - 6Q_4\right) + \\ & - \frac{1}{48}Q_5Q_8 - \frac{3}{32}Q_6Q_7 + \frac{5Q_6^2}{96} + \frac{1}{32}Q_8Q_9 + \frac{Q_8^2}{96} - \frac{1}{12}\hat{Q}_4 + \\ & - \frac{1}{12}\hat{Q}_5 - \frac{1}{12}\hat{Q}_8 - \frac{1}{24}\hat{Q}_9.\end{aligned} \tag{8.120}$$

8.4.3 Riemann tensor contractions

Besides the Weyl tensor, we can deduce constraints from the Riemann tensor itself by considering all its possible contractions with u^a, e^a, N^{ab}, ε^{ab} and imposing $R^a{}_{bcd} = 0$. Making use of the Ricci identity for the full connection,

$$[\nabla_c\nabla_d - \nabla_d\nabla_c]\, u_b = -R^a{}_{bcd}u_a, \qquad [\nabla_c\nabla_d - \nabla_d\nabla_c]\, e_b = -R^a{}_{bcd}e_a, \tag{8.121}$$

we deduce the following relations:

- $R^a{}_{bcd}u_a u_b u_c e^d = 0$,

$$\begin{aligned}&\hat{Q}_0 + 3\Sigma\left(\mathcal{A}^{(e)} - \mathcal{A}^{(u)}\right) - \frac{1}{3}\Theta\left(Q_4 + 2Q_5\right) - \Sigma\left(Q_4 - Q_5\right) + \\ &\quad + \mathcal{A}^{(u)}\left(Q_0 + 2Q_3\right) - \frac{1}{2}Q_0Q_4 - \frac{1}{2}Q_2Q_4 + Q_3Q_4 = 0;\end{aligned} \tag{8.122}$$

- $R^a{}_{bcd}e_a e_b u_c e^d = 0$,

$$\begin{aligned}&\hat{Q}_2 + 2\left(\Sigma + \frac{1}{3}\Theta\right)\left(\mathcal{A}^{(e)} - \mathcal{A}^{(u)} - \frac{1}{2}Q_1\right) + \\ &+ \mathcal{A}^{(u)}\left(Q_2 + 2Q_3\right) + -\frac{1}{2}Q_1Q_2 + Q_1Q_3 - \frac{1}{2}Q_2Q_4 = 0;\end{aligned} \tag{8.123}$$

- $R^a{}_{bcd}N_a{}^b u^c e^d = 0$,

$$\hat{Q}_6 + \tilde{A}Q_6 = 0; \tag{8.124}$$

- $R^a{}_{bcd}u_a e^b u^c e^d = 0$,

$$\hat{\mathcal{A}}^{(u)} + \mathcal{A}^{(u)2} + \frac{1}{2}\mathcal{A}^{(u)}Q_1 - \left(\Sigma + \frac{1}{3}\Theta\right)^2 - \frac{1}{2}\left(\Sigma + \frac{1}{3}\Theta\right)\left(Q_0 + 2Q_2 - 2Q_3\right) = 0; \tag{8.125}$$

- $R^a{}_{bcd}e_a u^b u^c e^d = 0$,

$$\hat{\mathcal{A}}^{(e)} + \mathcal{A}^{(e)}\mathcal{A}^{(u)} - \frac{1}{2}\mathcal{A}^{(e)}\left(Q_1 + 2Q_4\right) - \left(\Sigma + \frac{1}{3}\Theta\right)^2 + \frac{1}{2}\left(\Sigma + \frac{1}{3}\Theta\right)\left(Q_0 + 4Q_3\right) + \\ + \frac{1}{2}\phi\left(\mathcal{A}^{(e)} - \mathcal{A}^{(u)} + Q_5\right) - Q_3^2 - \frac{1}{2}Q_0 Q_3 = 0; \tag{8.126}$$

- $R^a{}_{bcd}N_a{}^c u^b u^d = 0$,

$$\left(\Sigma - \frac{2}{3}\Theta\right)^2 + \left(\Sigma - \frac{2}{3}\Theta\right)\left(Q_0 + 2Q_7\right) + \\ -2\phi\mathcal{A}^{(u)} + 2\phi Q_5 + 2\mathcal{A}^{(e)}Q_9 + Q_0 Q_7 + Q_7^2 = 0; \tag{8.127}$$

- $R^a{}_{bcd}N_a{}^c e^b e^d = 0$,

$$\hat{\phi} - \hat{Q}_9 + \frac{\phi^2}{2} + \frac{1}{2}Q_9^2 + \Sigma^2 - \frac{2}{9}\Theta^2 - \frac{1}{3}\Theta\Sigma + \frac{1}{2}\phi\left(Q_1 - 2Q_9\right) + \\ + \frac{1}{3}\Theta Q_7 + \Sigma Q_7 - \frac{1}{2}Q_1 Q_9 = 0; \tag{8.128}$$

- $R^a{}_{bcd}N_a{}^c u^b e^d = 0$,

$$\hat{\Sigma} - \frac{2}{3}\hat{\Theta} + \hat{Q}_7 + \frac{3}{2}\Sigma\phi - \frac{1}{2}\Sigma\left(Q_4 + 3Q_9\right) - \left(\phi - Q_9\right)\left(Q_3 - \frac{1}{2}Q_7\right) + \\ + \frac{1}{3}\Theta Q_4 - \frac{1}{2}Q_4 Q_7 = 0; \tag{8.129}$$

- $R^a{}_{bcd}N_a{}^c e^b u^d = 0$,

$$\frac{1}{2}\Sigma\phi - \frac{1}{3}\Theta\phi - \left(\Sigma - \frac{2}{3}\Theta\right)\left(Q_5 + \frac{1}{2}Q_9 + \mathcal{A}^{(e)}\right) + \\ -\frac{1}{2}\left(\phi - Q_9\right)\left(Q_2 - Q_7\right) - \mathcal{A}^{(u)}Q_7 = 0; \tag{8.130}$$

- $R^a{}_{bcd}N^{bc}u_a u^d = 0$,

$$\left(\Sigma - \frac{2}{3}\Theta\right)^2 - 2\phi\mathcal{A}^{(u)} - \left(\Sigma - \frac{2}{3}\Theta\right)\left(Q_0 + Q_6 - Q_7\right) = 0; \tag{8.131}$$

- $R^a{}_{bcd}N^{bc}e_a e^d = 0$,

$$\hat{\phi} + \frac{\phi^2}{2} + \Sigma^2 - \frac{2}{9}\Theta^2 - \frac{1}{3}\Theta\Sigma - \Sigma Q_3 + \frac{2}{3}\Theta Q_3 - \frac{1}{2}\phi\left(Q_1 - Q_8 + Q_9\right) = 0; \tag{8.132}$$

- $R^a{}_{bcd}N^{bc}u_a e^d = 0$,

$$\hat{\Sigma} - \frac{2}{3}\hat{\Theta} + \frac{3}{2}\Sigma\phi + \frac{1}{2}\left(\Sigma - \frac{2}{3}\Theta\right)\left(Q_4 + Q_8 - Q_9\right) = 0; \tag{8.133}$$

- $R^a{}_{bcd}N^{bc}e_a u^d = 0$,

$$\mathcal{A}^{(e)}\left(4\Theta - 6\Sigma\right) + \phi\left(-2\Theta + 3Q_2 - 3Q_6 + 3Q_7 + 3\Sigma\right) = 0; \tag{8.134}$$

All the remaining contractions, which are not explicitly expressed above, are identically null.

8.4.4 The final system of covariant equations

Before proceeding to write the final system of equations to be solved, some preliminary considerations are in order.

First of all, we are looking for a vacuum solution, so $\rho = p = q = \pi = 0$. Moreover, from the static nature of spacetime, we have already deduced the relations $\tilde{\Theta} = \tilde{\Sigma} = 0$. In addition to this, we decide to impose the conditions $\Theta = \Sigma = 0$ too. Both the total rate of expansion Θ and shear Σ are set to zero to simplify the mathematics and because we want a distortion-free spacetime as well. Hence, from Eq. (8.108) we get the identities

$$Q_2 = 2Q_3, \quad \text{and} \quad Q_6 = 2Q_7. \tag{8.135}$$

Furthermore, we require that the curves of the timelike congruence are autoparallel, i.e., $u^c \nabla_c u^a = 0$, which implies

$$Q_0 = 0, \quad \mathcal{A}^{(e)} = 0, \quad \text{and} \quad \mathcal{A}^{(u)} = Q_5. \tag{8.136}$$

Equations (8.135), (8.136), together with Eqs. (8.122)-(8.134), give rise to the identities

$$Q_2 = Q_3 = Q_6 = Q_7 = 0, \tag{8.137}$$

which are in agreement with $Q_0 + Q_2 - Q_6 = 0$ derived from the off diagonal equation (8.114) in vacuum.

By collecting all the obtained results, we end up with the final system of field equations:

$$\hat{\mathcal{A}} + \mathcal{A}^2 + \mathcal{A}\tilde{\phi} + \frac{1}{2}\frac{f}{f'} - \frac{1}{2}\mathcal{Q} - \frac{1}{2}\frac{f''}{f'}\left(Q_4 - 2Q_5\right)\hat{\mathcal{Q}} = 0, \tag{8.138}$$

$$\tilde{\mathcal{E}} + \mathcal{A}\tilde{\phi} + \frac{1}{3}\frac{f}{f'} - \frac{1}{3}\mathcal{Q} - \frac{1}{6}\frac{f''}{f'}Q_4\hat{\mathcal{Q}} + \frac{2}{3}\frac{f''}{f'}Q_5\hat{\mathcal{Q}} - \frac{1}{12}\frac{f''}{f'}Q_8\hat{\mathcal{Q}} - \frac{1}{6}\frac{f''}{f'}Q_9\hat{\mathcal{Q}} = 0, \tag{8.139}$$

$$\hat{\tilde{\phi}} + \frac{1}{2}\tilde{\phi}^2 - \mathcal{A}\tilde{\phi} - \frac{f''}{f'}Q_5\hat{\mathcal{Q}} + \frac{1}{2}\frac{f''}{f'}Q_8\hat{\mathcal{Q}} = 0, \tag{8.140}$$

$$\hat{Q}_5 + {Q_5}^2 + \frac{1}{2}Q_1 Q_5 = 0, \tag{8.141}$$

$$\hat{\phi} - \hat{Q}_9 + \frac{\phi^2}{2} + \frac{Q_9^2}{2} - \phi\left(Q_9 - \frac{Q_1}{2}\right) - \frac{1}{2}Q_1 Q_9 = 0, \tag{8.142}$$

$$Q_5\phi = 0, \tag{8.143}$$

$$\hat{\phi} + \frac{1}{2}\phi^2 - \frac{1}{2}\left(Q_1 - Q_8 + Q_9\right)\phi = 0, \tag{8.144}$$

$$\begin{aligned} 0 =& \tilde{\mathcal{E}} - \frac{1}{24}\mathcal{A}\left(6Q_1 + 2Q_4 + 10Q_5 - 2Q_8 - Q_9\right) + \frac{1}{48}\tilde{\phi}\left(6Q_1 - 4Q_4 + 10Q_5 - 5Q_8 - Q_9\right) + \\ & - \frac{1}{8}Q_1Q_4 + \frac{5}{24}Q_1Q_5 - \frac{1}{16}Q_1Q_8 + \frac{5}{48}Q_1Q_9 + \frac{5}{24}Q_4Q_5 + \frac{1}{24}Q_4Q_8 + \frac{1}{48}Q_4Q_9 - \frac{Q_4^2}{8} + \\ & - \frac{5}{48}Q_5Q_8 - \frac{3}{32}Q_8Q_9 + \frac{5Q_8^2}{96} + \frac{1}{6}\hat{Q}_4 - \frac{5}{12}\hat{Q}_5 + \frac{1}{12}\hat{Q}_8 - \frac{5}{24}\hat{Q}_9, \end{aligned} \tag{8.145}$$

$$\mathcal{Q} = \frac{1}{2}Q_1\left(-Q_5 + Q_9\right) - \frac{1}{2}Q_4\left(Q_5 - Q_8 + Q_9\right) - \frac{1}{8}Q_8\left(4Q_5 + Q_8\right). \tag{8.146}$$

Eq. (8.120) is not included in the final system of equations since one can prove that the combination (8.119)+(8.120) can always be obtained by suitable combinations of Eqs. (8.141)-(8.144).

