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RENORMALIZATION GROUP FLOWS & ENTROPY IN CURVED SPACETIMES

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Abstract

I discuss the derivation and applications of a non-perturbative Renormalization Group (RG) equation for gauge theories and quantum gravity in Lorentzian spacetimes.

A key ingredient to derive the RG equation in Lorentzian spacetimes is the use of a local regulator, acting as an artificial mass for correlation functions. A local regulator is compatible with the unitarity and Lorentz invariance of the theory, and it gives raise to a local RG equation. A Hadamard-type point-splitting regularisation guarantees that the RG equation is finite. The RG equation depends on the choice of a state, a distinctive feature that is not present in the Euclidean case.

If the effective average action describing the theory does not contain derivatives higher than second order, and it is local in the fields, an application of the renown Nash-Moser theorem proves that the RG equation for scalar fields admits local, exact solutions.

For gauge theories, the symmetries of the theory are controlled by an extended Slavnov-Taylor identity, that can be studied with the cohomology of the Batalin-Vilkovisky operator.

Assuming the Local Potential Approximation in the case of an interacting scalar field, in 3 dimensions the RG flow exhibits the well-known Wilson-Fisher fixed point. In 4 dimensions, the flow has no non-trivial fixed point in the vacuum, but it exhibits a novel non-trivial fixed point in the high-temperature limit of a thermal state for the free theory. Moreover, the Bunch-Davies state in de Sitter spacetime also has a non-trivial fixed point in the inflationary regime.

Finally, the flow is applied to the case of quantum gravity. Taking into account only state- and background-independent terms, the RG flow exhibits a non-trivial fixed point in the Einstein-Hilbert truncation, providing a mechanism for Asymptotic Safety in Lorentzian quantum gravity.

In the second part, I study the relative entropy for a free scalar field propagating over a dynamical, spherically symmetric black hole. Considering the back-reaction of the relative entropy on the black hole horizon, it is possible to write a flux balance equation for the generalised entropy of quantum and gravitational degrees of freedom. Similar considerations lead to a conservation law for the generalised entropy in de Sitter spacetime.

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Contents

1 *Introduction. The Universe in a glass of wine*

A poet once said that every glass of wine contains a Universe [114]; I wonder if she were not actually an expert of renormalization in disguise. In fact, you can learn a great deal on our Universe watching closely the light that reflects on the glass of wine. The scents from the wine are molecules that evaporate from its surface, heated by the Sun; the minerals that make the glass were mined from the sands of the oceans' beaches, but they were originated millions of years ago, in the spectacular explosions of dying stars. And the light, scattered form the glass, interacts with myriads of molecules in the most complicated ways, and after its journey through the density of atoms it give raise to this simple, burgundy translucency that shines in our eyes, exciting our nervous system.

Of the countless interactions among molecules and light, all that remains at the macroscopic level are just a handful of simple effects, that can be described by simple quantities like the temperature of the wine, or the intensity of light's radiation. We cannot comprehend the chaotic behaviour of an astronomical number of microscopic objects; but what we see and taste, what we measure with our macroscopic tools is simple, and tells us almost nothing of this microscopic cosmos: the temperature tells us a little about the molecules' average speeds, the volume about their repulsive interactions. If everything influenced everything else, we would need to discover the origin of the Universe before drinking a glass of wine! Instead, physical phenomena arrange themselves in a hierarchy of layers, roughly separated by their energy scale: from the lowest energies of light travelling through the Universe, to pulleys and heat engines and waterfalls and all the energy exchanges of animals and plants around us, to the dynamics of molecules and atoms and then to particles and even below that, to the highest energy scale that excites the quanta of spacetime itself.

The principal tool to investigate how the macroscopic world emerges from the microscopic one goes by the unfortunate name of *renormalization*. Together with gauge invariance, renormalization is one of the main organising principles behind modern physics.

In the Wilsonian picture of renormalization [249, 250], the basic building blocks are the microscopic degrees of freedom of the system one is interested in, and their symmetries. All possible interactions between the degrees of freedom are then constrained by the symmetries, and organised in inverse powers of some energy scale M, provided by some dimensionful, effective coupling constant. In principle, the degrees of freedom interact through *all* possible terms allowed by the symmetries; however, at a given energy scale k, only those terms whose inverse coupling is of order k contribute, while all the others are suppressed by increasing powers of M. It follows that any given system can be described by an infinite family of effective field theories, with different interactions emerging at different scales. The Wilsonian renormalization starts from a bare, microscopic action and arrives at a coarse-grained description of a system at macroscopic scales. This coarse-graining is described by a family of different effective field theories at different energy scales, and the *Renormalization Group* (RG) governs the flow of all possible theories under changes in the energy scale k.

The combination of the RG flow with gauge symmetry produced one of the most fruitful programs in modern physics, the renormalizability of the Standard Model of particle physics [1]. By now, the Standard Model shows some of the most accurate agreements of theory with experiments [145].

If possible, gauge symmetry is an even stranger foundation for modern physics than renormalization. More than describing a symmetry of Nature, gauge theory is a redundancy in our description of Nature. Yet, this redundancy is at the heart of Yang-Mills theories, describing the electromagnetic, weak, and strong forces forming our core understanding of fundamental interactions. In retrospect, it is also possible to interpret the invariance of General Relativity under diffeomorphisms as a gauge symmetry of gravity; so it seems that all the fundamental forces that we know of are invariant under some gauge symmetry.

The Functional Renormalization Group (fRG) represents one of the modern implementations of the Wilsonian RG [41, 102, 132, 201, 223]. In this approach, the insertion in the microscopic action of a quadratic regulator term, depending on an external mass scale k, provides the coarse-graining of correlation functions. In fact, the regulator term acts as an artificial mass term, effectively suppressing long-range modes. In the most studied Euclidean setting, the regulator usually depends on momentum, and it behaves as a momentum dependent mass, suppressing modes with $p \leq k$ and vanishing for higher modes. In general, however, even a simple momentum-independent mass term acts as an infra-red cut-off.

The main object of study in the fRG is the *effective average action*, defined as a shifted Legendre transform of the generating functional of connected Green's functions at scale k. In flat space, this is the generating functional for the 1PI Feynman diagrams [243]. In the limits $k \to \infty$ and $k \to 0$ the effective average action interpolates between the classical action and the full quantum action, and its flow under the scale k thus defines a Wilsonian RG flow. The effective average action acts as a microscope with variable resolution, which permits to move from the fine-grained, microscopic description to the rough, macroscopic view. The equation governing this flow is the RG flow equation. In the case of momentum-dependent regulator, this equation is usually called *Wetterich equation* or *Wetterich-Morris* equation [191, 221, 245, 247], and it has been developed from earlier ideas of Polchinski [208]. In the special case of a momentum-independent regulator, the Wetterich equation is also known as a *functional Callan-Symanzik equation* [5]. The Wetterich equation has been particularly useful as it is amenable to numerical manipulations. Moreover, contrary to the ill-defined path integral from which it is formally derived, the Wetterich equation is a well-defined functional differential equation.

The fRG has been successfully applied to many different physical situations, from condensed matter systems to high-energy physics, most notably QCD (see e.g. the reviews [41, 102] and references therein). In particular, since its first developments the Wetterich equation has been extensively used to study gauge theories [105, 178, 201, 217]. The extension of the Wetterich equation to gauge theories provided the principal tool of investigation of the *Asymptotic Safety* scenario in quantum gravity [103, 152, 202, 204, 205, 214, 219, 220, 226]. In fact, thanks to the structure of the Wetterich equation, the fRG admits non-perturbative approximation schemes that can go beyond usual perturbation theory, allowing the study of strongly coupled systems and perturbatively non-renormalizable theories such as quantum gravity.

Despite its successes, the fRG still lacks a mathematically rigorous formulation, in particular in Lorentzian spacetimes. In fact, most results within the fRG approach have been derived in Euclidean spaces. Moreover, the fRG usually starts from a formal path-integral representation of the regularised generating functional $Z_k(j)$; a finite solution of the Wetterich equation in the limit $k \to 0$ determines *a posteriori* a good definition of the path integral. Investigations on Lorentzian signature fRG flows, based on analytic continuation of Euclidean correlation functions, has been initiated in [115]. A different approach, based on real-time Schwinger-Keldysh formalism and the spectral representation of correlation functions in Minkowski spacetime, has been developed [40, 163, 164, 167, 170, 203] and is currently under investigation. Finally, the Asymptotic Safety scenario in Lorentzian quantum gravity has been studied assuming a 3+1 decomposition of the metric [184].

There are many advantages in working with Euclidean spaces. The main difference from the Lorentzian case is that the equations of motion in Euclidean spaces are elliptic, instead of hyperbolic: this implies that there is a unique Euclidean-invariant vacuum state, and thus a unique preferred inverse for the *quantum wave operator* $(\Gamma_k + Q_k)^{(2)}$, which is a fundamental ingredient of the Wetterich equation. In Euclidean space, the addition of a positive contribution $Q_k^{(2)}$ means that the quantum wave operator is non-vanishing, and thus always invertible. This regulates the UV divergences of the propagator in the Wetterich equation. Thanks to the regulator term, the Wetterich equation is then both ultraviolet (UV) and infrared (IR) finite. Finally, Euclidean signature allows for a representation of the singularity structure of th interacting propagator using heat kernel techniques, providing an algorithmic evaluation of the operators in the right-hand side (r.h.s.) of the Wetterich equation particularly efficient for numerical implementations.

The picture changes drastically when one switches to Lorentzian signature. First of all, due to the minus sign in the time-like direction of the momentum covector, a regulator term cannot completely eliminate UV divergences. Moreover, a momentum-dependent regulator can change the principal symbol of the equations of motion determining the interacting propagator: if the regularised equations of motion are not normally hyperbolic, a fundamental solution might even not exist.

Perhaps the most important conceptual difference from the Euclidean case is the fact that the quantum equations of motion, determining the interacting propagator, are hyperbolic, not elliptic. This implies that there is no distinguished fundamental solution, and different interacting propagators differ by the choice of an arbitrary smooth function. Physically, this corresponds to the arbitrariness in the choice of a state. Since the RG flow equations depend on the interacting propagator, in Lorentzian spacetimes the flow acquires a non-trivial state dependence.

Lorentzian RG flow equations

The derivation and applications of a mathematically rigorous RG flow for Lorentzian gauge theories is the main topic of this thesis. The Lorentzian RG flow equation is based on the functional Renormalization Group and the perturbative Algebraic approach to Quantum Field Theory on curved spacetimes (pAQFT) [56, 112, 118, 158, 160, 162, 211] in the functional approach [57]. Even though the approach is based on a representation of interacting observables in perturbative formal power series, the C^*−algebraic approach to AQFT [58–60, 68, 69] can provide a basis to formulate the results in an exact, non-perturbative setting. The approach is fully Lorentzian, and allows for a generalization of the Wetterich equation to generic Hadamard states and curved backgrounds, where there is no distinguished vacuum state. These Lorentzian RG flow equations exhibit a state dependent flow, and are based on a Hadamard regularisation.

In pAQFT, a quantum field theory model on a globally hyperbolic spacetime $\mathcal{M}$ is given as a net of formal power series (in the coupling constant λ, and potentially also in the reduced Planck constant $\hbar$) with coefficients in topological $*$−algebras assigned to relatively compact regions $\mathcal{O} \subset \mathcal{M}$. The net satisfies the axiom of *causality* ($[\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_2)] = 0$ if $\mathcal{O}_2$ is not causally connected to $\mathcal{O}_1$), and typically *isotony* (i.e., the algebra $\mathscr{A}(\mathcal{O}_1)$ associated to a subregion $\mathcal{O}_1 \subset \mathcal{O}$ of a larger region $\mathcal{O}$ is contained in the algebra of the larger region, $\mathscr{A}_1(\mathcal{O}_1) \subset \mathscr{A}(\mathcal{O})$) and the *time-slice axiom* (quantum version of the well-posedness of the Cauchy problem). Using the framework of locally covariant quantum field theory and the language of category theory, pAQFT can be formulated "on all spacetimes at once", complying with the principle of general covariance of General Relativity. More specifically, pAQFT is a functor from the category of globally hyperbolic spacetimes to the category of $*$−algebras satisfying Einstein causality [38, 66].

In the case of gauge theories, these axioms have to be slightly weakened, as algebras are replaced by differential graded algebras [35, 37, 39, 139]. The generalisation of pAQFT to gauge theories includes the BRST formalism for Yang-Mills theory [156] and the Batalin-Vilkovisky formalism for general gauge theories and effective quantum gravity [64, 65, 116, 119].

In pAQFT, interacting observables are represented as formal power series in a $*$−algebra. The construction uses the *Epstein-Glaser renormalization procedure*, which works in curved spacetimes and for perturbatively non-renormalizable theories, such as quantum gravity. Thanks to the Epstein-Glaser renormalization, every element of the theory is by construction ultraviolet regular.

After the construction of the interacting algebra, states are defined as normalised, positive, linear functionals, mapping an element of the algebra of observables to its expectation value $\omega : \mathscr{A} \longrightarrow \mathbb{C}$. Among all possible states, the *Hadamard condition* selects the state that have finite renormalized stress-energy tensor [171]. Since in general curved spacetimes there is no distinguished vacuum, the separation between states and observables allows to study the algebra of observables without referring to a particular state.

Given an algebra and a state, the Gelfand-Naimark-Segal (GNS) construction provides a representation of observables as operators on a Hilbert space. However, the representation depends on the state, and different representations are not unitarily equivalent. Therefore, pAQFT is a generalisation of flat space QFT, and it is well-suited to handle generic states on possibly curved Lorentzian spacetimes. Thanks to its generality and flexibility, pAQFT is a good candidate to overcome some of the difficulties of the standard fRG approach.

In the formal $*$−algebra of interacting observables, it is now possible to introduce the generalisation of generating functionals to curved spacetimes and of their regularised counterparts, and in particular to define an effective average action.

As we said, the construction of the effective average action requires the insertion of a regulator term in the generating functional for time-ordered correlation functions. However, thanks to the Epstein-Glaser renormalization procedure, the regulator needs not to regularise UV divergences, as these are already renormalized. Moreover, the Epstein-Glaser procedure only works for local functionals. Therefore, the regulator added to the generating functionals is local in position, preserving Lorentz symmetry. A local regulator Q_k acts as an artificial mass contribution to the field, to tame infrared problems.

Local regulators have been already used in the literature, in particular in the study of field theories at finite temperature, where thermal effects provide a natural UV cut-off [178, 179]. This regulator term has been called *Callan-Symanzik-type cut-off*, and the respective RG flow equation *functional Callan-Symanzik equations* [5]. A Callan-Symanzik regulator has also been used to study the flow of the spectral function of the graviton propagator in a Minkowski, flat background, one of the first applications of the fRG to Lorentzian quantum gravity [108].

The RG flow equation in Lorentzian spacetimes and for Hadamard states then is (5.6)

$$\partial_k \Gamma_k = \lim_{y \to x} \frac{\hbar}{2} \int_x \mathrm{Tr}\left\{ \partial_k q_k(x) \left[\left(\Gamma_k^{(2)} - q_k \right)^{-1} (x, y) - \tilde{H}_F(x, y) \right] \right\} .$$

The trace is over Lorentz as well as internal (such as colour) and field indices.

The presence of a local regulator leads to two main differences in the RG flow equations; first of all, the RG equation itself is local. The second, related difference is that the contribution of $(\Gamma_k^{(2)} + Q_k^{(2)})^{-1}(x, y)$ needs to be evaluated at coinciding points. This second modification would introduce UV divergences in standard treatments of flat space QFT; however, since the local fields in Q_k are normal-ordered, the coinciding point limit is finite without requiring extra regularisations. In fact, on the right-hand side $\tilde{H}_F(x, y)$ is a counter-term related to a Hadamard parametrix, and can be obtained just from the background geometry and the free (linearised) equation of motion of the theory [56, 66, 158]. Its subtraction is related to the known point-splitting regularisation, that gives the expectation values of Wick powers in curved backgrounds. A Hadamard parametrix constructed with local properties of the metric only, and not the subtraction of the two-point function of a state, keeps the theory covariant [66, 158].

The fundamental difference between the Euclidean and Lorentzian case is the state dependence of the RG equation, through the interacting propagator $\left(\Gamma_k^{(2)} - q_k \right)^{-1}$. State dependence and the necessity of the Hadamard condition on the background states appears also if one uses a regulator term that depends on the spatial momenta, which is particularly suited in cosmological models [16]. The RG equation provides a mathematically rigorous foundation to explore the RG in Lorentzian spacetimes, with non-perturbative approximations. In fact, assuming that Γ_k is at most quadratic in derivatives, and local in the fields, it is possible to prove that the RG equation admits local solutions, without truncation in field space. The result is based on an application in this novel framework of the Nash-Moser theorem.

Gauge theories

The generalisation of the RG flow equations to gauge theories requires a careful treatment of the quantum realisation of gauge symmetries along the flow.

In QFT, the gauge symmetry of the classical action gives raise, after gauge fixing, to the Becchi-Rouet-Stora-Tyutin (BRST) symmetry. The invariance of scattering amplitudes under BRST symmetry, captured by the local cohomology of the BRST operator, is the fundamental requirement to obtain gauge-invariant, physical observables from a gauge-fixed action.

In pAQFT, it is possible to give a rigorous description of gauge theories as a generalisation of the Batalin-Vilkovisky (BV) formalism to infinite dimensional configuration spaces [116, 119, 212, 213]. The BV formalism [26–28] is a powerful method to quantise theories with local symmetries. It is a generalisation of the BRST method [29–31], first developed in the context of perturbative Yang-Mills theories, to arbitrary Lagrangian gauge theories. The space of on-shell, gauge invariant observables is constructed as the 0-th order cohomology of the nilpotent BV operator s [22], and the gauge-independence of observables is guaranteed by the *Quantum Master Equation* (QME).

In turn, the QME imposes the *Slavnov-Taylor identities* on the effective average action, controlling its symmetries.

The main novelty in our treatment of gauge theories is the introduction of a new field that, together with the integral kernel of the cut-off regulator, forms a contractible pair in the cohomology of the BV operator. Thanks to this field, the symmetry constraint on the effective average action takes the same form as in the non-regularised case, of extended Slavnov-Taylor identity in Zinn-Justin form [256]: in terms of $\tilde{\Gamma}_k = \Gamma_k + Q_k(\phi)$, it reads

$$\int_x \left[\frac{\delta \tilde{\Gamma}_k}{\delta \phi(x)} \frac{\delta \tilde{\Gamma}_k}{\delta \sigma(x)} + q_k(x) \frac{\delta \tilde{\Gamma}_k}{\delta \eta(x)} \right] = 0 \,.$$

The symmetry constraint on the effective average action, which we call *effective master equation* (7.35), can be interpreted in terms of an *effective* BV formalism, in the space of effective fields ϕ and effective BRST sources σ. The construction of admissible average effective actions Γ_k then is interpreted as a cohomological problem, as in the non-regularised case [22].

In more practical terms, this means that the effective master equation (7.21) can be solved by cohomological means, providing the starting point for the Ansatz to be used in the solution of the flow equation.

Black holes

Even though the renormalization of gauge theories has little to do with them, everything started from black holes. Black holes are the best laboratories to test our theories on the Universe; thanks to their extreme conditions, we can hope to learn something on the breaking points of General Relativity and Quantum Field Theory, and their partial combination, QFT on curved spacetimes, by looking in some of their mysteries. Perhaps the most famous puzzle of theoretical physics of the last half century is the microscopic origin of black hole entropy.

In 1975, Hawking discovered that, if we perturb their event horizon with quantum effects, black holes are not entirely black, but emit a radiation of particles with a black body spectrum. This in turn gave firm ground on the hypothesis that black holes carry an entropy, proportional to their area [32]. In classical General Relativity, black holes are described only by a handful of parameters: their mass, their angular momentum, and their electric charge. However, their huge entropy reveals that, far from being the stable, simple objects we thought, black holes hide an enormous amount of microstates. In this respect, black holes are similar to the glass of wine that started this story, as they appears simple at our ordinary scales, but reveal a complex dynamics of many microscopic degrees of freedom if we look more closely. What are black holes made of? We can hope that the investigation of the renormalization group for quantum gravity will provide us at least some hints on this fundamental question.

The RG flow for quantum gravity is one of the main motivations to study the renormalization of Lorentzian gauge theories. However, even simple semi-classical arguments (the study of the propagation of quantum matter on a classical background, and the reaction of the background to quantum effects) can, as Hawking showed, provide many useful insights for our understanding of the fundamental theory of quantum gravity. In the last chapter, we replicate the gedanken experiments of Bekenstein and Hawking, of throwing a small quantum perturbation into a black hole, and analyse how the black hole reacts. In particular, we study how the black hole reacts to changes in entropy of the matter fields on its background. Since the quantum fields are dynamically evolving, we study the general case of a dynamical black hole, but preserving the important technical assumption of spherical symmetry. This allows us to show that, under changes in entropy of the matter fields, the black hole reacts with a change of one quarter of its dynamical horizon area. We are thus able to prove an entropy-area law, in semi-classical gravity, for dynamical black holes.

The back-reaction problem of relative entropy on horizons can be then generalised to a variety of contexts; in particular, using similar techniques it is possible to show that the back-reaction of relative entropy equals *minus* one-quarter the area of the cosmological horizon in de Sitter space.

Summary and results

The thesis is organised as follows. In the first two chapters, we review the theoretical foundations of the algebraic approach for gauge theories in the BV formalism, in globally hyperbolic spacetimes. In Chapter 2, we review the construction of the interacting algebra of observables, of Hadamard states, and the GNS representation of the algebra for scalar fields. We also discuss in some detail the connection of this formalism with the standard path-integral representation of flat space QFT. In Chapter 3, we review the classical and quantum theory of gauge theories in the BV formalism, their renormalization as effective field theories with the Epstein-Glaser renormalization, and the derivation of the Quantum Master Equation as a sufficient condition to have gauge-invariant, on-shell interacting observables.

We start the discussion of new results in Chapter 4, where we show how to construct the generating functionals for correlation functions in curved spacetimes and in general Hadamard states, providing the relevant definitions in the pAQFT formalism. We also discuss the introduction of a local regulator in the generating functionals, how it effectively acts as a mass parameter for correlation functions, and we prove many useful properties of the effective average action, that generalise known results in Euclidean settings to curved Lorentzian spacetimes.

In Chapter 5, we derive the RG flow equations for gauge theories in globally hyperbolic spacetimes and in generic Hadamard states. We show how to write the RG flow equations in order to have control on the state dependence of the flow, their connection with the standard form of the Wetterich equation, and we write them as partial differential equations for the effective average action, with initial data given by the background manifold, a Hadamard reference state for the free theory, and an initial data provided by a microscopic action in the infinite scale limit.

In Chapter 6, we show that if the effective average action does not contain derivatives higher than second order, and if it is a local functional of the fields, the RG flow equations admit local solutions. The main idea is to use the Nash-Moser theorem for the existence of an inverse of tame operators in tame Fréchet spaces.

In Chapter 7, we discuss a novel mechanism to derive Slavnov-Taylor identities for the effective average action, in Zinn-Justin form. The main idea is to introduce an additional auxiliary field, coupled with the BRST variation of the regulator term, and a scale-dependent BV differential. The original BV invariance of the classical action is generalised to a larger, scale-dependent symmetry for an extended action. At the quantum level, this larger symmetry gives the extended Slavnov-Taylor identities for the effective average action. Thanks to their linear structure, the extended Slavnov-Taylor identities can be solved in cohomology, providing an exact, non-perturbative constraint on the functional form of the effective average action. We then show that the symmetry is compatible with the flow, in the sense that, once it is satisfied at some scale $k = \bar{k}$, it is satisfied at all scales.

In Chapter 8, we discuss some applications of the general formalism. We start with an approximation scheme for the effective average action, called *Local Potential Approximation* for its similarity with the approximation of the same name that is widely studied in the Euclidean literature. We then recover known results on the fixed-point structure of the RG flow for scalar fields in Minkowski vacuum, in 3 and 4 dimensions. We then show that a thermal state introduces a new, non-trivial fixed point in the flow of the scalar field in 4-dimensional Minkowski space, demonstrating the relevance of the reference state in the RG flow. We then compute the RG flow for the scalar field, in the Bunch-Davies vacuum in de Sitter, showing that the general formalism can be successfully applied to curved spacetimes. We again find a non-trivial fixed in the flow.

As a last application, we study the flow of quantum gravity in the Einstein-Hilbert truncation. Instead of choosing a background and a state for the graviton, we are able to isolate background and state independent terms in the flow. Considering only these terms, the flow exhibit a non-trivial fixed point, supporting the evidence that quantum gravity is non-perturbatively renormalizable also in Lorentzian spacetimes.

Finally, in Chapter 9, we discuss the effects of relative entropy on the dynamics of dynamical black holes. In this case, we consider the simple case of a free, massless scalar field. However, even this simple model provides rich insights in the thermodynamics of black holes, and we prove an entropy-area law for the horizon of a dynamical black hole with spherical symmetry. Finally, we show that a similar result holds in the case of cosmological horizons of de Sitter space.

The main results are:

- ◇ The derivation of RG flow equations for gauge theories in curved spacetimes and Hadamard states, given in Eq. (5.6);
- ◇ The derivation of extended Slavnov-Taylor identities in Zinn-Justin form, that follow from the Quantum Master Equation, given in Eq. (7.35);
- ◇ The constructive proof of existence of local solutions of the RG flow equations, based on the Nash-Moser theorem and on the assumption that the ef-

fective potential does not contain derivatives of the Dirac delta, summarised in Theorem (6.14);

◇ The application of the RG flow to quantum gravity in the Einstein-Hilbert truncation, exhibiting a non-trivial fixed point, discussed in Section 8.4;

◇ The proof of the entropy-area law for dynamical, spherically symmetric black holes, computed from the modular theory for a scalar field, given in Eq. (9.65).

◇ A similar proof for an entropy-area law for the cosmological horizons in de Sitter spacetime, given in Eq. (9.84).

The main results presented in this thesis appeared in previous publications [80, 81, 83–86]. The section on the perturbative agreement for gauge theories, Section 3.5, and the study of the Wilson-Fisher fixed point, Section 8.2.3, did not appear elsewhere.

2 *Algebraic approach to QFT*

2.1 THE NECESSITY OF THE ALGEBRAIC APPROACH

The existence of particles is one of the fundamental predictions of Quantum Field Theory (QFT) in flat, Minkowski spacetime. QFT originated from the necessity of accommodating the principles of Special Relativity into Quantum Mechanics, which in essentially a unique way lead to QFT as we know it today [243]. In particular, the representations of the Poincaré group, which is the group of symmetries of Minkowski spacetime, plays a fundamental role in determining the particle content of QFT. As a matter of fact, at some point any theoretical physicist learns that particles *are* certain irreducible representations of the Poincaré group, according to Wigner's classification [248].

However, in order to generalise QFT to curved, dynamical spacetimes, we need to generalise the usual construction of flat space QFT to situations in which the background geometry is curved under the influence of gravity. On a practical level, such a generalisation is of crucial importance to understand those extreme situations in which quantum effects and the spacetime curvature are both important. Conceptually, the principles of General Relativity, and in particular general covariance, imply that the spacetime is a dynamical actor, that cannot be chosen a priori, but must be determined from the distribution of matter via the Einstein field equations. In order to incorporate general covariance in our theories, we need to study QFT in general curved spacetimes.

Moving away from flat space, Poincaré invariance is perhaps the most important concept that needs to be abandoned. The absence of Poincaré invariance implies that the particle concept loses its fundamental status in curved space: in fact, in general, different observers can detect different particle contents. The Unruh effect [238] and Hawking radiation [147] can both be interpreted as instances of this general phenomenon.

The ambiguity in the notion of particles is reflected in the mathematical structure of QFT in curved spacetimes. In QFT, observables form an algebra with an infinite numbers of generators, the smeared fields $\varphi(f)$ labelled by test functions, satisfying canonical commutation relations. As opposed to the quantum-mechanical case, in QFT there is an infinite number of unitarily inequivalent Hilbert space representations of the commutation relations, since in this infinite-dimensional setting the Stone-von Neumann theorem does not apply [190]. In flat space, Poincaré invariance picks a preferred vacuum state in a Hilbert space, selecting a preferred representation; particles are then states obtained exciting the Poincaré invariant vacuum. Even though not all states of physical interest are in this Hilbert space, such as thermal states, the vacuum representation of QFT forms the basis to compute scattering amplitudes, and gives predictions that are in astonishing agreement with experiment [145].

In the absence of Poincaré invariance, there is, in general, no preferred vacuum state; thus, there is no preferred Hilbert space representation of the quantum algebra of observables, and no preferred notion of particle. Therefore, instead of the Hilbert space of particle states, it is natural to take the algebra itself as the fundamental object; different algebra representations will describe different physical situations.

The *algebraic* approach takes as fundamental input the algebra of observables, generated by the smeared fields $\varphi(f)$, subject to commutation relations. It is then possible to discuss general properties of the QFT at hand from its algebra, without referring to a particular state. The choice of the state is introduced at a second stage, and it depends on the physical model under consideration: it is possible, for example, to evaluate the expectation value of some observable in a vacuum state or in a thermal state at finite temperature; the values will in general disagree, since they correspond to different physical situations, but the observable itself remains well-defined as an element of the underlying algebra.

None of these considerations are new. The foundations of the algebraic approach lie in Haag-Kastler axioms, first formulated in 1964 [142], that describe a free QFT in flat spacetime as the assignment of a von Neumann algebra to each open region in a spacetime [141]. The axioms were then modified to the assignment of algebras to any open region in a globally hyperbolic spacetime [61, 92]. More precisely, in order to have a QFT compatible not only with local covariance, but with the general covariance of General Relativity, a QFT must be defined on "all spacetimes at once": the mathematical formulation is that of a functor from the category of globally hyperbolic spacetimes to C^*-algebras, preserving isotony under inclusions of open regions and satisfying Einstein causality [66].

The inclusion of interactions in the algebraic approach is based on the observation [169] that renormalization theory, in the form of Epstein and Glaser renormalization procedure [106], produces local nets of algebras of interacting observables. However, these algebras are not C^*-algebras but only formal power series in a perturbative parameter λ, with coefficients in an involutive $*-$algebra. The Haag-Kastler axioms must then be relaxed to the assignment of a $*-$algebra to open spacetime regions.

Therefore, it is possible to formulate a QFT as a generally covariant theory, assigning a formal power series in some $*-$algebra to any globally hyperbolic spacetime. The result is a representation of interacting observables in the free algebra as formal power series. Causal perturbation theory allows to compute any interacting observable as quantum corrections to its free counterpart, at the desired order in the perturbative parameter.

Interacting QFTs in the algebraic approach were developed in detail around the early 2000s [61, 97], when the characterisation of the Hadamard condition from microlocal analysis by Radzikowski gave the final input for the construction of interacting observables [209, 210]. Thanks to the microlocal approach, it was possible to prove the existence of Wick polynomials in general curved spacetimes, study the renormalization group flow, and classify possible renormalization ambiguities [158–161]. Finally, based on causal perturbation theory, it was possible to prove the renormalization of quantum Yang-Mills theories [156] and of general gauge theories in the Batalin-Vilkovisky formalism [116, 119]. It is important to stress that the

Epstein-Glaser renormalization procedure works perturbatively, but it does not assume power-counting renormalizability; as such, it can be applied to any theory "renormalizable in the modern sense" [135], including gravity.

In this Chapter, we introduce the fundamental concepts and techniques of the algebraic approach to QFT on curved spacetimes for scalar field theories, while in the next one we discuss the BV formalism for gauge theories. The presentation follows the functional approach to algebraic QFT [56, 57]. We provide here a short summary of the main steps, postponing the definitions and details in the next sections. In this approach, classical observables are described by certain complex-valued functionals $F \in \mathcal{F}_{\mu c}$ over off-shell field configurations φ, which are sections of some vector bundle over the background manifold. The closure under pointwise product defines the classical algebra of observables; a typical example is the scalar field, which is a section of smooth functions on the manifold. Using the free equations of motion, and the requirement that the algebra should include non-linear observables, the commutative product is deformed in a non-commutative one, defining the quantum algebra of free observables. The Feynman propagator for the free theory provides a time-ordered product for *regular* functionals; time-ordered products between local functionals are defined axiomatically, and explicitly constructed extending the time-ordered products for regular functionals with the Epstein-Glaser renormalization procedure. From the time-ordered products for local observables it is possible to construct an algebraic $S-$matrix and the Bogoliubov map, which represents interacting observables as a formal power series in the free algebra. Any observable can then be evaluated in some Hadamard state, that in the algebraic approach is a functional from the algebra to complex numbers. The perturbative expansion of the state-evaluated Bogoliubov map of some observable corresponds to an expansion in Feynman diagrams.

2.2 CLASSICAL FIELD THEORY

We start introducing the kinematical data and the classical algebra of observables, with a particular focus on its properties underlying deformation quantization. The whole framework is based on the functional approach to pAQFT, which has been first introduced in [57].

In order to emphasise general covariance, it is possible to formulate pAQFT in the language of category theory, as a functor assigning an algebra to the category of globally hyperbolic spacetimes [66, 116, 119, 212]. Then, the construction is manifestly independent on the geometric background. Here, for concreteness, we fix an arbitrary, Hausdorff, second-countable, paracompact, orientable, smooth manifold $\mathcal{M}$ [148], with Lorentzian metric g with signature $(-,+,\ldots,+)$, and we call it a *spacetime* $(\mathcal{M},g)$. We further assume that the spacetime is time-orientable and globally hyperbolic, which means that there is no closed causal curve, and given any two points $p,q \in \mathcal{M}$, the intersection $J^+(p) \cap J^-(q)$ of the causal future $J^+(p)$ of p with the causal past $J^-(q)$ of q is compact.

The physical content of the theory is specified by a field configuration space $\mathcal{E}(\mathcal{M}) \ni \varphi$, assumed to be the space of real-valued smooth sections φ of some natural vector bundle with fibre V over $\mathcal{M}$, $\mathcal{E}(\mathcal{M}) := \Gamma(\mathcal{M},V)$. Typical examples of physical interest are scalar field theories, that have as configuration space real-valued smooth functions $C^\infty(\mathcal{M},\mathbb{R})$; Yang-Mills-type theories, with configuration space real-valued sections of the trivial principal bundle $\mathcal{E} = \Omega_1(\mathcal{M},\mathfrak{g})$, with $\mathfrak{g}$ some Lie algebra of some compact Lie group; and perturbative gravity, with configuration space consisting of sections of the space of symmetric bitensors $\mathscr{E} = \Gamma(T^*(\mathcal{M})^{\otimes 2})$. Generally speaking, the configuration space is modelled as some locally convex topological vector space.