8.5 Spherically symmetric solutions

In this section, we specialize the system of covariant equations (8.138)-(8.146) to a spherically symmetric scenario. For this purpose, we introduce the following expression of a static spherically symmetric metric,

$$ds^2 = -\mathbb{A}(r)dt^2 + \mathbb{B}(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{8.147}$$

where $\mathbb{A}(r)$ and $\mathbb{B}(r)$ are generic positive functions of the radial coordinate r. In this coordinate system, the 4-velocity u^a and the spatial 4-vector e^a are expressed as

$$u^a \equiv \left\{\frac{1}{\sqrt{\mathbb{A}}}, 0, 0, 0\right\}, \quad \text{and} \quad e^a \equiv \left\{0, \frac{1}{\sqrt{\mathbb{B}}}, 0, 0\right\}. \tag{8.148}$$

Using Eq. (8.148) and the definitions of $\tilde{\phi}$ and $\tilde{\mathcal{A}}$, we obtain the relations,

$$\frac{\mathbb{A}'(r)}{\mathbb{A}(r)} = \frac{4\tilde{\mathcal{A}}(r)}{r\tilde{\phi}(r)}, \tag{8.149}$$

$$\mathbb{B}(r) = \frac{4}{r^2}\frac{1}{\tilde{\phi}(r)^2}, \tag{8.150}$$

$$e^a\nabla_a\psi = e^a\tilde{\nabla}_a\psi = e^a\partial_a\psi = \frac{1}{2}\tilde{\phi}\, r\, \partial_r\psi, \tag{8.151}$$

where ψ is an arbitrary scalar function. From Eq. (8.143) two conditions arise:

$$\phi = 0 \qquad \text{or} \qquad Q_5 = 0. \tag{8.152}$$

To test the viability of both solutions (8.152), we consider the function $f(\mathcal{Q}) = \mathcal{Q}$ and verify that, in this case, the equations yield the Schwarzschild spacetime as a solution. The condition $\phi = 0$ leads to an unacceptable result for $f(\mathcal{Q}) = \mathcal{Q}$: all kinetic scalars are null. For this reason, we discard this branch of solutions, and we will narrow down our investigation to the branch $Q_5 = 0$. In such a circumstance, we easily obtain the following relations for $\tilde{\phi}$ and $\tilde{\mathcal{A}}$ when $f(\mathcal{Q}) = \mathcal{Q}$,

$$\tilde{\phi} = \frac{2}{r}\sqrt{1-\frac{r_s}{r}}, \qquad \tilde{\mathcal{A}} = -\frac{1}{2}Q_4 = \frac{r_s}{2r\sqrt{r^2 - r r_s}}, \tag{8.153}$$

which substituted into Eqs. (8.149) and (8.150) give the Schwarzschild metric as solution,

$$\mathbb{A} = \mathbb{B}^{-1} = 1 - \frac{r_s}{r}, \tag{8.154}$$

being r_s the Schwarzschild radius, as expected.

Coincident gauge

Before the resolution of the covariant equations, let us show that if we use the coincident gauge and the metric given in Eq. (8.147), either the function $f(\mathcal{Q})$ must be linear or the nonmetricity scalar must be a constant. In the coincident gauge, the nonzero components of the nonmetricity tensor are

$$\begin{gathered} Q_{111} = -\mathbb{A}', \qquad Q_{122} = \mathbb{B}', \qquad Q_{133} = 2r, \\ Q_{144} = 2r\sin^2\theta, \qquad Q_{244} = 2r^2\sin\theta\cos\theta. \end{gathered} \tag{8.155}$$

Now, using the field equations

$$E_{ab} = \frac{2}{\sqrt{-g}}\nabla_c\left(\sqrt{-g}f'P^c{}_{ab}\right) + \frac{1}{2}g_{ab}f + f'\left(P_{acd}Q_b{}^{cd} - 2Q^{cd}{}_aP_{cdb}\right) = 0, \tag{8.156}$$

it is straightforward to derive the off-diagonal equation,

$$E_{12} = -\frac{1}{2}f''\partial_r\mathcal{Q}\cot\theta = 0, \tag{8.157}$$

which proves the above statement. This is the reason why we chose to use the $1+1+2$ formalism in studying spherically symmetric metrics: it allows us a totally different covariant approach than the coincident gauge.

8.5.1 Schwarzschild-de Sitter solutions

For Eqs. (8.149) and (8.150) admit solutions of Schwarzschild-de Sitter type, from the well-known results of GR, we must have that

$$\hat{\tilde{\phi}} + \frac{1}{2}\tilde{\phi}^2 - \tilde{\mathcal{A}}\tilde{\phi} = 0. \tag{8.158}$$

Consequently, Eq. (8.140) and the condition $Q_5 = 0$ imply

$$f''Q_8\hat{\mathcal{Q}} = 0. \tag{8.159}$$

As we already know, Eq. (8.159) tells us that to have a Schwarzschild-de Sitter metric either the function $f(\mathcal{Q})$ must be linear or $\mathcal{Q}$ must be constant. But there is also a third possibility given by the requirement $Q_8 = 0$.

The condition $Q_8 = 0$ perfectly matches the gauge choice made in other works (e.g. [182, 183]) in which static spherically symmetric spacetimes are studied and the full connection is assumed to coincide with the Levi-Civita one of a Minkowski spacetime in spherical coordinates.

Now, we can solve the final system of equations with the constraint $Q_8 = 0$. From Eqs. (8.140), (8.142) and (8.144) we derive the relation

$$Q_1 = Q_4, \tag{8.160}$$

which substituted into Eq. (8.146) gives $\mathcal{Q} = 0$. The vanishing of nonmetricity scalar implies

$$\begin{gathered} f(\mathcal{Q})|_{\mathcal{Q}=0} = f_0, \qquad f'(\mathcal{Q})|_{\mathcal{Q}=0} = f_0', \\ f''(\mathcal{Q})|_{\mathcal{Q}=0} = f_0'', \end{gathered} \tag{8.161}$$

i.e., the function f and its derivatives are constant on shell. The remaining equations admit the solutions

$$\tilde{\phi} = \frac{2}{r}\sqrt{1 - \frac{r_s}{r} - \frac{1}{6}\frac{f_0}{f_0'}r^2}, \qquad \tilde{\mathcal{A}} = -\frac{1}{2}Q_4 = \frac{3f_0'r_s - f_0r^3}{6r^2\sqrt{f_0'^2\left(1 - \frac{r_s}{r} - \frac{1}{6}\frac{f_0}{f_0'}r^2\right)}}, \tag{8.162}$$

$$Q_9 = \frac{6f_0' r_s + f_0 r^3}{3r^2\sqrt{f_0'^2\left(1 - \frac{r_s}{r} - \frac{1}{6}\frac{f_0}{f_0'}r^2\right)}}, \quad \tilde{\mathcal{E}} = -\frac{r_s}{r^3}, \tag{8.163}$$

$$\mathbb{A} = \mathbb{B}^{-1} = 1 - \frac{r_s}{r} - \frac{1}{6}\frac{f_0 r^2}{f_0'}. \tag{8.164}$$

which represent the Schwarzschild-de Sitter spacetime.

8.5.2 $\mathcal{Q}$-Gravastars

Among the exotic objects that have been proposed as alternatives for black holes, there are the so-called gravitational vacuum condensate stars or Gravastars [196, 197]. A gravastar is essentially a compact object made of a dark energy condensate that represents an alternative to black holes as the final state of gravitational collapse. As the non-metricity terms in equations (6.73) can be considered as an effective fluid which can have negative pressure, one might ask if $f(\mathcal{Q})$ gravity can admit gravastar solutions without invoking explicitly the presence of a cosmological constant. We dub these solutions "Q-gravastars" and give their simplest realization in the following.

From the Einstein-like form of the metric field equations (8.110),

$$\tilde{R}_{ab} - \frac{1}{2}g_{ab}\tilde{R} = \frac{1}{f'}T_{ab} - \frac{1}{2}g_{ab}\left(\frac{f}{f'} - \mathcal{Q}\right) - 2\frac{f''}{f'}P^c{}_{ab}\partial_c\mathcal{Q}, \tag{8.165}$$

we have that, in the case $\mathcal{Q} = \mathcal{Q}_* = const.$, the effective energy-momentum tensor,

$$T_{ab}^{eff} = -\frac{1}{2}g_{ab}\left(\frac{f(\mathcal{Q}_*)}{f'(\mathcal{Q}_*)} - \mathcal{Q}_*\right) = -g_{ab}\Lambda_*, \tag{8.166}$$

can be thought of as a fluid characterized by a negative pressure,

$$\frac{1}{3}T_{ab}^{eff}h^{ab} = p_{eff} = -\rho_{eff} = T_{ab}^{eff}u^a u^b, \tag{8.167}$$

if

$$\Lambda_* = \frac{1}{2}\left(\frac{f(\mathcal{Q}_*)}{f'(\mathcal{Q}_*)} - \mathcal{Q}_*\right) > 0, \tag{8.168}$$

which corresponds to a function f, which grows slower than linear. It is now clear that, with our assumptions, considering a Schwarzschild radius equals zero, the solution of the field equations will be just the de Sitter solution,

$$\mathbb{A} = \mathbb{B}^{-1} = 1 - \frac{1}{3}\Lambda_* r^2. \tag{8.169}$$

Therefore, in this scenario, the nonmetricity tensor is the geometric object whose role is to give rise to the dark energy that fills compact objects when a gravitational collapse occurs.

8.6 Discussions

In this chapter, we have developed the $1+1+2$ covariant formalism for static LRS spacetimes of class II in the presence of nonmetricity. The resulting geometrical setting has been applied for studying the features of vacuum solutions arising in $f(\mathcal{Q})$ gravity, including the Birkhoff theorem and the presence of Gravastar solutions.

Our analysis shows that nonmetricity gives rise to significant kinematic and dynamical differences with the purely metric case. For example, the timelike and spacelike congruences can be associated with two different types of acceleration, provided by Eqs. (8.59) and (8.69), respectively. The difference between the accelerations is related to the geometric properties of the congruences themselves, like, e.g., autoparallelism.

As a first application of the general formalism, we have analyzed the case of static and spherically symmetric spacetimes. In particular, using some simplifying assumptions on the nonmetricity tensor connected with the requirement of autoparallelism for the timelike congruence, we have deduced a self-consistent system of algebraic/differential equations for the investigation of vacuum solutions. These equations allow the identification of the conditions under which the theory produces the same Einstein-like field equations as GR. In addition, we have derived sufficient conditions that ensure the existence of a Schwarzschild-de Sitter solution in vacuum, extending the ones presented so far in the literature. We also obtain a $\mathcal{Q}$-Gravastar solution in which nonmetricity generates the dark energy filling the compact object. A similar role for nonmetricity as dark energy is not new in literature, as it has been well explored in cosmology to justify the expansion of the universe (see Ch. 5).

III
Spinors and nonmetricity

9
Spinor fields in $f(\mathcal{Q})$ gravity

In the present chapter, we study $f(\mathcal{Q})$ gravity coupled to a spin-1/2 spinor field. A first step in this direction was given in [66], in the context of the STG.

As $f(\mathcal{Q})$ gravity is a metric-affine theory of gravitation where the dynamical connection is not metric compatible (but torsion-free and flat), the introduction of spinor fields in $f(\mathcal{Q})$ theory requires the definition of spinor covariant derivatives induced by a general affine connection, not necessarily metric compatible. The spinor covariant derivative can be obtained by using the Fock-Ivanenko coefficients with antisymmetric Lorentz indices [198]. As a result, the covariant derivatives of Dirac matrices are no longer zero, and this is reflected in the particular form of the spin conservation law.

Another peculiar aspect of the proposed approach to $f(\mathcal{Q})$ gravity coupled to spinors is the choice of the dynamical variables we used. Theories of gravity that deal with spinor fields employ tetrad fields to represent the metric tensor. In tetrad-affine theories, the additional gravitational degrees of freedom are usually incorporated by the spin connection: for instance, in Einstein-Cartan-like theories, the variational derivative of the action with respect to the spin connection induces the well-known coupling between spin and torsion. Instead, in this chapter, we make a somewhat different choice in which the dynamical variables are a tetrad field and an affine connection expressed in coordinate basis. With this choice, the Lagrange multipliers present in the action do not enter the Einstein-like equations, and thus we obtain more manageable field equations.

For this reason, in the following, emphasis has been given to the deduction and discussion of the gravitational field equations. Using our new formulation, we will see that the energy-momentum tensor acquires additional terms with respect to the standard Dirac tensor form. These additional terms directly involve covariant derivatives of the spin density related to its conservation law. The latter has been obtained by making use of the Dirac equations derived by variations. We found that even if the final expression of the spin conservation law is formally different from that holding in Einstein-Cartan-like theories, it still ensures that the antisymmetrized part of the Einstein-like field equations is identically zero, just as it happens in the theories with torsion and metricity [199]. This was an expected result that restores equivalence between metric-affine and tetrad-affine formulations of the theory.

In cosmology, spinor fields have been mostly considered since the 1990s. They can drive the universe into accelerated expansion at early and late times, thus offering theoretical frameworks for both inflation and dark energy. Cosmologies sourced by fermions are even known to be able to avoid the initial cosmological singularity [200–205]. Moreover, when dealing with spinor fields, anisotropic models of spacetime seem more appropriate (for example, think of the anisotropy induced by the spin four-vector). In connection with this we will study spinor fields in a BI spacetime arising from $f(\mathcal{Q})$ gravity.

Throughout the chapter, spacetime indices are indicated by Latin letters, while Lorentz[1] indices by Greek letters; both sets of indices run from 0 to 3. The metric signature is $(+,-,-,-)$.

The study performed in this chapter is based on the paper "Spinor fields in $f(\mathcal{Q})$-gravity" [77].

9.1 Spinor fields

Considering an arbitrary spacetime, we can define what is called tetrad field e_i^μ at each point [84]. Tetrad fields possess Lorentz indices as well as spacetime indices; they are defined by the relation

$$e^\mu = e_i^\mu \, dx^i, \tag{9.1}$$

and, together with their dual fields

$$e_\mu = e^i_\mu \frac{\partial}{\partial x^i}, \tag{9.2}$$

they satisfy the relations

$$e^i_\mu e_{i\nu} = \eta_{\mu\nu}, \qquad e^j_\mu e^\mu_i = \delta^j_i, \qquad \text{and} \qquad e^j_\mu e^\nu_j = \delta^\nu_\mu, \tag{9.3}$$

with $\eta_{\mu\nu}$ denoting the Minkowski metric with signature $(1,-1,-1,-1)$. The metric tensor g_{ij} can be described in terms of the tetrad field such that

$$g_{ij} = e^\mu_i e^\nu_j \eta_{\mu\nu}. \tag{9.4}$$

The same can be done for any generic tensor, that is, it can be represented by its components with respect to spacetime indices or Lorentz indices, as in the following

$$\mathbb{T}^{i_1\cdots i_n}{}_{j_1\cdots j_m} = e^{i_1}_{\mu_1}\cdots e^{i_n}_{\mu_n} e^{\nu_1}_{j_1}\cdots e^{\nu_m}_{j_m} \mathbb{T}^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} \tag{9.5}$$

$$\mathbb{T}^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} = e^{\mu_1}_{i_1}\cdots e^{\mu_n}_{i_n} e^{j_1}_{\nu_1}\cdots e^{j_m}_{\nu_m} \mathbb{T}^{i_1\cdots i_n}{}_{j_1\cdots j_m}. \tag{9.6}$$

Greek indices are raised and lowered by the Minkowski metric $\eta_{\mu\nu}$, whereas Latin indices by the metric tensor g_{ij}.

If we perform a tetrad change via the linear transformation

$$\check{e}^i_\mu = \Lambda^\nu{}_\mu e^i_\nu, \tag{9.7}$$

where $\Lambda^\nu{}_\mu$ is the matrix that define the transformation, from Eq. (9.4) we obtain the relation

$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}. \tag{9.8}$$

[1] The name will be clarified in the next section.

Assuming that the transformation (9.7) preserves the orthonormality between tetrads, we have that the matrices $\Lambda^{\nu}{}_{\mu}$ are elements of the Lorentz group [82, 84].

The adoption of a tetrad field may be seen as a change of local trivialization of the frame bundle over the spacetime [206]. As a consequence, the linear connection $\Gamma_{ij}{}^{h}$ gives rise to the corresponding spin connection

$$\omega_i{}^{\mu}{}_{\nu} = -e^k_{\nu}\partial_i e^{\mu}_k + e^k_{\nu}\Gamma_{ik}{}^{j} e^{\mu}_j. \tag{9.9}$$

Thanks to Eq. (9.9), we can define the full covariant derivative of an arbitrary tensor with respect to both Latin and Greek indices,

$$\begin{aligned}\nabla_i T^{\mu h\cdots}{}_{\nu k\cdots} =& \partial_i T^{\mu h\cdots}{}_{\nu k\cdots} + \omega_i{}^{\mu}{}_{\alpha} T^{\alpha h\cdots}{}_{\nu k\cdots} + \Gamma_{ip}{}^{h} T^{\mu p\cdots}{}_{\nu k\cdots} + \cdots + \\ &- \omega_i{}^{\alpha}{}_{\nu} T^{\mu h\cdots}{}_{\alpha k\cdots} - \Gamma_{ik}{}^{p} T^{\mu h\cdots}{}_{\alpha p\cdots}.\end{aligned} \tag{9.10}$$

From Eqs. (9.10) and (9.9), we have that the full covariant derivative of a tetrad is null,

$$\nabla_j e^{\mu}_i = \partial_j e^{\mu}_i + \omega_i{}^{\mu}{}_{\nu} e^{\nu}_i - \Gamma_{ji}{}^{k} e^{\mu}_k = 0, \tag{9.11}$$

which means that the conversion between spacetime and Lorentz indices commutes with the full covariant derivative.

To be consistent with the literature related to spinors, we once again give the decomposition of the affine connection but with some changed signs. Any given connection may be expressed as

$$\Gamma_{ij}{}^{h} = \tilde{\Gamma}_{ij}{}^{h} - K_{ij}{}^{h} - L_{ij}{}^{h}, \tag{9.12}$$

where

$$\tilde{\Gamma}_{ij}{}^{h} = \frac{1}{2} g^{hk}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right), \tag{9.13}$$

is the Levi-Civita connection induced by the metric g_{ij}, while

$$K_{ij}{}^{h} = \frac{1}{2}\left(-T_{ij}{}^{h} + T_j{}^{h}{}_i - T^{h}{}_{ij}\right), \quad \text{and} \quad L_{ij}{}^{h} = \frac{1}{2}\left(Q_{ij}{}^{h} + Q_{ji}{}^{h} - Q^{h}{}_{ij}\right), \tag{9.14}$$

are the contortion and the disformation tensors, respectively. As far as spin connection is concerned, the analogous decomposition of (9.12) assumes the form

$$\omega_i{}^{\mu}{}_{\nu} = \tilde{\omega}_i{}^{\mu}{}_{\nu} - K_i{}^{\mu}{}_{\nu} - L_i{}^{\mu}{}_{\nu}, \tag{9.15}$$

where

$$\tilde{\omega}_i{}^{\mu}{}_{\nu} = -e^k_{\nu}\partial_i e^{\mu}_k + e^k_{\nu}\tilde{\Gamma}_{ik}{}^{j} e^{\mu}_j, \tag{9.16}$$

$$K_i{}^{\mu}{}_{\nu} = K_{ij}{}^{h} e^{\mu}_h e^j_{\nu}, \tag{9.17}$$

$$L_i{}^{\mu}{}_{\nu} = L_{ij}{}^{h} e^{\mu}_h e^j_{\nu}. \tag{9.18}$$

9.1.1 Dirac matrices and spinors

We define the Dirac matrices as the $4x4$ matrices that satisfy the identities

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu} I_4, \tag{9.19}$$

and,

$$\gamma^i\gamma^j + \gamma^j\gamma^j = 2g^{ij}I_4, \tag{9.20}$$

with I_4 the unit matrix and $\gamma^i = \gamma^\mu e^i_\mu$. From the anticommutation relation (9.19), we derive the following identities

$$\begin{aligned}
\gamma^\mu\gamma_\mu =& 4I_4, &(9.21)\\
\gamma^\mu\gamma^\nu\gamma_\mu =& -2\gamma^\nu, &(9.22)\\
\gamma^\mu\gamma^\nu\gamma^\alpha\gamma_\mu =& 2\eta^{\nu\alpha}I_4, &(9.23)\\
\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma_\mu =& -2\gamma^\beta\gamma^\alpha\gamma^\nu, &(9.24)\\
\left\{\gamma^\mu, \gamma^{[\nu}, \gamma^{\alpha]}\right\} =& \gamma^{[\mu}\gamma^\nu\gamma^{\alpha]}. &(9.25)
\end{aligned}$$

The Dirac matrices transform under a tetrad change according to the relation

$$\check{\gamma}^\mu = \Lambda^\mu{}_\nu\gamma^\nu. \tag{9.26}$$

Let L be a matrix and L^{-1} its inverse such that

$$\gamma^\mu = \Lambda^\mu{}_\nu L\gamma^\nu L^{-1} = L\check{\gamma}^\mu L^{-1}, \tag{9.27}$$

then we define as spinor ψ the quantity that transforms under (9.7) as

$$\check{\psi} = L\psi, \tag{9.28}$$

while as adjoint spinor $\bar{\psi}$ the quantity that transforms as

$$\check{\bar{\psi}} = \bar{\psi}L^{-1}. \tag{9.29}$$

The product between adjoint spinor and spinor is a scalar since it is invariant under transformation, that is

$$\check{\bar{\psi}}\check{\psi} = \bar{\psi}\psi. \tag{9.30}$$

On the other hand, the product $\psi\bar{\psi}$ transforms as a Dirac matrix,

$$\check{\psi}\check{\bar{\psi}} = L\psi\bar{\psi}L^{-1}. \tag{9.31}$$

For an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$, with $\epsilon^\mu{}_\nu$ an infinitesimal quantity, from Eq (9.27) we find that L assumes the form

$$L = I_4 + \frac{1}{8}\epsilon_{\mu\nu}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right). \tag{9.32}$$

A special solution of Eq. (9.19) is given by the Dirac representation

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \gamma^A = \begin{pmatrix} 0 & \sigma^A \\ -\sigma^A & 0 \end{pmatrix}, \tag{9.33}$$

where I_2 is the $2x2$ unit matrix and $A = 1, 2, 3$. The quantities σ^A are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9.34}$$

which are traceless and Hermitian, that is,

$$\mathrm{tr}(\sigma^A) = 0 \qquad \text{and} \qquad \sigma^{A\dagger} = \sigma^A. \tag{9.35}$$

From these properties, we have that the Dirac matrices are traceless too, and that they satisfy the relations

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0, \qquad \gamma^{\mu *} = \gamma^2\gamma^\mu\gamma^2, \tag{9.36}$$

thereby, γ^0 is Hermitian, $\gamma^{0\dagger} = \gamma^0$, whereas the γ^A anti-Hermitian, $\gamma^{A\dagger} = -\gamma^A$. Let us consider the Hermitian conjugate of Eq. (9.32),

$$L^\dagger = I_4 + \frac{1}{8}\epsilon_{\mu\nu}\left(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger}\right), \tag{9.37}$$

and multiply it by γ^0,

$$L^\dagger\gamma^0 = \gamma^0 + \frac{1}{8}\epsilon_{\mu\nu}\left(\gamma^{\nu\dagger}\gamma^{\mu\dagger} - \gamma^{\mu\dagger}\gamma^{\nu\dagger}\right)\gamma^0 = \gamma^0 - \frac{1}{8}\epsilon_{\mu\nu}\gamma^0\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) = \gamma^0 L^{-1}, \tag{9.38}$$

where we used Eq. (9.36), thus the quantity $\psi^\dagger\gamma^0$ transforms as an adjoint spinor,

$$\psi^\dagger\gamma^0 \qquad \longrightarrow \qquad \psi^\dagger L^\dagger\gamma^0 = \psi^\dagger\gamma L^{-1}, \tag{9.39}$$

and we can give the following new definition

$$\bar{\psi} = \psi^\dagger\gamma^0. \tag{9.40}$$

Because of Eq. (9.19) we have that

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^{[0}\gamma^1\gamma^2\gamma^{3]}, \tag{9.41}$$

so we can define the matrix

$$\gamma^5 = -\frac{i}{24}\varepsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{9.42}$$

which in the Dirac representation is equal to

$$\gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{9.43}$$

and that has the properties

$$\mathrm{tr}\left(\gamma^5\right) = 0, \qquad \gamma^{5\dagger} = \gamma^5, \qquad \left\{\gamma^\mu, \gamma^5\right\} = 0, \qquad \left(\gamma^5\right)^2 = I_4. \tag{9.44}$$