Classical observables are smooth maps (in the sense of Bastiani [25]) from the configuration space to complex numbers, i.e. functionals $F : \mathscr{E}(\mathcal{M}) \to \mathbb{C}$. Notice that observables are defined as functionals of *off-shell* field configurations, that do not satisfy any equation of motion. The condition of being *on-shell* will be implemented at the level of states.

To define a class of physically interesting observables, these functionals need to satisfy a set of conditions. First, they must be smooth with respect to functional derivatives, defined as the Fréchet derivative.

Definition 2.1 (Fréchet derivative)**.** Let $\mathscr{E}(\mathcal{M})$ be the topological vector space of field configurations, and let $F : \mathcal{U} \subset \mathscr{E}(\mathcal{M}) \to \mathbb{C}$ be a classical observable. If it exists, the *Fréchet derivative* of F at the point $\varphi \in \mathcal{U}$ in the direction ψ is defined as the distribution $DF \in \mathscr{E}'$

$$DF(\varphi)(\psi) := \lim_{t\to 0} \frac{1}{t}(F(\varphi + t\psi) - F(\varphi)) \, .$$

$\mathscr{E}'(\mathcal{M})$ denotes the space of compactly supported distributions [166].

A functional is called *differentiable* in φ if $DF(\varphi)(\psi)$ exists for all ψ. Higher-order derivatives are defined by

$$D^n F(\varphi)(\psi_1, \dots, \psi_n) := \lim_{t\to 0} \frac{1}{t} \left[D^{n-1}F(\varphi + t\psi_n)(\psi_1, \dots, \psi_{n-1}) - D^{n-1}F(\varphi)(\psi_1, \dots, \psi_{n-1}) \right] \, .$$

Denoting the standard pairing on distributions $\langle \cdot, \cdot \rangle$, the Fréchet derivative is given by $D^n F(\varphi)(\psi_1, \dots, \psi_n) = \langle F^{(n)}(\varphi), \bigotimes_i^n \psi_i \rangle$.

Definition 2.2 (Smooth functional)**.** Let $\mathscr{E}(\mathcal{M})$ be the topological vector space of field configurations, and let $F : \mathcal{U} \subset \mathscr{E}(\mathcal{M}) \to \mathbb{C}$ be a classical observable. For every n, the n-th functional derivative of any *smooth functional* F must be a well defined, compactly supported, symmetric distribution $F^{(n)} \in \mathscr{E}'(\mathcal{M})$.

In the following, for notational simplicity we will often identify the distribution with its integral kernel, simply writing $F^{(n)}(x_1, \dots, x_n) := \frac{\delta^n F(\varphi)}{\delta\varphi(x_1)\dots\delta\varphi(x_n)}$. Moreover, we will write the action of linear operators as the directional derivative as $\langle F^{(1)}(\varphi), \psi \rangle = F^{(1)}(\varphi)\psi$, omitting the pairing.

To implement locality, we require that the functionals have compact support, where the support of a functional is defined as the support of its first derivative.

Definition 2.3 (Support of a functional)**.** Let $F : \mathscr{E}(\mathcal{M}) \to \mathbb{C}$ be a smooth functional on the configuration space $\mathscr{E}(\mathcal{M})$. The *support* of the functional F is defined as the set $\operatorname{supp} F \subseteq \mathcal{M}$ satisfying

$$\begin{aligned} \operatorname{supp} F := \{x \in \mathscr{M} \mid \forall \text{ neighbourhoods } U \text{ of } x \; \exists \, \varphi, \psi \in \mathscr{E}(\mathcal{M}), \\ \operatorname{supp} \psi \subseteq U : F(\varphi + \psi) \neq F(\varphi)\} \, . \end{aligned}$$

Finally, the set of observables of physical interest is that of *microcausal functionals*, defined by a certain condition on the wavefront set of their functional derivatives.

The wavefront set of a distribution encodes both its singular points and its singular directions. The basic intuition is that the more a distribution is regular, the faster it decays in Fourier space. The wavefront set thus encodes the directions in which a distribution does *not* decay fast.

Definition 2.4 (Wavefront set [166])**.** The *wavefront set* WF of a distribution $\varphi \in \mathscr{D}'(\mathcal{M})$ is defined as the complement of the set of all points $(x_0,\, p_0) \in T^*(\mathcal{M}),\ p_0 \neq 0$ in which φ decays sufficiently rapidly: that is, for any $n \in \mathbb{N}$ there exist a smooth function $f \in C_c^\infty(\mathcal{M})$ with $f(x_0) = 1$, an open conic neighbourhood C of p_0, and a constant c_n such that

$$\hat{\varphi}(f)(p) \leq c_n(1 + |p|)^{-n}\ \forall p \in C\,,$$

where the hat denotes the Fourier transform, $\hat{\varphi}(f)(p) = \varphi(e^{ip\cdot} f)$.

A conical neighbourhood of a point $p_0 \in \mathbb{R}^n \setminus \{0\}$ is a set $C \subset \mathbb{R}^n$ such that C contains the ball $B(p_0,\epsilon) = \{p \in \mathbb{R}^n || p_0 - p| < 0\}$ for some $\epsilon > 0$, and $\forall p \in C, \forall \alpha > 0\ \alpha p \in C$.

The set of microcausal functionals are now identified by the following condition on the wavefront set of their derivatives.

Definition 2.5 (Microcausal functionals)**.** The space of *microcausal functionals* $\mathscr{F}_{\mu c}$ is the subset of the space of smooth functionals with compact support F, satisfying

$$\mathrm{WF}(F^{(n)}) \cap (\overline{V}^n_+ \cup \overline{V}^n_-) = \emptyset\,.$$

$V_{+(-)}$ denotes the subset of the cotangent space with elements (x, p) such that p is contained in the future (past) light-cone of x, and $\overline{V}_{+(-)}$ denotes its closure. Moreover, a microcausal functional $F \in \mathscr{F}_{\mu c}$ has only a finite number of non-vanishing derivatives, $F^{(n+m)} = 0\ \forall \varphi \in \mathscr{E}(\mathcal{M}),\ \forall m > 0$, for some order $n \in \mathbb{N}^+$.

The vector space $\mathscr{F}_{\mu c}$ is equipped with a weak topology induced by the natural topologies of distributions. In fact, $A_l \in \mathscr{F}_{\mu c}$ converges to $A \in \mathscr{F}_{\mu c}$ for $l \to \infty$ if for every n and for every field configuration $\varphi \in \mathscr{E}(\mathcal{M})$, $A_l^{(n)}(\varphi)$ converges to $A^{(n)}(\varphi)$ in $\mathscr{E}'(\mathcal{M}^n)$ [211].

There are two important subsets of the space of microcausal functionals. The first is the set of *regular functionals* $\mathscr{F}_{\mathrm{reg}}$, satisfying $\mathrm{WF}(F^{(n)}) = \emptyset\ \forall n$; the prototype is the smeared field

$$\Phi(f) := \int_{x\in\mathcal{M}} f(x)\varphi(x)\mathrm{d}\mu(x)\,, \tag{2.1}$$

The notation for the integral is $\int_{x\in\mathcal{O}} \mathrm{d}\mu(x)$, with $\mu(x)$ the measure induced by the spacetime metric g on $\mathcal{M}$, and $\mathcal{O} \subset \mathcal{M}$ the region of integration. Whenever we integrate over the whole spacetime, we will omit $\mathcal{M}$ and the integration measure and simply write $\int_x$ for $\int_{x\in\mathcal{M}} \mathrm{d}\mu(x)$.

The second important subset is the set of *local functionals* $\mathscr{F}_{\mathrm{loc}}$, that are useful to describe local observables and interaction Lagrangians. They are defined by the conditions that i) their n-th derivatives $F^{(n)}$ are only supported on the diagonal $\mathcal{D}_n := \{(x_1, \ldots, x_n) \subset \mathcal{M}^n \mid \forall i, j\ x_i = x_j\}$ and ii) they satisfy $\mathrm{WF}(F^{(n)}) \perp T\mathcal{D}_n$, meaning that, for every covector p in $\mathrm{WF}(F^{(n)})$, $g^{-1}p$ has vanishing scalar product with any vector in $T\mathcal{D}_n$. The condition on the support of their derivatives implies that local functionals takes the familiar form

$$F(\varphi) = \int_x f(j_x(\varphi)) \,, \tag{2.2}$$

where f is a function on the jet space, and $j_x(\varphi) = (x, \varphi(x), \partial\varphi(x), \ldots$ is the k-jet space of φ evaluated at x. The condition on the wavefront set implies that local functionals coincide with the set of *additive functionals* [57, 100], where a functional is additive if, given $\varphi_1,\ \varphi_2,\ \varphi_3 \in \mathscr{E}(\mathcal{M})$ such that $\operatorname{supp}\varphi_1 \cap \operatorname{supp}\varphi_3 = \emptyset$, it holds that

$$F(\varphi_1 + \varphi_2 + \varphi_3) = F(\varphi_1 + \varphi_2) - F(\varphi_2) + F(\varphi_2 + \varphi_3) \,. \tag{2.3}$$

A typical example of a functional which is local but not regular are local powers of the field at a point, $\Phi^n(f) := \int_x f(x)\varphi^n(x)$.

Finally, we can define the space of off-shell classical observables equipping the space of microcausal functionals $\mathscr{F}_{\mu c}$ with a pointwise product and an involution.

Definition 2.6. The product on the space of microcausal functionals $\mathscr{F}_{\mu c}$ is defined as the pointwise product between $F,\ G \in \mathscr{F}_{\mu c}$ as

$$F \cdot G(\varphi) := F(\varphi)G(\varphi) \,.$$

Furthermore, on the space of microcausal functionals we define an involution $*$ as complex conjugation,

$$F^*(\varphi) := \overline{F(\varphi)} \,,$$

where the over-bar denotes complex conjugation. The space of microcausal functionals, equipped with the commutative product $\cdot$ and the involution $*$, defines the commutative, involutive, and unital $*-$algebra of off-shell classical observables $(\mathscr{F}_{\mu c}, \cdot, *)$.

2.3 CLASSICAL DYNAMICS, AND A TEASER ON SYMMETRIES

The algebra of classical observables defines the kinematical data of a classical field theory. The dynamics is defined providing second-order partial differential equations for the fields, which usually are derived from the Euler-Lagrange derivative of some action functional.

On globally hyperbolic spacetimes the definition of an action functional requires some care: in fact, globally hyperbolic spacetimes are non-compact. Since non-trivial solutions of globally hyperbolic wave equations have non-compact support, there is no non-trivial restriction on the support of field configurations. It follows immediately that the usual definition of the action as the integral over the whole spacetime of some Lagrangian density gives a divergent quantity. Instead, dynamics needs a description in terms of a *generalised Lagrangian*.

Definition 2.7 (Generalised Lagrangian [61])**.** A *Lagrangian* L is defined as a natural transformation from the space of compactly supported, smooth test functions $C_c^\infty(\mathcal{M})$ to the space of local functionals $\mathscr{F}_{\mathrm{loc}}$, satisfying $\operatorname{supp}(L(f)) \subseteq \operatorname{supp}(f)$ and the *additivity rule*

$$L(f+g+h) = L(f+g) + L(g+h) - L(g) \,,\ f, g, h \in C_c^\infty(\mathcal{M}) \,,\ \operatorname{supp} f \cap \operatorname{supp} h = \emptyset \,. \tag{2.4}$$

The additivity rule and the support properties imply that L is a local functional. The action $I(L)$ is defined as an equivalence class of Lagrangians, where two Lagrangians are equivalent if

$$\mathrm{supp}(L_1 - L_2)(f) \subset \mathrm{supp}\, \mathrm{d}f \,. \tag{2.5}$$

More explicitly, the function f in the Lagrangian acts as an IR cut-off: it is equal to 1 on the region of spacetime where we want to study our theory, so that the action integral is well-defined. This cut-off is eventually removed by taking the *adiabatic limit* $f \longrightarrow 1$ in a suitable way, typically at the level of expectation values of observables or of correlation functions, where the adiabatic limit gives finite results. Assuming without loss of generality that there exists a local chart on $\mathcal{M}$ such that the function f is equal to 1 in the vicinity of its origin, the adiabatic limit can be taken replacing the cut-off function with $f(x/n)$, and considering the limit $n \longrightarrow \infty$ of the expectation values.

There are various ways to introduce the IR cut-off in the typical Lagrangian densities of QFT. In Section 3.4.1 we will comment on an explicit construction of a generalised Lagrangian from the typical Lagrangian densities of QFT, such as the $\lambda\varphi^4$–theory for self-interacting scalar fields or Yang-Mills type theories, which preserves the symmetries of the original Lagrangian density.

The *Euler-Lagrange* derivative of I is defined by

$$\langle I^{(1)}(\varphi), h\rangle := \langle L^{(1)}(f)(\varphi), h\rangle \,, \tag{2.6}$$

where h is compactly supported and $f = 1$ on $\mathrm{supp}\, h$. The equations of motion (EOM) are

$$I^{(1)}(\varphi) = 0 \,. \tag{2.7}$$

The space of solutions of the equations of motion (2.7) is denoted by $\mathscr{E}_{\mathrm{os}}(\mathcal{M})$, and functionals on this subspace are called *on-shell* functionals, $\mathscr{F}_{\mathrm{os}}$. This space can be characterised as the quotient space $\mathscr{F}_{\mathrm{os}} = \mathscr{F}_{\mu c}/\mathscr{F}_0$, where $\mathscr{F}_0$ is the ideal generated by the equations of motion, i.e. the space of functionals that vanish on-shell [213].

In the case of an action I_0 quadratic in the fields, the second derivative of I_0 with respect to the fields defines an operator P_0, called the *free wave operator*, which is field-independent. P_0 is a second-order partial differential operator defining the equations of motion $P_0\varphi = I^{(1)}(\varphi) = 0$.

Definition 2.8 (Free wave operator)**.** Let I_0 be an action functional quadratic in the field configuration. The second derivative of the action I_0, with respect to the fields is defined by

$$\langle I_0^{(2)}, h_1 \otimes h_2\rangle := \langle L_0^{(2)}, h_1 \otimes h_2\rangle \,,$$

where $h_1,\ h_2$ are compactly supported and $f = 1$ on the $\mathrm{supp}\, h_1$ and $\mathrm{supp}\, h_2$.

By definition, $I_0^{(2)}$ defines a linear operator $I_0^{(2)} : \mathscr{E} \longrightarrow L(\mathscr{E}_c \times \mathscr{E}_c, \mathbb{C})$, and from the locality of the Lagrangian it follows that it can be extended to a linear operator on $\mathscr{E} \times \mathscr{E}_c$ [55]. Then, by Schwartz kernel theorem $I_0^{(2)}$ induces a continuous linear operator $P_0 : \mathscr{E} \longrightarrow \mathscr{E}^*$, where $E^* = \Gamma(\mathcal{M}, V^*)$ denotes the dual bundle, where V^* is the dual bundle of V. The *free wave operator* is thus determined by the second derivatives of the free action in the adiabatic limit, $P_0 = I^{(2)}|_{f=1}$.

The off-shell algebra of classical field theory incorporates the dynamics of the fields through the Peierls bracket, which is defined from the propagators of the free equations of motion. Moreover, the propagators also define the quantum product in deformation quantization. The propagators are fundamental solutions of the EOMs satisfying a certain condition on their supports:

Definition 2.9 (Advanced and retarded propagators)**.** If they exist, the *advanced* and *retarded* propagators $\Delta_{A,R}$ are two linear maps from the sections of the compactly supported dual bundle $\mathscr{E}^*_c = \Gamma(\mathcal{M}, V^*)$, to the configuration space $\Delta_{A,R} : \mathscr{E}^*_c(\mathcal{M}) \to \mathscr{E}(\mathcal{M})$ satisfying

$$P_0 \Delta_{A,R} = \mathrm{id}_{\mathscr{E}^*_c} \,, \tag{2.8}$$

$$\Delta_{A,R} P_0 = \mathrm{id}_{\mathscr{E}_c} \,, \tag{2.9}$$

and the support properties

$$\mathrm{supp}(\Delta_{A,R} f) \subset J^{+,-}(\mathrm{supp}\, f)\,, \forall f \in \mathscr{E}^*_c \,.$$

$J^{+(-)}(x)$ denotes the causal future (past) of x, and to the advanced propagator corresponds J^- in the above formula.

Normally hyperbolic operators, that are operators whose principal symbol is $g^{-1}(p,p)$, where g is the spacetime metric, on globally hyperbolic spacetimes admit advanced and retarded propagators, and they are unique. An example of normally hyperbolic operator is the d'Alembertian $\Box = g(\nabla, \nabla)$.

We can now give some motivation for the Batalin-Vilkovisky (BV) formalism, which provides a homological framework to quantize theories admitting local symmetries. In fact, an action admitting local symmetries cannot provide a normally hyperbolic operator. This is a consequence of Noether's identities for the classical action [116, 134]. An action invariant under some local transformation satisfies

$$\langle R(\varphi), I^{(1)} \rangle = 0 \,, \tag{2.10}$$

where $R : \mathscr{E}(\mathcal{M}) \to \mathscr{E}(\mathcal{M})$ are the infinitesimal generators of the symmetry. The above relation can be read formally as $\delta I = R(\varphi)\frac{\delta I}{\delta \varphi} = 0$, so it is nothing but the infinitesimal variation of the action under some local transformation. By differentiating this equation with respect to the fields we get

$$\langle R(\varphi), I^{(2)} \rangle + \langle I^{(1)}, R^{(1)}(\varphi) \rangle = 0 \,.$$

On the critical surface $I^{(1)} = 0$, the free wave operator $P_0 = I^{(2)}|_{f=1}$ admits nontrivial solutions R. If R is compactly supported, P_0 admits compactly supported solutions. However, a normally hyperbolic operator on globally hyperbolic spacetime do not admit compactly supported solutions, so P_0 cannot be a normally hyperbolic operator.

In the BV formalism, the procedure of gauge-fixing solves the problem by adding extra terms "by hand" to the action, breaking the symmetries of the theory so that the resulting wave operator is normally hyperbolic. The BV formalism provides control on the symmetries of the theory and guarantees that the gauge-fixed action describes the same physics as the original, gauge-invariant theory. We will discuss in detail the BV formalism in the next Chapter.

For now, we will simply assume that the free wave operator P_0 is normally hyperbolic, so that I can either be an action that does not admit local symmetries, or a gauge-fixed action for some gauge theory. An example is the free action for a real scalar field,

$$I_0(\varphi) = -\int_x \left(\frac{1}{2} \nabla_a \varphi \nabla^a \varphi + \frac{\xi}{2} R \varphi^2 + \frac{m^2}{2} \varphi^2 \right) f \,, \tag{2.11}$$

where m is the mass of the field and ξ its coupling to the scalar curvature R. In the above action we chose the IR cut-off function in the simplest possible way, as a multiplicative cut-off in front of the Lagrangian; we will see later that, in the case of gauge theories, a refinement is necessary in order to preserve local symmetries of the theory.

The Euler-Lagrange derivative of the free action for the scalar field gives the equations of motion (EOMs) $P_0\varphi = 0$, where P_0 is the linear, normally hyperbolic differential operator

$$P_0 = \Box - m^2 - \xi R \,.$$

From the advanced and retarded propagators $\Delta_{A,R}$ of P_0 we can now define the *Pauli-Jordan commutator function*, or *causal propagator*.

Definition 2.10 (Pauli-Jordan commutator function)**.** The *Pauli-Jordan commutator function*, or *causal propagator*, is a continuous linear map $\Delta : \mathscr{E}_c^*(\mathcal{M}) \to \mathscr{E}(\mathcal{M})$ defines as the difference between the retarded and advanced propagators,

$$\Delta := \Delta_R - \Delta_A \,.$$

For every solution u of the EOMs with compactly supported initial data on some Cauchy surface Σ, there exists a $f \in \mathscr{E}_c^*(\mathcal{M})$ such that $u = \Delta f$. Conversely, for every $f \in \mathscr{E}_c^*(\mathcal{M})$ such that $\Delta f = 0$, there exists a $g \in \mathscr{E}_c(\mathcal{M})$ such that $f = P_0 g$.

Thanks to the continuity and linearity of Δ, Schwartz kernel theorem guarantees that

$$\Delta(f, g) := \langle f, \Delta g \rangle$$

defines a distribution $\Delta \in \mathscr{E}_c^{*\prime}(\mathcal{M} \otimes \mathcal{M})$. We denote the bi-distribution $\Delta(f, g)$ and the operator Δf with the same symbol. Moreover, from the support properties of the advanced and retarded propagators, the commutator function $\Delta(f, g)$ vanishes when the supports of f and g are space-like separated. Additionally, again from the properties of $\Delta_{A,R}$, since its formal adjoint is $-\Delta$ the commutator function is antisymmetric, $\Delta(f, g) = -\Delta(g, f)$.

In the simple case of a scalar field theory, the configuration space is the space of real-valued smooth functions on the spacetime, $\mathscr{E}(\mathcal{M}) = C^\infty(\mathcal{M})$, and the propagators are maps $\Delta : C_c^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$.

The commutator function provides the basic ingredient for the Peierls bracket of classical field theory.

2.3.1 *Peierls bracket*

The $*-$algebra of classical field theory up to now does not contain any information about the dynamics of the fields. This is introduced by the additional structure of a symplectic form in the algebra, defining a *constant Poisson bracket* known as *Peierls bracket*. The Peierls bracket extends the canonical bracket on on-shell variables of classical mechanics to the off-shell algebra, and introduces a symplectic form on the phase space of classical solutions independently on the choice of canonical variables, thus preserving covariance [79]. Moreover, by the principle of correspondence, the Peierls bracket determines the canonical commutation relations between quantum fields.

Definition 2.11 (Peierls bracket)**.** Let $F,\ G \in \mathscr{F}_{\mu c}$ be two microcausal functionals. The *Peierls bracket* is defined as the bilinear operator

$$\lfloor F, G \rfloor := \langle F^{(1)}, \Delta G^{(1)} \rangle \,, \quad F,\ G \in \mathscr{F}_{\mu c} \,.$$

The Peierls bracket is antisymmetric and bilinear, by the properties of the commutator function, and it satisfies the Leibniz rule. While the space of local functionals is not closed under the Peierls bracket, the space of microcausal functionals is; for this reason, microcausal functionals are the most important class of functionals in classical field theory.

The Peierls bracket is equivalent, on the phase space of classical solutions, to the canonical symplectic form (see [56], Chapter 2). Fixing an arbitrary Cauchy surface Σ in the spacetime $\mathcal{M}$, the canonical phase space of classical solutions is identified by the smooth initial data f with space-like support on Σ, since any solution of the EOM can be written as $\mathscr{E}_{\mathrm{os}}\,(\mathcal{M}) \ni \varphi_f := \Delta f$. The phase space $\mathscr{E}_{\mathrm{os}}$ then admits a canonical symplectic structure

$$\sigma(\varphi_f, \varphi_g) = \int_\Sigma \left(\varphi_g \nabla_\mu \varphi_f - \varphi_f \nabla_\mu \varphi_g\right) \mathrm{d}\Sigma^\mu \,, \tag{2.12}$$

where $\mathrm{d}\Sigma^\mu$ is the directional volume element on Σ, $\mathrm{d}\Sigma^a = n^a |\det h|^{1/2}\, \mathrm{d}^3 y$, with n^a the unit, future-pointing vector normal to the hypersurface, h the induced 3-metric, and y^a the induced coordinates on the hypersurface. The field $\varphi_f|_\Sigma$, $n^a \nabla_a \varphi_f|_\Sigma$ can be regarded as the initial position and velocity of the field, and form a set of initial data for the propagation of the field φ_f. It is now possible to prove [56, 172] that the Peierls bracket, restricted to the space of classical solutions, is equivalent to the canonical symplectic form, in the sense that, given two linear functionals $\Phi(f)$, $\Phi(g)$, it holds

$$\lfloor \Phi(f), \Phi(g) \rfloor = \sigma(\varphi_f, \varphi_g) \,.$$

Naturally, by chain rule, the equivalence can be extended to generic functionals.

2.4 DEFORMATION QUANTIZATION

In the case of quantum theories satisfying linear hyperbolic equations of motions, the algebra of free, quantum observables is obtained from the classical one, $(\mathscr{F}_{\mu c}, \cdot, *)$, by deforming the pointwise product to a suitable non-commutative, associative product which encodes the canonical commutation relations. The non-commutative product is defined from the free equation of motion and the commutator function. In this sense, the quantum product encodes information on the causal structure of the background geometry (via the supports of the propagators) and on the free part of the action.

In order to quantize the theory, we assume that the free equations of motion, i.e., the equations generated by the Euler-Lagrange derivative of the quadratic part of the action I_0, are defined by a second-order, normally hyperbolic operator P_0.

The action I_0 could be the quadratic part of a scalar field theory or of the gauge-fixed action of some gauge theory, for example.

We now deform the commutative pointwise product of elements of $\mathcal{F}_{\text{reg}}$; by the principle of correspondence, the quantum product $\star$ must be in the form $F \star G = \sum \hbar^n p_n(F,G)$, with

$$p_0(F,G) = FG \text{ , and} \tag{2.13}$$

$$p_1(F,G) - p_1(G,F) = i\hbar \lfloor F, G \rfloor \, . \tag{2.14}$$

It is possible to find a product constructed from the commutator function Δ satisfying the principle of correspondence (2.13). Concretely, we define the quantum product $\star$ on $\mathcal{F}_{\text{reg}}$ as follows.

Definition 2.12 (Quantum product)**.** Let $F,\ G \in \mathcal{F}_{\text{reg}}$ be two regular functionals. The *quantum product* is defined by the formula

$$F \star G := \mathsf{M} \circ e^{\Gamma_{i\hbar\Delta/2}}(F \otimes G) \, ,$$
$$\Gamma_\Delta := \int_{x,y} \Delta(x,y) \frac{\delta}{\delta\varphi(x)} \otimes \frac{\delta}{\delta\varphi(y)} \qquad F, G \in \mathcal{F}_{\text{reg}},$$

where M maps the tensor product to the pointwise product, $\mathsf{M}(F \otimes G)(\varphi) = F(\varphi)G(\varphi)$. More explicitly,

$$F \star G = FG + \sum_{n\geq 1}^{\infty} \frac{1}{n!} \langle F^{(n)}, \left(\frac{i\hbar}{2}\Delta\right)^{\otimes n} G^{(n)} \rangle \, . \tag{2.15}$$

Such a product implements the canonical commutation relations between linear fields, in the sense that

$$[\varphi(f), \varphi(g)]_\star = \varphi(f) \star \varphi(g) - \varphi(g) \star \varphi(f) = i\hbar \langle f, \Delta\, g \rangle \, , \quad f, g \in \mathcal{E}^*_c(\mathcal{M})$$

and it is compatible with the involution $*$, $(F \star G)^* = G^* \star F^*$. Therefore, the off-shell algebra of regular observables is given by

$$\mathcal{A}_{\text{reg}} = (\mathcal{F}_{\text{reg}}, \star, *) \, .$$

The free algebra $\mathcal{A}_{\text{reg}}$ is in fact generated by the identity, together with all possible linear fields $\{\varphi(f) \mid f \in \mathcal{E}^*_c(\mathcal{M})\}$.

However, the algebra $\mathcal{A}_{\text{reg}}$ is too small to define a quantum theory, since it contains only linear combinations of smeared linear fields: it does not contain non-linear local observables; for example, powers of the field $\Phi^n(f)$ or the stress-energy tensor are not in $\mathcal{A}_{\text{reg}}$. The reason for this restriction is that the product of derivatives of local functionals with the commutator function is ill-defined, due to the structure of its wavefront set.

In order to include non-linear observables, it is necessary to further deform the product, so that it is well-defined between local functionals. This is done substituting a suitable bidistribution Δ_+ in place of $i\Delta/2$ in the construction of the $\star$-product, of the form

$$\Delta_+ := \Delta_S + \frac{i}{2}\Delta \, , \tag{2.16}$$

where Δ_S is a real and symmetric distribution, while Δ_+ solves the linear equation of motion P_0 in the weak sense, and its wave front set satisfies the *microlocal spectrum condition* [63, 210].

Definition 2.13 (Microlocal spectrum condition)**.** A bi-distribution $\Delta_+ : \mathscr{E}^*(\mathcal{M}) \times \mathscr{E}^*(\mathcal{M}) \to \mathbb{C}$ satisfies the *microlocal spectrum condition* if its wavefront is

$$\mathrm{WF}(\Delta_+) = \{(x, y; k_x, k_y) \in T^*(\mathcal{M}^2) \setminus \{0\} \mid (x, k_x) \sim (y, -k_y), k_x \triangleright 0\} ,$$

where $(x, k_x) \sim (y, -k_y)$ holds if x and y are joined by a null geodesic λ, $g^{-1}k_x$ is tangent to γ at x and $-k_y$ is the parallel transport of k_x along λ. $k_x \triangleright 0$ holds if $g^{-1}k_x$ is future pointing.

It is known that states that are quasifree and have a 2-point function ω_2 satisfying this condition exist, and their 2-point functions have an universal singular structure given by the Hadamard parametrix [171, 210].

Given Δ_+, we can equip $\mathscr{F}_{\mu c}$ with the quantum product $\star_{\Delta_+}$ defined by

$$F \star_{\Delta_+} G = \mathsf{M} \circ e^{\Gamma_{\hbar\Delta_+}}(F \otimes G) .$$

Of course, the canonical commutation relations between linear fields are preserved, since Δ_+ differs from the commutator function Δ by a symmetric bidistribution. Moreover, the $*$-subalgebra $(\mathscr{F}_{\mathrm{reg}}, \star_{\Delta_+}, *)$ is isomorphic to $\mathscr{A}_{\mathrm{reg}}$. The isomorphism $\alpha : \mathscr{A}_{\mathrm{reg}} \to (\mathscr{F}_{\mathrm{reg}}, \star_{\Delta_+}, *)$ is realised by

$$\alpha_{\Delta_S}(F) = e^{\tilde{\Gamma}_{\hbar\Delta_S}} F, \qquad \tilde{\Gamma}_{\hbar\Delta_S} = \frac{1}{2}\int_{x,y} \hbar\Delta_S(x, y)\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)} . \tag{2.17}$$

Equipping $(\mathscr{F}_{\mu c}, \star_{\Delta_+})$ with the involution defined by complex conjugation, $F(\varphi)^* = \overline{F(\varphi)}$, defines the free algebra of quantum observables

$$\mathscr{A}_{\Delta_+} := (\mathscr{F}_{\mu c}, \star_{\Delta_+}, *) ,$$

containing local as well as regular functionals. This algebra contains functionals of the fields φ, and it is sufficient to describe scalar field theories or, more generally, theories without local symmetries. In the case of gauge theories, the field configuration space must be enlarged to contain ghosts, antighosts, Nakanishi-Lautrup fields and their respective antifields. The algebra of free quantum observables is then constructed again by deforming the classical algebra of functionals on this extended configuration space via a bidistribution Δ_+. We will see in Chapter 3 how the BV formalism takes care of this issue.

The construction we have presented depends on the non-canonical choice of Δ_S (or equivalently, of Δ_+) in (2.16). However, different choices of Δ_+ produce isomorphic extended algebras: the isomorphism is defined by

$$\alpha_{\tilde{\Delta}_+ - \Delta_+} : (\mathscr{F}_{\mu c}, \star_{\Delta_+}, *) \to (\mathscr{F}_{\mu c}, \star_{\tilde{\Delta}_+}, *) ,$$

with α given in (2.17), and $\tilde{\Delta}_S$ is a different choice of symmetric distribution. Hence, $*$-algebras obtained with different symmetric distributions Δ_S are equivalent realizations of the same extended algebra of free observables, which we simply denote by $\mathscr{A}$.

In other words, for every choice of Δ_S, $(\mathscr{F}_{\mu c}, \star_{\Delta_+}, *)$ is a faithful representation of the same abstract $*$-algebra $\mathscr{A}$, and elements of $\mathscr{A}$ are represented in $(\mathscr{F}_{\mu c}, \star_{\Delta_+}, *)$ by means of α_{Δ_S}. A particular choice of representation $(\mathscr{F}_{\mu c}, \star_{\Delta_+}, *)$ of $\mathscr{A}$ is analogous to a choice of reference frame to represent quantum observables. However, since every representation is isomorphic, the choice does not play any role in the construction of observables, and in the following we drop the subscript Δ_+ from the definition of the quantum product.

2.5 STATES

In the algebraic formalism, states play the fundamental role of computing the expectation value of an observable from its corresponding element in the quantum algebra $\mathscr{A}$. Therefore, a state ω is defined as a positive, normalised, linear functional, initially given on $\mathscr{A}_{\text{reg}}$ and then extended to $\mathscr{A}$.

Remark 2.1. Thanks to linearity, a state ω is determined once its n-point *correlation functions* are known as distributions on compactly supported smooth functions, defined by

$$\omega_n(f_1, \ldots, f_n) := \omega(\varphi(f_1) \star \ldots \star \varphi(f_n)) \, .$$

Moreover, the collection of all Wick products $\varphi(f_1) \star \ldots \star \varphi(f_n)$ generates the free algebra of observables; it follows that knowledge of all correlation functions allows for the reconstruction of the state evaluation of any interacting observable, represented in the free algebra. ■

Since the algebra of observables is constructed as functionals over off-shell field configurations, states need to be compatible with the equations of motion: the kernel of the state must contain the field configurations that solve the equations of motion:

$$\mathscr{F}_{\mu c} \cdot \varphi(P_0 f) \subset \operatorname{Ker}(\omega) \, , \qquad f \in \mathscr{E}_c^*(\mathcal{M}) \, . \tag{2.18}$$

This guarantees that the n-point functions ω_n are weak solutions of the linear equation of motion in any of their entries.

A particularly useful class of states are the *quasifree* or *Gaussian* states.