With the introduction of the gamma matrix γ^5 we also have the following identity

$$\gamma^\mu\gamma^\nu\gamma^\alpha = \eta^{\mu\nu}\gamma^\alpha + \eta^{\nu\alpha}\gamma^\mu - \eta^{\mu\alpha}\gamma^\nu - i\varepsilon^{\mu\nu\alpha\beta}\gamma_\beta\gamma^5. \tag{9.45}$$

9.1.2 Spinor connection

Because the ordinary derivative of a spinor is not a spinor,

$$\partial_i \check{\psi} = \partial_i L \psi + L \partial_i \psi, \tag{9.46}$$

we have to introduce a spinor connection Ω_i, which transforms in the following way

$$\check{\Omega}_i = L \Omega_i L^{-1} - \partial_i L L^{-1}, \tag{9.47}$$

so that we can define the covariant derivative of a spinor

$$\nabla_i \psi = \partial_i + \Omega_i \psi. \tag{9.48}$$

From the derivative of the scalar $\bar{\psi}\psi$,

$$\nabla_i \left(\bar{\psi}\psi\right) = \partial_i \left(\bar{\psi}\psi\right), \tag{9.49}$$

we derive the covariant derivative of the adjoint spinor too,

$$\nabla_i \bar{\psi} = \partial_i \bar{\psi} - \bar{\psi}\Omega_i. \tag{9.50}$$

We decide to use as spinor connection Ω_i the Fock-Ivanenko coefficients

$$\Omega_i = \frac{1}{8} \omega_i{}^{\mu\nu} \left[\gamma_\mu, \gamma_\nu\right]. \tag{9.51}$$

The covariant derivate of the product $\psi\bar{\psi}$ is equal to

$$\nabla_i \left(\psi\bar{\psi}\right) = \left(\nabla_i \psi\right) \bar{\psi} + \psi \nabla_i \bar{\psi} = \partial_i \left(\psi\bar{\psi}\right) + \Omega_i \psi\bar{\psi} - \psi\bar{\psi}\Omega_i = \partial_i \left(\psi\bar{\psi}\right) + \left[\Omega_i, \psi\bar{\psi}\right]. \tag{9.52}$$

Therefore, since the Dirac matrices transform as $\psi\bar{\psi}$, the covariant derivative of γ^μ is given by

$$\nabla_i \gamma^\mu = \partial_i \gamma^\mu + \omega_i{}^\mu{}_\nu \gamma^\nu + \left[\Omega_i, \gamma^\mu\right] = \omega_i{}^\mu{}_\nu \gamma^\nu + \left[\Omega_i, \gamma^{mu}\right]. \tag{9.53}$$

Making use of Eq. (9.51) as well as of the algebraic identities

$$\left[\gamma^\mu, \left[\gamma^\sigma, \gamma^\tau\right]\right] = 4 \left(\gamma^\tau \eta^{\mu\sigma} - \gamma^\sigma \eta^{\mu\tau}\right), \tag{9.54}$$

from Eq. (9.53) we get the relation

$$\nabla_i \gamma^\mu = \omega_i{}^{(\mu\nu)} \gamma_\nu = -L_i{}^{(\nu\mu)} \gamma_\nu, \tag{9.55}$$

with $\gamma_\nu = \gamma^\mu \eta_{\mu\nu}$. Considering the identity (9.11), we also have

$$\nabla_i \gamma^j = -L_i{}^{(hj)} \gamma_h = -\frac{1}{2} Q_i{}^{jh} \gamma_h, \tag{9.56}$$

where $\gamma_h = \gamma^k g_{kh}$. Therefore, in the case of nonmetricity, Dirac matrices have in general non-zero covariant derivatives.

9.2 $f(\mathcal{Q})$ gravity coupled to a spinor field

In this section, we shall consider $f(\mathcal{Q})$ gravity coupled with a spinor field ψ. In order to deal with spinor fields, we shall assume as gravitational fields a tetrad e^μ_i and an affine connection $\Gamma_{ij}{}^h$, defined on the spacetime.

The action functional of the theory is

$$\mathcal{A}(e^\mu_i, \Gamma_{ij}{}^h, \psi) = \int \left[\sqrt{-g} f(\mathcal{Q}) + \lambda_h{}^{kij} R^h{}_{kij} + \lambda_h{}^{ij} T_{ij}{}^h + \mathcal{L}_D\right] d^4x, \tag{9.57}$$

where $\mathcal{L}_D$ is the Dirac Lagrangian

$$\mathcal{L}_D = \sqrt{-g}\left[\frac{i}{2}\left(\bar{\psi}\gamma^i\nabla_i\psi - \nabla_i\bar{\psi}\gamma^i\psi\right) - m\bar{\psi}\psi\right]. \tag{9.58}$$

Equation (9.58) is the simplest Lagrangian density we can construct for a spinor field that contains the first derivatives of spinors and it is real. This is because the term

$$i\left(\bar{\psi}\gamma^i\nabla_i\psi - \nabla_i\bar{\psi}\gamma^i\psi\right) \tag{9.59}$$

is the simplest real linear combination of derivative that is not a total divergence, whereas $\bar{\psi}\psi$ is the simplest quadratic scalar we can use to construct the mass term.

The field equations are derived by varying the action (9.57) with respect to tetrad, affine connection, spinor field, and Lagrange multipliers. More specifically, variations with respect to the Lagrange multipliers give rise to the two well known constraints

$$R^h{}_{kij} = 0 \qquad \text{and} \qquad T_{ij}{}^h = 0. \tag{9.60}$$

In order to carry out variations with respect to the spinor field, it is convenient to make use of the identity (9.15) to express the covariant spinor derivative in the form

$$\nabla_i\psi = \tilde{\nabla}_i\psi + \frac{1}{8}L_i{}^{\nu\mu}\left[\gamma_\mu, \gamma_\nu\right]\psi \qquad \text{and} \qquad \nabla_i\bar{\psi} = \tilde{\nabla}_i\bar{\psi} - \frac{1}{8}\bar{\psi}L_i{}^{\nu\mu}\left[\gamma_\mu, \gamma_\nu\right] \tag{9.61}$$

where the constraint $T_{ij}{}^h = 0$ has been considered. For the variation with respect to $\bar{\psi}$, we have to further elaborate the derivative term $\nabla_i\bar{\psi}$ into the Dirac Lagrangian (9.58),

$$\begin{aligned}
-\frac{i}{2}\sqrt{-g}\nabla_i\bar{\psi}\gamma^i\psi =& -\frac{i}{2}\sqrt{-g}\tilde{\nabla}_i\bar{\psi}\gamma^i\psi + \frac{i}{16}\sqrt{-g}\bar{\psi}L_i{}^{\nu\mu}\left[\gamma_\mu, \gamma_\nu\right]\gamma^i\psi = \\
=& -\tilde{\nabla}\left(\frac{i}{2}\sqrt{-g}\bar{\psi}\gamma^i\psi\right) + \frac{i}{2}\sqrt{-g}\bar{\psi}\tilde{\nabla}_i\left(\gamma^i\psi\right) + \\
&+ \frac{i}{16}\sqrt{-g}\bar{\psi}L^{\alpha\nu\mu}\gamma_\alpha\left[\gamma_\mu, \gamma_\nu\right]\psi - \frac{i}{4}\sqrt{-g}\bar{\psi}L^{\alpha\nu\mu}\gamma_\nu\eta_{\alpha\mu}\psi + \\
&+ \frac{i}{4}\sqrt{-g}\bar{\psi}L^{\alpha\nu\mu}\gamma_\mu\eta_{\alpha\nu}\psi = \\
=& -\tilde{\nabla}\left(\frac{i}{2}\sqrt{-g}\bar{\psi}\gamma^i\psi\right) + \frac{i}{2}\sqrt{-g}\bar{\psi}\gamma^i\nabla_\psi + \frac{i}{4}\sqrt{-g}\bar{\psi}\left(q^h - Q^h\right)\gamma_h\psi,
\end{aligned} \tag{9.62}$$

where the first term can be neglected being a boundary term, and we used the identity (9.54). The same evaluation must be made in the variation with respect to ψ, this time considering the derivative $\nabla_i\psi$. Therefore, the Dirac equations are equal to

$$\begin{aligned} i\gamma^h\nabla_h\psi + \frac{i}{4}q_h\gamma^h\psi - \frac{i}{4}Q_h\gamma^h\psi - m\psi = 0, \quad \text{and} \\ i\nabla_h\bar{\psi}\gamma^h + \frac{i}{4}q_h\bar{\psi}\gamma^h - \frac{i}{4}Q_h\bar{\psi}\gamma^h + m\bar{\psi} = 0. \end{aligned} \tag{9.63}$$

By varying with respect to the connection, we obtain the equations

$$2\nabla_p\lambda_h{}^{jip} + 2\lambda_h{}^{ij} - 4\sqrt{-g}f'P^{ij}{}_h = \Phi^{ij}{}_h \tag{9.64}$$

where $f' = \frac{\partial f}{\partial \mathcal{Q}}$ and $\Phi^{ij}{}_h = -\frac{\delta\mathcal{L}_D}{\delta\Gamma_{ij}{}^h}$. As we already know from Sec. 3.1, it is not necessary to elaborate further on this last equation.

In order to perform the variation with respect to the tetrad field, we exploit the results already present above together with the identities

$$\frac{\partial e}{\partial e^\mu_i} = e e^i_\mu \qquad \text{and} \qquad \frac{\partial e^j_\nu}{\partial e^\mu_i} = -e^i_\nu e^j_\mu \tag{9.65}$$

where $e = \sqrt{-g}$. On one hand, we have

$$\begin{aligned} \frac{\delta\sqrt{-g}f(\mathcal{Q})}{\delta g^{ij}}\delta g^{ij} =& \sqrt{-g}\left[\frac{2}{\sqrt{-g}}\nabla_k\left(\sqrt{-g}f'P^k{}_{ij}\right) + \frac{1}{2}g_{ij}f(\mathcal{Q}) + \right. \\ & \left. + f'\left(P_{iab}Q_j{}^{ab} - 2Q^{ab}{}_iP_{abj}\right)\right]\delta g^{ij}. \end{aligned} \tag{9.66}$$

Making use of Eqs. (9.65), it is easily seen that

$$\frac{\delta g^{ij}}{\delta e^\tau_h} = -2g^{jh}e^i_\tau. \tag{9.67}$$

Collecting the above results, the variation of the gravitational Lagrangian with respect to the tetrad field may be expressed as

$$\begin{aligned} \frac{\delta\sqrt{-g}f(\mathcal{Q})}{\delta e^\tau_h} =& -2\sqrt{-g}\left[\frac{2}{\sqrt{-g}}\nabla_k\left(\sqrt{-g}f'P^k{}_{ij}\right) + \frac{1}{2}g_{ij}f(\mathcal{Q}) + \right. \\ & \left. + f'\left(P_{iab}Q_j{}^{ab} - 2Q^{ab}{}_iP_{abj}\right)\right]g^{jh}e^i_\tau. \end{aligned} \tag{9.68}$$

For later use, saturating Eq. (9.68) by e^τ_q and lowering the index h, we obtain the equivalent identity

$$\begin{aligned} \frac{\delta\sqrt{-g}f(\mathcal{Q})}{\delta e^\tau_h}e^\tau_q g_{hs} =& -2\sqrt{-g}\left[\frac{2}{\sqrt{-g}}\nabla_k\left(\sqrt{-g}f'P^k{}_{qs}\right) + \frac{1}{2}g_{qs}f(\mathcal{Q}) + \right. \\ & \left. + f'\left(P_{qab}Q_s{}^{ab} - 2Q^{ab}{}_qP_{abs}\right)\right]. \end{aligned} \tag{9.69}$$

The variation of Dirac Lagrangian with respect to e^τ_h deserves a little more attention. First, we observe that Eq. (9.9) allows us to represent the full covariant derivative of the variation δe^τ_h in the form

$$\nabla_i\left(\delta e^\tau_h\right) = \partial_i\left(\delta e^\tau_h\right) - \partial_i\left(e^\tau_p\right)e^p_\gamma\delta e^\gamma_h - \Gamma_{ih}{}^s\delta e^\tau_s + \Gamma_{ip}{}^q e^\tau_q e^p_\gamma\delta e^\gamma_h. \tag{9.70}$$

With the identity (9.70) in mind, after performing the variation of the Dirac Lagrangian with respect to the tetrad field, we get the expression

$$
\begin{aligned}
\frac{\delta \mathcal{L}_D}{\delta e^\tau_h}\delta e^\tau_h =& e^h_\tau \mathcal{L}_D \delta e^\tau_h - \frac{i}{2} e \left[\bar{\psi}\gamma^\mu e^h_\mu e^i_\tau \nabla_i \psi - \nabla_i \bar{\psi}\gamma^\mu e^h_\mu e^i_\tau \psi \right] \delta e^\tau_h + \\
& - \frac{i}{16} e \nabla_i \left(\delta e^\tau_h\right) \bar{\psi} \left\{ \gamma^i, \left[\gamma_\tau, \gamma^h \right] \right\} \psi,
\end{aligned}
\tag{9.71}
$$

where, on the r.h.s., the presence of the last term is due to the fact that we are using the tetrad field and the affine connection to represent the spin connection (9.9), involved in the spinor covariant derivative (9.48). Now, considering Eqs. (9.12), (9.15), (9.16) and (9.18) as well as the constraint ${T_{ij}}^h = 0$, we have the identity

$$
\nabla_i \left(\delta e^\tau_h\right) = \tilde{\nabla}_i \left(\delta e^\tau_h\right) + {L_{ih}}^s \delta e^\tau_s - {L_i}^\tau{}_\sigma \delta e^\sigma_h . \tag{9.72}
$$

In view of Eq. (9.72), after some calculations we end up with the further identity

$$
\begin{aligned}
-\frac{i}{16} e \nabla_i \left(\delta e^\tau_h\right) \bar{\psi} \left\{ \gamma^i, \left[\gamma_\tau, \gamma^h \right] \right\} \psi = & \\
& - \frac{i}{16} e \tilde{\nabla}_i \left(\delta e^\tau_h \bar{\psi} \left\{ \gamma^i, \left[\gamma_\tau, \gamma^h \right] \right\} \psi \right) + \frac{i}{16} e \tilde{\nabla}_i \left(\bar{\psi} \left\{ \gamma^i, \left[\gamma_\tau, \gamma^h \right] \right\} \psi \right) \delta e^\tau_h \\
& - \frac{i}{16} e {L_{is}}^h \bar{\psi} \left\{ \gamma^i, \left[\gamma_\tau, \gamma^s \right] \right\} \psi \delta e^\tau_h + \frac{i}{16} e {L_i}^\sigma{}_\tau \bar{\psi} \left\{ \gamma^i, \left[\gamma_\sigma, \gamma^h \right] \right\} \psi \delta e^\tau_h .
\end{aligned}
\tag{9.73}
$$

Notice that the first addendum on the right-hand side of Eq. (9.73) is a divergence that leads to a boundary term. Moreover, still using Eqs. (9.12), (9.15), (9.16) and (9.18), it is an easy matter to verify the relation

$$
\begin{aligned}
\frac{i}{16} e \tilde{\nabla}_i \left(\bar{\psi} \left\{ \gamma^i, \left[\gamma_\tau, \gamma^h \right] \right\} \psi \right) - \frac{i}{16} e {L_{is}}^h \bar{\psi} \left\{ \gamma^i, \left[\gamma_\tau, \gamma^s \right] \right\} \psi + \frac{i}{16} e {L_i}^\sigma{}_\tau{}^\sigma \bar{\psi} \left\{ \gamma^i, \left[\gamma_\sigma, \gamma^h \right] \right\} \psi = & \\
\frac{i}{16} e \nabla_i \left(\bar{\psi} \left\{ \gamma^i, \left[\gamma_\tau, \gamma^h \right] \right\} \psi \right) + \frac{i}{16} e {L_{is}}^i \bar{\psi} \left\{ \gamma^s, \left[\gamma_\tau, \gamma^h \right] \right\} \psi . &
\end{aligned}
\tag{9.74}
$$

Replacing the content of Eqs. (9.73) and (9.74) into Eq. (9.71), we obtain the identity

$$
\begin{aligned}
\frac{\delta \mathcal{L}_D}{\delta e^\tau_h} =& e^h_\tau \mathcal{L}_D - \frac{i}{2} e \left[\bar{\psi}\gamma^\mu e^h_\mu e^i_\tau \nabla_i \psi - \nabla_i \bar{\psi}\gamma^\mu e^h_\mu e^i_\tau \psi \right] + \\
& + \frac{i}{16} e \nabla_i \left(\bar{\psi} \left\{ \gamma^i, \left[\gamma_\tau, \gamma^h \right] \right\} \psi \right) + \frac{i}{16} e {L_{is}}^i \bar{\psi} \left\{ \gamma^s, \left[\gamma_\tau, \gamma^h \right] \right\} \psi ,
\end{aligned}
\tag{9.75}
$$

which, saturated by e^τ_q, yields

$$
\begin{aligned}
\frac{\delta \mathcal{L}_D}{\delta e^\tau_h} e^\tau_q =& \delta^h_q \mathcal{L}_D - \frac{i}{2} e \left[\bar{\psi}\gamma^h \nabla_q \psi - \nabla_q \bar{\psi}\gamma^h \psi \right] + \\
& + \frac{i}{16} e \nabla_i \left(\bar{\psi} \left\{ \gamma^i, \left[\gamma_q, \gamma^h \right] \right\} \psi \right) + \frac{i}{16} e {L_{is}}^i \bar{\psi} \left\{ \gamma^s, \left[\gamma_q, \gamma^h \right] \right\} \psi .
\end{aligned}
\tag{9.76}
$$

The expression (9.76) can be further elaborated and simplified, making use of the Dirac equations (9.63). First, it is an easy matter to verify that the Dirac Lagrangian $\mathcal{L}_D$ vanishes on shell. Thus, the first addendum to the r.h.s. of Eq. (9.76) may be omitted. Moreover, once the index h is lowered, the antisymmetric part of the whole r.h.s. of Eq. (9.76) vanishes as well. To see this point, we analyze in detail the divergence term

$$\begin{aligned}\frac{i}{16}e\nabla_i\left(\bar\psi\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\psi\right) &= \frac{i}{16}e\left(\nabla_i\bar\psi\right)\gamma^i\left[\gamma_q,\gamma^h\right]\psi+\frac{i}{16}e\bar\psi\gamma^i\left[\gamma_q,\gamma^h\right]\left(\nabla_i\psi\right)+\\ &+\frac{i}{16}e\left(\nabla_i\bar\psi\right)\left[\gamma_q,\gamma^h\right]\gamma^i\psi+\frac{i}{16}e\bar\psi\left[\gamma_q,\gamma^h\right]\gamma^i\left(\nabla_i\psi\right)+\frac{i}{16}e\bar\psi\left(\nabla_i\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\right)\psi.\end{aligned} \tag{9.77}$$

By adding and subtracting the terms

$$\frac{i}{16}e\bar\psi\left[\gamma_q,\gamma^h\right]\gamma^i\left(\nabla_i\psi\right), \tag{9.78}$$

and

$$\frac{i}{16}e\left(\nabla_i\bar\psi\right)\gamma^i\left[\gamma_q,\gamma^h\right]\psi \tag{9.79}$$

in Eq. (9.77), we obtain the following expression

$$\begin{aligned}\frac{i}{16}e\nabla_i\left(\bar\psi\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\psi\right) &= \frac{1}{8}e\left\{\left(i\nabla_i\bar\psi\gamma^i\right)\left[\gamma_q,\gamma^h\right]\psi+\bar\psi\left[\gamma_q,\gamma^h\right]\left(i\gamma^i\nabla_i\psi\right)\right\}+\\ &+\frac{i}{2}e\left\{-\frac{1}{8}\left(\nabla_i\bar\psi\right)\left[\gamma^i,\left[\gamma_q,\gamma^h\right]\right]\psi+\frac{1}{8}\bar\psi\left[\gamma^i,\left[\gamma_q,\gamma^h\right]\right]\left(\nabla_i\psi\right)\right\}+\\ &+\frac{i}{16}e\bar\psi\left(\nabla_i\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\right)\psi.\end{aligned} \tag{9.80}$$

By employing the Dirac equations (9.63) and the identity (9.54), from Eq. (9.80) we easily get

$$\begin{aligned}\frac{i}{16}e\nabla_i\left(\bar\psi\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\psi\right) &= \frac{i}{4}e\left(\bar\psi\gamma^h\nabla_q\psi-\bar\psi\gamma_q\nabla^h\psi-\nabla_q\bar\psi\gamma^h\psi+\nabla^h\bar\psi\gamma_q\psi\right)\\ &+\frac{i}{32}e\left(Q_{ji}{}^{j}-Q_{ij}{}^{j}\right)\bar\psi\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\psi+\frac{i}{16}e\bar\psi\left(\nabla_i\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\right)\psi.\end{aligned} \tag{9.81}$$

Except for inessential multiplying factors, Eq. (9.81) represents the conservation law of the spin density in the current theory. Compared to that holding in Einstein-Cartan-like theories (for example, see [199]), there are evident differences due to the explicit presence of the nonmetricity tensor and to the fact that the covariant derivatives of Dirac matrices are no longer zero.

To proceed further, we must elaborate the term

$$\frac{i}{16}e\bar\psi\left(\nabla_i\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\right)\psi. \tag{9.82}$$

To this end, making use of relation (9.56), we have

$$\begin{aligned}&\frac{i}{16}e\bar\psi\left(\nabla_i\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\right)\psi=\\ &\quad-\frac{i}{32}eL_i{}^{pi}\bar\psi\left\{\gamma_p,\left[\gamma_q,\gamma^h\right]\right\}\psi-\frac{i}{32}eL_i{}^{ip}\bar\psi\left\{\gamma_p,\left[\gamma_q,\gamma^h\right]\right\}\psi+\\ &\quad-\frac{i}{32}eQ_{ipq}\bar\psi\left\{\gamma^i,\left[\gamma^h,\gamma^p\right]\right\}\psi-\frac{i}{32}eQ_{ip}{}^{h}\bar\psi\left\{\gamma^i,\left[\gamma_q,\gamma^p\right]\right\}\psi.\end{aligned} \tag{9.83}$$

At this point, given Eqs. (9.81) and (9.83) as well as the identities $L_{is}{}^{i} = \frac{1}{2}Q_{si}{}^{i}$ and $L_{i}{}^{i}{}_{p} = Q_{i}{}^{i}{}_{p} - \frac{1}{2}Q_{pi}{}^{i}$, we can express the sum of the last two terms on the r.h.s. of Eq. (9.76) in the form

$$\begin{aligned}\frac{i}{16}e\nabla_i\left(\bar{\psi}\left\{\gamma^i,\left[\gamma_q,\gamma^h\right]\right\}\psi\right)+\frac{i}{16}eL_{is}{}^{i}\bar{\psi}\left\{\gamma^s,\left[\gamma_q,\gamma^h\right]\right\}\psi = \\ =\frac{i}{4}e\left(\bar{\psi}\gamma^h\nabla_q\psi-\bar{\psi}\gamma_q\nabla^h\psi-\nabla_q\bar{\psi}\gamma^h\psi+\nabla^h\bar{\psi}\gamma_q\psi\right) \\ -\frac{i}{32}eQ_{ipq}\bar{\psi}\left\{\gamma^i,\left[\gamma^h,\gamma^p\right]\right\}\psi-\frac{i}{32}eQ_{ip}{}^{h}\bar{\psi}\left\{\gamma^i,\left[\gamma_q,\gamma^p\right]\right\}\psi.\end{aligned} \tag{9.84}$$