Definition 2.14 (Quasifree states)**.** A *quasifree state* is a state satisfying

i. Every odd n-point function vanishes, and
ii. even n-point functions can be computed from the 2-point function according to Wick's rule [56]

$$\omega_n(f_1, \ldots f_n) = \sum_{\text{partitions}} \omega_2(f_{i_1}, f_{i_2}) \ldots \omega(f_{i_{n-1}}, f_{i_n})$$

where the sum over partitions refers to all possible decompositions of the set $\{1, \ldots, n\}$ into $n/2$ pairwise disjoint subsets of two ordered elements $\{i_1, i_2\} \ldots \{i_{n-1}, i_n\}$, with $i_{2k-1} < i_{2k}$ for $k = 1, 2, \ldots n/2$ and $i_{2k-1} < i_{2k+1}$.

It follows that fixing the symmetric part Δ_S of the 2-point function uniquely identifies a quasifree state.

If we are interested in computing expectation values in a state ω of $\mathscr{A}$ whose 2-point function is Δ_+, it is particularly useful to represent $\mathscr{A}$ with $(\mathscr{F}_{\mu c}, \star_{\Delta_+}, *)$ where the $\star$-product is constructed with Δ_+. In this case, the expectation value of $F \in \mathscr{A}$ in the state ω is simply the evaluation of $\alpha_{\Delta_S}(F)$ on the vanishing configuration, namely $\omega(F) = \alpha_{\Delta_S}(F)(0)$.

2.5.1 *Hadamard condition*

We now need to extend states on the algebra of regular functionals $\mathscr{A}_{\text{reg}}$ to $\mathscr{A}$ by continuity, characterising the extended states by the same n-point functions. In order to do so, among all possible states, we need to select a sufficiently regular class of states [211]. These states shall provide physically sensible expectation values for the observables that are of most interest, such as correlation functions or the renormalized stress-energy tensor.

The most widely accepted regularity condition on states is the *Hadamard condition*. It was first rigorously formulated by Kay and Wald [171], with the idea of generalising the local structure of the Minkowski vacuum 2-point function to curved spacetimes.

Apart from more exotic, non-differential objects that can arise in non-perturbative quantum gravity [11], Lorentzian spacetimes are always modelled after some curved manifold which, by definition, locally resembles flat Minkowski spacetime. It is thus reasonable to assume that the local 2-point function of any physically sensible state resembles the divergences of the vacuum state in Minkowski.

The Hadamard condition thus requires that the 2-point function exhibits a universal singular structure. Explicitly, for y in a normal neighbourhood of x, the integral kernel of the 2-point function ω_2 must satisfy the *Hadamard condition*.

Definition 2.15 (Hadamard condition)**.** Given a bidistribution $\omega_2(f,g)$ with integral kernel $\omega_2(x,y)$, we say that ω_2 is of *Hadamard form* if, for any $x \in \mathcal{M}$ and any y in a normal convex neighbourhood of x, the integral kernel $\omega_2(x,y)$ takes the following expression

$$\omega_2(x,y) = \lim_{\epsilon\to 0^+}\left[\frac{u(x,y)}{\sigma_\epsilon(x,y)} + v(x,y)\log\left(\frac{\sigma_\epsilon(x,y)}{\mu^2}\right)\right] + w(x,y) \tag{2.19}$$

$$:= H_S(x,y) + \frac{i}{2}\Delta(x,y) + w(x,y) \tag{2.20}$$

$$:= H(x,y) + w(x,y)\,. \tag{2.21}$$

where u, v, and w are smooth functions, $\sigma_\epsilon(x,y) = \sigma(x,y) + i\epsilon(t(x) - t(y))$ with t a generic time function, and σ, known as the Synge world function, is one half of the squared geodesic distance taken with sign. The function u is the square root of the van-Vleck-Morette determinant [207], so it is a purely geometric object; v is uniquely fixed by geometry, the coupling constants and mass parameters of the theory, and can be expanded in a formal power series of σ:

$$v(x,y) = \sum_{n\geq 0} v_n(x,y)\sigma^n(x,y)\,,$$

so that only v_0 is relevant in the coincidence limit without derivatives. The series for v converges on analytic spacetimes, while in general smooth Lorentzian spacetimes, the series is only asymptotic [128, 143]. Finally, w remains an arbitrary, smooth function, containing the residual freedom in the choice of the state. The additional freedom in the constant μ is required to have a dimensionless argument in the logarithm. In the coincidence limit, the divergent part of the 2-point function is encoded in the *Hadamard function* $H(x,y)$ and in the commutator function Δ, which are known a priori.

Non quasi-free states are Hadamard states if their 2-point function satisfies the Hadamard condition, and the other truncated n-point functions are smooth.

Many states of physical interest satisfy the Hadamard condition. Known examples, apart naturally from the Minkowski vacuum, are KMS states for free theories in flat spacetime, the Bunch-Davies states for linear fields on De Sitter spacetime, and the Unruh state in Schwarzschild [88] and Kerr-de-Sitter black holes [173].

The Hadamard condition allows for the construction of Wick polynomials, and Hadamard states are those with finite renormalised stress-energy tensor; thus, they are now regarded as the physically acceptable states [113].

The Hadamard condition (2.19) stems from the requirement that the UV divergence of the 2-point function is no worse than the Minkowski vacuum. However, the vacuum in flat space QFT itself is usually not defined by a condition on its UV behaviour, but rather by imposing that it is the state with minimal energy in a Hilbert space representation (originating from the GNS construction, see Section 2.10). In turn, this requires the positivity of the energy spectrum, or more precisely on the requirement that the energy spectrum must be bounded from below, called *spectrum condition*.

In flat space QFT, quantization proceeds by constructing a Fock space and an operator-valued distribution $\pi(\Phi)(f)$. As we will explain later in Section 2.10, the operator $\pi(\Phi)(f)$ is the GNS representation of the element of the algebra Φ as an operator on a Hilbert space, and it coincides with the familiar field operator introduced in flat space QFT. The field operator $\pi(\Phi)(f)$ is subject to Poincaré covariance. In particular, this means that translations, $x \longrightarrow x + a,\ a \in \mathbb{R}^4$, are represented in the Hilbert space by a strongly continuous unitary group of operators $U(a) := e^{ia\cdot p}$. The *spectrum condition* on the positivity of the energy requires that the generator $p \in T^*(\mathbb{R}^4)$, called the energy-momentum covector, lies in the future light-cone. The field operator $\pi(\Phi)(f)$ can be represented in terms of creation and annihilation operators $a,\ a^{\ddagger}$ on the Hilbert space,

$$\pi(\Phi)(f) = \int_{\mathbf{p}} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (e^{ip\cdot x} a(f) + e^{-ip\cdot x} a^{\dagger}(f)) \,.$$

The vacuum state $|\Omega\rangle$ in the Hilbert space is then defined by the requirement that it is annihilated by the annihilation operator, $a(f)\,|\Omega\rangle = 0$. This in particular implies that the action of the field on the state $\pi(\Phi)(f)\,|\Omega\rangle$ contains only negative frequencies.

Of course, general curved spacetimes are not translation invariant. Therefore, there is no natural action of the Poincaré subgroup of translations, and there is no reason to ask for a Poincaré invariant vacuum state.

However, using microlocal analysis [166], and in particular the wavefront set, Definition 2.4, it is still possible to give a condition on the positivity of the energy-momentum covector p to select a ground state, in analogy with the flat space case.

In terms of the wavefront set, the fundamental condition that translates the spectrum condition to curved spacetimes is the microlocal spectrum condition [63, 210], Definition 2.13. The definition requires that only positive frequencies enters the wavefront set, and thus it generalises to curved spacetimes, the requirement that the energy-momentum four-vector p lies in the future light-cone. In turn, it is possible to prove that a state satisfies the microlocal spectrum condition if and only if its GNS representation $|\Omega\rangle$ satisfies [231]

$$\mathrm{WF}(\pi(\Phi)(f)\,|\Omega\rangle) \in V^- \,.$$

The last condition requires the natural generalisation of the wavefront set to operator-valued distributions. It generalises the vacuum condition in Minkowski $a\,|\Omega\rangle = 0$. Notice however that, while the wavefront set of the 2-point function Δ_+ satisfying the microlocal spectrum condition must not contain positive frequencies, the state

$\pi(\Phi)\,|\Omega\rangle$ can still be a sum of positive and negative modes. The Hadamard condition thus is not a direct translation of the vacuum, but rather a generalisation; for example, thermal states for free theories are Hadamard, but still exhibit both negative and positive frequencies.

The microlocal spectrum condition is a natural generalisation of energy positivity, and it selects ground states in general curved spacetimes. We can now go full circle, and connect it with the Hadamard condition: in fact, based on previous work by Duistermaat and Hörmander [101, 165], Radkizowski showed that if a state is such that: i) the 2-point function satisfies the microlocal spectrum condition 2.13, and ii) the one-point function and the truncated n-point functions, with $n > 2$, are smooth, then it locally satisfies the Hadamard condition (2.19) [209, 210, 225]. Imposing the microlocal spectrum condition is thus equivalent to require the Hadamard condition on the UV behaviour of the 2-point function.

2.6 NORMAL ORDERING

Consider a representation $(\mathscr{F}_{\mu c}, \star_{\Delta_+}, *)$ of $\mathscr{A}$. The deformed $\star$-product on $\mathscr{F}_{\mu c}$ automatically implements the Wick theorem for the product of non-linear observables; in fact, consider for example the field square Φ^2:

$$\Phi^2(x) = \lim_{y\to x}\left[\Phi(x)\star_{\Delta_+}\Phi(y) - \Delta_+(x,y)\right] ,$$

where $\Phi(x)\star\Phi(y)$ is the integral kernel of $\Phi(f)\star\Phi(g)$, seen as a distribution tested on $f\otimes g$. The above expression for Φ^2 is finite by construction. It is easy to see that the same holds for higher polynomials. This means that local functionals like $\Phi^n(f)$ in $(\mathscr{F}_{\mu c}, \star_{\Delta_+}, *)$ are Wick-ordered monomials of the fields, where the Wick ordering is with respect to Δ_+. More explicitly,

$$:\Phi^n(f):_{\Delta_+} := \int_x :\Phi^n:_{\Delta_+} f \in \mathscr{A}$$

where

$$:\Phi^n(f):_{\Delta_+} := \alpha_{-\Delta_+}(\Phi(f)).$$

In this way $:\Phi^n(f):_{\Delta_+} \in \mathscr{A}$ is represented in $(\mathscr{F}_{\mu c}, \star_{\Delta_+}, *)$ as $\alpha_{\Delta_S} :\Phi^n(f):_{\Delta_+} = \alpha_{\Delta_S}\alpha_{-\Delta_+}\Phi^n(f) = \Phi^n(f)$.

However, such normal ordering is not covariant, because Δ_+ is globally defined, since it depends on the non canonical choice of its symmetric part Δ_S. The quasifree state constructed with Δ_+ would represent a preferred reference state, in contradiction with the requirements of the Equivalence Principle.

A possibility to perform normal ordering with local quantities only is to use the Hadamard function H given in (2.19): since H contains the singularities of Δ_+, a new normal ordering prescription again implements the Wick theorem,

$$:\Phi^2:_H(x) = \alpha_{-H}\Phi(x)^2 = \lim_{y\to x}\left[\Phi(x)\Phi(y) - \hbar H(x,y)\right] , \tag{2.22}$$

which is local and generally covariant [158–160]. The drawback is that now one needs to pay attention to the correction introduced in representing Wick-ordered polynomials with respect to H, in the algebra constructed with the $\star$-product defined by $\Delta_+ = H + w$. For example, in view of (2.22) we have that $:\Phi^2:_H = \alpha_{-H}\Phi^2 \in \mathscr{A}$ is represented in $\mathscr{F}_{\mu c}$ as

$$\alpha_{\Delta_S} :\Phi^2:_H(x) = \Phi^2(x) + w(x,x) .$$

It is easy to see that this normal ordering implements the Wick's theorem with respect to the Hadamard function H; in fact, consider for example the representation in the quantum algebra of the product $\Phi^2(x)\Phi^2(y)$: in the algebraic setting, this is understood as the quantum product between two normal-ordered quantities, so that

$$\begin{aligned}\alpha_{\Delta_S}\left(:\Phi^2:_H(x)\star:\Phi^2:_H(y)\right)=&\alpha_{\Delta_S}(:\Phi^2:_H(x))\star_{\Delta_+}\alpha_{\Delta_S}(:\Phi^2:_H(y))\\=&(\Phi^2(x)+w(x,x))(\Phi^2(y)+w(y,y))+\\&+4\hbar\Delta_+(x,y)\Phi(x)\Phi(y)+2\hbar^2\Delta_+(x,y)^2\,,\end{aligned}\tag{2.23}$$

and so its expectation value, in the quasifree state ω whose 2-point function is Δ_+, is

$$\omega(:\Phi^2:_H(x)\star:\Phi^2:_H(y))=w(x,x)w(y,y)+2\hbar\Delta_+^2(x,y)\,.$$

The square of the distribution $\Delta_+(x,y)$ is well-defined thanks to the microlocal spectrum condition on its wavefront set. The requirement of local covariance leaves some residual freedom in the construction of normal-ordering quantities. First of all, there is some freedom in the choice of H, as the length scale μ in the logarithmic contribution to H. Secondly, adding a covariantly constructed smooth function to H does not break general covariance, and preserve the finiteness of normal-ordered quantities. However, general covariance strongly constrains the form of these possible smooth additions, and the freedom in the choice of the Wick powers has been completely classified [158, 160, 161]. For example, at the level of the Wick square this freedom reduces to the choice of two real "regularisation" constants c_1 and c_2

$$:\Phi^2:_H=:\Phi^2:_{\tilde{H}}+c_1m^2+c_2R\,.\tag{2.24}$$

In equation (2.23), we kept explicit both the normal ordering prescription and the dependence on Δ_+ in the $\star$-product, to clarify their relationship. In the usual QFT notation, one would leave implicit the $\star$-product, writing explicitly the normal-ordering prescription; in what follows, adopting the more usual notation in the mathematical physics literature, we will keep the $\star$-products explicit but, without referring to a particular representation, we drop the subscript Δ_+; at the same time, if not strictly necessary, we keep the covariant normal ordering implicit.

2.7 THE WEYL ALGEBRA OF A FREE SCALAR FIELD THEORY

Before discussing interacting QFTs we take here a moment to introduce an alternative construction of the algebra of observables for the free, quantum scalar field, which will be the basis for the Tomita-Takesaki modular theory and the definition of relative entropy relevant in Chapter 9.

Let's consider a scalar field configuration $\varphi\in C^\infty(\mathcal{M})$ on some globally hyperbolic spacetime $\mathcal{M}$, with action $I(f)=-\frac{1}{2}\int_x f\nabla_a\varphi\nabla^a\varphi$; since the mass will not play a role in the following discussion, we consider a massless field for simplicity. The function f as usual is the IR cut-off function, that renders the action functional finite. From the action, we derive the Klein-Gordon equation for the massless scalar field

$$\Box\varphi=0\,,\tag{2.25}$$

where $\Box=g(\nabla,\nabla)$. The free wave operator $P_0=\Box$ is normally hyperbolic, and thus admits advanced and retarded propagators $\Delta_{A,R}$. From the commutator function Δ we can construct the algebra of classical observables $(\mathcal{F}_{\text{reg}},\cdot,*)$ equipped with the Peierls bracket $\lfloor F, G \rfloor := \langle F^{(1)}, \Delta G^{(1)} \rangle$. As we discussed in Section 2.3.1, the Peierls bracket is equivalent to a symplectic form on the space of solutions with space-like compact initial data, defined by

$$\sigma(\varphi_f, \varphi_g) = \int_{x \in \Sigma} \left(\varphi_g \nabla_\mu \varphi_f - \varphi_f \nabla_\mu \varphi_g \right) \mathrm{d}\Sigma^\mu \,,$$

where $\varphi_f = \Delta f$, and $\varphi_g = \Delta g$.

Since the field is free, and we are not interested in non-linear observables for the moment, we consider here only regular functionals $\mathcal{F}_{\mathrm{reg}}$. The $*-$algebra of quantum observables then is generated by the symbols $1,\ \Phi(f)$, together with the relations

$$\Phi(f)^* = \Phi(f) \,, \tag{2.26}$$
$$\Phi(af + bg) = a\Phi(f) + b\Phi(g) \,, \tag{2.27}$$
$$\Phi(P_0 f) = 0 \,, \tag{2.28}$$
$$[\Phi(f), \Phi(g)] = i\hbar\Delta(f, g) \cdot 1 \,. \tag{2.29}$$

The first relation encodes reality of the field, where $\bar{f}$ denotes complex conjugation; the second encodes linearity for every $a, b \in \mathbb{C}$ and real-valued test functions $f, g \in C_c^\infty$; the third relation encodes the EOM, and the last the canonical commutation relations.

In the case of the free scalar field, there is however an equivalent construction which admits the structure of C^*-algebra, called *Weyl algebra* and denoted $\mathscr{W}$. The Weyl algebra is generated by the symbols $\mathbf{1}$ and $W(f)$, labelled by real-valued test functions $f \in C_c^\infty(\mathcal{M})$ and subject to relations

$$W(0) = \mathbf{1} \tag{2.30}$$
$$W(f)^* = W(-f) = W(f)^{-1} \tag{2.31}$$
$$W(f)W(g) = e^{-\frac{i}{2}\langle f, \Delta g \rangle} W(f+g) \,, \tag{2.32}$$

for every $f,\ g \in C_c^\infty(\mathcal{M}, \mathbb{R})$. The Weyl algebra is unique up to isomorphisms [190], and can be considered as the "exponentiated" version of the infinitesimal relations (2.26), by the formal identification

$$W(f) \text{``} = \text{''} e^{i\Phi(f)} \,. \tag{2.33}$$

Notice that the above relation is only formal, since there is no notion of functional calculus in a $*-$algebra.

On the Weyl algebra $\mathscr{W}$, quasifree states are determined by their action on the Weyl operator: it must be equal to

$$\omega(W(f)) = e^{-\frac{1}{2}\Delta_S(f, f)} \,. \tag{2.34}$$

$\Delta_S(f, f)$ is a symmetric bi-distribution that uniquely determines the state, and it corresponds to the 2-point function of the $*-$algebra for $\Phi(f)$. In fact, it is possible to check that $\Delta_+(f_1, f_2) = \omega(\Phi(f_1)\Phi(f_2))$, explicitly computing $\omega(W(f)) = \omega\left[\sum_n \frac{i^n}{n!}\Phi(f)^n\right]$. A quasi-free state for the Weyl algebra $\mathscr{W}$ is Hadamard if and only if Δ_+ satisfies the Hadamard condition.

From the Weyl C^*-algebra $\mathscr{W}$, the GNS reconstruction theorem recovers the usual description of a QFT as a theory of linear operators acting on a Hilbert space.

If ω is a quasifree state, it is possible to give a Fock representation of the C^*−algebra, in which $|\Omega_\omega\rangle$ is the vacuum vector of the Fock space. The double commutant of the representation of the algebra on a Hilbert space defines a *von Neumann algebra* $v\mathcal{N} = \pi_\omega(\mathscr{A})''$. We will discuss the GNS theorem in more detail in Section 2.10.

2.8 INTERACTIONS

2.8.1 *Time-ordered products between regular functionals*

Up to now, the discussion took into account only quantum algebras associated with free (quadratic) actions I_0, whose equations of motion are normally hyperbolic and linear in the field φ, and that admit exact propagators. It is now time to introduce interacting QFTs, defined by equations of motion that are non-linear in the fields, or, equivalently, by an action that is more than quadratic in the fields. In general, solutions to non-linear, second-order partial differential equations can be found only perturbatively. The algebra of interacting fields will then be constructed via perturbation theory from elements of the algebra of free observables. However, in the typical QFTs that are of physical interest, such as those appearing in the Standard Model, the perturbative series defining interacting observables do not converge. Interacting observables can only be represented as a formal power series, in the coupling constant λ governing the strength of the interaction, in the algebra of free observables. If the observables are non-polynomial in the fields, interacting observables are a formal power series also in the reduced Planck constant $\hbar$.

Let's start with an action functional, $I(\varphi)$, which is more than quadratic in the fields. In the case of gauge theories, the action will depend on the gauge fields $\mathcal{A}$ as well as on the ghosts, antighosts, Nakanishi-Lautrup fields and their respective antifields; we denote the collection of the fields by the field multiplet $\varphi = \{\varphi_i\}_i$, where i runs over the field species. In this way, formulas for the scalar case and the gauge field case are formally identical. For example, the action of $\lambda\varphi^4$−scalar field theory takes the form

$$I(\varphi) = I_0 + \lambda V = -\int_x \left(\frac{1}{2}\nabla_a\varphi\nabla^a\varphi + \frac{\xi}{2}R\varphi^2 + \frac{m^2}{2}\varphi^2 + \lambda\frac{\varphi^4}{n!}\right) f\,, \tag{2.35}$$

where f is a compactly supported IR cut-off function introduced to keep $I(\varphi) \in \mathcal{F}_{\mu c}$, and equals 1 on the causal completion of the region where we want to test our theory.

In order to proceed to perturbation theory, the action I must be split in a free, quadratic part I_0 and an interaction term V. This can be done on general grounds as follows. Let's shift the field configuration, writing $\varphi \longrightarrow \bar{\varphi} + \varphi$ as a sum over a non-dynamical background $\bar{\varphi}$ and a fluctuation field φ. The expansion of the action in Taylor series up to second order is

$$I(\bar{\varphi} + \varphi) = I(\bar{\varphi}) + \langle I^{(1)}(\bar{\varphi}), \varphi\rangle + \frac{1}{2}\langle I^{(2)}(\bar{\varphi}), \varphi \otimes \varphi\rangle + V(\bar{\varphi}, \varphi)\,,$$

where $V(\bar{\varphi}, \varphi)$ contains all higher-order terms. Notice that, when the background field satisfies the equations of motion (i.e., for *on-shell backgrounds*) the linear term vanishes and the action reduces to the sum of a quadratic part and interactions, since the field-independent constant $I(\bar{\varphi})$ does not play any role in the equations of motion nor in quantization. However, in certain situations it is desirable to keep the background arbitrary: this is particularly important in any candidate theory of non-perturbative quantum gravity, where the background geometry cannot be fixed a priori but it is the variable to quantise. In this case, it is possible to proceed with the algebraic approach treating the linear term as an external source in the equations of motion, and constructing the propagators for these equations. Alternatively, it is also possible to include the source term in the interaction part of the action, and construct solutions of the full equations of motion perturbatively.

Either way, in the following we will assume that the action takes the general form

$$I(\varphi) = I_0(\varphi) + V(\varphi) ,$$

where I_0 is a quadratic action providing normally hyperbolic equations of motion, and V remains an unspecified, interacting term, which is overall proportional to some coupling constant λ.

Thanks to the construction of the last sections, we can associate the free algebra $\mathscr{A}$ to the free action I_0; interacting observables will be represented as a formal power series in the coupling constant λ with coefficients in the free algebra $\mathscr{A}$.

The perturbative construction of interacting fields makes use of a new operation, the time-ordered product T. In order to understand the algebraic structure, we first define the T-product in the subset of regular functionals $\mathscr{F}_{\text{reg}}$, in a way similar to the quantum product (2.15).

Definition 2.16 (Time-ordered products)**.** Consider two regular functionals in the algebra of regular observables, $F,\ G \in \mathscr{A}_{\text{reg}}$. The *time-ordered product* between any two elements of $\mathscr{A}_{\text{reg}}$ is defined by the formula

$$F \cdot_T G := \mathsf{M} \circ e^{\Gamma_{\Delta_F}} (F \otimes G)\, F, G \in \mathscr{F}_{\text{reg}} ,$$

where we recall that

$$\Gamma_{\Delta_F} := \int_{x,y} \Delta_F(x,y) \frac{\delta}{\delta\varphi(x)} \otimes \frac{\delta}{\delta\varphi(y)} ,$$

and M maps the tensor product to the pointwise product, $\mathsf{M}(F \otimes G)(\varphi) = F(\varphi)G(\varphi)$.

More explicitly,

$$F \cdot_T G = FG + \sum_{n\geq 1}^{\infty} \frac{1}{n!} \langle F^{(n)}, \Delta_F^{\otimes n} G^{(n)} \rangle ,$$

where Δ_F is a Feynman propagator associated with Δ_+,

$$\Delta_F := \Delta_+ + i\Delta_A = \Delta_S + \frac{i}{2}(\Delta_R + \Delta_A) . \tag{2.36}$$

Notice the formal similarity between the time-ordered and the quantum product, Definition 2.12. Thanks to the properties of the Feynman propagator, the time-ordered product is associative and commutative.

2.8.2 *Algebraic renormalization*

Even if we are working with normal-ordered quantities, the T-product defined on $\mathscr{F}_{\text{reg}}$ cannot be extended to $\mathscr{F}_{\mu c}$. The basic reason is the incompatibility of the wavefront set of the Feynman propagator with the wavefront set of the functional derivatives of local functionals: the Hörmander criterion is not satisfied, and thus the product of these distributions is ill-defined. Physically, the divergences that arise in the products of local functionals coincide with the well-known UV divergences of QFT arising in the loop contributions of Feynman diagrams to correlation functions. The simplest of such divergences is the one-loop contribution to the 2-point function of a scalar field theory, which arises from the product of two Feynman propagators at the same spacetime point. In Fourier space, the divergence arises due to the ill-defined convolution of two Feynman propagators, which diverges because of their behaviour at arbitrarily high momentum p.

When the support of two functionals F, G is non-overlapping, the time-ordered product in Definition 2.16, $F \cdot_T G$, is everywhere well-defined and produces a new distribution. More precisely, given two distributions with integral kernel $F^{(n)}(x_1, \dots, x_n)$ and $G^{(n)}(y_1, \dots, y_n)$, the time-ordered product $F \cdot_T G$ is ill-defined at order n only when two points coincide, $x_i = y_i$ for some $i \in \{1, \dots n\}$. The problem of *renormalization* in QFT is the problem of dealing with the UV divergences arising in Feynman diagrams. In the context of distribution theory, the renormalization procedure is cast as an extension of the products of distributions to the diagonal, in order to have a well-defined product among local functionals at the same spacetime point.

In pAQFT, the preferred method of renormalization is based on the Epstein-Glaser procedure, which was developed on ideas of Stuckelberg and Bogoliubov [106]. The Epstein-Glaser procedure is based on a local treatment in position space, in contrast with the most used techniques of renormalization in physics that are developed in momentum space. This allows for a straightforward generalization to curved spacetimes. Moreover, even if the Epstein-Glaser renormalization procedure works perturbatively, it does not assume power-counting renormalizability, and so it can be applied to any effective field theory. In fact, Epstein-Glaser renormalization has been first developed for scalar theories [57], and then extended to Yang-Mills theories [156], general gauge theories [116, 119] and perturbative quantum gravity [65].

The Epstein-Glaser renormalization procedure starts with an axiomatic prescription for the T-product on local functionals, seen as a symmetric and multilinear map from multilocal functionals $\mathcal{F}_{\text{loc}}^{\otimes n}$ to $\mathcal{A}$, satisfying a set of conditions [57, 61, 158, 160, 161]. First, we define the family of linear maps $T_n : \mathcal{F}_{\text{loc}}^{\otimes n} \to \mathcal{F}_{\mu c}[[\hbar]]$, with

$$\operatorname{supp} T_n(F_1, \dots, F_n) \subset \bigcup \operatorname{supp} F_i \,,$$

and satisfying the following set of five axioms.

- A1 *Causal factorization property*: whenever there is a Cauchy surface Σ such that the functionals $F_1, \dots, F_k$ are localized in the future of Σ, and the functionals $F_k, \dots, F_n$ in its past, we have

$$T_n(F_1, \dots, F_n) = T_k(F_1, \dots, F_k) \star T_{n-k}(F_{k+1}, \dots, F_n) \,. \tag{2.37}$$

- A2 Identity map: we define the map $T_1 : \mathcal{F}_{\text{loc}}^{1} \to F_{\mu c}[[\hbar]]$ as $T_1 := e^{\Gamma_w}$, where w is the smooth part of Δ_+. Up to an arbitrary length scale T_1 is uniquely determined by this formula, and using w ensures good covariance properties of T_1. Notice that, for Minkowski vacuum, $w = 0$ and T_1 coincides with the identity. More details can be found in [159].

- A3 symmetry: if the configuration space contains only bosons, the time-ordered products T_n must be symmetric in their arguments; if fermions are also present, the time-ordered products must be graded symmetric;
- A4 field-independence: as an element of microcausal functionals $\mathcal{F}_{\mu c}$, the time-ordered products depend on the field φ only through their arguments, so that

$$\frac{\delta}{\delta\varphi(x)}T_n(F_1,\dots F_n) = \sum_i^n T_n\left(F_1,\dots \frac{\delta F_i}{\delta\varphi},\dots F_n\right);$$

- A5 Field locality:

$$T_n(F_1,\dots F_n) = T_n\left(F_1^{[N]},\dots,F_n^{[N]}\right) + \mathcal{O}(\hbar^{N+1})\,,$$

  where $F^{[N]}$ denotes the covariant Taylor expansion of F with respect to φ up to order N.

Using locality of the factors, multilinearity and field independence, the problem of constructing the time-ordered product with n elements reduces to the problem of extending suitable distributions $t_n \in C_c^\infty(\mathcal{M}^n \setminus d_n)$ defined outside the thin diagonal d_n, that is, outside the coincidence limit of any two points, to the whole $\mathcal{M}^n$ [61, 158].

The Epstein-Glaser renormalization procedure provides a concrete, inductive method to construct renormalized time-ordered products between local functionals satyisfying axioms A1-A5 [158, 160]. The existence of a starting element T_1 from axiom A2 is straightforward. From the starting element, the Epstein-Glaser procedure inductively constructs the T_n map from T_{n-1} maps. For arguments with pairwise non-overlapping supports, the causal factorization property and the symmetry of T_n determine that T_n is given by Def. 2.16. The extension of T_n to local functionals with overlapping supports proceeds recursively on the number of factors, extending the maps T_n to the thin diagonal d_n. The extension can be done assuming fixed the Steinmann scaling degree of the distribution [230], up to some known ambiguities. These correspond to the well-known renormalization freedom, which is parametrised by an $n-$linear map

$$z_n : \mathcal{F}_{\text{loc}}^{\otimes n} \longrightarrow \mathcal{F}_{\text{loc}}[[\hbar]]\,. \tag{2.38}$$

By the Main Theorem of Renormalization [57, 100], the map z_n belongs to the Petermann-Stückeberg Renormalization Group, and if two renormalization prescriptions provide two different renormalized $S-$matrices, there must be an element of the renormalization group connecting the two $S-$matrices.

The time-ordered products of n elements of $\mathcal{F}_{\text{loc}}[[\hbar]]$ then is defined by

$$A_1 \cdot_T \dots \cdot_T A_n := T_n(T_1^{-1}A_1,\dots,T_1^{-1}A_n)\,. \tag{2.39}$$

Notice that the action of T on local functionals maps local functionals to covariant normal ordered ones [158], so that $T(F) =: F :_H$.

Epstein-Glaser renormalization is perturbative in nature: in the construction of Feynman diagrams, Epstein-Glaser renormalization extends the time-ordered products of n factors of the interaction term V, and each factor carries a power of the coupling constant in the interaction term. However, it does not assume power-counting renormalizability: this means that, while the renormalization procedure does not need to finish at any point in the iteration procedure, it is applicable to effective field theories, such as quantum gravity. For more details, see [57, 61, 116, 118, 119, 158, 160, 161, 211].

2.8.3 *Relative S-matrix and interacting observables*

In the following, we assume that the procedure of renormalization has been carried out and that the time-ordered products T_n are well-defined maps $T_n : \mathscr{F}_{\mathrm{loc}}^{\otimes n} \to \mathscr{A}[[V]]$ in the formal power series $\mathscr{A}[[V]]$ with coefficients in the free algebra $\mathscr{A}$. Time-ordered products are now the fundamental operation defining the *formal S-matrix*.

Definition 2.17 (Formal S-matrix)**.** Let $V \in \mathscr{A}[[\lambda]]$ be a microcausal functional in the space of formal power series in the coupling constant λ with coefficients in the free algebra of observables $\mathscr{A} = (\mathscr{F}_{\mu c}, *, \star)$. The *formal S−matrix* (or simply the S−matrix) is a map $S : \mathscr{A}[[\lambda]] \to \mathscr{A}[[\lambda]]$, defined by

$$S(V) := e_{\cdot_T}^{i\lambda TV} = Te^{i\lambda V} = \sum_n \frac{i^n \lambda^n}{\hbar^n n!} T(\underbrace{V \dots V}_{n \text{ times}}) \,.$$

The S-matrix is an element of $\mathscr{A}[[\lambda]]$ satisfying

i. Causality: $S(A+B+C) = S(A+B) \star S(B)^{-1} \star S(B+C)$ if $J^+(\mathrm{supp}\, A) \cap J^-(C) = \emptyset$;
ii. $S(0) = 1, \quad S^{(1)}(0) = 1$;
iii. Formal unitarity: $S(V)^{-1} = S(V)^*$, where the inverse is with respect to the star product, for any real local interaction V;
iv. Field independence: $S^{(1)}(V) = \frac{i}{\hbar} S(V) \cdot_T \lambda T V^{(1)}$.

In the following, we include the coupling constant λ into the definition of the interaction term V, to keep the notation reasonably compact, denoting the algebra of formal power series simply as $\mathscr{A}[[\lambda]] = \mathscr{A}[[V]]$.

Notice that a sequence in $\mathscr{A}[[V]]$ converges if the coefficients of the formal power series converge in the weak topology of $\mathscr{F}_{\mu c}$. See [211] for further details.

Remark 2.2. Since the microcausal functionals we work with have only a finite number of non-vanishing derivatives by definition, the S−matrix is a formal power series in λ, but not in $\hbar$. If, for the sake of generality, one is interested in studying non-polynomial microcausal functionals, the non-commutative $\star$−product would be defined as a formal power series in $\mathscr{A}[[\hbar]]$, and the S−matrix would be a formal power series in λ and a Laurent series in $\hbar$. ■

Contrary to the usual definition in QFT, the S−matrix is a local quantity defined at the level of the algebra. Evaluating the S-matrix on a state reproduces the S−matrix that is more familiar in standard QFT, seen as the collection of transition elements from the initial to final state.

The main purpose of the S−matrix is to provide a way to construct interacting correlation functions. In fact, from the S-matrix, we define the *relative* S-matrix as

$$S_V(F) := S(V)^{-1} \star S(V+F) \,. \tag{2.40}$$

Finally, the *Bogoliubov map* (also called *quantum Møller map*) represents interacting fields in the free algebra.

Definition 2.18 (Bogoliubov map)**.** Consider an element of the (abstract, unreachable) interacting algebra $F_{\mathrm{int}} \in \mathscr{A}_{\mathrm{int}}$. The *Bogoliubov formula* represents the interacting observable F_{int} as a formal power series in λ (and $\hbar$, if the functional is non-polynomial in the fields) with coefficients in the free algebra, via the *Bogoliubov* *map* $R_V(F_{\mathrm{int}})$ applied to a corresponding element of the free algebra $F \in \mathscr{A}$. The Bogoliubov map is defined by

$$R_V(F_{\mathrm{int}}) := \frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}t} S_V(tT^{-1}F)\bigg|_{t=0} = S(V)^{-1} \star [S(V) \cdot_T F] \, .$$

Since it is in general not possible to explicitly construct the interacting observable F_{int}, the above formula is often taken as a *definition* for F_{int} as a perturbative series in terms of the free observable, and we interpret the Bogoliubov map as a map that takes a free observable and gives the corresponding interacting observable as a perturbative series $R_V : \mathscr{A} \to \mathscr{A}[[V]]$, defined by

$$F_{\mathrm{int}} = R_V(F) := S(V)^{-1} \star [S(V) \cdot_T F] \, .$$

However, the Bogoliubov map actually intertwines between the free and interacting algebra: the next lemma proves that the *argument* of the Bogoliubov map satisfies the *interacting* equations of motion, if the corresponding free observable φ satisfies the *free* equations of motion. Loosely speaking, the next lemma allows to interpret $R_V(\varphi)$ as the interacting field.

Lemma 2.1. *$R_V(\varphi)$ weakly satisfies the equation of motion,*

$$R_V(P_0\varphi) + R_V(TV^{(1)}) = P_0\varphi \, .$$

Proof. The proof follows by the observation that, thanks to the relation between the Feynman propagator and the 2-point function Δ_+ in equation (2.36), if A is a functional linear in the field it holds that

$$A \cdot_T B = AB + \hbar\langle A^{(1)}, (\Delta_+ + i\Delta_A)B^{(1)}\rangle = A \star B + i\hbar\langle A^{(1)}, \Delta_A B^{(1)}\rangle \, . \qquad (2.41)$$

In particular, if $A = P_0\varphi$, it follows immediately that

$$P_0\varphi \cdot_T B = P_0\varphi \star B + i\hbar B^{(1)} \, ,$$

since Δ_A is a propagator for the wave operator P_0. Then, by direct inspection

$$\begin{aligned} &R_V(P_0\varphi) \\ &= S(V)^{-1} \star S(V) \cdot_T P_0\varphi = S(V)^{-1} \star S(V) \star P_0\varphi - S(V)^{-1} \star S(V) \cdot_T TV^{(1)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = P_0\varphi - R_V(TV^{(1)}) \, . \end{aligned}$$

□

Evaluating the above relation on a state, since the state is on-shell we have

$$\omega\left(R_V(P_0\varphi + TV^{(1)})\right) = 0 \, . \qquad (2.42)$$

2.9 PERTURBATIVE AGREEMENT

Deformation quantization depends on the split $I = I_0 + V$ into quadratic terms and interactions. The classical action is clearly invariant under this split: the solutions of EOMs and the Poisson bracket describing the dynamics do not depend on it. In deformation quantization, however, the quadratic part and the interacting terms in the action play very different roles, since the quadratic part determines the propagators and therefore the products of the algebra, while the interacting term determines the $S-$matrix and the Bogoliubov map that represents interacting observables. The quantum theory should be independent on the non-canonical split between free theory and interactions, but this is not clear a priori; for example, choosing a different split (including, for example, the mass term in the interactions instead of in the quadratic part) could describe different quantum theories.

The requirement that the quantum theory is independent on the split is called *Principle of (generalised) perturbative agreement* (PPA) [95, 161, 254]. In this short Section we do not review the PPA, but we only recall some basic facts that will play a role in next sections. We mainly follow the notation of Drago, Hack, and Pinamonti [95].

As usual, the PPA can be rigorously formulated for scalar field theories, with extensions to gauge theories that are complicated by controlling the symmetries at the quantum level. Here we discuss only the properties of the PPA for the fields φ, which may be the scalar field or the multiplet of gauge fields, ghosts, Nakanishi-Lautrup fields and antighosts, $\varphi = \{\mathcal{A}, b, c, \bar{c}\}$. In the case of gauge theories, the PPA additionally requires that the Quantum Master Equation (the equation governing the symmetries at the quantum level) is also independent on the split. For Yang-Mills-type theories (that are, theories that have actions linear in antifields, such as Yang-Mills theories and gravity), this requirement involves terms that are linear in the antifields. The properties we prove here concern antifield-independent terms, only; we will prove the equivalence between QMEs coming from different splits in Section 3.5.

Let's now start the discussion considering an action I admitting a split $I = I_0+V$ in a free, quadratic part I_0 and interactions V. Moreover, let Q_k be a quadratic potential depending on a parameter k

$$Q_k(\varphi) = -\frac{1}{2}\int \mathrm{d}^d x q_k(x)\varphi^2(x), \tag{2.43}$$

where $q_k = k^2 f \in C_c^\infty(\mathcal{M})$, and $f \in C_c^\infty(\mathcal{M})$ is the usual infrared cut-off. Q_k describes a mass term, before the adiabatic limit, with mass k.

In what follows, we consider

$$I_k := I + Q_k = I_0 + (V + Q_k) = I_{0k} + V = I_0 + V_k\,,$$

and we denote with $\star$, $\cdot_T$, $\mathscr{A}$, S, R (*resp.* $\star_k$, $\cdot_{T_k}$, $\mathscr{A}_k$, S_k, R_k) the star product, time-ordered product, algebra of observables, S-matrix and Bogoliubov map associated with I_0 (*resp.* I_{0k}).

Considering the (abstract, unreachable) interacting algebra $\mathscr{A}_{I_k}$, we have two maps defining a perturbative representation of $\mathscr{A}_{I_k}$ in either $\mathscr{A}[[V_k]]$ or $\mathscr{A}_k[[V]]$, depending on whether Q_k is considered as part of I_0 or of V. These are the Bogoliubov maps given in Def. 2.18 R_{V_k} and $R_{k,V}$, which represent the generators of the interacting algebra in $\mathscr{A}[[V_k]]$ or in $\mathscr{A}_k[[V]]$.

The PPA proceeds by constructing an isomorphism between the two algebras $\mathscr{A}_k$ and $\mathscr{A}$ and the time-ordered products, mapping classical observables into quantum (time-ordered) observables.

First, we define an isomorphism $r_{Q_k} : \mathscr{A}_k \to \mathscr{A}$, called *classical Møller map*.

Definition 2.19 (Classical Møller map)**.** Consider an element $F \in \mathscr{A}_k$, the algebra associated with the free action $I_0 + Q_k$ on the field configuration space $\mathscr{E}(\mathcal{M}) \ni \varphi$. The *Møller map* is a map $r_{Q_k} : \mathscr{A}_k \longrightarrow \mathscr{A}$ satisfying

$$(r_{Q_k} F)(\varphi) = F(r_{Q_k}\varphi)\,, \qquad r_{Q_k}\varphi := (1 - \Delta_{R,k} q_k)\varphi\,,$$

where $\Delta_{R,k}$ is the retarded operator associated to I_{0k}.

Since Q_k is local and does not contain second derivatives, it does not change the principal symbol of the operator $P_0 + Q_k^{(2)}$, which remains normally hyperbolic; it follows that $\Delta_{R,k}$ is well-defined. Moreover, r_{Q_k} intertwines between I_{0k} and I_0: in fact, $I_{0k}^{(1)} r_{Q_k} = I_0^{(1)}$.

Besides the algebras, there is an isomorphism between the time-ordered products $\cdot_T$ and $\cdot_{T_k}$ as well. In fact, using $\tilde{\Gamma}$ given in equation (2.17), we can define the isomorphism

$$\rho_k : \mathscr{F}_{\text{loc}} \longrightarrow \mathscr{F}_{\text{loc}}\,, \qquad \rho_k F := e^{\tilde{\Gamma}_{\Delta_{F,k} - \Delta_F}} F\,, \tag{2.44}$$

such that

$$\rho_k(F \cdot_T G) = \rho_k(F) \cdot_{T_k} \rho_k(G)\,. \tag{2.45}$$

Remark 2.3. The map ρ_k can also be used to define a Wick-ordering map for $\mathscr{A}_k$ [95, 161]. In fact, given the Wick ordering map $T : \mathscr{F}_{\text{loc}} \longrightarrow \mathscr{F}_{\text{loc}}$ for the algebra $\mathscr{A}$, we consider $\rho_k \circ T$. It is then possible to prove [161] that the renormalization ambiguities of the Wick ordering map T can be chosen so that $T_k = \rho_k \circ T : \mathscr{F}_{\text{loc}} \longrightarrow \mathscr{F}_{\text{loc}}$ is a normal-ordering map for $\mathscr{A}_k$. ■

It is not difficult to show that, in a perturbative sense,

$$\rho_k = r_{Q_k}^{-1} \circ R_{Q_k}\,, \tag{2.46}$$

where $R_{Q_k} : \mathscr{A}_k \longrightarrow \mathscr{A}[[Q_k]]$ is the Bogoliubov map representing $\mathscr{A}_k$ in $\mathscr{A}[[Q_k]]$. This relation can be lifted to the interacting case as an equivalence between the Bogoliubov maps R_{V_k} and $R_{k,V}$; in fact, the following lemma holds.

Lemma 2.2. *Consider the algebras $\mathscr{A}, \mathscr{A}_k$, the respective products $\star$, $\star_k$ and the Bogoliubov map R (resp. R_k) representing the interacting algebra $\mathscr{A}_{I_k}$ in $\mathscr{A}$ (resp. in $\mathscr{A}_k$. Then the following holds.*

$$R_{V_k} = r_{Q_k} \circ R_{k,\rho_k V} \circ \gamma_k\,, \tag{2.47}$$

and similarly

$$S_{V_k}(F) = r_{Q_k} S_{k,\rho_k V}(\rho_k F)\,. \tag{2.48}$$

Furthermore, $\rho_k V$ and V differs only by a different choice of renormalization constants.

Proof. To prove Eq. (2.47) and (2.48) we proceed by direct inspection: in fact, we have

$$\begin{aligned} r_{Q_k} S_{k,\rho_k V}(\rho_k F) &= r_{Q_k}\left[S_k(\rho_k V)^{-1} \star_k S_k(\rho_k(F+V))\right] \\ &= \left[r_{Q_k} S_k(\rho_k V)\right]^{-1} \star r_{Q_k} \circ \gamma_k S(F+V) \\ &= \left[r_{Q_k} S_k(\rho_k V)\right]^{-1} \star R_{Q_k} S(F+V) \\ &= \left[r_{Q_k} S_k(\rho_k V)\right]^{-1} \star S(Q_k)^{-1} \star S(F+V_k)\,. \end{aligned}$$

Moreover, by direct computation we get

$$r_{Q_k} S_k(V) = r_{Q_k} \rho_k S(V) = R_{Q_k} S(V) = S(Q_k)^{-1} \star S(V_k)\,,$$

so that overall we have

$$\begin{aligned} r_{Q_k} S_{k,\rho_k V}(\rho_k F) &= \left[r_{Q_k} S_k(\rho_k V)\right]^{-1} \star S(Q_k)^{-1} \star S(F+V_k) \\ &= \left[S(Q_k)^{-1} \star S(V_k)\right]^{-1} \star S(Q_k)^{-1} \star S(F+V_k) = S_{V_k}(F)\,. \end{aligned}$$

This proves Equation (2.48).

Equation (2.47) follows from equation (2.48) as

$$R_{V_k} F = \hbar \frac{\mathrm{d}}{i\mathrm{d}\mu} S_{V_k}(\mu F)\bigg|_{\mu=0} = \hbar \frac{\mathrm{d}}{i\mathrm{d}\mu} r_{Q_k} S_{k,\rho_k V}(\mu \gamma_k F)\bigg|_{\mu=0} = r_{Q_k} R_{k,\rho_k V} \rho_k F\,.$$

□

2.9.1 *Perturbative agreement for linear terms*

Finally, we briefly comment here on the case in which the action $I(\varphi) = J(\varphi) + I_0(\varphi) + V(\varphi)$ also contains a linear term $J(\varphi) = \int_x j(x)\varphi(x)$. As we commented at the beginning of Section 2.8, a linear term can arise when the action is expanded around a background that does not solve the EOMs; it also appears in cases where the fields are coupled to an external current, such as a quantum particle interacting with an external potential.

Just as for the quadratic terms, the linear term can be considered either in the free or in the interacting part of the action. In the former case, it acts as a source in the free equations of motion, modifying the propagators accordingly. When the linear term is considered in the Bogoliubov map, as part of the interaction, it acts as a translation on the observables, shifting the field configuration $\varphi \to \varphi + i\Delta_F j$. In the case of a quasi-free state, the net result of the presence of currents then is to shift the value of the one-point function to a non-vanishing value $\omega(\varphi) = i\Delta_F j$.

The statement follows by evaluating the product of the time ordered exponential of local currents with a local field.

Lemma 2.3. *For all $F \in \mathscr{A}$ we have*

$$[S(J) \cdot_T F](\varphi) = e^{-\Delta_F(j,j)/2} e^{iJ(\varphi)} F(\varphi + i\Delta_F j)\,. \tag{2.49}$$

Moreover, it also holds that

$$S(J)^{-1} \star (S(J) \cdot_T F(\varphi)) = F(\varphi - \Delta_R j)\,.$$

Proof. By direct inspection we have

$$S(J) = e^{iJ}_{\cdot_T} = T(e^{iJ}_{\otimes}) = \sum_{n\geq 0} \frac{1}{n! 2^n} \Delta_F^{\otimes n} (ij)^{\otimes 2n} e^{iJ} = e^{-\Delta_F(j,j)/2} e^{iJ}\,. \tag{2.50}$$

In the above relation, the time-ordered operator on the tensor products is defined in the obvious way, as the tensor product of the time-ordered operators applied to each factor. Moreover, for all $F \in \mathscr{A}$ we have

$$(e^{iJ} \cdot_T F)(\varphi) = \sum_{n\geq 0} \frac{1}{n!} e^{iJ(\varphi)} \langle (i\,j)^{\otimes n}, \Delta_F^{\otimes n} F^{(n)}(\varphi)\rangle = e^{iJ(\varphi)} F(\varphi + i\Delta_F j)\,. \tag{2.51}$$

Combining these results leads to equation (2.49).

To prove the second part of the statement, we notice that by a computation similar to Eq. (2.50) we get

$$S(J) = e^{-\frac{i}{2}\Delta_A(f,f)} e_\star^{iJ} .$$

From the relation $S(J)^{-1} \star S(J) = 1$, it follows that

$$S(J)^{-1} = e_\star^{-iJ} e^{\frac{i}{2}\Delta_A(f,f)} = e^{-iJ} e^{-\frac{1}{2}\Delta_+(f,f)} e^{\frac{i}{2}\Delta_A(f,f)} = e^{-iJ} e^{-\frac{1}{2}\bar{\Delta}_F(f,f)} .$$

Again with a reasoning similar to the time-ordered case, it is easy to see that

$$e^{-iJ} \star F(\varphi) = e^{-iJ} F(\varphi - i\bar{\Delta}_+ j) ,$$

and so in particular, using Eq. (2.49),

$$e^{-iJ} \star (e^{iJ} \cdot_T F) = e^{-iJ} e^{iJ(\varphi - i\bar{\Delta}_+ j} F(\varphi + i(\Delta_F - \bar{\Delta}_+)j) .$$

The statement follows by recalling that $\Delta_F - \bar{\Delta}_+ = i\Delta_R$. □

2.10 HILBERT SPACE REPRESENTATION

The great advantage of the algebraic approach lies in its clear separation between the different operations that construct correlation functions of interacting observables. The construction of the algebra of free observables, in particular, proceeds taking as input the field content of the theory and the causal structure of the spacetime only. This abstract point of view allows to prove very general properties which are model independent; for example, it is possible to prove perturbative renormalizability of Yang-Mills theories [156] without referencing to specific states or backgrounds.

On the other hand, the choice of a state often involves subtle considerations on the physical properties of the specific model of interest; more than a technique dictated by theory, it is an art inspired by physical intuition [110].

Once a state is chosen, it is possible to recover the standard approach to QFT, based on the construction of quantum fields as operators acting on a Hilbert space. The construction is based on the Gel'fand, Naimark, and Segal (GNS) theorem, or Wightman reconstruction argument [141]. Given a $*-$algebra $\mathscr{A}$, a representation of the algebra $(\pi, \mathcal{H}, \mathcal{D})$ is a Hilbert space $\mathcal{H}$, a dense subspace $\mathcal{D} \subset \mathcal{H}$, and a map $\pi : \mathscr{A} \longrightarrow \pi(\mathscr{A})$ from the algebra $\mathscr{A}$ to the space of closable operators over $\mathcal{D}$, satisfying, for any $A,\ B \in \mathscr{A}$, and any $a,\ b \in \mathbb{C}$,

i. compatibility with unity: $\pi(1_{\mathscr{A}}) = 1_{\mathcal{D}}$;
ii. compatibility with the product: $\pi(A)\pi(B) = \pi(AB)$;
iii. linearity: $\pi(aA + bB) = a\pi(A) + b\pi(B)$;
iv. compatibility with the involution: $\pi(A^*) = \pi(A)^*|_{\mathcal{D}}$.

Theorem 2.4 (GNS theorem). *Let $\omega : \mathscr{A} \longrightarrow \mathbb{C}$ be a state on a unital $*-$algebra $\mathscr{A}$. Then, there exists a representation π of the algebra by linear operators acting on a dense subspace $\mathcal{D}$ of some Hilbert space $\mathcal{H}$, and a unit vector $|\Omega\rangle \in \mathcal{D}$ such that*

$$\omega(A) = \langle\Omega| A |\Omega\rangle \ \forall A \in \mathscr{A} ,$$

and $\mathcal{D} = \{\pi(A)\Omega,\ A \in \mathscr{A}\}$. Moreover, the representation associated to each given state ω is unique up to unitary transformations.

The proof can be found e.g. in Ref. [211].

Thus, any state ω on $\mathscr{A}$ identifies a unique GNS representation $(\pi_\omega, \mathcal{H}_\omega, \mathcal{D}_\omega, |\Omega\rangle)$ up to unitary transformations. The bracket on the vector $|\Omega\rangle$ of the operator representing A on $\mathcal{D}$ represents the expectation value $\omega(A)$

$$\omega(A) = \langle\Omega|\, \pi_\omega(A)\, |\Omega\rangle \ . \tag{2.52}$$

Since vectors of the form $|A\rangle = \pi_\omega(A)\,|\Omega\rangle$ are dense in $\mathcal{H}_\omega$, the vector $|\Omega\rangle$ itself is *cyclic*.

The GNS representation shows that, in fact, the algebraic approach encompasses the Hilbert space representation of QFT, with the important generalisation that it does not refer to a particular state from the outline.

2.11 CONNECTION WITH THE PATH INTEGRAL

The pAQFT formalism in the functional approach closely resembles the usual pQFT formalism preferred in the physics literature.

In fact, the time-ordered expectation value of some interacting observable in the vacuum state in the algebraic setting can be regarded as the generalization of the (often ill-defined) path integral approach in usual QFT.

To make the connection apparent, let's focus for the moment to the subset of regular functionals, where the Feynman propagator is sufficient to construct the time-ordering operator. A state-evaluated time-ordered functional in a quasi-free state is

$$TF(0) = \omega\left(TF(\varphi)\right) = \omega\left(\mathsf{M} e^{\frac{1}{2\hbar}\int \frac{\delta}{\delta\varphi}\Delta_F \frac{\delta}{\delta\varphi}} F\right) = e^{\frac{1}{2\hbar}\int \Delta_F \frac{\delta^2}{\delta\varphi\delta\varphi}} F(\varphi)\Big|_{\varphi=0} \ .$$

It is possible to associate to the above formula a path-integral, heuristic representation of the state. In fact, the time-ordering operator T acts as an integral measure, with covariance Δ, in the space of the *Fourier transform* of the field φ; formally rewriting $j = -i\frac{\delta}{\delta\varphi}$, the time-ordering operator can be written by a formal anti-Fourier transform as

$$TF(\varphi) = \int \mathcal{D}j e^{-\frac{1}{2\hbar}\int j\Delta_F j} e^{i\int j\varphi} \hat{F}(j) \ , \tag{2.53}$$

where we denote the convolution as the distribution multiplication, and we introduced the formal Fourier integral measure $\mathcal{D}j$ and the formal Fourier transform

$$\hat{F}(j) = \int \mathcal{D}\varphi e^{-i\int j\varphi} F(\varphi) \ .$$

Substituting the definition of the Fourier transform $\hat{F}(j)$ into formula (2.53), the time-ordering operator is formally equivalent to

$$TF(\varphi) = \int \mathcal{D}j e^{-\frac{\hbar}{2}\Delta_F(j,j) + i\int j\varphi} \int \mathcal{D}\tilde{\varphi} e^{-i\int j\tilde{\varphi}} F(\tilde{\varphi}) \ .$$

The functional integral in $\mathcal{D}j$ can be computed as a Gaussian integral with covariance Δ_F; since the Feynman propagator is a fundamental solution of the free equations of motion, $P_0\Delta_F = i\delta$, we can write

$$TF(0) = \int \mathcal{D}\varphi e^{\frac{i}{\hbar}I_0(\varphi)} F(\varphi) \ ,$$

where $I_0(\varphi) = \frac{1}{2}\varphi P_0 \varphi$.

Of course, all these steps are only formal, and in particular the functional integral measures $\mathcal{D}j$, $\mathcal{D}\varphi$ are ill-defined, since there is no notion of functional calculus in general infinite-dimensional configuration spaces.

The evaluation on a quasi-free state, that is, on the vanishing configuration, becomes a path-integral centred in $\phi = 0$ and with covariance the inverse of the free action

$$TF(0) = \int \mathrm{d}\mu_{i\hbar\Delta_F}(\varphi)F(\varphi) = \int \mathcal{D}\varphi e^{\frac{i}{\hbar}I_0}F(\varphi)\,. \tag{2.54}$$

More generally, choosing as state the evaluating functional $\omega(TF(\varphi)) = TF(\varphi)\big|_{\varphi=\phi}$, where ϕ is the state 1-point function, the state evaluation of a time-ordered functional reads

$$TF(\phi) = \int \mathrm{d}\mu_{i\hbar\Delta_F}(\varphi)F(\phi-\varphi) = \int \mathcal{D}\varphi e^{\frac{i}{\hbar}I_0}F(\phi-\varphi)\,.$$

It is possible to find similar connections for interacting observables. In fact, the interacting field $R_V(\varphi)$ defined by the Bogoliubov formula is nothing but the field in the interaction picture,

$$R_V(\varphi) = S(V)^{-1} \star [S(V) \cdot_T \varphi] = T(e^{iV})^{-1}T(e^{iV}\varphi)\,,$$

where in the last equality we dropped the $\star$-product as it is common in the physics literature. By the same reasoning on the time-ordering operator as before, it follows that expectation values of interacting observables are analogous to

$$R_V(F)(0) = \int \mathcal{D}\varphi e^{\frac{i}{\hbar}I_c(\varphi)}F(\varphi)\,,$$

where $I_c(\varphi)$ is the action functional integrated along a closed contour around the real line in time, known as Keldysh contour, corresponding to the representation of interacting observables in the in-in formalism. In fact, the two directions of integration along the closed contour corresponds to the two operators $S(V)$ and $S(V)^{-1}$ in the Bogoliubov formula.

The algebraic formalism thus provides a generalisation of the path integral approach, applicable to curved spacetimes and when the state is not quasifree, nor it admits a path integral representation. In fact, the main difference from the usual construction is that this formalism does not make use of the vacuum representations of fields. At the same time, local observables like the interaction Lagrangian are normal-ordered in a covariant way.

However, the generalisation to the Bogoliubov map has relevant physical consequences in curved spacetimes or in generic states, due to the extra factor $S(V)^{-1}$.

When dealing with e.g. the scattering theory in QFT on flat spacetime, one usually takes expectation values in the vacuum state on Minkowski; in this case, the Gell-Mann-Low formula allows to factorise the $\star$-product present in the Bogoliubov map,

$$\omega(S(V)^{-1}\star S(V)\cdot_T\varphi) = \omega(S(V)^{-1})\omega(S(V)\cdot_T\varphi) = \omega(S(V))^{-1}\omega(S(V)\cdot_T\varphi)\,, \tag{2.55}$$

at least when the support of V tends to the entire Minkowski spacetime, namely in the adiabatic limit. A discussion about the validity of (2.55) for the case of massive field can be found in Section 6.2 of [97], which makes use of estimates given in the appendix of [99]. In this case, we have that

$$\omega(R_V(\varphi)) = \frac{\omega(T(e^{iV}\varphi))}{\omega(Te^{iV})} ,$$

with analogous formulas for the n-point functions; in the perturbative expansion of the right-hand side, only time-ordered products appear.

However, it is known that, for example for thermal states, the path integral must be computed along the Keldysh contour, taking into account both the upper and the lower path of integration. The Keldysh contour can be seen as a consequence of the fact that the Gell-Mann-Low formula fails for thermal states. In fact, the Gell-Mann-Low formula has been proven only for the Minkowski vacuum with a mass gap in the adiabatic limit [97, 99]. For general states, or on curved backgrounds, the Gell-Mann-Low formula does not hold, and the factor $S(V)^{-1}$ needs to be taken into account. When a Keldysh contour representation is not available, the Bogoliubov formula and the algebraic approach provides the most general and explicit way of writing interacting observables.

The Bogoliubov formula can be expanded as usual in perturbation theory, in terms of Feynman diagrams. However, in the perturbative expansion the $\star$-product plays the important role of a new, oriented (as the product is non-commutative) internal line in Feynman diagrams. Physically, this corresponds to the fact that vacuum bubbles, which are captured by $S(V)$, are now connected with diagrams entering scattering amplitudes, due to curvature effects or state effects that mix positive and negative frequencies.

The discussion in this Section serves to provide a map between the heuristic formulas that are more familiar in the physics literature, with the rigorous formulas of pAQFT. In fact, the functional approach to AQFT provides a way to rigorously define the heuristic formulas of the path-integral formalism. We stress here once again that the correspondence is only formal, since there is no rigorous formulation of functional calculus in the infinite-dimensional configuration spaces of interest in QFT.

However, far from being a simple “formalization” of the path-integral formulation, pAQFT also represents an important generalisation of QFT on flat space in the vacuum representation to curved spacetimes and generic states. In particular, while quasi-free, vacuum states, or thermal states for the free theory, can admit a heuristic path-integral representation, this is not true any more for more complicated states, such as thermal states at equilibrium for the interacting theory. Moreover, while the Gell-Mann-Low formula allows for a path-integral representation of interacting correlation functions, when it does not hold the Bogoliubov formula cannot be easily represented in a path-integral formulation.

The algebraic approach thus provides a unified framework that vastly generalises QFT on flat spacetimes, at the same time clarifying conceptual issues and distinguishing between states and algebras, in a mathematically sound language. It takes directly into account all the effects due to curved backgrounds and nontrivial states, providing a tool to investigate physical consequences of these more general situations.

3 BV formalism

3.1 THE PROBLEM OF QUANTUM GAUGE THEORY

Deformation quantization relies on the assumption that the action provides normally hyperbolic equations of motion, a property that, as we have seen in Section 2.3, is not fulfilled by an action possessing local (gauge) symmetries. In the case of gauge theories, therefore, extra work is needed in order to quantise the theory. The general strategy is based on the procedure of *gauge fixing*, consisting in introducing additional terms in the action that explicitly breaks gauge symmetry, and the equations of motion are again normally hyperbolic. However, the fundamental requirement is that the gauge-fixing procedure does not change the physical predictions of the theory. In other words, while gauge-fixing is necessary, quantum observables should not depend on the specific choices made to fix the gauge, a requirement known as *gauge independence*.

The heavy-lifting to guarantee the equivalence between the two theories is done by the BV formalism. The fundamental idea is to add a gauge-fixing term so that, even if the gauge symmetry is broken, the gauge-fixed action preserves a global (super-)symmetry, known as *BRST symmetry*. The BRST symmetry is described by the action of a differential, the BRST differential, on the fields. Since the symmetry is described by a differential, it is possible to construct a differential complex, whose cohomology describes various physical properties of the theory. The key property of local BRST cohomology is that the zeroth cohomology of the BRST differential describes gauge invariant observables: in this way, it is possible to recover the physical space of gauge-invariant observables from gauge-fixed quantities. From a mathematical perspective, the BRST differential coincides with the *Chevalley-Eilenberg* differential γ_{ce} introduced in Lie algebra cohomology.

On-shell functionals can be characterised in a similar way, using homological algebra. In fact, modulo some technical assumptions, on-shell observables can be described as the zeroth homology of another differential, the *Koszul differential* δ [153]. These technical assumptions can be roughly summarised with the requirement that, *after gauge fixing*, the equations of motion are normally hyperbolic. It follows that the homological language allows for a description of the on-shell observables that do not require normal hyperbolicity of the original equations of motion.

From the Koszul differential it is possible to construct a homology complex, the Koszul complex, so that its zeroth homology describes on-shell observables. The sum of the Koszul and Chevalley-Eilenberg differentials defines the BV differential, $s := \delta + \gamma$. The main theorem of local BRST cohomology is that on-shell, gauge invariant observables, which can be characterised by $H^0(H_0(\delta), \gamma)$, coincides with the zeroth cohomology of the BV differential [22]

$$H^0(s) = H^0(H_0(\delta), \gamma) \ .$$

This allows for a complete cohomological characterisation of the on-shell, gauge invariant observables, in terms of the cohomology of the BV differential s.

While most of the physical literature on the BV formalism uses formal arguments based on the path integral, as the original papers [26–28], and the mathematical literature usually assumes a finite dimensional setting, as in a recent review by Cattaneo, Mnev, and Schiavina [74], the BV formalism can be adapted to the infinite-dimensional framework of QFT [116, 119], generalising the geometric intuition to infinite-dimensional manifolds. In this setting, symmetries are described as directional derivatives leaving the space of functionals invariant. At the infinitesimal level, this requires the introduction of *vector fields* on the space of functionals, using infinite-dimensional calculus. The tangent space to the field configuration space is generated by what in the physics literature are known as *antifields*. The homological description of gauge theories requires the introduction of a new bracket in the classical field theory, the *antibracket*, which coincides with the Schouten bracket among multilocal vector fields.

Having symmetries and equations of motion under control, it is now possible to proceed with deformation quantization of the gauge-fixed action, in order to construct the formal power series of quantum, interacting observables. The time-ordering operator T maps the classical BV differential s in its quantum, interacting counterpart $\hat{s}$.

As the algebra of classical observables, quantum observables must be *gauge-independent*. The gauge independence of the quantum theory follows requiring that the functionals describing the quantum observables are in the zeroth cohomology of the quantum BV differential $\hat{s}$. Moreover, gauge-independence of the $S-$matrix is guaranteed if the action satisfies the *Quantum Master Equation* (3.35).

In the following, we review the BV formalism for the classical and quantum theories possessing gauge symmetries, based on the introduction of infinite-dimensional calculus in the framework of pAQFT [116, 119, 211, 213].

3.2 CLASSICAL THEORY

The BV formalism provides a description of the classical algebra of observables in terms of the cohomology of the Koszul map and the Chevalley-Eilenberg differential. The equations of motion are derived as the Euler-Lagrange equations for an action I_{inv} invariant under some group of symmetries. The classical description will provide a basis to gauge-fix the theory, producing well-defined propagators for quantization.

From now on, when talking about gauge fields we denote the elements of the field configuration space $\mathscr{E}(\mathcal{M})$ with $\mathcal{A} \in \mathscr{E}(\mathcal{M})$. This change is to accommodate the gauge fields $\mathcal{A}$, ghosts c, antighosts $\bar{c}$, and Nakanishi-Lautrup fields b in a field multiplet $\varphi := \{\mathcal{A}, b, c, \bar{c}\}$, an element of the extended configuration space $\overline{\mathscr{E}}$. In fact, in gauge theories, deformation quantization is performed on the odd cotangent bundle of the extended configuration space; with this change in notation, the field configuration subject to quantization is φ both in the scalar and in the gauge field case. Since in the quantum theory the propagators, the states, and, later, the generating functionals are defined with respect to the field multiplet φ, this choice makes most equations formally identical between the scalar field and the gauge field case. Of course, in the case of scalar fields, the field multiplet reduces to the single scalar $\varphi = \{\varphi\}$.

Let's now start with the description of the additional classical structures required by the BV formalism. The first objects we need are *vector fields* $X \in \mathcal{V}(\mathcal{M})$, defined as smooth sections on the tangent bundle of the configuration space. The tangent space to a locally convex topological vector space can be simply identified with the space itself, $T\mathscr{E} := \mathscr{E} \times \mathscr{E}$, and so vector fields are smooth functions from the configuration space to itself, $X \in \Gamma(T\mathscr{E}) \simeq C^\infty(\mathscr{E}, \mathscr{E})$. Vector fields act as derivations of the space of functionals on the configuration space, and can thus be interpreted as directional derivatives on the space of observables.