Inserting Eq. (9.84) into Eq. (9.76) and lowering the index h, we obtain the final expression

$$\begin{aligned}\frac{\delta\mathcal{L}_D}{\delta e^\tau_h}e^\tau_q g_{hs} = -\frac{i}{2}e\left(\bar{\psi}\gamma_{(s}\nabla_{q)}\psi-\nabla_{(q}\bar{\psi}\gamma_{s)}\psi\right)+ \\ -\frac{i}{32}eQ_{ipq}\bar{\psi}\left\{\gamma^i,\left[\gamma_s,\gamma^p\right]\right\}\psi-\frac{i}{32}eQ_{ips}\bar{\psi}\left\{\gamma^i,\left[\gamma_q,\gamma^p\right]\right\}\psi.\end{aligned} \tag{9.85}$$

To conclude, by equating Eq. (9.69) with Eq. (9.85) and dividing by $\sqrt{-g} = e$, we get the explicit form of the energy-momentum tensor

$$\Sigma_{ij} = \frac{i}{4}\left(\bar{\psi}\gamma_{(s}\nabla_{q)}\psi-\nabla_{(q}\bar{\psi}\gamma_{s)}\psi\right)+\frac{i}{32}Q_{ip(q}\bar{\psi}\left\{\gamma^i,\left[\gamma_{s)},\gamma^p\right]\right\}\psi, \tag{9.86}$$

and of the field equations deduced by varying with respect to the tetrad field, namely

$$\begin{aligned}\frac{2}{\sqrt{-g}}\nabla_k\left(\sqrt{-g}f'P^k{}_{qs}\right)+\frac{1}{2}g_{qs}f(\mathcal{Q})+f'\left(P_{qab}Q_s{}^{ab}-2Q^{ab}{}_qP_{abs}\right) = \\ =\frac{i}{4}\left(\bar{\psi}\gamma_{(s}\nabla_{q)}\psi-\nabla_{(q}\bar{\psi}\gamma_{s)}\psi\right)+\frac{i}{32}Q_{ip(q}\bar{\psi}\left\{\gamma^i,\left[\gamma_{s)},\gamma^p\right]\right\}\psi,\end{aligned} \tag{9.87}$$

where both sides of the equation (9.87) are symmetric in the indices q and s. Further evaluation of the r.h.s. of (9.87) shows that, by explicating the Levi-Civita derivative of the spinors, the terms due to nonmetricity cancel out. Indeed, using Eqs. (9.61), from the derivative terms we obtain

$$\frac{i}{4}\bar{\psi}\gamma_{(s}\nabla_{q)}\psi = \frac{i}{4}\bar{\psi}\gamma_{(s}\tilde{\nabla}_{q)}\psi+\frac{i}{32}\bar{\psi}\left(Q_{pi(q}\gamma_{s)}\gamma^i\gamma^p-Q_{ip(q}\gamma_{s)}\gamma^p\gamma^i\right)\psi, \tag{9.88}$$

and

$$-\frac{i}{4}\nabla_{(q}\bar{\psi}\gamma_{s)}\psi = -\frac{i}{4}\tilde{\nabla}_{(q}\bar{\psi}\gamma_{s)}\psi+\frac{i}{32}\bar{\psi}\left(Q_{pi(q}\gamma^i\gamma^p\gamma_{s)}-Q_{ip(q}\gamma^p\gamma^i\gamma_{s)}\right)\psi, \tag{9.89}$$

whereas the remaining terms give

$$\frac{i}{32}Q_{ip(q}\bar{\psi}\left\{\gamma^i,\left[\gamma_{s)},\gamma^p\right]\right\}\psi = \frac{i}{32}Q_{ip(q}\bar{\psi}\left(\gamma_{s)}\gamma^p\gamma^i-\gamma_{s)}\gamma^i\gamma^p+\gamma^p\gamma^i\gamma_{s)}-\gamma^i\gamma^p\gamma_{s)}\right)\psi. \tag{9.90}$$

Therefore, Eq. (9.87) can be recast as

$$\begin{aligned}\frac{2}{\sqrt{-g}}\nabla_k\left(\sqrt{-g}f'P^k{}_{qs}\right)+\frac{1}{2}g_{qs}f(\mathcal{Q})+f'\left(P_{qab}Q_s{}^{ab}-2Q^{ab}{}_qP_{abs}\right) = \\ =\frac{i}{4}\left(\bar{\psi}\gamma_{(s}\tilde{\nabla}_{q)}\psi-\tilde{\nabla}_{(q}\bar{\psi}\gamma_{s)}\psi\right).\end{aligned} \tag{9.91}$$

The same is true for the Dirac equations (9.63), which can be written as

$$\begin{aligned} i\gamma^h\tilde{\nabla}_h\psi - m\psi = 0, \quad \text{and} \\ i\tilde{\nabla}_h\bar{\psi}\gamma^h + m\bar{\psi} = 0, \end{aligned} \tag{9.92}$$

in which we used the identity (9.54). Thus, both the energy-momentum tensor and Dirac equations are what we would have in GR, which means that spinors seem to be unaffected by nonmetricity. However, in the next section, we will use Eqs. (9.87), and (9.63) as they make the math easier in the case where coincident gauge is used.

For the sake of completeness, we also write the fully developed expression of Eq. (9.64),

$$2\nabla_p\lambda_h{}^{jip} + 2\lambda_h{}^{ij} - 4\sqrt{-g}f'P^{ij}{}_h = -\frac{i\sqrt{-g}}{16}\bar{\psi}\left\{\gamma^i, \left[\gamma^j, \gamma_h\right]\right\}\psi. \tag{9.93}$$

9.3 Bianchi type-I cosmological models

In the coincidence gauge $\Gamma_{ij}{}^h = 0$, we assume a BI metric of the form

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2, \tag{9.94}$$

describing a homogeneous and anisotropic universe[2]. The components of the tetrad field associated with the line element (9.94) are expressed as

$$e^\mu_0 = \delta^\mu_0, \quad e^\mu_1 = a(t)\delta^\mu_1, \quad e^\mu_2 = b(t)\delta^\mu_2, \quad e^\mu_3 = c(t)\delta^\mu_3 \qquad \mu = 0,1,2,3 \tag{9.95a}$$

with inverse relations given by

$$e^0_\mu = \delta^0_\mu, \quad e^1_\mu = \frac{1}{a(t)}\delta^1_\mu, \quad e^2_\mu = \frac{1}{b(t)}\delta^2_\mu, \quad e^3_\mu = \frac{1}{c(t)}\delta^3_\mu \qquad \mu = 0,1,2,3 \tag{9.95b}$$

Moreover, homogeneity and coincidence gauge assumptions together with Eqs. (9.9), (9.48), (9.51) and (9.95) yield the identities

$$\nabla_0\psi = \dot{\psi}, \quad \nabla_A\psi = 0 \qquad A = 1,2,3. \tag{9.96}$$

After inserting the content of Eq. (9.96) into Eqs. (9.63), Dirac equations assume the form

$$i\gamma^0\partial_0\psi + \frac{i}{2}\frac{\dot{\tau}}{\tau}\gamma^0\psi - m\psi = 0 \tag{9.97a}$$

$$i\partial_0\bar{\psi}\gamma^0 + \frac{i}{2}\frac{\dot{\tau}}{\tau}\bar{\psi}\gamma^0 + m\bar{\psi} = 0. \tag{9.97b}$$

Equations (9.97) can be easily integrated; adopting the Dirac representation (9.33) for the matrices γ^μ, they possess solutions of the form

$$\psi = \frac{1}{\sqrt{\tau}}\begin{pmatrix} c_1e^{-imt} \\ c_2e^{-imt} \\ c_3e^{imt} \\ c_4e^{imt} \end{pmatrix}, \tag{9.98}$$

[2] In this chapter, we use the same conditions on the 4-velocity of the observers given in Ch. 5

where c_i, $i = 1, \ldots, 4$, are suitable integration constants. Moreover, from Dirac equations (9.97) or also from their solutions (9.98), it is easily seen the following relation necessarily holds

$$\frac{d}{dt}\left(\tau\bar{\psi}\psi\right) = 0 \quad \Longleftrightarrow \quad \bar{\psi}\psi = \frac{K}{\tau}, \tag{9.99}$$

with $K = |c_1|^2 + |c_2|^2 - |c_3|^2 - |c_4|^2$.

As for Eqs. (9.87), due to Eqs. (9.96) and (9.97), the only non-zero component of the tensor Σ_{ij} is given by

$$\Sigma_{00} = \frac{i}{4}\left(\bar{\psi}\gamma_0\nabla_0\psi - \nabla_0\bar{\psi}\gamma_0\psi\right) = \frac{1}{2}m\bar{\psi}\psi. \tag{9.100}$$

In view of this, a direct calculation shows that field equations (9.87) assume the explicit form

$$\frac{1}{2}f + 2f'\left(\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{a}\dot{c}}{ac}\right) = \frac{1}{2}m\bar{\psi}\psi, \tag{9.101a}$$

$$\dot{f}'\left(-\frac{\dot{a}}{a} + \frac{\dot{\tau}}{\tau}\right) + f'\left(\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + 2\frac{\dot{b}\dot{c}}{bc}\right) + \frac{1}{2}f = 0, \tag{9.101b}$$

$$\dot{f}'\left(-\frac{\dot{b}}{b} + \frac{\dot{\tau}}{\tau}\right) + f'\left(\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + 2\frac{\dot{a}\dot{c}}{ac}\right) + \frac{1}{2}f = 0, \tag{9.101c}$$

$$\dot{f}'\left(-\frac{\dot{c}}{c} + \frac{\dot{\tau}}{\tau}\right) + f'\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} + 2\frac{\dot{a}\dot{b}}{ab}\right) + \frac{1}{2}f = 0, \tag{9.101d}$$

$$\left(\dot{a}b - a\dot{b}\right)\bar{\psi}\gamma^5\gamma_3\psi = 0, \tag{9.102a}$$

$$\left(\dot{a}c - a\dot{c}\right)\bar{\psi}\gamma^5\gamma_2\psi = 0, \tag{9.102b}$$

$$\left(\dot{b}c - b\dot{c}\right)\bar{\psi}\gamma^5\gamma_1\psi = 0, \tag{9.102c}$$

where Eqs. (9.101) and (9.102) derive from the diagonal and the off-diagonal part of Eqs. (9.87), respectively. It is interesting to note that the conditions (9.102) are identical to those that arise also in $f(R)$ theories with torsion [205]. However, in that case, these conditions stem directly from the Dirac tensor, while in the present case, they originate from the additional terms appearing in the r.h.s. of Eq. (9.87).

Eqs. (9.102) are automatically satisfied in the case of an isotropic universe. Instead, anisotropic spacetimes imply stringent constraints on the spinor field i.e.,

$$\bar{\psi}\gamma^5\gamma^1\psi = \bar{\psi}\gamma^5\gamma^2\psi = \bar{\psi}\gamma^5\gamma^3\psi = 0. \tag{9.103}$$

In this case, the orthogonality between the current four-vector $\bar{\psi}\gamma^\mu\psi$ and the spin four-vector $\bar{\psi}\gamma^5\gamma^\mu\psi$ implies that the time component $\bar{\psi}\gamma^5\gamma^0\psi$ has to be zero. In fact, if $\bar{\psi}\gamma^0\psi$ were allowed to vanish, then the whole spinor field would be zero, but the vanishing of the entire spin four-vector implies the condition $\bar{\psi}\psi = 0$. It follows that, in the anisotropic case, the spinor field does not enter the gravitational equations (9.101) which become identical to the ones we would have in vacuum. Of course, there are also intermediate situations where, in the face of partial isotropy, some constraints persist on the spin four-vector: for instance, the case $a = b$ and $\bar{\psi}\gamma^5\gamma^1\psi = \bar{\psi}\gamma^5\gamma^2\psi = 0$. Anyway, any constraints imposed on the four-vector $\bar{\psi}\gamma^5\gamma^\mu\psi$ translate into restrictions on the admissible values of the integration constants c_i appearing in Eq. (9.98). For example, the sets of constants

$(c_1 = e^{i\theta}c_4, c_2 = c_3 = 0)$ or $(c_2 = e^{i\theta}c_3, c_1 = c_4 = 0)$ make the requirement $\bar{\psi}\gamma^5\gamma^\mu\psi = 0$ satisfied; the less restrictive choice $(c_2 = c_3 = 0)$ ensure the weaker condition $\bar{\psi}\gamma^5\gamma^1\psi = \bar{\psi}\gamma^5\gamma^2\psi = 0$.