Definition 3.1 (Vector fields)**.** Let $F \in \mathscr{F}_{\mu c}(\mathcal{M})$ be a microcausal functional on the configuration space $\mathscr{E}(\mathcal{M}) \ni \mathcal{A}$. We define a *vector field* $X \in \mathcal{V}(\mathcal{M})$ as a smooth map $X : \mathscr{E}(\mathcal{M}) \to \mathscr{E}_c(\mathcal{M})$, acting as a derivation on microcausal functionals: if $F \in \mathscr{F}_{\mu c}$, we have

$$\partial_X F(\mathcal{A}) := \langle F^{(1)}(\mathcal{A}), X(\mathcal{A})\rangle \,.$$

The support of a vector field X is defined as

$$\begin{aligned}\operatorname{supp} X := \{x \in \mathcal{M} | \forall \text{ neighbourhood } \mathcal{U} \text{ of } x\ \exists F \in \mathcal{F}_{\mu c}(\mathcal{M}),\ \operatorname{supp} F \subset \mathcal{U} \\ \mid \partial_X F \neq 0 \text{ or } \exists \varphi,\ \psi \in \mathcal{E}(\mathcal{M}),\operatorname{supp}\psi \subset \mathcal{U} \mid X(\varphi+\psi) \neq X(\varphi)\}\,.\end{aligned}$$

The action of vector fields as derivations, and the Lie bracket between two vector fields can be generalized to the *Schouten bracket* between *alternating multivector fields*, defined as smooth, compactly supported multilocal maps from $\mathscr{C}(\mathcal{M})$ to $\Lambda\mathscr{C}^*(\mathcal{M})' = \bigoplus \Lambda^n\mathscr{C}^*(\mathcal{M})'$, where

$$\Lambda^n\mathscr{C}^*(\mathcal{M})' \subset \Gamma((V^*)^{\boxtimes n} \to M^n)'\,,$$

by a slight abuse of notation, denotes the space of compactly supported distributional sections which are totally antisymmetric under permutations of their arguments. Here $\boxtimes$ is the exterior tensor products of vector bundles and V^* is the bundle dual to V. We set $\Lambda^0 C_0^\infty(\mathcal{M}) = \mathbb{R}$ (for more details, see Section 3.4 of Ref. [211]). The space of alternating multivector fields forms a graded commutative algebra $\Lambda\mathcal{V}(\mathcal{M})$ with respect to the product $X \wedge Y(\mathcal{A}) = X(\mathcal{A}) \wedge Y(\mathcal{A})$.

The Schouten bracket is an odd Poisson bracket on this algebra,

$$\{\cdot,\cdot\} : \Lambda^n\mathcal{V}(\mathcal{M}) \times \Lambda^m\mathcal{V}(\mathcal{M}) \to \Lambda^{m+n-1}\mathcal{V}(\mathcal{M})\,.$$

It satisfies the following properties:

i. Graded antisymmetry: $\{Y, X\} = -(-1)^{(n-1)(m-1)}\{X, Y\}$;
ii. Graded Leibniz rule: $\{X, Y \wedge Z\} = \{X, Y\} \wedge Z + (-1)^{nm}\{X, Z\} \wedge Y$;
iii. Graded Jacobi rule:

$$\{X, \{Y, Z\}\} - (-1)^{(n-1)(m-1)}\{Y, \{X, Z\}\} = \{\{X, Y\}, Z\}\,.$$

In the standard approach to the BV formalism, vector fields are identified with *antifields*. The action of vector fields on classical observables can be formally written as

$$\partial_X F(\mathcal{A}) = \int_x X(\mathcal{A}) \frac{\delta F(\mathcal{A})}{\delta \mathcal{A}(x)}\,. \tag{3.1}$$

In the following, whenever a field-dependent expression is integrated over the same spacetime point x, we will always implicitly assume a summation on the field species, Lorentz, and internal indices as well. For example, in non-abelian Yang-Mills theories, the gauge field $\mathcal{A}^a_\mu$ carries both a color index a and a Lorentz index μ. Moreover, the field configuration space must contain also ghosts, antighosts, and Nakanishi-Lautrup fields, so that the field configuration multiplet $\varphi = \{\mathcal{A}^a_\mu, b^a, c^a, \bar{c}^a\}$ replaces the single field φ of scalar theories. More abstractly, the field configuration is a field multiplet $\varphi = \{\varphi_a\}_a$, for some multi-index a that includes internal (color) and Lorentz indices, as well as the species index indicating if the field is the gauge, ghost, antighost, or Nakanishi-Lautrup field. By our convention, it follows that, for example, the last equation can be spelled out as

$$\int_x X(\varphi)\frac{\delta F(\varphi)}{\delta\varphi(x)} := \sum_a \int_x X(\varphi_a)\frac{\delta F(\varphi)}{\delta\varphi_a(x)} ,$$

with the property $\dfrac{\delta\varphi_b(y)}{\delta\varphi_a(x)} = \delta_{ab}\delta(x-y)$, and where δ_{ab} is the Kronecker delta. This very compact notation prevents from a flourishing of indices in the formulas, and it also makes many equations for gauge fields formally identical to the scalar field case. This will help in recognising that equations for the gauge fields are immediate generalisations of those for the scalar field, such as the ones displayed in Section 4.4.

Identifying the functional derivatives $\frac{\delta}{\delta\mathcal{A}}$ with the antifields $\mathcal{A}^\ddagger$, the algebra of alternating multivector fields is generated by fields and antifields, so that a generic element of $\Lambda^k\mathcal{V}(\mathcal{M})$ can be written as

$$X(\mathcal{A}) = \int_{x_1,\dots,x_n} X(\mathcal{A})(x_1,\dots,x_n)\mathcal{A}^\ddagger(x_1)\dots\mathcal{A}^\dagger(x_n) \in \Lambda^k\mathcal{V}(\mathcal{M}) .$$

The Schouten bracket between two multivector fields X, Y, with degree respectively $n,\ m$, is interpreted in the physics literature as the *antibracket*:

$$\{X,Y\} = -\int_x \left(\frac{\delta X}{\delta\mathcal{A}(x)}\frac{\delta Y}{\delta\mathcal{A}^\ddagger(x)} + (-1)^n\frac{\delta X}{\delta\mathcal{A}^\ddagger(x)}\frac{\delta Y}{\delta\mathcal{A}(x)}\right) . \tag{3.2}$$

The BV formalism aims at a homological description of on-shell, gauge-invariant functionals. Thanks to the introduction of the antifields, we can take the first step in the BV construction, giving a geometrical description of the EOMs and of on-shell functionals. In fact, these are characterised by the zeroth homology of a certain differential, called the *Koszul map*.

Definition 3.2 (Koszul map)**.** Let I_{inv} be an action functional invariant under some local symmetry. The *Koszul map* is defined as the operator acting on alternating multivector fields $X \in \Lambda\mathcal{V}(\mathcal{M})$ as

$$\delta_K(X) := \{X, I_{inv}\} ,\ X \in \Lambda\mathcal{V}(\mathcal{M}) .$$

The image of δ_K is contained in the space of functionals that vanish on-shell, $\mathcal{F}_0$. Under technical assumptions, it is possible to prove that $\mathrm{Im}\delta_K = \mathcal{F}_0$ [153], and the space of on-shell functionals $\mathcal{F}_{\mathrm{os}}$ is characterised by the 0th-homology of the Koszul operator $H_0(\delta_K)$ [213].

The Koszul map δ_K defines a complex in $\Lambda\mathcal{V}(\mathcal{M})$,

$$\ldots \to \underset{2}{\Lambda^2\mathcal{V}(\mathcal{M})} \to \underset{1}{\Lambda\mathcal{V}(\mathcal{M})} \to \underset{0}{\mathcal{F}(\mathcal{M})} \to 0\,. \tag{3.3}$$

Local symmetries of the action are described by the directions in which the action does not change, i.e., the space of symmetries is the Lie subalgebra $\mathfrak{s}(\mathcal{M})$ of $\mathcal{V}(\mathcal{M})$ such that

$$X \in \mathfrak{s}(\mathcal{M}) \Leftrightarrow \partial_X I_{inv}(\mathcal{A}) = \delta_K X(\mathcal{A}) = 0\,\forall \mathcal{A} \in \mathcal{E}(\mathcal{M})\,, \tag{3.4}$$

that is, $\mathfrak{s}(\mathcal{M}) = \mathrm{Ker}\delta_K$. A symmetry is called *trivial* if it vanishes on-shell, namely if $X(\mathcal{A}) = 0\,\forall \mathcal{A} \in \mathcal{E}_{\mathrm{os}}(\mathcal{M})$. It is possible to show that trivial symmetries are contained in the image of δ_K, so the first homology of the Koszul complex $H_1(\Lambda\mathcal{V}(\mathcal{M}), \delta_K)$ contains the non-trivial local symmetries of the theory.

As a consequence, for a theory without non-trivial local symmetries the graded algebra $(\Lambda\mathcal{V}, \delta_K)$ is a resolution of $\mathcal{F}_{\mathrm{os}}(\mathcal{M})$, called the *Koszul resolution*. It can be proven that it is sufficient for $I^{(2)}_{inv}$ to be a normally hyperbolic operator in order to have no non-trivial local symmetries. In the presence of local symmetries, one needs to assume regularity conditions on the action, implying that $I^{(2)}_{inv}$ is a normally hyperbolic operator *after gauge fixing*. Then, the Koszul complex must be modified to the *Koszul-Tate complex*, an iterative procedure (which needs not to terminate at some finite order) constructs the *Koszul-Tate* resolution of $\mathcal{F}(\mathcal{M})$. For the details of the construction, we refer to [213].

3.2.1 *Symmetries*

The *Chevalley-Eilenberg cochain complex* provides a geometrical interpretation of the space of gauge-invariant functionals [48, 185, 227]. To start, we notice that the invariant action I_{inv} reflects an invariance of the field configuration space under the action of some group transformation. The field configuration space thus is a section $\mathcal{E}(\mathcal{M}) = \Gamma(\mathcal{M}, V)$ of some principal $G-$bundle $\pi : P \to \mathcal{M}$ with fibre V and with structure group G. The *gauge group* $\mathcal{G}$ is the infinite-dimensional Lie group of gauge transformations on P,

$$\mathcal{G} := \{\zeta : P \to G \mid \zeta(p \cdot M) = M^{-1}\zeta(p)M,\ p \in P,\ M \in G\}\,,$$

and its Lie algebra $\mathcal{g}$ is the infinite-dimensional algebra of infinitesimal transformations on P,

$$\mathcal{g} := \{\xi : P \to \mathfrak{g} \mid \xi(p \cdot M) = \mathrm{Ad}_{M^{-1}}\xi(p),\ p \in P,\ M \in G\}\,.$$

$\mathfrak{g}$ is the Lie algebra of the Lie group G. For a trivial bundle P, the gauge group is simply $\mathcal{G} \simeq C^\infty_c(\mathcal{M}, G)$, the space of smooth sections with values in the Lie group, and $\mathcal{g} = C^\infty(\mathcal{M}, \mathfrak{g})$. The gauge fields $\mathcal{A} \in \Lambda^1(P, \mathfrak{g}) = \Omega_1(\mathcal{M}, \mathfrak{g})$ are thus the space of 1-form connections. We denote $\omega \in \Lambda(P, \mathfrak{g}) := \sum_{n\geq 0} \Lambda^n(P, \mathfrak{g})$ the space of ad-equivariant $k-$forms ω.

The gauge group $\mathcal{G}$ has a natural representation ρ on $\mathcal{A} \in \Omega_1(P, \mathfrak{g})$ by pullback, that is

$$\rho(\zeta)\mathcal{A} := (\zeta^{-1})^*\mathcal{A}\,, \zeta \in \mathcal{G},\ \mathcal{A} \in \Omega_1(P, \mathfrak{g})\,.$$

The derived representation ρ' of $\mathcal{g}$ on $\Lambda^0(P, \mathfrak{g}) \simeq \mathcal{g}$ is the adjoint representation of $\mathcal{g}$,

$$\rho'(\xi)\eta = [\xi, \eta]\,, \xi,\ \eta \in \mathcal{g}\,.$$

The induced action ρ' of the Lie algebra $\mathfrak{g}$ on $\mathcal{A} \in \Lambda^1(P,\mathfrak{g})$ is defined by the infinitesimal action of the exponential map on $\mathcal{A}$, so that

$$\rho'(\xi)\mathcal{A} := \frac{\mathrm{d}}{\mathrm{d}t}_{t=0}\rho'(\exp\{t\xi\})\mathcal{A} = \frac{\mathrm{d}}{\mathrm{d}t}_{t=0}\exp^*(-t\xi) = \mathcal{L}_{Z_\xi}\mathcal{A} = \mathrm{d}\xi + [\mathcal{A},\xi] := D_{\mathcal{A}}\xi\,.$$

In the last expression, $\mathcal{L}_{Z_\xi}$ denotes the Lie derivative along the fundamental vector Z_ξ on P associated with ξ. ρ' thus associates the vector field $\rho'(\xi)$ to the gauge parameter ξ; in other words, it defines a map from the Lie algebra $\mathfrak{g}$ to the space of vector fields on $\mathscr{E}(\mathcal{M})$.

The action of symmetries on the field configuration space $\mathscr{E}$ is thus a subalgebra of directional derivatives on $\mathscr{E}$: we see that the space of antifields $\mathcal{A}^\ddagger \in \mathscr{V}(\mathcal{M})$ plays a fundamental role both in the homological description of the EOMs, and in the cohomological description of symmetries.

Symmetries form a Lie-subalgebra of $\mathscr{V}$ and we assume that each symmetry can be written as $X = \rho'(\xi)+X_0$, where X_0 is a symmetry vanishing on-shell, $\xi \in \mathfrak{g}$, where $\mathfrak{g}$ is a Lie-algebra that can be expressed as a space of smooth sections of some vector bundle $\xrightarrow{\pi} \mathcal{M}$ and $\rho : \mathfrak{g} \to \mathscr{V}$ is a Lie-algebra morphism.

The action I_{inv} is thus invariant under some Lie group $\mathscr{G}$, such that the initial value problem is well-posed in the space of gauge orbits $\mathscr{E}/\mathscr{G}$. Examples of invariant actions are the Einstein-Hilbert or the Yang-Mills actions.

The space of invariant functionals $\mathscr{F}_{inv} = \mathscr{F}(\mathscr{E}/\mathscr{G})$ has a cohomological description in terms of the Chevalley-Eilenberg cochain complex. This is the Chevalley-Eilenberg cohomology of the Lie algebra $\mathfrak{g}$ with respect to the derived representation ρ'. The underlying algebra to the Chevalley-Eilenberg complex is the algebra of microcausal functionals on $\mathscr{E} \oplus \mathfrak{g}[1]$,

$$\mathscr{C}\mathscr{E}(\mathcal{M}) := \mathscr{F}_{\mu c}(\mathscr{E} \oplus \mathfrak{g}[1]))\,, \tag{3.5}$$

with the external differential with respect to the action of the group γ_{ce}, called the *Chevalley-Eilenberg differential*. The grading is called the *pure ghost number* #pg. The Chevalley-Eilenberg complex can be equivalently defined as the microcausal sections from the field configuration space to the exterior product of the dual gauge algebra, $C^\infty_{\mu c}(\mathscr{E}, \Lambda\mathfrak{g}')$, where $\Lambda\mathfrak{g}'$ denotes the dual of $\Gamma(\mathscr{k}^{\boxtimes n} \to \mathcal{M}^n)$.

The Chevalley-Eilenberg differential is defined by its action on the forms $\omega \in \Lambda(P,\mathfrak{g})$ as

$$\begin{aligned}\gamma_{ce}\omega(\xi_1,\dots,\xi_n) &= \sum_{j=0}^{n}(-1)^j\rho'(\xi_i)\omega(\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_n)\\ &\quad+\sum_{i<j}(-1)^{i+j}\omega([\xi_i,\xi_j],\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_{j-1},\xi_{j+1},\dots,\xi_n)\,.\end{aligned} \tag{3.6}$$

Notice that $\gamma_{ce}^2 = 0$. It is extended to a differential on the Chevalley-Eilenberg cochain complex by defining its action on linear functionals as $\gamma_{ce}F(\omega) := \langle F^{(1)}, \gamma_{ce}\omega\rangle$, and on generic microcausal functionals by the graded Leibniz rule and linearity.

The Chevalley-Eilenberg differential generates the Chevalley-Eilenberg cochain complex,

$$0 \to \mathscr{F}(\mathscr{E}) \xrightarrow{\gamma_{ce}} C^\infty(\mathscr{E},\Lambda\mathfrak{g}[1]) \xrightarrow{\gamma_{ce}} \mathscr{C}^\infty(\mathscr{E},\Lambda^2\mathfrak{g}[1])\dots\,.$$

Now, the *Maurer-Cartan form* is the particular one-form $c \in \mathfrak{g}'$ such that $c(\xi) = \xi$ for every element $\xi \in \mathfrak{g}$. It follows from its definition that the action of the Chevalley-Eilenberg differential on the gauge fields $\mathcal{A}$ is $\gamma_{ce}\mathcal{A} = D_{\mathcal{A}}c$, and on the Maurer-Cartan form is $\gamma_{ce}c = \frac{1}{2}[c, c]$.

Notice in particular that for $F \in \mathcal{F}(\mathcal{E})$ it holds

$$\gamma_{ce}F(\mathcal{A}) = \langle F^{(1)}, D_{\mathcal{A}}c\rangle \, . \tag{3.7}$$

The forms $c \in \Lambda \mathfrak{g}'$ are known in the physics literature as *ghosts*.

An element of $\mathcal{CE}$ can be written in two equivalent ways. First, as a functional on $\mathcal{E} \oplus \mathfrak{g}$, it can formally be expressed as the integral over its distributional kernel, as

$$F(\mathcal{A}, \xi) = \sum_{a_1,\ldots,a_n} \int_{x_1,\ldots,x_n} f(\mathcal{A})(x_1, \ldots, x_n)_{a_1,\ldots,a_n} \xi^{a_1}(x_1) \wedge \ldots \wedge \xi^{a_n}(x_n) \, .$$

Secondly, we can use the Maurer-Cartan form and re-write it as a section $F \in C^\infty(\mathcal{E}, \Lambda \mathfrak{g}')$, as a sum of local functionals of the gauge fields, and the exterior product of the ghosts in the form

$$F(\mathcal{A}, c) = \sum_{a_1,\ldots,a_n} \int_{x_1,\ldots,x_n} f(\mathcal{A})(x_1, \ldots, x_n)_{a_1,\ldots,a_n} c^{a_1}(x_1) \wedge \ldots \wedge c^{a_n}(x_n) \, .$$

c^a are the coefficients of the Maurer-Cartan form on $\mathcal{G}$. They are elements of the Chevalley-Eilenberg complex, and they can be seen as formal generators of the algebra $\mathcal{CE}(\mathcal{M})$.

Finally, from the action of the Chevalley-Eilenberg differential on a functional of the field configuration space, Eq. (3.7), we see that the kernel of γ_{ce} in degree 0 characterises the gauge-invariant functionals. On the other hand, the image of γ_{ce} in degree 0 is simply the 0 term, since $\gamma_{ce}0 = 0$. It follows that the cohomology in degree 0 of the Chevalley-Eilenberg complex describes the gauge-invariant functionals, $H^0(\gamma_{ce}) = \mathcal{F}_{inv}$.

3.2.2 *Batalin-Vilkovisky algebra*

The Batalin-Vilkovisky algebra $\mathcal{BV}$ is the graded symmetric tensor algebra of graded compactly-supported microcausal derivations of $\mathcal{CE}$, i.e., it is the algebra of microcausal functionals acting on

$$\mathcal{E}[0] \oplus \mathfrak{g}[1] \oplus \mathcal{E}'_c[-1] \oplus \mathfrak{g}'_c[-2] \, .$$

This is the odd cotangent bundle of the extended configuration space $\overline{\mathcal{E}} := \mathcal{E}[0] \oplus \mathfrak{g}[1]$, where the number in brackets denotes the pure ghost number. Elements of the extended configuration space will be collectively denoted by φ in the following, indicating both the original field configurations $\mathcal{A} \in \mathcal{E}[0]$ and the ghosts $c \in \mathfrak{g}[1]$. The elements of the tangent space are the antifields for the fields and the ghosts, and we identify the functional derivatives $\varphi^\ddagger := \frac{\delta}{\delta\varphi}$ as the "basis" for the fibre $\mathcal{E}'_c[-1] \oplus \mathfrak{g}'_c[-2]$, where the prime indicates to take sections of the corresponding dual bundles.

The $\mathcal{BV}$ algebra has two gradings, the ghost number $\#gh$ and the antifield number $\#af$, related to the pure ghost number via $\#gh = \#pg - \#af$. Functionals of the physical fields have both numbers equal to zero; functionals of ghosts have $\#gh = \#pg$, and vector fields have non-zero antifield number $\#gh = -\#af$. Seen as the space of graded multivector fields, $\mathscr{BV}$ is equipped with a generalised graded Schouten bracket, which can be shown to be an odd Poisson bracket.

In analogy with the Koszul map, Definition 3.2, it is possible to find a natural transformation I_{af} so that

$$\gamma_{ce}(X) := \{X, I_{af}\} . \tag{3.8}$$

The BV differential is defined as the sum of the Koszul map and the Chevalley-Eilenberg differential,

$$s_{BV} := \{\cdot, I_{cl}\} = \delta_K + \gamma_{ce} + \dots , \tag{3.9}$$

where $I_{cl} := I_{inv} + I_{af}$. The dots represent the fact that, in the most general case, $\delta_K + \gamma_{ce}$ fails to be a nilpotent operator. The formula can then be understood as an expansion of s in antifield number, and higher order terms are needed in order to ensure the crucial property $s_{BV}^2 = 0$ [134]. In the case of Yang-Mills theories and gravity, the sum contains the first two terms only, and the action I_{cl} is at most linear in the antifields. Theories that are linear in the antifields are collectively known as *Yang-Mills-type theories*.

The nilpotency of s_{BV} is ensured by the *Classical Master Equation* (CME), which in this formalism must be understood at the level of natural transformations. For scalar field theories, generalised Lagrangians are natural transformations to the space of microcausal functionals, that are, to observables of the field configurations $\mathcal{A}$. In the BV formalism, generalised Lagrangians are now natural transformations from the space of test functions to the BV algebra, in order to include also ghosts and antifields. The classical Lagrangian $L_{cl} := L_{inv} + L_{af}$ is such an example. In the space of generalised Lagrangians, it is possible to introduce an equivalence relation in analogy with (2.5), and define the action as the equivalence class of generalised lagrangians. The CME is then the condition

$$\{I_{cl}, I_{cl}\} = 0 . \tag{3.10}$$

The cohomology of γ_{ce} characterises invariant functionals, and the homology of δ_K the on-shell functionals. The main theorem of homological perturbation theory guarantees that the gauge-invariant, on-shell observables of the theory can be characterised by the zeroth cohomology of the BV differential, $\mathscr{F}_{\text{os}}^{\text{inv}} = H^0(\mathscr{BV}, s_{BV})$ [22, 116, 119, 213]. In fact, it can be proven that

$$H^0(\mathscr{BV}, s_{BV}) = H^0(H_0(\mathscr{BV}, \delta_K), \gamma_{ce}) . \tag{3.11}$$

3.2.3 *Gauge fixing and non-minimal sector*

In order to obtain an action I that provides normally hyperbolic EOMs, the gauge invariant action I_{cl} must be deformed via the procedure known as *gauge-fixing*, so that its EOMs are normally hyperbolic. The main goal of the BV formalism is to provide a deformation of the classical action, in such a way that the quantum observables do not depend on the gauge-fixing term.

In the BV formalism, the gauge fixing is performed in two steps. First, we need to enlarge the BV complex to include the *antighosts* $\bar{c} \in \mathfrak{g}'[-1]$ and the *Nakanishi-Lautrup fields* $b \in \mathfrak{g}'[0]$. For notational simplicity, we again denote the space of field configurations $\overline{\mathscr{E}} := \mathscr{E} \oplus \mathfrak{g}[1] \oplus \mathfrak{g}'[0] \oplus \mathfrak{g}'[-1]$, so that now the field multiplet φ includes the antighosts and the Nakanishi-Lautrup fields too,

$$\varphi = \{\mathcal{A}, b, c, \bar{c}\} ,$$

and the extended BV complex as $\mathcal{BV}$.

On the extended BV complex, we need to defined the action of the BV differential on Nakanishi-Lautrup fields and antighosts. This is defined so that they form a contractible pair,

$$s_{BV}\bar{c} = ib \text{ , and } s_{BV}b = 0 \text{ .}$$

It follows that they do not contribute to the BRST cohomology [22].

The gauge-fixing is performed as an automorphism of the BV complex, leaving the antibracket invariant, and such that the transformed part of the action that does not contain antifields has a well posed Cauchy problem. First, we introduce the gauge-fixing Fermion ψ as a fixed element of the algebra with $\#gh = -1$ and $\#af = 0$. The automorphism then is

$$\alpha_\psi(F) := \sum_{n=0}^{\infty} \frac{1}{n!}\{ \underbrace{\psi, \dots}_{n-1 \; times} , \{\psi, F\} \dots\} \text{ .} \tag{3.12}$$

Notice that the sum is actually finite, since the antibracket with ψ preserves the ghost number and lowers the antifield number by 1. Moreover, it preserves the product, as well as the Schouten bracket.

The automorphism $\alpha_\psi(F)$ performs the usual gauge-fixing, seen as a canonical transformation to the "gauge-fixed basis" [134]

$$\alpha_\psi(F) = F\left(\varphi, \varphi^\ddagger + \frac{\delta\psi}{\delta\varphi}\right) \text{ .} \tag{3.13}$$

The gauge-fixed action is now $I := \alpha_\psi(I_{cl})$. For theories that are linear in the antifields, as Yang-Mills and gravity, the gauge-fixed action I takes the usual form

$$I := \alpha_\psi(I_{cl}) = I_{inv} + I_{af} + s\Psi = I_{inv} + I_{af} + I_{gh} + I_{gf} \text{ ,} \tag{3.14}$$

where I_{inv} is the original, gauge-invariant action, $\Psi = \int_x f\psi$, for some test function f, is the gauge-fixing term, and the antifield, ghost, and gauge-fixing sectors are respectively I_{af}, I_{gh}, and I_{gf}.

The gauge-fixing automorphism acts on the BV differential s_{BV} as well. Together with $\bar{c}$ and b, we introduce their corresponding antifields, so that the transformed BV differential $s := \alpha_\psi \circ s_{BV} \circ \alpha_\psi^{-1}$ now is

$$s = \{\cdot, I\} = \delta + \gamma \text{ .} \tag{3.15}$$

δ is the Koszul operator for the EOMs derived from I, and γ is the BRST operator.

The space of gauge-invariant, on-shell functionals is isomorphic to $H^0(\mathcal{BV}, s)$, where the isomorphism is given by α_ψ. Finally, physical observables are obtained by setting $\varphi^\ddagger = 0$; in particular, the corresponding gauge-fixed BV differential simplifies into the BRST operator $s(\varphi^\ddagger = 0) = \gamma$. In our formalism, we set the antifields to zero when we evaluate the observables on some state functional, to obtain gauge-invariant, on-shell correlation functions.

3.2.4 *Example: Yang-Mills theory*

Let's discuss the BV construction in the concrete example of Yang-Mills (YM) theories [156]. These are defined by a globally hyperbolic spacetime $(\mathcal{M}, g)$, and a principal bundle $P \to \mathcal{M}$, where $G = SU(N)$ is a compact Lie group. In a local trivialisation of the bundle, a connection $\mathcal{A}$ is a $\mathfrak{g}$−valued 1-form, and the Yang-Mills configuration space $\mathscr{E} = \Omega_1(\mathcal{M}, \mathfrak{g})$ are sections of 1-forms with values in the Lie algebra $\mathfrak{g} = \mathfrak{su}(N)$, for some N. The fundamental invariant associated with the connection $\mathcal{A}$ is the curvature $F := \mathrm{d}\mathcal{A} + \lambda_{YM}\frac{1}{2}[\mathcal{A}, \mathcal{A}]$, where λ_{YM} is the Y-M coupling constant, and the invariant action for Yang-Mills theories is

$$I_{inv}(f_{\mathcal{A}}) = -\frac{1}{2}\int_x \mathrm{Tr}\{F(f_{\mathcal{A}}\mathcal{A}) \wedge \star F(f_{\mathcal{A}}\mathcal{A})\}\,,$$

where $\star$ is the Hodge operator and Tr is the trace in the adjoint representation of $\mathfrak{su}(N)$ defined by the Killing-Cartan metric. For notational simplicity, in the following we omit the IR cut-off function $f_{\mathcal{A}}$. This action is the most general action containing up to two derivatives compatible with the experimental observation of force-carrying bosons, massless bosons of spin 1 associated with the electroweak and strong forces [244]. The action I_{inv} is invariant under the action of a G-valued function g of the form

$$\mathcal{A} \to \mathcal{A}_g = -\frac{i}{\lambda_{YM}} g\,\mathrm{d}\big(g^{-1}\big) + g\mathcal{A}g^{-1}\,,$$

which infinitesimally takes the form of standard gauge transformations

$$\delta_\xi \mathcal{A} = D_{\mathcal{A}}\xi = \mathrm{d}\xi + i\lambda_{YM}[\mathcal{A}, \xi],$$

where ξ is a gauge parameter. $D_{\mathcal{A}}$ is the covariant extension of the exterior derivative, which acts as $D_{\mathcal{A}} = \mathrm{d} + i\lambda_{YM}\mathcal{A}$ on the fundamental representation and as in the formula above in the adjoint representation of the group $SU(N)$. The EOM derived from the invariant YM action are

$$D_{\mathcal{A}} \star F = 0\,. \tag{3.16}$$

The Chevalley-Eilenberg complex is the algebra $\mathscr{CE} = \mathscr{F}(\mathscr{E} \oplus \mathfrak{g}[1])$ of microcausal functionals of the gauge fields $\mathcal{A}$ and ghosts c, which are functions with values on the Lie algebra $\mathfrak{su}(N)$. The Chevalley-Eilenberg differential is defined by its action on the fields as

$$\gamma_{ce}\mathcal{A} = D_{\mathcal{A}}c\,, \quad \gamma_{ce}c = -\frac{i\lambda_{YM}}{2}[c, c]\,.$$

The BV algebra is the algebra of microcausal functionals on the odd cotangent bundle of the extended configuration space $T^*(\overline{\mathscr{E}})$. Identifying the elements of the tangent space with antifields, their basis is given by the derivatives $\mathcal{A}^\ddagger = \frac{\delta}{\delta\mathcal{A}}$ and $c^\ddagger = \frac{\delta}{\delta c}$. The Koszul map now by definition acts on a generic functional X as

$$\delta_K(X) = \{X, I_{inv}\} = \int_x \frac{\delta X}{\delta\mathcal{A}} D_{\mathcal{A}} \star F\,.$$

On the other hand, it is immediate to find the antifield contribution to the action I_{af}; from the action of the Chevalley-Eilenberg differential it follows that

$$I_{af} = \int_x \left(\mathcal{A}^\ddagger D_{\mathcal{A}}(f_c c) - \frac{i\lambda_{YM}}{2} c^\ddagger [f_c c, f_c c]\right)\,.$$

Again, in the following we omit the IR cut-off function f_c.

The action I_{cl} and the BV differential s_{BV} are determined by the CME, $\{I_{cl}, I_{cl}\} = 0$. We can now check by direct computation that the combination $I_{inv} + I_{af}$ actually satisfies the CME,

$$\{I_{inv} + I_{af}, I_{inv} + I_{af}\} = 2\{I_{inv}, I_{af}\} + \{I_{af}, I_{af}\} .$$

For the quadratic term we have $\{I_{inv}, I_{inv}\} = 0$ since I_{inv} does not contain antifields; the contribution $\{I_{inv}, I_{af}\}$ vanishes thanks to gauge invariance of the Y-M action. It remains to check that the last term vanishes as well. This is given by

$$\{I_{af}, I_{af}\} = 2\int_x \left(\frac{\delta D_{\mathcal{A}} c}{\delta \mathcal{A}} D_{\mathcal{A}} c - \frac{i\lambda_{YM}}{2} \frac{\delta}{\delta c} (D_{\mathcal{A}} c + [c, c])[c, c] \right) .$$

Now, we have

$$\begin{aligned} \frac{\delta D_{\mathcal{A}} c}{\delta \mathcal{A}} D_{\mathcal{A}} c &= i\lambda_{YM} [D_{\mathcal{A}} c, c] \, , \text{ and} \\ \frac{i\lambda_{YM}}{2} \frac{\delta}{\delta c} (D_{\mathcal{A}} c + \frac{1}{2}[c, c])[c, c] &= \frac{i\lambda_{YM}}{2} D_{\mathcal{A}} [c, c] + [c, [c, c]] \, . \end{aligned}$$

Since $[c, [c, c]] = 0$ by Jacobi identity, the two remaining terms cancel; therefore the classical action $I_{cl} = I_{inv} + I_{af}$ satisfies the CME. The BV differential thus contains only two terms, $s_{BV} = \delta_K + \gamma_{ce}$, and it is nilpotent.

The EOM (3.16) are not normally hyperbolic, since the action satisfies the Noether identities associated with gauge invariance,

$$\int_x \frac{\delta I_{inv}}{\delta \mathcal{A}(x)} D_{\mathcal{A}} c(x) = 0 .$$

Therefore, we need to perform gauge fixing. This is an automorphism of the BV algebra that leaves the antibracket invariant, and such that its action on I_{cl} produces a gauge-fixed action that satisfies the CME and have normally hyperbolic EOM. For a generic gauge-fixing function $\mathcal{G}(\mathcal{A})$ and a gauge parameter ξ, the gauge-fixing Fermion takes the form

$$\Psi = \int_x \psi = i \int_x \bar{c} \left(\xi \frac{b}{2} + \mathcal{G}(\mathcal{A}) \right) .$$

The gauge-fixed action $I = \alpha_\psi(I_{cl})$ is the sum of two terms only, since the classical action is linear in the antifields:

$$I = I_{cl} \left(\varphi, \varphi^\ddagger + \frac{\delta \psi}{\delta \varphi} \right) = I_{cl} + s\Psi .$$

It follows by direct computation that the gauge-fixed action for Yang-Mills theories takes the familiar expression

$$I(\varphi, \varphi^\ddagger = 0) = -\frac{1}{2} \int_x \mathrm{Tr}(F(f) \wedge *F(f)) - i \int_x \mathrm{Tr}[d\bar{c}, D_A c] - \int_x \frac{\xi}{2} b^2 + b\mathcal{G}(\mathcal{A}) . \tag{3.17}$$

3.3 QUANTUM BV DIFFERENTIAL AND BV LAPLACIAN

From the gauge-fixed action I, the quantization of gauge theories proceeds straightforwardly as for the scalar field case, presented in Section 2.4. We split the action in $I = I_0 + V$, where I_0 is a term quadratic in the fields, with $\#af = 0$, and V is the remaining, interacting term. There is a corresponding decomposition of the Koszul operator, which decomposes into $\delta = \delta_0 + \delta_V$, where δ_0 characterises the solutions of the free equations of motion. The main difference from the scalar case is that the space of field configurations must be extended to contain the multiplet φ of gauge, ghosts, antighosts, and Nakanishi-Lautrup fields.

The Euler-Lagrange derivative of the quadratic action $I_0(\varphi)$ gives the free equations of motion $I_0^{(1)}(\varphi) = 0$. By definition, thanks to gauge-fixing these equations are normally hyperbolic they admit advanced and retarded propagators, and thus it is possible to define the Pauli-Jordan commutator function. The addition of a symmetric distribution Δ_S defines the 2-point function $\Delta_+ = H + w$, so that Δ_+ satisfies the Hadamard condition, just as in the scalar case. The difference is that the equations of motion depends also on ghosts, antighosts, and the Nakanishi-Lautrup field, and so Δ_+ is a matrix-valued distribution in field space, carry internal (color) and Lorentz indices as well, which can be explicitly computed in specific models; for example, in the case of the free electromagnetic field [212]. It follows that both Δ_+ and, accordingly, the Feynman propagator $\Delta_F = \Delta_+ + i\Delta_A$ includes contributions for the gauge field propagators as well as the ghost propagators, corresponding to internal lines in the graphical expansion in Feynman diagrams.

Since the Nakanishi-Lautrup field is non-dynamical (there are no derivative terms in the action), its equations of motion are algebraic, rather than differential. In the on-shell BV formalism, the Nakanishi-Lautrup field is then substituted with the solution of its EOM, that is, the gauge-fixing condition $b = \mathcal{G}(\mathcal{A})$. However, in this case, the BV operator is nilpotent only on-shell, since $sb = \mathcal{G}(s\mathcal{A})$.

To keep the off-shell nilpotency of the BV differential s, the Nakanishi-Lautrup field b must be taken into account in the quantum theory as well.

Example 3.1 (Yang-Mills theories)**.** We recall that the gauge-fixed Yang-Mills action takes the form

$$I = I_{cl} + I_{af} + I_{gf} + I_{gh} \,.$$

In this Section we choose the gauge-fixing functional

$$\mathcal{G}(\mathcal{A}) = \star \mathrm{d} \star \mathcal{A} = \nabla_\mu \mathcal{A}^\mu \,,$$

known as Lorenz gauge. The linearisation of the action into $I_0 + V$ around the trivial field configuration $\varphi = 0$ produces the free wave operator P_0; in the case $\xi = 1$ it can be written in terms of the Hodge laplacian $\Box_H := -(\tilde{\delta}\mathrm{d} + \mathrm{d}\tilde{\delta})$, where $\tilde{\delta}$ is the co-differential, as [211]

$$P_0 = \begin{pmatrix} \Box_H + \mathrm{d}\tilde{\delta} - \mathrm{d} & 0 & 0 & 0 \\ \tilde{\delta} & -1 & 0 & 0 \\ 0 & 0 & 0 & i\Box_H \\ 0 & 0 & -i\Box_H & \end{pmatrix} .$$

■

In addition to the extension of the equations of motion and propagators to the full field configuration space, the classical theory of gauge fields also contains the antibracket and the BV differential s. These structures must be deformed in a way compatible with the quantum and time-ordered products.

The quantum and time-ordered products on the extended configuration space take the same expressions as in the scalar case, Eqs. (2.12) and (2.16) [119]. The only difference is that we need to include appropriate signs so that the fermionic variables (the ghosts) anti-commute. We now extend the definitions of the quantum and time-ordered products from the extended configuration space to the antifields, so that the products can be defined on the whole BV algebra. As for the functionals of the physical field configurations, we first restrict our attention to the *regular antifields* $\mathcal{V}_{reg}$, that are, the vector fields whose nth-derivatives are test functions in $\Gamma_c(M^{n+1}, V^{\otimes(n+1)})$. If we think of $X \in \mathcal{V}_{reg}$ as a section, i.e., as maps from $\mathcal{E}(\mathcal{M})$ to $\mathcal{E}_c(\mathcal{M})$, since T acts as a differential operator it is natural to set

$$T(X) := \int_x T(X(x)) \frac{\delta}{\delta\varphi(x)} . \tag{3.18}$$

The non-commutative product is extended to vector fields as

$$X \star Y := e^{\hbar\Gamma_{\Delta+}}(X \wedge Y) . \tag{3.19}$$

The BV structures of the classical algebra gets translated in structures of the quantum algebra by means of the time-ordered product. The graded algebra of antifields is transformed into $T(\mathcal{V}_{reg})$, with the *time-ordered antibracket*

$$\{X, Y\}_T := T\{T^{-1}X, T^{-1}Y\} . \tag{3.20}$$

The free equations of motion are mapped, under time-ordering, in the image of the *time-ordered Koszul operator*

$$\delta_0^T := T^{-1}\delta_0 T^{-1} = \{\cdot, I_0\}_T . \tag{3.21}$$

Since I_0 is quadratic in the fields, and since Δ is a solution of the equations of motion, we have a relation between the time-ordered and the classical free equations of motion

$$\{F, I_0\} = \{F, I_0\}_T - i\hbar \vartriangle F , \tag{3.22}$$

where the *BV laplacian* $\vartriangle$ is a nilpotent operator that acts on regular multi-vector fields as a divergence:

$$\vartriangle X := (-1)^{1+|X|} \int_x \frac{\delta^2 X}{\delta\varphi(x)\delta\varphi^\ddagger(x)} . \tag{3.23}$$

Equation (3.22) is a Dyson-Schwinger-type equation for gauge theories (see Eq. (4.29)), where the classical EOMs are corrected by $\mathcal{O}(\hbar)-$contributions into the quantum ones.

The BV laplacian combines well with the antibracket; in particular, the two following formulas hold:

$$\{X, Y\} = \vartriangle(PQ) - \vartriangle(X)Y - (-1)^{|X|} \vartriangle (X)Y \quad , \text{ and} \tag{3.24}$$

$$\{X, Y\}_T = \vartriangle(X \cdot_T Y) - \vartriangle(X) \cdot_T Y - (-1)^{|X|} \vartriangle (X) \cdot_T Y . \tag{3.25}$$

The above two formulas suggest the introduction of a $\star-$antibracket, defined analogously:

$$\{X, Y\}_\star := \triangle(X \star Y) - \triangle(X) \star Y - (-1)^{|X|} \triangle(X) \star Y\,. \tag{3.26}$$

It can also be written more explicitly as

$$\{X, Y\}_\star = -\int_x \frac{\delta X}{\delta\varphi(x)} \star \frac{\delta Y}{\delta\varphi^\ddagger(x)} + (-1)^{|X|} \frac{\delta Y}{\delta\varphi^\ddagger(x)} \star \frac{\delta Y}{\delta\varphi(x)}\,. \tag{3.27}$$

Since $\{\cdot, I_0\} = \{\cdot, I_0\}_\star$, we can now write the relation between the classical (time-ordered) and quantum equations of motion, Eq. (3.22) as a relation between the time-ordered and the quantum antibracket

$$\{F, I_0\}_\star = \{F, I_0\}_T - i\hbar \triangle F\,. \tag{3.28}$$

Note that $\{\cdot, I_0\}_\star$ is not a derivation with respect to the time-ordered product; instead, it holds that

$$\{X \cdot_T Y, I_0\}_\star - \{X, I_0\}_\star \cdot_T Y - (-1)^{|X|} X \cdot_T \{Y, I_0\}_\star = -i\hbar\{X, Y\}_T\,. \tag{3.29}$$

Finally, we introduce the *quantum BV operator* as the deformation of the operator $\{\cdot, I_0\}_\star$ under the Bogoliubov map

$$\hat{s} := R_V^{-1} \circ \{\ , I_0\}_\star \circ R_V\,, \tag{3.30}$$

which can be rewritten in a more standard form [116] as

$$\hat{s}F = \{F, I\}_T - i\hbar \triangle F\,. \tag{3.31}$$

3.4 QUANTUM MASTER EQUATION

The gauge independence of on-shell interacting functionals is encoded in the Quantum Master Equation (QME). Such a condition suffices to establish the on-shell gauge independence of the $S-$matrix and of the physical observables, which are in the cohomology of the interacting BRST operator $\hat{s}$[119].

The QME follows from the requirement that the $S-$matrix does not depend on the gauge condition. First, one defines an automorphism of the algebra $T(\mathcal{BV}_{reg})$ by

$$\alpha_\psi(F) = T(\alpha_{T^{-1}\psi}(T^{-1}F))\,. \tag{3.32}$$

Denoting $\tilde{F} = \alpha_{\xi\psi}(F)$, the condition for the on-shell gauge independence of the $S-$matrix and of interacting fields can be stated as

$$\frac{\mathrm{d}}{\mathrm{d}\xi} S(\tilde{V}) \overset{\text{o.s.}}{\approx} 0\,,\ \text{ and} \tag{3.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} R_{\tilde{V}}(\tilde{F}) \overset{\text{o.s.}}{\approx} 0\,. \tag{3.34}$$

It is possible to show that a sufficient condition for the on-shell gauge independence of the $S-$matrix is the *Quantum Master Equation* (QME) [119],

$$\{S(V), I_0\}_\star = 0\,. \tag{3.35}$$

Moving to the time-ordered antibracket, we get

$$S(V)\cdot_T\left[\{V,I_0\}_T+\frac{1}{2}\{V,V\}_T-i\hbar\Delta V\right]=0\,. \tag{3.36}$$

Recalling that I_0 does not contain antifields, the above can be rewritten in the more standard form

$$\{I,I\}_T-2i\hbar\vartriangle I=0\,. \tag{3.37}$$

Following a similar derivation, from the second requirement (3.34) one obtains the condition

$$R_{\tilde{V}}\left(\psi\cdot_T\alpha_{\lambda\psi}(\hat{s}F)\right)=0\,. \tag{3.38}$$

Therefore, if an observable F is in the cohomology of the quantum BRST operator $\hat{s}$, the corresponding interacting observable $R_V(F)$ is gauge independent.

3.4.1 *IR regularisation and the adiabatic limit*

The motivation for the construction of generalised Lagrangians is the basic observation that, since a globally hyperbolic spacetime cannot be compact, an action integral $I=\int_x L(\varphi)$ is necessary divergent. A straightforward procedure to build a generalized Lagrangian from the Lagrangian density of some theory is to consider

$$I=\int_x fL\,,$$

for some IR cut-off function function f. The cut-off function is assumed to be equal to 1 in some finite region of spacetime, so that the action functional is well-defined. However, this choice introduces technical difficulties in the proof of the Quantum Master Equation (3.35), the necessary condition to ensure that the $S-$matrix is gauge independent.

Here, we choose a different regularisation for the generalized Lagrangian. We introduce a pair $f=(f_{\mathcal{A}},f_{gh})$ of test functions, and, given a particular Lagrangian density, we define a generalised Lagrangian as

$$I(f_{\mathcal{A}},f_{gh})=\int_x L(f_{\mathcal{A}}\mathcal{A},f_{gh}c)\,. \tag{3.39}$$

In other words, we introduce a set of regularised, compactly supported variables $(f_{\mathcal{A}}\mathcal{A},f_{gh}c)$, substituting the original gauge fields $\mathcal{A}$ and ghosts c. Now, when we consider the antibracket between the generalized Lagrangian and a compactly support functional F, we will always implicitly assume that $f_{\mathcal{A}}=f_{gh}=1$ on the support of F, so that the test functions do not influence the gauge transformation properties of F.

With this choice of IR regularisation, the QME equation holds exactly. In fact, the support of the field configuration does not play a role in its derivation; since the QME holds for generic field configuration, it also holds in particular for the compactly supported, regularised configurations $\varphi=(f_{\mathcal{A}}\mathcal{A},f_{gh}c)$.

If we had chosen a generalised Lagrangian $I=\int_x fL$, the same would not immediately hold. In fact, one can explicitly check that, e.g. for Yang-Mills-type theories, the QME would hold only up to a functional supported on the support of $\mathrm{d}f$, i.e., we would have, for some local density ϕ,

$$\{I,I\}_T-i\hbar\vartriangle I=\int_x\rho(\varphi,\varphi^{\ddagger})\mathrm{d}f\,.$$

The QME would only hold in the adiabatic limit $f \longrightarrow 1$ over the whole $\mathcal{M}$.

In the following, for notational convenience, we are always implicitly assuming a regularisation $F(f\varphi)$.

3.4.2 *Algebraic renormalization and the Master Ward Identity*

Just as in the scalar case, having defined the time-ordered products for regular quantities, the procedure of algebraic renormalization provides an extension to local functionals.

In particular, given that the renormalized time-ordered products on local functionals of the fields are constructed as in Section 2.8.2, the renormalized time-ordered products between vectors can be defined as in equation (3.18),

$$T_r(X) := \int_x T_r(X(x)) \frac{\delta}{\delta\varphi(x)} .$$

From the renormalized time-ordered products one can define the renormalized $S-$matrix $S_r(V) := e_{T_r}^{\frac{i}{\hbar}V}$ and the renormalized Bogoliubov map.

The classical BV structure is then translated at the quantum level by the use of the renormalized time-ordered product T_r, and we can define the renormalized time-ordered antibracket and the renormalized time-ordered Koszul map in analogy with their non-renormalized counterparts.

The main problem in the extension of the BV formalism to local functionals is the BV laplacian $\triangle$, which give raise to a divergence proportional to $\delta(0)$ when it acts on local functionals. Thus, it must be replaced by a more regular operator on local functionals. The relation between the $\star-$product and the T_r-antibrackets is thus substituted by the *Master Ward Identity* (MWI) [54, 96, 156]

$$\{S_r(V)\cdot_{T_r}, I_0\}_\star = \frac{i}{\hbar} S_r(V) \cdot_{T_r} \left[\{V, I_0\}_{T_r} + \frac{1}{2}\{V, V\}_{T_r} + i\hbar \triangle (V) \right] , \tag{3.40}$$

where S_r is the renormalized S-matrix. Here $\triangle$ is understood as the *anomaly* and is related to the *renormalized BV Laplacian* $\triangle_V$ by means of:

$$\triangle_V(X) \doteq \frac{d}{d\lambda} \triangle (V + \lambda X)\big|_{\lambda=0} .$$

The MWI implicitly defines the *renormalized BV laplacian* $\triangle_V$ as a linear map $\triangle_V : T_r(\mathcal{BV}_{loc}[[\lambda]]) \longrightarrow \mathcal{BV}_{loc}[[\lambda]]$ with $\operatorname{supp} \triangle_V(X) \subset \operatorname{supp} X \cap \operatorname{supp} V$.

In the following, to avoid heavy notation we will denote the renormalized time-ordered product and the renormalized BV laplacian as the non-renormalized ones, $\cdot_T$ and $\triangle$; whenever we are dealing with local functionals, it will always implicitly assumed that we am using the renormalized definitions.

The derivation of the QME (3.35) holds also for local functionals, since it follows from properties of the $\star-$products alone. However, Eq. (3.37) needs to be modified, due to the appearance of the singular BV laplacian. From the MWI (3.40), it follows that

$$\{S(V), I_0\}_\star = \{S(V), I_0\}_{T_r} - i\hbar \triangle_V (V) .$$

By manipulations similar to the non-renormalized case, we find that a sufficient condition to satisfy the QME is given by the *renormalized QME* (rQME), taking the simple form

$$\frac{1}{2}\{I, I\}_{T_r} = i\hbar \triangle_V (V) . \tag{3.41}$$

Notice that the rQME can be re-expressed in an equivalent formulation as [212]

$$S(-V) \cdot_T \{S(V), I_0\}_\star = 0\,. \tag{3.42}$$

3.5 PERTURBATIVE AGREEMENT FOR YANG-MILLS-TYPE THEORIES

In Yang-Mills theories and gravity, the BV differential can be canonically split in two terms

$$s = \delta + \gamma\,.$$

When dealing with perturbative quantization one needs to split the action in a quadratic and an interacting part; however, just as for the scalar case, this split is non-canonical, and so it is the corresponding split of the BV differential. Here, we followed the convention [116, 119]

$$s = \delta_0 + s_V\,,$$

that is, the linearised BV differential is defined by $\delta_0 = \{\cdot, I_0\}$, and I_0 is the quadratic part of the action that *does not contain antifields*; it follows that the linearised BV differential simply corresponds to the linearised Koszul map, the Koszul map for the free EOMs. The split corresponds to the split in the action $I = I_0 + V$, where I_0 is quadratic and does not contain antifields. On the other hand, the most common choice is the split [212]

$$s = \delta_0 + \gamma_0 + \tilde{s}_V\,,$$

that is, the linearised BV differential $s_0 := \delta_0 + \gamma_0$ actually contains two terms: the first is the linearised Koszul map $\delta_0 = \{\cdot, I_0\}$ as before, while the second term corresponds to the linearised BRST differential and it equals the contribution to the action that is linear in the antifields and linear in the fields. It thus corresponds to a split of the action into

$$I = I_0 + \theta_0 + \tilde{V}\,,$$

where $\theta_0 = \int_x \varphi^\ddagger K\varphi$ is a quadratic term linear in the antifields.

The ambiguity in the choice of the BV split is analogous to the non-canonical choice of quadratic action that appears already for scalar theories. In that case, the perturbative agreement [95, 161, 254] guarantees that the quantum theory does not depend on the split. In the case of gauge theories, even if the same argument can be applied for terms that are quadratic in the fields, the situation is more complicated by the compatibility of the split with gauge symmetries at the quantum level. The possible source of issues in this case is a term that is linear in the fields and linear in the antifields, as θ_0.

In the following, we prove that the gauge symmetries at the quantum level are independent on the choice of the split of the action, either into $I = I_0 + V$ or $I = I_0 + \theta_0 + V$. In fact, the renormalized Quantum Master Equation (rQME) for one split implies the validity of the rQME for the other. Notice that the proof is based on the rQME, so it holds for local functionals as well as regular functionals. The result represents an extension of the perturbative agreement to Yang-Mills-type theories, including Yang-Mills and gravity, since it proves that the quantum algebra of observables is independent on the non-canonical choice of keeping a term linear in the antifields in the free or in the interaction term in the action.

The proof is based on the property of the time-ordered product of the S−matrix of a linear observable with a generic functional F, Eq. (2.51). We re-express it in this case as follows: consider a functional linear in the fields $\theta_0 = \int \varphi^\ddagger K\varphi$, for some K; then from equation (2.51) we have

$$S(\theta_0) \cdot_T F = e^{-\frac{1}{2}\Delta_F(\varphi^\ddagger K, \varphi^\ddagger K)} e^{\frac{i}{\hbar}\theta_0} F(\varphi + i\Delta_F(K\varphi^\ddagger)) \,. \tag{3.43}$$

Proposition 3.1. *The rQME (3.42) for the action $I = I_0 + V = I_0 + \theta_0 + \tilde{V}$ can be written in two, equivalent ways, depending on the choice of the split between quadratic and interaction terms:*

$$S(-(\tilde{V} + \theta_0)) \cdot_T \{S(\tilde{V} + \theta_0), I_0\} = S(-\tilde{V}) \cdot_T \{S(\tilde{V}), I_0 + \theta_0\} \,. \tag{3.44}$$

The l.h.s is the rQME in which the quadratic action is I_0, while the r.h.s is the rQME for the quadratic action $I_0 + \theta_0$.

Proof. The proof works by direct inspection. We first rewrite

$$\{S(\tilde{V} + \theta_0), I_0\} = \int_x \frac{\delta S(\tilde{V} + \theta_0)}{\delta \varphi^\ddagger} \frac{\delta I_0}{\delta \varphi} = \frac{i}{\hbar} \int_x S(\tilde{V} + \theta_0) \cdot_T \frac{\delta(\tilde{V} + \theta_0)}{\delta \varphi^\ddagger} \frac{\delta I_0}{\delta \varphi} \,.$$

Now, we can use Eq. (3.43) identifying $F(x) = S(\tilde{V}) \cdot_T \frac{\delta}{\delta \varphi^\ddagger(x)}(\tilde{V} + \theta_0)$, obtaining

$$\{S(\tilde{V} + \theta_0), I_0\} = \frac{i}{\hbar} e^{\frac{i}{\hbar}\theta_0 - \frac{1}{2}\Delta_F(\varphi^\ddagger K, \varphi^\ddagger K)} \int_x F(x) \frac{\delta I_0}{\delta \varphi^\ddagger(x)} \,.$$

Then we have

$$\begin{aligned} - i\hbar S(-(\tilde{V} + \theta_0)) \cdot_T \{S(\tilde{V} + \theta_0), I_0\} = \\ S(-\tilde{V}) \cdot_T S(-\theta_0) \cdot_T \left[e^{\frac{i}{\hbar}\theta_0 - \frac{1}{2}\Delta_F(\varphi^\ddagger K, \varphi^\ddagger K)} \int_x F(x) \frac{\delta I_0}{\delta \varphi^\ddagger(x)} \right] \,. \end{aligned}$$

We can now apply again Eq. 3.43, identifying the functional F with the expression in square brackets, to get

$$\begin{aligned} &S(-(\tilde{V} + \theta_0)) \cdot_T \{S(\tilde{V} + \theta_0), I_0\} \\ &= \frac{i}{\hbar} S(-\tilde{V}) \cdot_T \left[e^{\frac{i}{\hbar}\left(\theta_0(\varphi) - \theta_0(\varphi - i\Delta_F \varphi^\ddagger K)\right)} \int_x S(\tilde{V}) \cdot_T \frac{\delta(\tilde{V} + \theta_0)}{\delta \varphi^\ddagger} \frac{\delta}{\delta \varphi} \left(I_0 - iP_0 \Delta_F \varphi^\ddagger K \right) \right] , \end{aligned}$$

where we used the fact that $\frac{\delta I_0}{\delta \varphi} = P_0 \varphi$. To analyse the last expression, we first notice that

$$\theta_0(\varphi - i\Delta_F \varphi^\ddagger K) = \int_x \varphi^\ddagger K(\varphi - i\Delta_F \varphi^\ddagger K) = \theta_0 - i\Delta_F \varphi^\ddagger K K^2 = \theta_0 \,,$$

because K does not act on the antifields and $K^2 = 0$ because it is the linearised BRST operator, which is nilpotent by construction. Since $P_0 \Delta_F = -i\delta$ we then have

$$\begin{aligned} &S(-(\tilde{V} + \theta_0)) \cdot_T \{S(\tilde{V} + \theta_0), I_0\} \\ &\qquad = \frac{i}{\hbar} S(-\tilde{V}) \cdot_T \int_x S(\tilde{V}) \cdot_T \frac{\delta(\tilde{V} + \theta_0)}{\delta \varphi^\ddagger} \frac{\delta}{\delta \varphi} (I_0 + \theta_0) \,. \end{aligned} \tag{3.45}$$

We finally have to check the term

$$\int_x S(\tilde{V}) \cdot_T \frac{\delta\theta_0}{\delta\varphi^\ddagger}\frac{\delta}{\delta\varphi}(I_0+\theta_0)$$
$$= S(\tilde{V}) \cdot_T \int_x \frac{\delta\theta_0}{\delta\varphi^\ddagger}\frac{\delta}{\delta\varphi}(I_0+\theta_0) + iS(\tilde{V}) \cdot_T \int_x \frac{\delta\tilde{V}}{\delta\varphi}\Delta_F\frac{\delta^2\theta_0}{\delta\varphi\delta\varphi^\ddagger}\frac{\delta}{\delta\varphi}(I_0+\theta_0)\,.$$

In the above expression, we expanded the time-ordered product, recalling that θ_0 is linear in the fields φ.

The first term in the above expression vanishes, since

$$\int_x \frac{\delta\theta_0}{\delta\varphi^\ddagger}\frac{\delta}{\delta\varphi}(I_0+\theta_0) = \frac{1}{2}\{I_0+\theta_0, I_0+\theta_0\} = 0\,,$$

where the last expression is vanishing for the linearised CME. Using $\Delta_F\frac{\delta I_0}{\delta\varphi} = -i\varphi$ and $K^2 = 0$ we then have

$$i\int_x S(\tilde{V}) \cdot_T \frac{\delta\theta_0}{\delta\varphi^\ddagger}\frac{\delta}{\delta\varphi}(I_0+\theta_0) = S(\tilde{V}) \cdot_T \int_x \frac{\delta\tilde{V}}{\delta\varphi}\frac{\delta\theta_0}{\delta\varphi^\ddagger}\,.$$

Substituting the above equality in (3.45) we finally arrive at the result:

$$S(-(\tilde{V}+\theta_0)) \cdot_T \{S(\tilde{V}+\theta_0), I_0\} = S(-\tilde{V}) \cdot_T \{S(\tilde{V}), I_0+\theta_0\}\,. \tag{3.46}$$

□

4 Generating functionals in pAQFT

4.1 GENERATING FUNCTIONALS FOR SCALAR THEORIES

In the last two chapters, we reviewed the construction of interacting observables as formal power series in the free algebra, both for scalar and gauge theories. Now, we start discussing the new results of this thesis. In this Chapter, we introduce the generating functionals for (connected) Green's functions and the effective action on globally hyperbolic spacetimes and generic Hadamard states. We show that they share many of the important properties of their flat space counterparts. After that, we introduce a local regulator term, we define the regularised generating functionals, and we prove some properties of the effective average action. Again, we find generalisations of the known properties in flat space.

For simplicity, we start with scalar field theories, to focus on the main points in the discussion. The generalisation to gauge theories is straightforward, and discussed at the end of the Chapter, in Section 4.4.

In the last two chapters, we have seen that interacting correlation functions are given by the action of the Bogoliubov map,

$$\langle T : \varphi(x_1) \dots \varphi(x_n) :\rangle_\omega := \omega \left(R_V(\varphi(x_1) \cdot_T \dots \cdot_T \varphi(x_n)) \right) ,$$

where the angle bracket denotes the interacting expectation value in some state, in physicists' notation.

The computation of Green's functions in the pAQFT formalism now proceeds similarly to standard QFT computations: first choose a state, that in turn defines the propagators of the theory when the algebra is realised with the corresponding $\star-$product; then expand the Bogoliubov map in Feynman diagrams, where now the internal lines include also an oriented line corresponding to $\star-$product contractions, up to the desired order; and finally evaluate the Feynman diagrams on the chosen state.

In the case of quasi-free states for the free theory, the computation is completely analogous to the Minkowski case: the choice of the state reduces to the choice of propagators; computing the products is equivalent to the Wick's theorem for normal-ordering; and the evaluation on the state means evaluating on the vanishing field configuration $\varphi = 0$, so that only completely contracted quantities contribute. The end result is a sum of Feynman diagrams.

In flat space QFT, mainly developed in the vacuum state representation, the generating functionals are objects that compactly store the information on correlation functions. An exact knowledge of the generating functional of all correlation functions would allow for a reconstruction of the state evaluation of any observable (cf. Remark 2.1). Naturally, the computation of generating functionals is almost never possible; their evaluation proceeds via some approximation scheme, e.g. a perturbative expansion in the coupling constant λ or a loop expansion in the Planck constant $\hbar$. In this regard, the objective of the functional Renormalization Group is precisely the non-perturbative evaluation of one of these generating functionals, the effective action, as a solution of a functional differential equation, usually the Wetterich equation. However, the fRG simply transfers the difficulties of the generating functionals in the difficulties of the Wetterich equation, which can almost never be exactly solved; the advantage of the fRG lies in the crucial fact the Wetterich equation allows for approximation schemes which are non-perturbative in the coupling constant λ, and thus can access non-perturbative effects that are out of reach in perturbation theory.

Let's start with a review of the standard definitions [244]. On flat Minkowski spacetime, consider a theory described by an action I, coupled with arbitrary, classical sources j via the linear term $J = \int_x j(x)\varphi(x)$. In our notation, the classical sources corresponds to the smeared linear field $\varphi(j)$ with some compactly supported smooth function $j \in C_c^\infty(\mathcal{M})$. The complete vacuum-to-vacuum amplitude in the heuristic path-integral formulation is

$$\langle\Omega|\Omega\rangle_j \text{“} := \text{”} \int \mathcal{D}\varphi e^{\frac{i}{\hbar}(I(\varphi)+J(\varphi))} \,,$$

where the l.h.s. denotes the inner product between the GNS vector for the vacuum state in some Hilbert space representation of the $*-$algebra. The generating functional for correlation functions is usually defined as the normalised vacuum-to-vacuum amplitude,

$$\mathcal{Z}(j) := \frac{1}{\langle\Omega|\Omega\rangle_{j=0}}\langle\Omega|\Omega\rangle_j \,.$$

The functional derivatives of $\mathcal{Z}(j)$, for vanishing sources, give the correlation functions in the theory described by the action I,

$$\hbar^n \frac{\delta^n}{i^n \delta j(x_1)\cdots\delta j(x_n)} \mathcal{Z}(j)\Big|_{j=0} \text{“} = \text{”} \frac{1}{\int \mathcal{D}\varphi e^{\frac{i}{\hbar}I}} \int \mathcal{D}\varphi e^{\frac{i}{\hbar}I(\varphi)}\varphi(x_1)\ldots\varphi(x_n) = \langle T\left(\varphi(x_1)\ldots\varphi(x_n)\right)\rangle \,.$$

It is easy to translate the above formulas to the algebraic setting, based on our discussion in Section 2.11. Notice that the path integral automatically provides the time ordered correlation functions, or *Green's functions*. In the algebraic formalism, this will require the use of the $S-$matrix. Moreover, the path integral formulation almost always assumes a vacuum state in Minkowski background; therefore, to make the connection clearer, we first consider this special case, and then we generalise to Hadamard states on curved spacetimes.

Denoting by ω_0 the vacuum state on Minkowski spacetime, by direct analogy with the path-integral formulas, the generating functional for the interacting time-ordered products in Minkowski vacuum can be written as

$$\mathcal{Z}(j) := \frac{\omega_0(S(V+J))}{\omega_0(S(V))} \,. \tag{4.1}$$

The above formula satisfies the defining condition for the generating functional: its functional derivatives at vanishing sources are the time-ordered interacting correlation functions. To see this, by direct computation we get

$$\hbar^n \frac{\delta^n}{i^n \delta j(x_1) \cdots \delta j(x_n)} \mathcal{Z}(j)\Big|_{j=0} = \frac{1}{\omega_0(S(V))} \omega_0 \left(S(V) \cdot_T \varphi(x_1) \cdot_T \cdots \cdot_T \chi(x_n)\right) .$$

Using the Gell-Mann-Low formula (2.55), the generating functional $\mathcal{Z}$, in Minkowski vacuum and in the adiabatic limit, fulfils the defining property that its functional derivatives give time-ordered, interacting correlation functions,

$$\hbar^n \frac{\delta^n}{i^n \delta j(x_1) \cdots \delta j(x_n)} \mathcal{Z}(j)\Big|_{j=0} = \omega_0 \circ R_V(\varphi(x_1) \cdot_T \cdots \cdot_T \chi(x_n)) . \tag{4.2}$$

However, as already discussed, the Gell-Mann-Low formula holds only in the very particular case of the adiabatic limit of the massive vacuum state in Minkowski. It greatly reduces the complexity in the evaluation of $\omega_0(R_V(\varphi))$, as it requires to compute only time-ordered products, but it fails in generic states and on general backgrounds. It follows that $\mathcal{Z}$ cannot be a suitable definition for the generating functional in the general case.

The two key properties of a generating functional $Z(j)$ are that i) it should fulfil the defining property (4.2), on any Hadamard state and in curved spacetimes, and ii) it should reduce to formula (4.1) for the case of the vacuum state on Minkowski spacetime.

Definition 4.1 (Generating functional of Green's functions)**.** Consider an action functional $I(\varphi) = I_0(\varphi) + V(\varphi)$, and a $*-$algebra $\mathcal{A}$ on some globally hyperbolic spacetime $(\mathcal{M}, g)$, with $\star-$product arising from the propagators for the linearised action I_0. Let $\omega : \mathcal{A} \to \mathbb{C}$ be an arbitrary Hadamard state on the free algebra, and let $S(V)$ be the renormalized $S-$matrix associated with the interacting Lagrangian V. The *generating functional for interacting time-ordered correlation functions (interacting Green's functions)* $Z(j)$ is defined by

$$Z(j) := \omega(S_V(J)) = \omega[S(V)^{-1} \star S(V+J)] = \omega[R_V S(J)] .$$

Due to its analogy with the partition function of statistical mechanics, the generating functional is denoted by Z from the German word *Zustandssumme*.

Eq. (4.2) is verified by direct inspection. For $\omega = \omega_0$ (Minkowski vacuum) and in the adiabatic limit, the definitions given in Eq. (4.1) and in Definition 4.1 coincide thanks to the Gell-Mann-Low formula.

To justify our definition of Z, recall that on regular functionals, one can introduce the interacting star product as [117]

$$F \star_V G := R_V^{-1}(R_V(F) \star R_V(G)) .$$

On the other hand, for a given state ω of the free theory, the interacting state is defined by $\omega_V := \omega \circ R_V$, so that the $n-$point correlation functions for interacting fields in the interacting state ω_V are given by:

$$\omega_V(\varphi(x_1) \star_V \ldots \star_V \varphi(x_n)) = \omega(R_V(\varphi(x_1)) \star \ldots \star R_V(\varphi(x_n)) .$$

However, the time-ordered version of $\star_V$ coincides with $\cdot_T$ (see e.g. [95]), and so the time-ordered $n-$point functions in the interacting theory are

$$\omega_V(\varphi(x_1) \cdot_T \ldots \cdot_T \varphi(x_n)) = \omega \circ R_V(\varphi(x_1) \cdot_T \ldots \cdot_T \varphi(x_n)) ,$$

which is proportional to the nth-functional derivative with respect to j of Z,

$$\hbar^n \left.\frac{\delta^n Z}{i^n \delta j(x_1)\ldots\delta j(x_n)}\right|_{j=0} = \omega \circ R_V\left(\varphi(x_1)\cdot_T \ldots \cdot_T \varphi(x_n)\right) . \tag{4.3}$$

It follows that the functional derivatives of Definition 4.1 exactly give the interacting expectation value of the Green's functions. In this sense, the defining property of Z as the generating functional for the Green's functions is satisfied also by our definition, which generalises the usual approach to generic states and possibly curved spacetimes.