To discuss and solve Eqs. (9.101), we preliminarily observe that Eq. (9.101a) can be rewritten as

$$\frac{1}{2}f(\mathcal{Q}) - f'(\mathcal{Q})\mathcal{Q} = \frac{mK}{2\tau}, \tag{9.104}$$

where

$$\mathcal{Q} = -2\left(\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc}\right), \tag{9.105}$$

and Eq. (9.99) have been employed. Equation (9.104) highlights how the contribution of the spinor field reduces to that of cosmological dust.

Now, given the function $f(\mathcal{Q})$ and except for some pathological cases, in general from Eq. (9.104) we may derive the expression of the nonmetricity scalar in terms of τ, i.e. $\mathcal{Q} = \mathcal{Q}(\tau)$. In view of this, by subtracting Eq. (9.101b) from Eq. (9.101c) and from Eq. (9.101d) separately, we obtain the two equations[3]

$$\frac{d}{dt}\left[f'\tau\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)\right] = 0 \tag{9.106a}$$

$$\frac{d}{dt}\left[f'\tau\left(\frac{\dot{a}}{a} - \frac{\dot{c}}{c}\right)\right] = 0, \tag{9.106b}$$

which in turn implies the relations

$$\frac{a}{b} = e^{d_2}\exp\int\frac{d_1}{f'\tau}dt, \tag{9.107a}$$

and

$$\frac{a}{c} = e^{g_2}\exp\int\frac{g_1}{f'\tau}dt, \tag{9.107b}$$

with d_1, d_2, g_1 and g_2 suitable integration constants and where $f'(\mathcal{Q}(\tau))$ is a function of τ.

If we could now get an equation for the variable τ alone, the systems of field equations would be entirely worked out. This goal may be achieved through a suitable combination of Eqs. (9.101). Indeed, by summing Eqs. (9.101b), (9.101c) and (9.101d) each to the other and subtracting Eq. (9.101a) multiplied by 3, we get the final dynamical equation for the unknown τ

$$\frac{2}{\tau}\frac{d}{dt}\left(f'\dot{\tau}\right) + 3f'\mathcal{Q} = -\frac{3mK}{2\tau}. \tag{9.108}$$

Once Eq. (9.108) was solved, from Eqs. (9.107) together with the relation $\tau = abc$, we would have the expressions for the scale factors:

$$a(t) = \sqrt[3]{\tau}\exp\left[\frac{(d_1 + g_1 + d_2 + g_2)}{3}\int_{t_0}^{t}\frac{dt}{f'\tau}\right], \tag{9.109a}$$

$$b(t) = \sqrt[3]{\tau}\exp\left[\frac{(-2d_1 + g_1 - 2d_2 + g_2)}{3}\int_{t_0}^{t}\frac{dt}{f'\tau}\right]. \tag{9.109b}$$

[3]The following procedure is similar to that described in Sec. 4.4.

$$c(t) = \sqrt[3]{\tau} \exp\left[\frac{(d_1 - 2g_1 + d_2 - 2g_2)}{3} \int_{t_0}^{t} \frac{dt}{f'\tau}\right]. \tag{9.109c}$$

With the only exception $f(\mathcal{Q}) = \alpha\sqrt{\mathcal{Q}}$, the previous argument applies also to the case $\bar{\psi}\psi = 0$ $(K = 0)$ giving rise to the condition $\mathcal{Q} = \text{const.}$.

At this point, the last step would be to verify that the relation (9.105) is preserved over time. This final requirement is seen to select suitable relationships between the admissible integration constants appearing in the found solutions.

To show how the above outlined procedure works, we consider the model $f(\mathcal{Q}) = \alpha\mathcal{Q}^n$ (with n natural odd number for brevity) as an example. In this case, from Eq. (9.104) we deduce the relation

$$\mathcal{Q} = \left[\frac{H}{(1-2n)\,\alpha}\right]^{\frac{1}{n}} \left(\frac{1}{\tau}\right)^{\frac{1}{n}}, \tag{9.110}$$

and thus

$$f(\mathcal{Q}(\tau)) = \frac{H}{(1-2n)\,\tau}, \quad \text{and} \quad f'(\mathcal{Q}(\tau)) = \alpha n \left[\frac{H}{(1-2n)\,\alpha}\right]^{\frac{n-1}{n}} \left(\frac{1}{\tau}\right)^{\frac{n-1}{n}}, \tag{9.111}$$

where we have set $H := mK$ for simplicity. If we require expansion in all three spatial directions, we must impose the condition

$$\frac{H}{(1-2n)\,\alpha} < 0. \tag{9.112}$$

Under the same hypothesis, if the exponent n had been even, we would have had to define the quantity (9.110) with the opposite sign, demanding

$$\frac{H}{(1-2n)\,\alpha} > 0 \tag{9.113}$$

as well.

Inserting Eqs. (9.110) and (9.111) into Eq. (9.108) and after the first integration step, we end up with the differential equation

$$\dot{\tau} = \frac{1}{\alpha n}\left[\frac{\alpha\,(1-2n)\,\tau}{H}\right]^{\frac{n-1}{n}} \left[-\frac{3Ht}{4\,(1-2n)} + H_0\right], \tag{9.114}$$

with H_0 an integration constant. Eq. (9.114) admits exact solutions of the form

$$\tau(t) = \left\{-\frac{[\alpha\,(1-2n)]^{-\frac{1}{n}}}{n^2}\left[H_0\,(2n-1)\,H^{\frac{1-n}{n}} + \frac{3}{8}H^{\frac{1}{n}}t\right]t + H_1\right\}^n, \tag{9.115}$$

where H_1 is again an integration constant. As already mentioned, the integration constants H_0, H_1, d_i, and g_i have to be chosen in such a way that the relation (9.105) is preserved over time. For instance, in this specific case, it is easily seen that the conditions $H_0 = 0$ and $H_1 = 0$ are only compatible with the choice $d_1 = g_1 = 0$, namely with a totally isotropic universe; instead, the values

$$H_0 = 0 \quad \text{and} \quad H_1 = \frac{2g_1^2}{3n^2\alpha^2}\left(\frac{H}{\alpha\,(1-2n)}\right)^{\frac{1-2n}{n}} \tag{9.116}$$

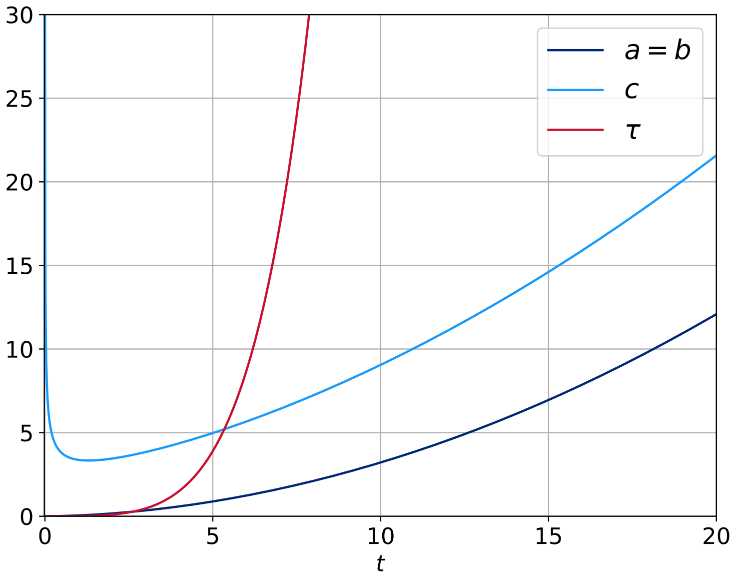

Figure 9.1: Plot of the scale factors (9.119) for $H = 3$, $\alpha = 2$, $n = 3$, $g_1 = 1$, $g_2 = 0$. The plot has been translated in such a way that the scale factors are zero at $t = 0$.

are compatible with a partially isotropic universe $a = b$ ($d_1 = d_2 = 0$).

As a last remark, it is worth noticing that the following identity holds

$$f'\left(\mathcal{Q}\left(\tau\right)\right)\tau = \alpha n\left[\frac{H}{\left(1-2n\right)\alpha}\right]^{\frac{n-1}{n}}\tau^{\frac{1}{n}}, \tag{9.117}$$

and therefore the scale factors (9.109) necessarily isotropize at late cosmological time. Moreover, by suitably choosing the value of the exponent n, still at late time we may have accelerated expansion for all the scale factors. This result is confirmed by solving for the scale factors. Indeed, setting

$$A^2 = \frac{3}{8n^2}\left[\frac{H}{\alpha(2n-1)}\right]^{\frac{1}{n}}, \quad B^2 = \frac{2g_1^2}{3n^2\alpha}\left[\frac{H}{\alpha(2n-1)}\right]^{\frac{1-2n}{n}}, \quad C = \alpha n\left[\frac{H}{\alpha(1-2n)}\right]^{\frac{n-1}{n}}, \tag{9.118}$$

from Eqs. (9.109), (9.115) and (9.117) we have[4]

$$a(t) = b(t) = \left[A^2t^2 - B^2\right]^{\frac{n}{3}} e^{\frac{g_2}{3}}\left[\frac{t-(B/A)}{t+(B/A)}\right]^{\left(\frac{g_1}{6ABC}\right)}, \tag{9.119a}$$

$$c(t) = \left[A^2t^2 - B^2\right]^{\frac{n}{3}} e^{\frac{-2g_2}{3}}\left[\frac{t-(B/A)}{t+(B/A)}\right]^{\left(\frac{-g_1}{3ABC}\right)}. \tag{9.119b}$$

9.4 Discussions

In this chapter, we have presented a tetrad-affine approach to $f(\mathcal{Q})$ gravity coupled to spinor fields of spin-1/2. The proposed formulation relies on the adoption of unusual

[4]It is the same solution we found in Sec. 5.2

pairs of dynamical variables $(e_i^\mu, \Gamma_{ij}{}^h)$, consisting of a tetrad field and an affine connection. This choice has been motivated at first by the necessity to have more treatable field equations, but then it has revealed some interesting features of $f(\mathcal{Q})$ gravity.

The use of the affine connection $\Gamma_{ij}{}^h$, instead of the commonly used spin connection $\omega_i{}^\mu{}_\nu$, implies the appearance of additional terms in the energy-momentum tensor which modify the standard Dirac tensor. These additional terms involve the covariant derivatives of the spin density and can be elaborated by using the conservation law for the spin, directly deduced by the Dirac equations.

Afterward, we have shown that in $f(\mathcal{Q})$ gravity the energy-momentum tensor and Dirac equations are unaffected by the nonmetricity tensor. Moreover, we have analyzed BI cosmologies with the use of the coincident gauge. As in the case of $f(R)$ theories with torsion, we have that the off-diagonal part of the gravitational field equations imposes restrictions on both the geometry and the spinor field.

In order for the constraints mentioned above to be satisfied, three different scenarios are possible: i) an isotropic spacetime where no further restrictions are imposed on the spinor field; ii) an anisotropic spacetime where both the spin four-vector and the scalar $\bar{\psi}\psi$ are zero, and thus a spacetime where the spinor field does not contribute to cosmological dynamics; iii) a universe where two scale factors are identical, and only one spatial component of the spin four-vector does not vanish.

Finally, we have proposed a general procedure to solve the resulting field equations, reducing the dynamical problem to a single differential equation for the spatial volume τ. To show how the given procedure works, we have considered gravitational Lagrangians of the kind $f(\mathcal{Q}) = \alpha\mathcal{Q}^n$. The corresponding dynamical problem has been analytically solved, showing that such models can give rise to initially anisotropic universes that isotropize with accelerated expansion.

IV
Final remarks

10 Conclusions

In this thesis, we investigated the role of nonmetricity in metric affine theories of gravity, mainly focusing on the $f(\mathcal{Q})$ theory.

After some preliminary chapters that serve as theoretical background (Chs. 1, 2, 3, and 4), we considered spatially homogeneous cosmological models within $f(\mathcal{Q})$ gravity. In particular, our attention was focused on studying the spatially flat FLRW and BI metrics.

In Ch. 5, we obtained exact solutions of the field equations using the reconstruction technique. We proved that nonmetricity can drive an accelerated expansion phase and, therefore, act as an effective cosmological content. The most evident example is the exponential solution given in the section 5.3.2, where it is possible to observe all three different phases of the evolution of the universe: inflation, matter-dominated era with decelerated expansion, and an accelerated expansion phase. However, working with coordinates does not always provide a clear understanding of how nonmetricity affects isotropy in anisotropic metrics like BI. For this reason, in Ch. 6, we introduced the $1+3$ covariant formalism, where the anisotropy is represented via one of the kinematical quantities related to cosmological observables, namely the shear tensor. Here, it was shown that the shear is affected by the tensor components of the nonmetricity that are symmetric and traceless. In Ch. 7, we applied the DSA to the cosmological BI equations in the $1+3$ formalism so that an analysis of the stability of the solutions has been performed. Among the four different models we have considered, the ones of the Secs. 7.2.1, 7.2.3, and 7.2.4 are important for finding solutions that isotropize. In these examples, by setting the right parameters of the theory, we always found that the isotropic phase of the universe constitutes an attractor to which the universe tends. Instead, in the model of Sec. 7.2.2, we see that the isotropic phase constitutes a transient phase of the universe because of the anisotropic pressure.

Instead, in Ch. 8, we focused on astrophysical aspects by considering static and LRS metrics. Using the $1+1+2$ formalism, we identified the components of the various geometric quantities relevant for describing LRS spacetimes. As in the $1+3$ case, we applied the formalism to a spacetime with a torsion-free connection. The geometry of nonmetric spacetimes presents several differences from GR. For example, we have two different accelerations for the vectors u^a and e^a. Also, stationary metrics can still present non-zero expansion θ and shear Σ due to the effects of the nonmetricity tensor. Therefore, in STG, it is crucial to clearly state the meaning of the commonly employed terms in GR. In studying $f(\mathcal{Q})$ gravity in vacuum, we found that Schwarzschild-de Sitter-type solutions can only be obtained under specific conditions on the nonmetricity tensor. In this context, the most significant result has been to provide a new perspective on spherically symmetric solutions in $f(\mathcal{Q})$ gravity through the $1+1+2$ formalism. This perspective might lead to new insights in studying compact objects in $f(\mathcal{Q})$ theory.

Finally, in Ch. 9, we studied how to incorporate spinors within the $f(\mathcal{Q})$ gravity via the use of an affine tetrad approach. As might be expected, due to the lack of torsion, the Dirac energy-momentum tensor and the Dirac equations result unaffected by nonmetricity, retaining the same form as in GR.

The $f(\mathcal{Q})$ gravity is still in its embryonic state as its theoretical development has come to the scientific community's attention only in recent years. Although interesting results have already been obtained, $f(\mathcal{Q})$ gravity is only one of the possible approaches to incorporate nonmetricity among the geometrical aspects that drive gravity. A limitation in the early/current research on the $f(\mathcal{Q})$ theory is the most used choice of the so-called coincident gauge. This gauge seems to have less potential but is practical. In fact, the vanishing of the connection coefficients that characterize this gauge shifts the focus to Levi-Civita quantities, thus confining the nonmetricity tensor as an additional field mediating gravity.

Overall, the nonmetricity tensor is a promising geometric quantity for considering alternative theories of gravity due to its versatility in describing phenomena not well explained by GR (e.g., the natural introduction of dark energy). However, nonmetricity introduces changes in parallel transport and conservation of norms that are cornerstones of GR: in nonmetric theories, parallel transport does not preserve the norm of a vector. In literature, for example, it was proposed to change the definition of parallel transport to incorporate the length change [207], thereby deviating from the typical geometric definitions. In this thesis, we chose to use the usual definition of parallel transport, which so far in the literature is the most widely used to study nonmetricity properties and to preserve the normalization of the tangent vectors of the timelike and spacelike congruences (see Chs. 5, 6, 8, and 9). However, this choice imposes constraints on the nonmetricity tensor with a consequent reduction of the additional degrees of freedom that distinguish nonmetric theories from GR [208].

A systematic characterization of transport rules in the presence of a nonmetric connection would be crucial for a more complete understanding of the nonmetricity tensor, as it could also be a promising way to assess nonmetric theories through observational data.

V
Appendices

A

Boundary term in the Einstein-Hilbert action

In Ch. 2, we said that in the variation of the Einstein-Hilbert action, the boundary term is not zero and we need to add an additional term so that they cancel each other out. Therefore, the correct Einstein-Hilbert action is given by

$$A_{E-H} = \frac{1}{16\pi G_N}\int \tilde{R}\sqrt{-g}\mathrm{d}^4x + \int \mathcal{L}_m\sqrt{-g}\mathrm{d}^4x + \frac{1}{8\pi G_N}\oint_{\partial\mathcal{V}} \varepsilon\tilde{K}\sqrt{h}\mathrm{d}^3y, \tag{A.1}$$

where $\partial\mathcal{V}$ is the boundary surface of the integration domain $\mathcal{V}$, with $\delta g_{ab}|_{\partial\mathcal{V}} = 0$, $\tilde{K}$ is the trace of the extrinsic curvature of $\partial\mathcal{V}$, h is the determinant of the induced metric on $\partial\mathcal{V}$, and ε is equal to -1 or 1 depending on if $\partial\mathcal{V}$ is a timelike or spacelike hypersurface, respectively. Both extrinsic curvature and induced metric are defined as in Ch. 6, with u^a substituted by the vector n^a orthogonal to $\partial\mathcal{V}$, $n^a n_a = \varepsilon$.

The boundary term obtained by the variations of the Ricci scalar $\tilde{R}$ with respect to g^{ab} is equal to

$$\partial_c\left(\sqrt{-g}g^{ab}\delta\tilde{\Gamma}_{ba}{}^{c} - \sqrt{-g}g^{cb}\delta\tilde{\Gamma}_{ab}{}^{a}\right). \tag{A.2}$$

Using the Stokes' theorem we have

$$\int \partial_c\left(\sqrt{-g}g^{ab}\delta\tilde{\Gamma}_{ba}{}^{c} - \sqrt{-g}g^{cb}\delta\tilde{\Gamma}_{ab}{}^{a}\right)\mathrm{d}^4x = \oint_{\partial\mathcal{V}} \varepsilon\left(g^{ab}\delta\tilde{\Gamma}_{ba}{}^{c} - g^{cb}\delta\tilde{\Gamma}_{ab}{}^{a}\right)n_c\sqrt{h}\mathrm{d}^3y. \tag{A.3}$$

Then, considering the variation of the Levi-Civita connection,

$$\delta\tilde{\Gamma}_{ab}{}^{c} = \frac{1}{2}g^{cd}\left(\partial_a\delta g_{bd} + \partial_b\delta g_{ad} - \partial_d\delta g_{ab}\right), \tag{A.4}$$

it follows

$$\begin{aligned}\left(g^{ab}\delta\tilde{\Gamma}_{ba}{}^{c} - g^{cb}\delta\tilde{\Gamma}_{ab}{}^{a}\right)n_c =& g^{ab}\left(\partial_a\delta g_{bc} - \partial_c g_{ab}\right)n^c = \\ =& \left(\varepsilon n^a n^b + h^{ab}\right)\left(\partial_a\delta g_{bc} - \partial_c g_{ab}\right)n^c = \\ =& h^{ab}\left(\partial_a\delta g_{bc} - \partial_c g_{ab}\right)n^c.\end{aligned} \tag{A.5}$$

Since δg_{ab} is null on $\partial\mathcal{V}$, its projected derivatives must vanish as well, i.e., $h^{ab}\partial_a \delta g_{bc} = 0$. Hence, we find

$$\begin{aligned}\delta A_{E-H} =& \frac{1}{16\pi G_N}\int G_{ab}\delta g^{ab}\sqrt{-g}\mathrm{d}^4x - \frac{1}{16\pi G_N}\oint_{\partial\mathcal{V}} \varepsilon h^{ab}n^c\partial_c\delta g_{ab}\sqrt{h}\mathrm{d}^3y + \\ &+ \int\left(\mathcal{L}_m\sqrt{-g}\right)\mathrm{d}^4x + \frac{1}{8\pi G_N}\oint_{\partial\mathcal{V}}\varepsilon\tilde{K}\sqrt{h}\mathrm{d}^3y.\end{aligned} \tag{A.6}$$

From the definition (6.31), we know that

$$\tilde{K} = h^{ab}\tilde{\nabla}_a n_b = h^{ab}\left(\partial_a n_b - \tilde{\Gamma}_{ab}{}^c n_c\right), \tag{A.7}$$

so its variation with respect to the metric tensor is

$$\delta\tilde{K} = -h^{ab}\delta\tilde{\Gamma}_{ab}{}^c n_c = \frac{1}{2}h^{ab}n^c\partial_c\delta g_{ab}, \tag{A.8}$$

where we used again that the projected derivatives of δg_{ab} vanish on $\partial\mathcal{V}$. Finally, we obtain

$$\begin{aligned}\delta A_{E-H} =& \frac{1}{16\pi G_N}\int G_{ab}\delta g^{ab}\sqrt{-g}\mathrm{d}^4x - \frac{1}{16\pi G_N}\oint_{\partial\mathcal{V}} \varepsilon h^{ab}n^c\partial_c\delta g_{ab}\sqrt{h}\mathrm{d}^3y + \\ &+ \int\mathcal{L}_m\sqrt{-g}\mathrm{d}^4x + \frac{1}{16\pi G_N}\oint_{\partial\mathcal{V}}\varepsilon h^{ab}n^c\partial_c\delta g_{ab}\sqrt{h}\mathrm{d}^3y = \\ =& \frac{1}{16\pi G_N}\int G_{ab}\delta g^{ab}\sqrt{-g}\mathrm{d}^4x + \int\delta\left(\mathcal{L}_m\sqrt{-g}\right)\mathrm{d}^4x,\end{aligned} \tag{A.9}$$

which provides the Einstein field equations [1].

B
Frobenius' theorem

In deriving homogenous cosmologies (Ch. 4), and in applying both $1+3$ and $1+1+2$ formalism (Chs. 6 and 8), we discussed congruence of timelike curves that are hypersurface orthogonal. Here, we will prove Frobenius' theorem, which provides the constraint for congruence to be orthogonal to hypersurfaces [87, 89, 163]. Since our studies mainly concern torsion-free connection, we prove the theorem by imposing the condition $T_{ab}{}^{c} = 0$.

A congruence is hypersurface orthogonal if the tangent vectors u^a of the curves are proportional to the vector n^a normal to the hypersurfaces. Hence, if the hypersurfaces are described by equations $\Phi(x^a) = c$, with c a constant that labels each hypersurface, then

$$u_a = -\mu\partial_a\Phi, \tag{B.1}$$

where μ is a generic function. Let us consider the completely antisymmetric tensor

$$u_{[c}\nabla_b u_{a]} = \frac{1}{6}\left(u_c\nabla_b u_a + u_b\nabla_a u_c + u_a\nabla_c u_b - u_c\nabla_a u_b - u_b\nabla_c u_a - u_a\nabla_b u_c\right). \tag{B.2}$$

Using Eq. (B.1) and that $(\nabla_a\nabla_b - \nabla_b\nabla_a)\Phi = 0$, we have

$$u_{[c}\nabla_b u_{a]} = 0. \tag{B.3}$$

The inverse proof that from Eq. (B.3) we obtain (B.1) is trivial since we have zero torsion.

In this way, we have proved Frobenius' theorem which states that a congruence of curves is hypersurface orthogonal if and only if $u_{[c}\nabla_b u_{a]} = 0$, with u^a the tangent vectors of the curves.

Now, if we consider Eq. (6.18), the following relation holds,

$$u_{[c}\nabla_b u_{a]} = \frac{1}{3}\left(u_c\omega_{ba} + u_b\omega_{ac} + u_a\omega_{cb}\right). \tag{B.4}$$

Therefore, from Frobenius' theorem, we have that $\omega_{ab} = 0$ is a condition that ensures the congruence to be hypersurface orthogonal if Eq. (B.3) is true.

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