4.1.1 *Generating functional for connected Green's functions*

The generating functional of time-ordered correlation functions contains redundant information, in that it counts as different those Feynman diagrams that differ only by a permutation of vertices. A more efficient way of storing the information on correlation functions, counting only connected diagrams, is through the *generating functional of connected Green's functions*, defined as the natural logarithm of Z.

Definition 4.2 (Generating functional of connected Green's functions)**.** From the generating functional $Z(j)$ of Definition 4.1, the *generating functional of connected Green's functions* $W(j)$ is defined by the formula

$$e^{\frac{i}{\hbar}W(j)} := Z(j)\,.$$

The definition gives connected correlation functions also in the algebraic approach, since the connected part of ω is defined by

$$\omega^c(\varphi(x_1)\star\cdots\star\chi(x_n)) := \frac{\hbar^n}{i^n}\frac{\delta^n}{\delta j(x_1)\cdots\delta j(x_n)}\log\omega[\exp_\star(i\chi(j))]\Big|_{j=0} . \tag{4.4}$$

Similarly, the connected time-ordered functions of ω are defined by

$$\omega^c(\varphi(x_1)\cdot_T\cdots\cdot_T\chi(x_n)) := \frac{\hbar^n}{i^n}\frac{\delta^n}{\delta j(x_1)\cdots\delta j(x_n)}\log\omega[S(\varphi(j))]\Big|_{j=0} . \tag{4.5}$$

It follows that W is actually the generator of interacting, connected Green's functions, since its functional derivatives at vanishing sources give

$$\hbar^n \left.\frac{\delta^n}{i^n\delta j(x_1)\ldots\delta j(x_n)}W(j)\right|_{j=0} = (\omega^c\circ R_V)(\varphi(x_1)\cdot_T\ldots\cdot_T\chi(x_n))\,. \tag{4.6}$$

A remarkable property of the generating functional in Minkowski vacuum, $\mathcal{Z}$, defined in Eq. (4.1) is that Eq. (4.6) still makes sense for $j\neq 0$; in fact, thanks to the Gell-Mann-Low formula (2.55),

$$\frac{\hbar^n}{i^n}\frac{\delta^n}{\delta j(x_1)\cdots\delta j(x_n)}\log\mathcal{Z}(j) = (\omega_0^c\circ R_{V+J})(\varphi(x_1)\cdot_T\cdots\cdot_T\chi(x_n))\,. \tag{4.7}$$

Unfortunately, $Z(j)$ defined in Definition 4.1 (or, equivalently, W) does not satisfy the property (4.7), as it is possible to see already for the first derivative:

$$\begin{aligned}\hbar\frac{\delta}{i\delta j(x)}\log Z(j) &= \frac{\omega(S(V)^{-1}\star[S(V+J)\cdot_T\chi(x)])}{\omega(S_V(J))}\\ &= \frac{\omega\left(S_V(J)\star R_{V+J}(\varphi(x))\right)}{\omega(S_V(J))} =: \omega_J\left(R_{V+J}(\varphi(x))\right)\,,\end{aligned} \tag{4.8}$$

where $\omega_J : \mathcal{A}[[V]] \to \mathbb{C}[[V]]$ is a well-defined linear functional which, however, fails to be positive. In fact, in the next Lemma we show that property (4.7) cannot be fulfilled in states that does not fulfil the Gell-Mann-Low formula, and therefore it is not a sensible condition to require for general states.

Lemma 4.1. *If ω does not fulfil the Gell-Mann-Low formula given in Eq. (2.55), there is no functional $\zeta(j)$ satisfying Eq. (4.7).*

Proof. Let $\zeta(j)$ be any generating functional fulfilling (4.7) for all $n \in \mathbb{N}$. For $n = 1$ we have

$$-i\hbar \log \zeta(j)^{(1)}(x) = \omega^c \circ R_{V+J}(\varphi(x)) \,. \tag{4.9}$$

By direct inspection we have

$$\begin{aligned}
A(x_1, x_2) &:= \hbar^2 \frac{\delta^2}{i^2 \delta j(x_1) \delta j(x_2)} \log \zeta(j) \\
&= \frac{1}{2} \frac{\delta}{\delta j(x_1)} \omega(R_{V+J}\varphi(x_2)) + x_1 \leftrightarrow x_2 \\
&= \frac{1}{2} [\omega(R_{V+J}[\varphi(x_1) \cdot_T \chi(x_2)]) \\
&\qquad - \omega(R_{V+J}\varphi(x_1) \star R_{V+J}\varphi(x_2))] + x_1 \leftrightarrow x_2 \\
&= \frac{1}{2} \operatorname{sign}(t(x_1) - t(x_2)) \omega([R_{V+J}\varphi(x_1), R_{V+J}\varphi(x_2)]_\star) \,,
\end{aligned}$$

where we used the symmetry of the left-hand side in x_1, x_2. By Eq. (4.7) for $n = 2$ the right-hand side should be equal to

$$B(x_1, x_2) := \omega(R_{V+J}[\varphi(x_1) \cdot_T \chi(x_2)]) - \omega(R_{V+J}\varphi(x_1))\omega(R_{V+J}\varphi(x_2)) \,,$$

which in general is not the case: actually, at zeroth order in the perturbation parameter, for quasifree states and for $t(x_1) > t(x_2)$

$$A(x_1, x_2) = \frac{i}{2} \operatorname{sign}(t(x_1) - t(x_2)) \Delta(x_1, x_2)), \qquad B(x_1, x_2) = \Delta_F(x_1, x_2),$$

and Δ_S, the symmetric part of the two-point function, is present in B but not in A. □

Even though Eq. (4.7) does not hold for Z, the non-positive functionals ω_J defined in Eq. (4.8) behave as j–dependent states on an algebra $\mathcal{A}_\circledast[[V]]$ isomorphic to $\mathcal{A}[[V]]$. In other words, in the presence of non-vanishing sources, the generating functional Z produces the Green's functions on a deformed algebra, in a j–dependent state, as can be seen by the following proposition.

Proposition 4.2. *Let $U := S_V(J)/\omega(S_V(J)) \in \mathcal{F}_{\mu c}[[V]]$. Let $\mathcal{A}_\circledast$ be the $*$-algebra obtained equipping $\mathcal{F}_{\mu c}[[V]]$ with the product $\circledast$ and the $*$-involution $*_\circledast$*

$$A \circledast B := A \star U \star B \,, \qquad A^{*_\circledast} := U^* \star A^* \star U^* \,. \tag{4.10}$$

Then, $\mathcal{A}_\circledast$ is a unital $*$-algebra and ω_J, defined by equation (4.8), is a state on $\mathcal{A}_\circledast$. Moreover, the map $\varsigma : \mathcal{A} \to \mathcal{A}_\circledast$ defined by $\varsigma(A) := U^* \star A$ is a $*$-isomorphism, and $\varsigma^* \omega_J = \omega$.

Proof. By direct inspection, $\circledast$ is associative with unit given by $\mathbf{1}_\circledast := U^*$ — notice that U is unitary as $V, J \in \mathscr{F}_{\mathrm{loc}}$. Moreover $\circledast$ and $*_\circledast$ are compatible, meaning that $(A \circledast B)^{*_\circledast} = B^{*_\circledast} \circledast A^{*_\circledast}$. Since $*_\circledast$ is an involution, we have that $\mathscr{A}_\circledast$ is a unital $*$-algebra.

Now, let $\varsigma \colon \mathscr{A} \longrightarrow \mathscr{A}_\circledast$ be defined by $\varsigma(A) := U^* \star A$. Then ς is linear and invertible, and it holds that

$$\varsigma(A) \circledast \varsigma(B) = U^* \star A \star U \star U^* \star B = \varsigma(A \star B) \,.$$

It follows that ς is a $*$-isomorphism. Finally

$$\varsigma^* \omega_J(A) := \omega_J(\varsigma(A)) = \omega(A) \,.$$

□

Proposition 4.2 shows that the j-functional derivatives of the generating functional $Z(j)$, are physically meaningful for $j \neq 0$: they coincide with the Green's functions for the state ω_J on $\mathscr{A}_\circledast$. Notice that, as $\mathscr{A}_\circledast[[V]] \simeq \mathscr{A}[[V]]$, the latter state can be interpreted as a state on $\mathscr{A}[[V]]$ too.

4.1.2 *Effective action*

It is usually convenient to take a step further in the construction of the generating functionals, defining the Legendre transform of W, the *effective action*.

Starting from the generating functional $W(j)$, from (4.2), we have

$$\left.\frac{\delta W}{\delta j(x)}\right|_{j=0} = \frac{1}{Z(0)} \omega(R_V(\varphi(x))) \,.$$

We can now define the *mean field* as a function of the classical sources j, as

$$\phi(j) := \frac{\delta W}{\delta j(x)} \,.$$

Proposition 4.3 shows that the relation between j and ϕ can be inverted, to get the current $j = j_\phi$ as a function of ϕ, at least in perturbation theory. Therefore, it is possible to implicitly define the ϕ−dependent current j_ϕ as the solution of the last equation, and define the mean fields as independent variables:

$$\phi := \left.\frac{\delta W}{\delta j}\right|_{j=j_\phi} \,. \tag{4.11}$$

Since the relation between ϕ and j is invertible, it is possible to compute the Legendre transform of W with respect to j, defining the effective action.

Definition 4.3 (Effective action)**.** The *effective action* is defined as the Legendre transform of W with respect to the sources j, taking the mean fields ϕ as independent variables:

$$\tilde{\Gamma}(\phi) = W(j_\phi) - J_\phi(\phi) \,,$$

where $J_\phi(\phi) = \int_x j_\phi(x)\varphi(x)$.

The main object of investigation in the fRG is a deformation of the effective action, the *effective average action*, by an infra-red regulator term that cuts off fields with momenta lower than some scale k.

The effective action satisfies many important properties, that makes it a fundamental object to investigate the quantum properties of a given theory. However, since most of these properties are inherited from the effective average action, and the proofs in both cases follow the same steps, we will directly discuss these properties for the effective average action.

4.2 REGULARISED GENERATING FUNCTIONALS

4.2.1 *Regulator*

A direct computation of the generating functional $Z(j)$ or of the effective action is usually not possible. The main idea of the Wilsonian renormalization group is to introduce a cut-off in the generating functional, so that it generates only short-distance (or high-energy) correlation functions. This is usually done introducing an artificial scale k, so that the modes with energy $E < k$ are suppressed. The cut-off is then lowered, so that the generating functional progressively takes into account correlation functions at longer and longer distances. In this way, the original bare action I generates an entire family of effective theories, each at a different scale k, described by a corresponding effective average action Γ_k. Although the physical interpretation of theories at finite k is not entirely clear, especially in curved spacetimes, it is possible to prove (see Section 4.3.2) that in the limit of infinite scale $k \longrightarrow \infty$, all quantum correlations are suppressed, and the effective average action reduces to the bare action, modulo an irrelevant (and finite) constant $\Gamma_\infty = I + c$. Removing the regulator gives the quantum effective action in which all quantum fluctuations are taken into account, $\Gamma_0 = \Gamma$. The effective average action thus interpolates between the classical and quantum actions. The derivative of the effective average action with respect to the scale k then defines the RG flow.

In order for the RG flow equation to be useful, the regulator Q_k needs to have certain properties [178, 179]:

- ◇ it should vanish in the limit $k \longrightarrow 0$, so that the original theory is recovered in that limit;
- ◇ it should suppress all the quantum fluctuations in the limit $k \longrightarrow \infty$, so that in that limit one obtains a theory governed by a classical action;
- ◇ at finite k, it should behave as an effective mass term to control potential infrared divergences;
- ◇ at finite k, it should vanish at high momentum to not alter drastically the short distance behaviour of the correlation functions.

In the original approach and for Euclidean field theories, Q_k is chosen as a momentum cut-off [246]. One of the most used sharp cut-offs assumes a simple expression in the Fourier transform of its second functional derivative, as $\hat{Q}_k^{(2)}(p) = -(k^2 - p^2)\theta(k^2 - |p|^2)$, where θ is the Heaviside step function. This regulator meets all the requirements listed above, and furthermore it permits to keep the technical difficulties in practical computations under control [177].

When such a regulator is used, the Wetterich equation fir the effective average action Γ_k has a peak in a vicinity of $|p|^2 \sim k^2$, while both high and low momentum modes are suppressed. This gives rise to a flow in Wilsonian sense, for which at scale k only the spectrum of the various propagators at momentum squared equal to k^2 matters, thus providing an interpretation of the regularisation at scale k as a coarse-graining procedure.

Unfortunately, a regulator local in momentum space is non-local in position space. For this reason, it is difficult to extend similar techniques to field theories on generic curved backgrounds. Similarly, if the state in which the theory is constructed is not a vacuum, it is not clear if this choice of regulator completely regularises the theory. This happens, for example, with the Wetterich equation in the case of thermal fields [178, 236].

Moreover, a non-local regulator in spacetimes with Lorentzian signature can alter the principal symbol of the equation of motion governing the evolution. If this is the case, and the regularised equations of motion are not normally hyperbolic, the propagators of the regularised theory do not even exist in general, even in the non-interacting case.

In Lorentzian spacetimes, even in the case when the Fourier transform is possible, the energy-momentum covector p has a norm that is not positive definite, but it can be time-like, space-like, or null. Therefore, the addition of a positive regulator, local in momentum space, would not be sufficient to shift the energy spectrum away from zero, as the combination $p^2 + k^2$ still has zeroes. A local regulator in momentum thus does not completely remove the IR divergences in Lorentzian spacetimes. It is still possible to regulate only the time-like or only the space-like [16] components of the momentum, or even more complicated, mixed schemes [109], but the physical interpretation of the regulator in this cases becomes less clear.

Instead of a non-local regulator, it is possible to use a local, mass term regulator $Q_k^{(2)} \sim k^a$ at the price of introducing a different regularisation procedure of the UV divergences [179]. However, this last requirement is not an issue in the algebraic approach to interacting field theories: thanks to normal-ordering and the Epstein-Glaser renormalization, the time-ordered products are automatically ultraviolet finite [57].

If Q_k is local, the S-matrix (used to build interacting fields needed to describe the generating functional $Z(j)$) is formally unitary. This implies in particular that, in the Lorentzian case, the effective action obtained from that $W(j)$ is real-valued. On the contrary, the S-matrix constructed with non local regulators is in general non-unitary (for states which are not the Minkowski vacuum, see for example [240]) and thus the corresponding effective action could be complex-valued, with an imaginary contribution due to the form of the non-local regulator, and not to intrinsic properties of the investigated physical system.

Finally, a mass term regulator appears to be useful whenever one is interested in preserving the analytical structure of the propagator.

A regulator local in position space is non-local in momentum. It follows that the source term of the flow equation for the effective action is not peaked around momenta of scale k any more, and the picture of progressive coarse-graining of Wilsonian renormalization is less clear. In the simplest case of a regulator that, modulo an IR cut-off, is just a mass term, all quantum fluctuations are suppressed by the same artificial mass term. The net effect of the regulator is to lift the mass-shell away from zero, regularising IR divergences or improving convergences properties if the theory were already massive.

In the important limit of $k \to \infty$, all quantum fluctuations are suppressed, also with the use of a local regulator. Intuitively, the reason is that changes in the scale k effectively behave as changes in the mass parameter in correlation functions:

the flow of the effective average action can then be interpreted as a flow in the mass parameter space. Since the propagators are proportional to the inverse of the mass, so that, in the limit of infinite mass, propagators collapse to a point and the quantum theory reduces to the classical one. We will prove this result more rigorously in Section 4.3.2.

Therefore, in the limit $k \longrightarrow \infty$, correlation functions at large distances are effectively suppressed; as k is lowered, the effective correlation length of correlation functions becomes larger, as it is inversely proportional to the effective mass. Thus, the flow in the mass parameter space can still be interpreted as an RG flow, in which correlations at larger distances are progressively taken into account.

The renormalization scale k drives the theory from the microscopic, or bare, action, at $k \longrightarrow \infty$, to the macroscopic one, where all quantum fluctuations are taken into account, at $k = 0$. While the interpretation of the theory at intermediate k is unclear (just as in the Euclidean case), the two important limits of vanishing and infinite scale k are unambiguous. For these reasons, even if the regulator is local in position, the scale derivative of the effective average action still represents an RG flow.

We can now discuss the precise implementation of the regulator term in our formalism. The key property of the regulator is that it is quadratic in the fields, so that the RG flow will be proportional to the interacting propagator. Instead of introducing it directly into the action, the regulator is inserted in the definition of generating functionals, acting as an effective mass on the correlation functions.

Definition 4.4 (Regulator term)**.** The *regulator term* in the generating functional Z is defined as a functional $Q_k \in \mathscr{F}_{\mu c}$ quadratic in the fields, smeared by a function $q_k \in C_c^\infty(\mathcal{M})$:

$$Q_k = -\frac{1}{2}\int_x q_k(x) T\chi(x)^2 \, .$$

The simplest choice for the regulator function is $q_k(x) = k^n f(x)$, where f is a compactly supported smooth function equal to 1 on a large region of the spacetime, and which plays the role of adiabatic cut-off. The regulator is then simply an artificial mass term with mass k adiabatically turned on in the region of spacetime of interest. The power n of the mass depends on the field species: usually, $n = 2$ for bosons and $n = 1$ for fermions.

In the adiabatic limit the cut-off function tends to 1 over the whole spacetime, and Q_k tends to a mass contribution to the field.

Now, the *regularised generating functional* $Z_k(j)$ is defined as

$$Z_k(j) := \omega(S(V)^{-1} \star S(V + J + Q_k)) \, . \tag{4.12}$$

It reduces to $Z(j)$ (4.1) in the limit $k \longrightarrow 0$, since Q_k vanishes.

Remark 4.1. Even if we are changing the effective mass of Green's functions computed from Z_k, we are not changing the propagators of the free theory. In other words, $\mathscr{A}$ still denotes the $*$-algebra associated with the action I_0, so that Q_k acts as a mass deformation on quantum observables computed through the generating functionals, but the $\star$ and T products are k−independent. ■

This regularisation scheme is consistent with the usual IR regularisation one can find in the fRG literature [41]. Actually, if the Gell-Mann-Low formula holds, we can factor the definition of the generating functional into

$$Z_k(j) = \frac{1}{\omega(S(V))}\omega(S(V+J+Q_k)) =: \frac{\mathcal{Z}_k(j)}{\omega(S(V))} , \tag{4.13}$$

which, apart from a normalization constant $\omega(S(V))$, coincides with the usual formulation.

From the definition of Z_k, it is possible to define the effective action in complete analogy with the $k = 0$ case. First, the *regularised generating functional for connected Green's functions* is

$$W_k(j) = -\frac{i}{\hbar} \log Z_k(j) . \tag{4.14}$$

The first derivative of W_k defines the *mean field* $\phi_j(x)$ as a spacetime function functionally depending on j, which we can write in several equivalent ways as

$$\phi_j(x) := \frac{\delta W_k(j)}{\delta j(x)} \tag{4.15}$$

$$= e^{-\frac{i}{\hbar}W_k}\omega\left(S(V)^{-1} \star S(V+Q_k+J) \cdot_T \varphi(x)\right) \tag{4.16}$$

$$= \langle \varphi \rangle . \tag{4.17}$$

In the last relation, the angle brackets (chevrons) denote the weighted expectation value of an interacting operator F, for non-vanishing sources and regulator, which we call *mean value operator* $\langle \cdot \rangle$:

$$\langle F \rangle = e^{-\frac{i}{\hbar}W_k}\omega\left(S(V)^{-1} \star [S(V+Q_k+J) \cdot_T F]\right) . \tag{4.18}$$

The mean value operator takes a single input $\langle \cdot \rangle$, while the angle brackets between two objects $\langle \cdot, \cdot \rangle$ denotes the standard pairing on $\mathcal{M}^{\otimes 2}$, so no confusion should arise from the notation. The mean value is a state evaluation only in the limit $k \to 0$ and for $j = 0$.

The mean value of an observable $\langle F \rangle$ depends on j and k. However, the relation between j and ϕ can be inverted, at all orders in perturbation theory, giving j as a functional of ϕ. The following proposition proves this result. In the Euclidean setting, the same result holds non-perturbatively [255].

Proposition 4.3. *Let ω be a state on $\mathscr{A}$ and let $\phi \in C^\infty(M)$ be such that $\phi = i\Delta_F \tilde{j}_0$, for some $\tilde{j}_0 \in C^\infty_c(M)$, while Δ_F is the Feynman propagator of the theory we are considering. Then there exists a unique $j_\phi \in C^\infty_c(M)[[V]]$ which solves equation (4.11), and it is given by the solution of*

$$-j_\phi = P_0\phi + Q_k^{(1)}(\phi) + \langle TV^{(1)} \rangle , \tag{4.19}$$

obtained by induction on the perturbative order.

Proof. First of all, recall that, by definition,

$$\phi(x) = \langle \varphi(x) \rangle .$$

The action of P_0 in the above relation gives

$$P_0\phi = \frac{1}{Z_k(j)}\omega\big(S(V)^{-1} \star [S(V+J+Q_k) \cdot_T P_0\varphi]\big) , \tag{4.20}$$

since P_0 acts on spacetime-dependent quantities only. Now, the proof proceeds similarly to the case of the Lemma on the interacting field weakly satisfying the interacting EOMs, Lemma 2.1. In fact, the relation between $T-$ and $\star-$products of linear observables in Eq. (2.41), for $A = P_0\varphi$ and $B = S(V + J + Q_k)$, gives

$$P_0\phi(x) = \frac{1}{Z_k(j)}\Big(\omega\big(S(V)^{-1} \star S(V + J + Q_k) \star P_0\varphi(x)\big) + i\omega\big(S(V)^{-1} \star S(V + J + Q_k)^{(1)}(x)\big)\Big) .$$

The first term in parenthesis vanishes, because ω is on-shell and so $\omega(A \star P_0\varphi) = 0$ (see Eq. (2.18)). On the other hand, the derivative of S in the second term can be computed explicitly, leading to

$$P_0\phi = -\langle TV^{(1)} + Q_k^{(1)} + J^{(1)}\rangle .$$

Since J is linear in the field φ, its first derivative gives the classical current $j(x)$, while $\langle Q_k^{(1)}(\varphi)(x)\rangle = Q_k^{(1)}(\phi(x)) = -q_k\phi$ because $Q_k^{(1)}(\varphi)(x) = -q_k(x)\varphi(x)$; it follows that the definition of the mean field, Definition (4.11) is equivalent to Eq. (4.19),

$$-j_\phi = (P_0 - q_k)\phi + \langle TV^{(1)}\rangle .$$

This equation can now be used to obtain j_ϕ from ϕ as a formal power series in V.

Notice that the obtained solution is unique and lies in $C_c^\infty(M)[[V]]$. In fact, at zeroth order in perturbation series, the equation simply gives

$$j_{\phi,0}(x) = -P_0\phi(x) - Q_k^{(1)}(\phi)(x) = -(P_0 - q_k)\phi(x) , \tag{4.21}$$

which is nothing but the free, regularised equation of motion with sources. Moreover, $j_{\phi,0} \in C_c^\infty(M)$ because $P_0\phi = -\tilde{j}_0 \in C_c^\infty(M)$ by hypothesis, and q_k is smooth and with compact support.

If j_ϕ is compactly supported up to order V^{n-1}, it must be smooth at order V^n: in fact, denoting by $j_{\phi,n}$ the solution up to order V^n,

$$j_{\phi,n} = j_{\phi,0} - \frac{1}{Z_k(j_{\phi,n-1})}\omega\big(S(V)^{-1} \star (S(V + Q_k + J_{\phi,n-1}) \cdot_T TV^{(1)})\big)$$

and $TV^{(1)}$ has compact support thanks to the cut-off functions. It follows that Eq. (4.19) can be solved by induction on the perturbative order. □

Remark 4.2. In the limit $k \longrightarrow 0$ the previous proposition implies that the relation between j and ϕ is invertible also in the unregularised case. ■

Since the relation between j and ϕ can be inverted to get the current $j = j_\phi$ as a function of ϕ, we can think the mean value operator as equivalently depending on ϕ and k, where the dependence on ϕ comes through $j(\phi)$.

The second derivative of $W_k^{(2)}$ is proportional to the connected, interacting Feynman propagator

$$-i\hbar W_k^{(2)}(j) = \langle\varphi(x) \cdot_T \varphi(y)\rangle - \phi(x)\phi(y) , \tag{4.22}$$

where we can read either j as a function of ϕ in the l.h.s, or ϕ as a function of j in the r.h.s.

The above relation follows by a direct computation: in fact,

$$\begin{aligned}
-i\hbar\frac{\delta^2 W_k(j)}{\delta j(x) j(y)} &= \frac{1}{Z_k(j)}\omega\Big(S(V)^{-1}\star[S(V+J+Q_k)\cdot_T\varphi(x)\cdot_T\varphi(y)]\Big) \\
-\frac{1}{Z_k(j)^2}\omega\Big(S(V)^{-1}&\star[S(V+J+Q_k)\cdot_T\varphi(x)]\Big)\,\omega\Big(S(V)^{-1}\star[S(V+J+Q_k)\cdot_T\varphi(y)]\Big) \\
&= \frac{1}{Z_k(j)}\omega\Big(S(V)^{-1}\star[S(V+J+Q_k)\cdot_T\varphi(x)\cdot_T\varphi(y)]\Big) - W_k^{(1)}(x)W_k^{(1)}(y)\,.
\end{aligned} \tag{4.23}$$

Inverting the relation between j and ϕ, and finding j_ϕ as a function of the mean fields, the Legendre transform of W_k gives a $k-$dependent effective action,

$$\tilde{\Gamma}_k(\phi) := W_k(j_\phi) - J_\phi(\phi)\,, \tag{4.24}$$

where $J_\phi(\varphi) = \Phi(j_\phi) = \int \mathrm{d}x j_\phi(x)\varphi(x)$. By definition of the Legendre transform, the derivative of $\tilde{\Gamma}_k$ gives the *quantum equations of motion*,

$$\frac{\delta\tilde{\Gamma}_k}{\delta\phi} = \frac{\delta(\Gamma_k + Q_k)}{\delta\phi} = -j_\phi\,. \tag{4.25}$$

Finally, we can translate $\tilde{\Gamma}_k$ subtracting the classical contribution $Q_k(\phi)$, to get the *effective average action*.

Definition 4.5 (Effective average action)**.** The *effective average action* is defined as a modified Legendre transform of W_k with respect to ϕ; in fact, inverting the relation between j and ϕ, considering ϕ as an independent variable, the effective average action is

$$\Gamma_k(\phi) := W_k(j_\phi) - J_\phi(\phi) - Q_k(\phi) = \tilde{\Gamma}_k(\phi) - Q_k(\phi)\,.$$

It is a consequence of the relation between W_k and Γ_k that their second functional derivatives are inverse one of the other.

Lemma 4.4. *Consider $\Gamma_k(\phi)$ as defined in Definition 4.5; it holds that*

$$(\Gamma_k^{(2)} - q_k)W_k^{(2)} = -\delta\,.$$

Proof. From (4.25) and (4.11), it follows that

$$\begin{aligned}
\delta(x,y) = \frac{\delta j_\phi(x)}{\delta j_\phi(y)} &= -\frac{\delta}{\delta j_\phi(y)}\frac{\delta}{\delta\phi(x)}(\Gamma_k + Q_k) \\
&= -\int \mathrm{d}z\frac{\delta\phi(z)}{\delta j_\phi(y)}\frac{\delta}{\delta\phi(z)}\frac{\delta}{\delta\phi(x)}(\Gamma_k + Q_k) \qquad (4.26)\\
&= -\int \mathrm{d}z(\Gamma_k^{(2)} + Q_k^{(2)})(x,z)\frac{\delta^2 W_k}{\delta j(z)\delta j(y)}\,. \qquad (4.27)
\end{aligned}$$

□

We call $\Gamma_k^{(2)} - q_k$ the *quantum wave operator*, in analogy with the free wave operator P_0.

The above relation shows that $W_k^{(2)}$ is a fundamental solution of the quantum wave operator, and therefore we call it the *interacting propagator*. In Lemma 5.3, using the principle of perturbative agreement discussed in Section 2.9, we will see that the second derivative of W_k is, in certain limits, the Feynman propagator for the regularised theory.

A major consequence of working in Lorentzian spacetimes is that the quantum wave operator is, in general, of hyperbolic type, just as the free wave operator P_0. It follows that $W_k^{(2)}$ is non-unique as a propagator of the quantum wave operator. Its non-uniqueness is directly related to the choice of a state for the free theory, as can be seen in perturbation theory. We will show the connection between the two objects, writing $W_k^{(2)}$ in terms of the free Feynman propagator Δ_F and the advanced and retarded propagator, in Section 5.2.3.

Notice that, since $W_k^{(2)}$ is a propagator for the wave operator $\Gamma_k^{(2)} - q_k$, we can subtract its divergences and define its normal-ordered counterpart $: W_k^{(2)} :_{\tilde{H}_F}$ that satisfies the wave equation up to smooth terms,

$$(\Gamma_k^{(2)} - q_k) : W_k^{(2)} :_{\tilde{H}_F} = \sigma \,. \tag{4.28}$$

For now, we leave the counterterm $\tilde{H}_F$ implicit; we will discuss its explicit expression in the next sections.

We finally notice that, as in the k-independent case, for finite j as well as finite k, the functional derivatives of $W_k(j)$ are not the connected correlation functions; only taking the limits $j \to 0$ and $k \to 0$ one recovers the meaning of W_k as a generating functional of the truncated time-ordered correlation functions.

Remark 4.3. A well-defined S-matrix arising from the algebraic approach and Epstein-Glaser renormalization is unitary. It follows that $Z_k(j)$ is pure phase because it is the expectation value of a product of unitary operators; thus, $W_k(j) := -i \log Z_k(j)$ must be real. This in turn ensures that the effective average action is real, implying that quantum contributions to the action cannot give rise to complex couplings. ■

4.3 PROPERTIES OF THE EFFECTIVE AVERAGE ACTION

4.3.1 *Schwinger-Dyson equations*

In order to have the correct physical interpretation, the effective action needs to satisfy a set of important properties: in particular, it is known that the effective action reduces to the classical action in the classical $\hbar \to 0$ limit, it satisfies the same linear, global symmetries as the classical action, and it satisfies the Schwinger-Dyson equation in functional form, relating the quantum equations of motion (the equations of motion satisfied by the Green's functions, and in particular the equations for which the interacting propagator is a fundamental solution) with the classical ones.

Moreover, it is known that the effective average action, defined for the vacuum state in Euclidean space, in the limit of infinite regulator scale $k \to \infty$ reduces to the classical action. This is an important requirement, since it means that the regulator actually cuts off quantum fluctuations below its scale k; when the scale goes to infinity, all quantum fluctuations are suppressed and the effective action, describing the dynamics of quantum degrees of freedom, must coincide with the classical one. Intuitively, the same should hold also for a local regulator: in fact, correlation functions are suppressed, in momentum space, roughly by a factor $1/M_k^2$, where

M_k is the mass of the theory taking into account the regulator. In the limit $k \to \infty$, the artificial mass diverges. The Feynman propagator and the Pauli-Jordan function (associated with the time-ordered and $\star$−product, respectively) should go to zero; the perturbative expansion of interacting correlation functions thus reduces to the classical contributions only.

Even if, on general grounds, we could expect that the effective average action still reduces to the classical action in the limit $k \to \infty$, the proof of this property requires some detailed knowledge on the structure of the propagator. For this reason, we prove the classical limit of the effective average action when $k \to \infty$ only in the ground state of a scalar field propagating over ultra-static spacetimes with bounded curvature. The proof relies on a bound on the Feynman propagator and the Pauli-Jordan function, which is known to hold in more general scenarios; for example, for the Bunch-Davies vacuum in de Sitter space.

In this Section we prove that our definition of the effective average action in the general case satisfies its known properties in Euclidean vacuum.

First of all, Eq. (4.19) of Proposition 4.3 can be cast in the form of a functional Dyson-Schwinger equation. In fact, using the quantum equation of motion, Eq. (4.25), and recalling that $P_0(\phi) = \langle I_0^{(1)}(\varphi)\rangle$ thanks to linearity, it is immediate to get

$$-\frac{\delta}{\delta\phi}(\Gamma_k + Q_k) = \langle \frac{\delta}{\delta\varphi}(I_0 + TV + Q_k)\rangle \, ,$$

and simplifying the regulator terms gives the *Dyson-Schwinger equation* (DSE),

$$-\frac{\delta}{\delta\phi}\Gamma_k(\phi) = \langle \frac{\delta}{\delta\varphi}(I_0 + TV)\rangle \, . \tag{4.29}$$

Remarkably, the effective average action at finite k satisfies a DSE identical to the DSE for the effective action at $k = 0$, thanks to the subtraction of the classical term $Q_k(\phi)$ in its definition (4.5).

The DSE can be used to obtain the form of the effective average action Γ_k in perturbation theory. For example, at linear order in V and in the limit $k \to 0$, using Lemma 2.1, Equation (4.19) reduces to

$$\begin{aligned}
\frac{\delta\Gamma_k}{\delta\phi} &= -P_0\phi - \frac{1}{\omega(S(J_\phi))}\omega\left(S(J_\phi)\cdot_T TV^{(1)}\right) \quad \text{mod } O(V^2) \\
&= -P_0\phi - TV^{(1)}(i\Delta_F j_\phi) \quad \text{mod } O(V^2) \\
&= -P_0\phi - TV^{(1)}(\phi) \quad \text{mod } O(V^2) \, .
\end{aligned}$$

Thus, up to normal ordering (that is, up to field-independent constants), at leading order the effective action coincides with the classical action I.

The expression for the Dyson-Schwinger equation in equation (4.29) is analogous to the usual expressions in terms of the effective action that can be found in the physics literature, e.g. in [6, 201]. In the pAQFT literature [57, 211] the DSE is usually stated as a relation between the classical and time-ordered equations of motion as Eq. (2.41)

$$A \cdot_T \varphi(P_0 f) = A \star \varphi(P_0 f) + i\hbar\langle A^{(1)}, f\rangle \, ,$$

where A is any functional in $\mathcal{F}_{\mu c}$ and f is a test function.

The relation with the DSE in terms of the effective (average) action, Eq. (4.29), becomes apparent choosing as functional $A = S(V+J+Q_k) \cdot_T O$ for some operator O in the equation above, so that we have

$$\begin{aligned} S(V+J+Q_k) \cdot_T O \cdot_T P_0\varphi &= S(V+J+Q_k) \cdot_T O \star P_0\varphi \\ &+ i\hbar \left(S(V+J+Q_k) \cdot_T O\right)^{(1)} \\ &= S(V+J+Q_k) \cdot_T O \star P_0\varphi \\ &+ i\hbar \left[S(V+J+Q_k) \cdot_T \left(\frac{\delta O}{\delta\varphi} + \frac{i}{\hbar}\frac{\delta}{\delta\varphi}(V+J+Q_k) \cdot_T O\right)\right] . \end{aligned}$$

Multiplying both sides by $S(V)^{-1}$ from the left and rearranging terms give

$$\begin{aligned} &S(V)^{-1} \star \left[S(V+J) \cdot_T \frac{\delta}{\delta\varphi}(I+J) \cdot_T O\right] \\ &\quad = i\hbar S(V)^{-1} \star \left[S(V+J) \cdot_T \frac{\delta O}{\delta\varphi}\right] + S(V)^{-1} \star \left[S(V+J) \cdot_T O \star P_0\varphi\right] . \end{aligned}$$

Evaluating on an on-shell state ω annihilates the last term, so we arrive at

$$\langle \frac{\delta}{\delta\varphi}(I+J+Q_k) \cdot_T O\rangle = i\hbar\langle\frac{\delta O}{\delta\varphi}\rangle .$$

Since J is linear in the fields φ, $\dfrac{\delta J}{\delta\varphi} = j$, so that it can be pulled out of the mean value. Recalling that $j_\phi = -\dfrac{\delta(\Gamma_k + Q_k)}{\delta\phi}$, we have

$$\langle\frac{\delta}{\delta\varphi}(I+Q_k) \cdot_T O\rangle - \langle\frac{\delta O}{\delta\varphi}\rangle\frac{\delta}{\delta\phi}(\Gamma_k + Q_k) = i\hbar\langle\frac{\delta O}{\delta\varphi}\rangle .$$

which is the functional DSE for any functional O. Setting $O = 1$ gives the DSE (4.29).

Remark 4.4 (Classical limit)**.** In the classical limit $\hbar \to 0$, the non-commutative and time-ordered products reduce to the point-wise product. The Bogoliubov map in turn reduces to the classical Møller operator, which acts on functionals of the field by pull-back, so that $\langle F(\varphi)\rangle = F(\phi) + \mathcal{O}(\hbar)$. From the DSE (4.29) it follows that

$$\Gamma_k(\phi) = I(\phi) + \mathcal{O}(\hbar) . \tag{4.30}$$

■

4.3.2 *Infinite mass limit*

The DSE (4.29) can be used to compute the effective average action in certain approximations; we already commented on its perturbative expansion, showing that the effective (average) action equals the classical action at tree level, and on the classical limit $\hbar \to 0$. Now, we would like to compute the limit $k \to \infty$. In the Euclidean case, in this limit the effective average action and the bare action differ by the infinite mass limit of a one-loop determinant [218].

To study the limit in our formalism, it is convenient to apply the principle of perturbative agreement [95, 161, 254], which we recalled in Section 2.9, and in particular Eq. (2.47) and Eq. (2.48). In fact, it is particularly useful to convert the non-commutative products $\star$ to k-dependent products $\star_k$, so that, instead of taking the $k \longrightarrow \infty$ limit in the Bogoliubov map, we can consider the limit in the propagators.

As discussed in the introduction to this Section, although the $k \longrightarrow \infty$ limit can be considered in greater generality, we make some simplifying assumptions that allow to be very precise in the statement of the limit. In particular, the following theorem on the infinite scale limit of Γ_k holds in the case of quasi-free states on ultrastatic spacetimes. Before proving the theorem, we rewrite the effective average action in terms of $k-$dependent products with the principle of perturbative agreement.

Lemma 4.5. *Let ω be a quasifree state on $\mathscr{A}$ (associated to I_0), and consider $\star$ and $\cdot_T$ constructed out of the two-point function Δ_+ of ω. Let $\star_k$ and $\cdot_{T_k}$ the star and time ordered products of $\mathscr{A}_k$, which descends from the action $I_0 + Q_k$, and constructed out of $\Delta_{+,k} = r_{Q_k}\Delta_+ r^*_{Q_k}$: The classical Møller map r_{Q_k} is given in equation (2.19) of Section 2.9, and $r^*_{Q_k} f := f - q_k \Delta_{A,k} f$. Then, it holds that for every $A \in \mathscr{F}_{\text{loc}}$*

$$\begin{aligned}\omega\big(R_V(S(Q_k + J) \cdot_T A)\big) &= S(V)^{-1} \star [S(V + Q_k + J) \cdot_T A]\Big|_{\varphi=0} \\ &= S_k(\rho_k V - \rho_k Q_k)^{-1} \star_k [S_k(\rho_k V + J) \cdot_{T_k} \rho_k A]\Big|_{\varphi=0} .\end{aligned} \tag{4.31}$$

where S_k is the S-matrix constructed with $\cdot_{T_k}$, while the map ρ_k is given in (2.44).

Proof. It is useful to first consider the functional $S(V)^{-1} \star S(V + Q_k + J + \mu A)$, since its derivative with respect to μ, at vanishing μ, and evaluated on the state ω, equals the l.h.s of (4.31). Inserting the identity $1 = S(V + Q_k) \star S(V + Q_k)^{-1}$, the functional $S(V)^{-1} \star S(V + Q_k + J + \mu A)$ is equivalent to

$$\begin{aligned}&S(V)^{-1} \star S(V + Q_k + J + \mu A) \\ &\quad = S(V)^{-1} \star S(V + Q_k) \star S(V + Q_k)^{-1} \star S(V + Q_k + J + \mu A) \\ &\quad = R_V(Q_k) \star S_{V+Q_k}(J + \mu A) .\end{aligned}$$

The first factor can be rewritten using Eq. (2.47) of Section 2.9 as

$$R_V(Q_k) = r_{Q_k} R_{k,\rho_k(V-Q_k)}(\rho_k Q_k).$$

On the other hand, using Eq. (2.48) of Section 2.9, the second factor is

$$S_{V+Q_k}(J + \mu A) = r_{Q_k}\left[S_k(\rho_k V)^{-1} \star_k S_k(\rho_k V + J + \rho_k \mu A)\right] ,$$

where we also used the identity $\rho_k J = J$ because J is φ-linear. Since r_{Q_k} intertwines $\star$ to $\star_k$ and since $r_{Q_k} B\big|_{\varphi=0} = B \circ r_{Q_k}\big|_{\varphi=0} = B\big|_0$ because $r_{Q_k}(\varphi)\big|_0 = 0$, Eq. (4.31) follows. □

The last Lemma allows to rewrite the effective average action (or, more generally, the Bogoliubov map) in terms of $k-$dependent products. The following Theorem gives the $k \longrightarrow \infty$ limit of the effective average action.

Theorem 4.6. *Let $(\mathcal{M}, g)$ be an ultrastatic spacetime with bounded curvature, let ω be the ground state on $\mathscr{A}$, and consider the limit where the support of q_k tends to $\mathcal{M}$, namely where $q_k = k^2$. Then the effective average action Γ_k coincides with the classical action up to a constant, in the limit where $q_k = k^2$ and $k \longrightarrow \infty$; in other words, it holds that*

$$\Gamma_k^{(1)}(\phi) \underset{k\to\infty}{\longrightarrow} I^{(1)}(\phi)$$

in the sense of pointwise converges of functions, at any order in the coupling constant.

Proof. We start writing the DSE (4.29) more explicitly as

$$\Gamma_k^{(1)}(\phi) = P_0\phi + \frac{1}{Z_k(j)}\omega\big(R_V(S(Q_k + J) \cdot_T TV^{(1)})\big)\,.$$

Using Lemma 4.5, we may rewrite the Bogoliubov map as

$$\Gamma_k^{(1)}(\phi) = P_0\phi + \frac{1}{Z_k(j)}\omega_k\left(S_k(\rho_k V - \rho_k Q_k)^{-1} \star_k [S_k(\rho_k V + J) \cdot_{T_k} \rho_k TV^{(1)}]\right)$$

where $\omega_k = \omega \circ r_{Q_k}$.

When q_k tends to k^2, ω_k tends to the ground state related to the equation $P_0\phi - k^2\phi = 0$. The proof, for the case of Minkowski background, can be found in Lemma D.1 of Ref. [95]; it can be generalized to generic ultrastatic spacetime with bounded scalar curvature, and to the case of thermal states [93].

Since the spacetime $\mathcal{M}$ is ultrastatic, it admits a natural notion of time, and it can decomposed into a product $\mathcal{M} = \mathbb{R} \times \Sigma$. Thanks to the split between time and spatial coordinates, the free wave operator admits a decomposition

$$P_0 = -\partial_t^2 - B - k^2\,,$$

where B is a self-adjoint operator on $L^2(\Sigma)$, whose spectrum is bounded from below by $m^2 - \xi\|R\|_\infty$. Hence, if k is sufficiently large, $-k$ is in the resolvent set of B: it follows that, for large k, $(B + k^2)^{-1}$ is a bounded positive operator. Furthermore, notice that, for any $\mathbb{N} \ni l > 0$ and $k^2 > r$, the inverse $(B + k^2)^{-l}$ is well-defined, and

$$\|(B + k^2)^{-l}\psi\|_2 \leq \frac{1}{(k^2 - r)^l}\|\psi\|_2, \qquad \psi \in L^2(\Sigma),$$

where r is a positive constant such that $(m^2 - \xi R) \geq -r$ uniformly on $\mathcal{M}$.

For sufficiently large k, the spectrum of $B + k^2$ is contained in $\mathbb{R}^+$. Hence, by standard functional calculus we can construct the operators $\tilde{\Delta}_{+,k}(t)$ and $\tilde{\Delta}_{k,F}(t)$, used as integral kernels of $\Delta_{F,k}$ and of $\Delta_{+,k}$. For every t it holds that

$$\tilde{\Delta}_{+,k}(t) = \frac{e^{it\sqrt{B+k^2}}}{2\sqrt{B + k^2}}\,,$$

and

$$\tilde{\Delta}_{F,k}(t) := \theta(t)\Delta_{+,k}(t) + \theta(-t)\tilde{\Delta}_{+,k}(-t)\,,$$

and both are elements of $B(L^2(\Sigma, \mathrm{d}x))$. Their operator norms are such that

$$\|\tilde{\Delta}_{+,k}(t)\| \leq \frac{1}{2\sqrt{k^2 - r}}, \qquad \|\tilde{\Delta}_{F,k}(t)\| \leq \frac{1}{2\sqrt{k^2 - r}}. \tag{4.32}$$

For every $f \in C_c^\infty(\mathcal{M})$ we can write the action of the 2-point function distribution Δ_+ on f

$$(\Delta_{+,k}f)(t_x,\mathbf{x}) = \int_{\mathbb{R}} \mathrm{d}t' \Big(\tilde{\Delta}_{+,f}(t_x - t')f(t',\cdot)\Big)(\mathbf{x})\,,$$

and similarly for $\Delta_{F,k}f$. Given the estimates of $\tilde{\Delta}_{+,k}$ and $\tilde{\Delta}_{F,k}$, valid uniformly in time, and the last observation, we can now estimate the distributions $\Delta_{+,k}^{\otimes n}$ and $\Delta_{F,k}^{\otimes n}$ on $\mathcal{M}^{2n}$.

For every $h, g \in C_c^\infty(\mathcal{O})$ and with $f \in C_c^\infty(\mathcal{M})$ which is 1 on $\mathcal{O}$, and for every $l \in \mathbb{N}$, with C a suitable constant, it holds that

$$|\langle h, \Delta_{+,k}g\rangle| \leq |\langle h, \frac{P_0^l}{k^{2l}}\Delta_{+,k}g\rangle| \leq \frac{1}{k^{2l}}\|P_0^l h\|_2 \|f\Delta_{+,k}g\|_2 \leq \frac{C}{k^{2l}}\|P_0^l g\|_2\|f\|_2\|g\|_2 \tag{4.33}$$

where now the $\|\cdot\|_2$ norms act on $L^2(\mathcal{M}, \mathrm{d}x)$. We used the fact that $\Delta_{+,k}$ is a weak solution of $P_0 - k^2$ in the first inequality, Cauchy-Schwartz inequality in the second step, and the uniform estimates in Eq. (4.32) in the third one.

We can now generalise this observation to estimate

$$|(F^{(n)}\Delta_{+,k}^{\otimes n}G^{(n)})| \leq \frac{C_l}{k^{2l}} \tag{4.34}$$

for F,G obtained as tensor product of local functionals and valid for large k and for every $l \in \mathbb{N}$. In fact, observe that as an operator on $L^2(\Sigma) \otimes L^2(\Sigma)$, the 2-point function satisfies

$$\frac{-i}{\sqrt{B_1 + k^2} + \sqrt{B_2 + k^2}}\partial_{t_k}\Delta_{+,k}(t_x - t_y)\Delta_{+,k}(t_x - t_z) = \Delta_{+,k}(t_x - t_y)\Delta_{+,k}(t_x - t_z).$$

With estimates analogous to Eq. (4.33), we get Eq. (4.34). With this at disposal, the $\star_k$-product reduces to the point-wise product in the limit $k \to \infty$ even if in one of the factors Q_k appears.

Using also Lemma 2.3 we actually get

$$\lim_{k\to\infty} \frac{\omega_k\Big(S_k(\rho_k V - \rho_k Q_k)^{-1} \star_k [S_k(\rho_k V + J) \cdot_{T_k} \rho_k T V^{(1)}]\Big)}{Z_k(j)} = \lim_{k\to\infty}\frac{\omega_k\Big(S_k(\rho_k V_{\phi_0}) \cdot_{T_k} \rho_k T V^{(1)}_{\phi_0}\Big)}{\omega_k\Big(S_k(\rho_k V_{\phi_0})\Big)}$$

where $F_{\phi_0}(\varphi) = F(\varphi + \phi_0)$ and where $\phi_0 = i\Delta_{F,k}j_\phi$. In the limit $k \to \infty$, thanks to the estimate in Eq. (4.32) the T_k-product among local functionals reduces to a pointwise product.

Furthermore, $\rho_k T V^{(1)} = T_k V^{(1)}$ (*cf.* Remark 2.3 in Section 2.9), and under the same limit $T_k V^{(1)} \xrightarrow{k\to\infty} V^{(1)}$. Finally, using Eq. (4.19) we get that in the limit $k \to \infty$, ϕ_0 converges to ϕ; hence, since V as a function of ϕ is smooth, we obtain

$$\lim_{k\to\infty}\frac{\omega_k\Big(S_k(\rho_k V_{\phi_0}) \cdot_{T_k} \rho_k T V^{(1)}_{\phi_0}\Big)}{\omega_k\Big(S_k(\rho_k V_{\phi_0})\Big)} = V^{(1)}(\phi)\,,$$

and from the DSE the $k \to \infty$ limit of the effective average action reads

$$\Gamma_k^{(1)}(\phi) \xrightarrow[k\to\infty]{} P_0\phi + V^{(1)}(\phi) = I^{(1)}(\phi) ,$$

where $I(\phi)$ is the classical action and the limit holds in the sense of pointwise convergence of functions. □

Although Theorem 4.6 is proved for the case of ultrastatic spacetimes with bounded curvature and for ground states, its thesis holds in a more general setup. Indeed, the generalisation to the case of states satisfying a similar bound as those given in Eq. (4.32) is straightforward. Notice that equilibrium states on flat spacetimes or Bunch Davies states in the case of de Sitter backgrounds satisfy a similar estimate.

This shows that the effective average action and the classical action coincide, up to a constant, in the limit $k \to \infty$; more precisely, the expansion at $k \to \infty$ coincides with the semiclassical approximation described in Remark 4.2.

4.3.3 *Parity of the effective average action*

Since is usually impossible to compute the effective average action exactly, it is crucial to derive some of its general properties without explicit computations. For example, standard methods of solving the RG flow equations consist in guessing an Ansatz on the functional dependence of Γ_k on the mean fields ϕ, usually in the form of a finite sum of polynomials in the fields with $k-$dependent coefficients. It is clear that projecting Γ_k on a complete basis of field functionals is not possible; therefore, any constraint on the functional dependence of Γ_k on the fields greatly improves any approximation scheme, while also helping in the identification of scheme-independent properties of the RG flow.

We thus study how global symmetries of the classical action I affect Γ_k. For simplicity, we only consider the $O(1)$ scalar model with an interaction term V even in the field φ. The classical action of this scalar field theory is invariant under a global $\mathbb{Z}_2$ symmetry, that is, the symmetry transformation $\varphi \to -\varphi$.

Naturally, since the effective average action depends on the state as well as the algebra, there can be state effects that break the symmetry. Therefore, to prove the invariance of the effective average action we need to choose a state that does not spontaneously break the classical global symmetry. In this simple case this amounts to the requirement that the one-point function $\omega(\varphi)$ is even under the symmetry $\varphi \to -\varphi$, which is true whenever the state is quasifree for the free theory, $\omega(\varphi) = 0$.

If the state ω is quasifree, the next proposition proves that Γ_k is invariant under the same global symmetry $\phi \to -\phi$.

Proposition 4.7. *Let V be even with respect to $\varphi \to -\varphi$, so that it contains an even number of fields only. If ω is quasifree, the effective average action Γ_k is even.*

Proof. Since Q_k is ϕ-even, it suffices to study the parity of $\tilde{\Gamma}_k$. We analyse the ϕ-parity of $\tilde{\Gamma}_k$ through its Definition (4.3):

$$\tilde{\Gamma}_k(\phi) = W_k(j_\phi) - J_\phi(\phi) = -i \log Z_k(j_\phi) - J_\phi(\phi) , \quad Z_k(j_\phi) = \omega\left[R_V(S(Q_k + J_\phi)\right] .$$

First of all, we analyse the φ-parity of Z_k and W_k.

Since V is φ-even and $\star, \cdot_T$ preserve φ-parity (because they acts as functional derivatives twice on the factors), it follows that $R_V F$ has the same φ-parity of F.

Since Q_k is φ-even and ω corresponds to evaluation at $\varphi = 0$, it follows that the contribution

$$\omega\left[R_V(S(Q_k + J_\phi))\right] = \omega\left[R_V(S(Q_k) \cdot_T S(J_\phi))\right] ,$$

contains only even powers of J_ϕ, so this contribution is j_ϕ-even. It then follows that $W_k(j_\phi) = -i \log Z_k(j_\phi)$ is j_ϕ-even.

We now prove that j_ϕ is ϕ-odd. With the observations already made, this will imply that $J_\phi(\phi)$ is ϕ-even and so is $\tilde{\Gamma}_k$, because of Eq. (4.3). Recall Eq. (4.19),

$$j_\phi = -P_0\phi - Q_k^{(1)}(\phi) - \frac{1}{Z_k(j_\phi)}\omega\left[R_V\left(S(Q_k + J_\phi) \cdot_T TV^{(1)}\right)\right] .$$

It follows that $j_{\phi,0}$ (the 0-th order in V of j_ϕ) is ϕ-odd. By induction, assume that the expansion $j_{\phi,n}$ up to order V^n is ϕ-odd. For the expansion $j_{\phi,n+1}$ up to V^{n+1} we have

$$j_{\phi,n+1} = -\frac{1}{Z_k(j_{\phi,n})}\omega\left[R_V\big(S(Q_k + J_{\phi,n}) \cdot_T TV^{(1)}\big)\right] .$$

$TV^{(1)}$ is φ-odd, because $V^{(1)}$ is odd and the map T preserves the φ-parity. It follows that

$$\omega\left[R_V\big(S(Q_k + J_{\phi,n}) \cdot_T TV^{(1)}\big)\right]$$

contains only odd powers of $j_{\phi,n}$, because only φ-even terms in $S(Q_k + J_{\phi,n+1}) \cdot_T TV^{(1)}$ provide a non-vanishing contribution to the expectation value. Finally, $Z_k(j_{\phi,n})$ is $j_{\phi,n}$-even and thus ϕ-even. By the inductive assumption, we find that $j_{\phi,n+1}$ is ϕ-odd. □

Hence, quantum contributions cannot violate the parity of symmetry of the starting classical action, at least in this simple example. This observation will justify the ansatz in Eq. (8.11) in actual computations of the flow for the scalar theory.

4.3.4 *Towards a non-perturbative formulation of the effective action*

Before discussing the extension to gauge theories, we discuss here an alternative formulation of the effective average action, which could form the basis for a non-perturbative formulation in the C^*−algebraic approach to interacting field theories, first formulated for scalar fields by Buchholz and Fredenhagen [68]. The idea is to start from a definition of the effective (average) action directly from an expression in terms of the Bogoliubov map, and then to derive its relevant properties, without referring to the Z or W [53]. Since we work in the off-shell formalism, Γ_k will be implicitly defined by a solution of some integro-differential equation. The advantage is that the Bogoliubov map and interacting observables admit a treatment as elements of an abstract C^*−algebra, instead as formal power series [68]; formulating the effective average action in the same C^*−algebraic context will provide the non-perturbative generalisation of the effective average action.

We discuss this equivalent formulation for the scalar field only. From Definition 4.5 as the modified Legendre transform and from the definition of W_k in Eq. (4.14), it follows that

$$e^{\frac{i}{\hbar}(\Gamma_k + Q_k(\phi) + J_\phi(\phi))} = \omega \circ R_V(S(J_\phi(\varphi) + Q_k(\varphi)) \Rightarrow$$
$$e^{\frac{i}{\hbar}\Gamma_k} = \omega \circ R_V(S(J_\phi(\varphi) - J_\phi(\phi) + Q_k(\varphi) - Q_k(\phi)) .$$

Now, we have

$$J_\phi(\varphi) - J_\phi(\phi) = J_\phi(\varphi - \phi) = -\int_x \frac{\delta \Gamma_k}{\delta \phi}(\varphi - \phi) + \frac{\delta Q_k}{\delta \phi}(\varphi - \phi)\,.$$

On the other hand,

$$Q_k(\varphi) - Q_k(\phi) - \int_x \frac{\delta Q_k}{\delta \phi}(\varphi - \phi) = -\frac{1}{2}\int_x q_k(x)\left[\varphi^2 - \phi^2 - 2\phi(\varphi - \phi)\right] = Q_k(\varphi - \phi)\,,$$

so we arrive at

$$e^{\frac{i}{\hbar}\Gamma_k} = \omega \circ R_V\left[S\left(-\int_x \frac{\delta \Gamma_k}{\delta \phi}(\varphi - \phi) + Q_k(\varphi - \phi)\right)\right]\,.$$

Thus, we can define the effective average action as the solution of the above equation,

Definition 4.6 (Non-perturbative definition of the effective average action)**.** The effective average action $\Gamma_k : \mathbb{R}^+ \times C^\infty(\mathcal{M}) \to \mathbb{C}$ is defined as the solution of the following equation:

$$e^{\frac{i}{\hbar}\Gamma_k(\phi)} = \omega \circ R_V\left[S\left(-\int_x \frac{\delta \Gamma_k}{\delta \phi}(\varphi - \phi) + Q_k(\varphi - \phi)\right)\right]\,.$$

We call this definition *non-perturbative* because, giving a representation of the Bogoliubov map and the S−matrix as abstract generators of a C^*−algebra [68], the above formula translates to a non-perturbative, C^*−definition of the effective average action.

We can now show that from Definition (4.6) Γ_k satisfies the same properties as the modified Legendre transform.

First of all, the derivative with respect to ϕ of (4.6) gives

$$\begin{aligned} e^{\frac{i}{\hbar}\Gamma_k}\frac{\delta \Gamma_k}{\delta \phi(x)} &= \\ &= \omega \circ R_V\left[S\left(-\int_x \frac{\delta \Gamma_k}{\delta \phi}(\varphi - \phi) + Q_k(\varphi - \phi)\right)\cdot_T\right. \\ &\left.\left(\frac{\delta \Gamma_k}{\delta \phi(x)} - \frac{\delta Q_k(\varphi - \phi)}{\delta \phi(x)} - \int_y \frac{\delta^2 \Gamma_k}{\delta \phi(x)\delta \phi(y)}(\varphi - \phi)(y)\right)\right]\,. \end{aligned}$$

Now, it holds that $\frac{Q_k}{\delta \phi(x)}(\varphi - \phi) = \int_y \frac{\delta^2 Q_k}{\delta \phi(x)\delta \phi(y)}(\varphi - \phi)(y)$. We now denote

$$\langle F \rangle := e^{-\frac{i}{\hbar}\Gamma_k}\omega \circ R_V\left[S\left(-\int_x \frac{\delta \Gamma_k}{\delta \phi}(\varphi - \phi) + Q_k(\varphi - \phi)\right)\cdot_T F\right]\,.$$

Recognising that $\Gamma_k(\phi)$ and $Q_k^{(2)}$ do not depend on φ we get

$$\int_y \frac{\delta^2}{\delta \phi(x)\delta \phi(y)}(\Gamma_k + Q_k)(\langle \varphi(y)\rangle - \phi(y)) = 0\,.$$

Since this equation must hold for any fluctuation field $\langle \varphi \rangle - \phi$, and in particular when it is not a solution of the QEOM with vanishing sources, we recover the standard definition of ϕ:

$$\phi = \langle \varphi \rangle\,.$$

We now prove the relation between the quantum wave operator and the interacting propagator. Deriving the last expression with respect to ϕ gives

$$\frac{\delta\langle\varphi(x)\rangle}{\delta\phi(y)} = \delta(x,y)\ .$$

Writing explicitly the derivative of the mean value, we get

$$\delta(x,y) = \frac{\delta}{\delta\phi(y)}\left\{e^{-\frac{i}{\hbar}\Gamma_k}\omega\circ R_V\left[S\left(-\int_z \frac{\delta\Gamma_k}{\delta\phi(z)}(\varphi-\phi)(z) + Q_k(\varphi-\phi)\right)\cdot_T \varphi(x)\right]\right\}.$$

The derivative produces three terms by Leibniz rule, that are

$$\begin{aligned}\delta(x,y) = &-\frac{i}{\hbar}\int_z\left(\Gamma_k^{(2)}(\phi) + Q_k^{(2)}(z,y)\right)\langle(\varphi-\phi)(z)\cdot_T\varphi(x)\rangle\\ &+\frac{i}{\hbar}\frac{\delta\Gamma_k}{\delta\phi(y)}\phi(x)\\ &-\frac{i}{\hbar}\frac{\delta\Gamma_k}{\delta\phi(y)}\phi(x)\ .\end{aligned}$$

Rearranging terms, we get

$$\int_z(\Gamma_k(\phi) + Q_k(\phi)^{(2)}(x,z)\langle\varphi(z)\cdot_T\varphi(y)\rangle_c = i\hbar\delta(x,y)\ ,$$

so that the quantum wave operator is the inverse of the connected interacting two-point function, with the $i\hbar$ factor converting it to the interacting propagator.

Finally, we can compute the flow equations. The computation is much more direct than in the modified Legendre transform perspective, that we assume for the rest of the thesis; here we show only the main steps, while we present a detailed derivation in Section 5.1. We start computing the derivative with respect to k of Definition (4.6). The derivative produces a term proportional to $\partial_k\Gamma_k^{(1)}(\langle\varphi\rangle-\phi)$, which vanishes since it is proportional to the difference $\langle\varphi\rangle-\phi=0$. We are left with

$$\partial_k\Gamma_k(\phi) = \langle Q_k(\varphi-\phi)\rangle\ .$$

After a short computation, the above gives the RG flow equations (5.6), written as

$$\partial_k\Gamma_k(\phi) = -\frac{1}{2}\int_x \partial_k q_k(x)\langle T(\varphi-\phi)^2\rangle\ .$$

To write the flow equations as closed differential equations for the effective average action, we denote : $G_k:(x,x) := -i\hbar\langle T(\varphi-\phi)^2\rangle$ and we get the RG flow equations (5.6), with the additional condition that G_k is an inverse for the quantum wave operator.

Definition 4.6 thus reproduces all the key properties of the Legendre effective average action. The only relation which cannot be proven without assuming the perturbative expression of the quantum and time-ordered products given in Definitions 2.12 and 2.16, and thus of the Bogoliubov maps and of the effective average action itself, is the Schwinger-Dyson equation, Eq. (4.29). The generalisation of this formalism to the non-perturbative C^*−algebraic thus requires some Dyson-Schwinger-type relation relating the effective average action with the bare action. It should be possible to start from the one proposed by Buchholz and Fredenhagen [68] and derive from it the relation between Γ_k and the bare action. We leave this important issue for future investigations.

4.4 EXTENSION TO GAUGE THEORIES

Before moving on with the derivation of RG flow equations for the effective average action, we extend the definitions of generating functionals to the case of gauge theories. The extension requires two modifications to the generating functionals: the first is straightforward, since it consists in the inclusion of the field multiplet φ in the source and regulator terms J and Q_k.

The second modification consists in the introduction of classical sources σ for the BV variations of the fields φ, through the insertion of a term Σ in Z_k, and for the BV variations of Q_k, through an additional term H. These terms do not play a role in the flow equations, since they are $k-$independent, but they will be necessary to control the local symmetries of the effective average action. The sources σ are known in the literature as classical BRST sources, and were first introduced by Zinn-Justin [256]. In our formalism, they correspond to a classical evaluation of the antifields $\varphi^\ddagger$ on some antifield configuration σ, and so they are analogous to a gauge-fixing term.

We will discuss the inclusion of the H and Σ terms in Section 7.1. Here, we focus on the extension needed to derive the RG flow equation for gauge theories.

So, first of all, the source term $J(\varphi)$ is now an element of the BV algebra $\mathscr{BV}$, i.e. it is a linear functional of the field multiplet $\varphi = \{\mathcal{A}, b, c, \bar{c}\}$. In turn, this implies that the generating functional $Z(j)$ in Definition 4.1 depends on a collection of sources $j = \{j_{\mathcal{A}}, j_b j_c, j_{\bar{c}}\}$ via the linear coupling $J(\varphi) = \int_x j\varphi$, where, as always in our notation, the integration of a field-dependent term also implicitly includes a summation over the field species and internal and Lorentz indices. Just as the field configuration, different sources can actually be different geometrical objects: for example, for Yang-Mills theories, the gauge field is a Lorentz vector, while the other fields are Lorentz scalars; it follows that $j_{\mathcal{A}}$ must be a Lorentz covector, while the other sources are Lorentz scalars.

The generating functional $Z(j)$ defines the generating functional for connected Green's functions $e^{\frac{i}{\hbar}W(j)} = Z(j)$, just as in the scalar case.

The Legendre transform of W with respect to all sources gives the effective action, $\Gamma(\phi)$, that now depends on the field multiplet $\phi = \{\phi_{\mathcal{A}}, \phi_b, \phi_c, \phi_{\bar{c}}\}$; they are defined as in the scalar case by $\dfrac{\delta W}{\delta j} = \phi$. The perturbative inversion of this relation gives the quantum equation of motion, and its derivative with respect to j_ϕ shows that the interacting propagator is a fundamental solution of the quantum wave operator:$\int_{x,y} \Gamma^{(2)}(\phi) W^{(2)}(j_\phi) = -\delta$. The difference from the scalar case is that, together with integration over spacetime points, we now need to sum over the field indices of j_ϕ and ϕ.

The regularised generating functionals now depend on a quadratic term which is formally identical to the scalar case,

$$Q_k(\varphi) = -\frac{1}{2} T \int_x \varphi(x) q_k(x) \varphi(x) \,. \tag{4.35}$$

However, the above relation must be read in matrix notation as a quadratic form, where q_k is a block diagonal matrix. Since it must act as an effective mass for the non-trivial propagators, it is explicitly given by

$$q_k(x) = f(x) \begin{pmatrix} q_k & q_k^b & 0 & 0 \\ q_k^b & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{q}_k \\ 0 & 0 & -\tilde{q}_k & 0 \end{pmatrix} ,$$

where q_k^a is some function of the scale k; in the simplest case, $q_k^a = k^2$ for bosons and $q_k^a = k$ for fermions. The matrix rows and columns are ordered alphabetically, as $(\mathcal{A}, b, c, \bar{c})$. The regulator is thus the sum of three contributions,

$$Q_k(\varphi) = -\frac{1}{2} \int_x f(x) \left[q_k T \mathcal{A}^2 + 2 q_k^b T \mathcal{A} b - \tilde{q}_k T \left(\bar{c} c - c \bar{c} \right) \right] . \tag{4.36}$$

Clearly, in the scalar field case the above definition reduces to what we already discussed in Section 4.2. The regulator acts as an effective mass term both for gauge and ghost fields. Since the regulator term does not contain antifields, it is possible to use the formulas of perturbative agreement, Eqs. (2.47) and (2.48) to prove that the regulator acts as an effective mass in the propagators, thus suppressing long-range correlations. It follows that Q_k behaves as an IR regulator in gauge theories as well.

The connected, interacting propagators now can be organised in a matrix,

$$- i\hbar W_k^{(2)} = \\ \begin{pmatrix} \langle \mathcal{A}(x) \cdot_T \mathcal{A}(y) \rangle_c & \langle \mathcal{A}(x) \cdot_T b(y) \rangle_c & 0 & 0 \\ \langle \mathcal{A}(x) \cdot_T b(y) \rangle_c & 0 & 0 & 0 \\ 0 & 0 & 0 & \langle c(x) \cdot_T \bar{c}(y) \rangle_c \\ 0 & 0 & \langle \bar{c}(x) \cdot_T c(y) \rangle_c . & \end{pmatrix} \tag{4.37}$$

With the extensions of the source and regulator terms to gauge theories, we can now discuss the derivation of the RG flow equations.

5 Functional renormalization in curved spacetimes

5.1 RG FLOW EQUATIONS

We are finally in the position of deriving one of the main results of this thesis. The RG flow equations govern the flow of the effective average action under infinitesimal re-scaling of the parameter k. They take the form of the Wetterich equation with a local regulator (also known as functional Callan-Symanzik equation), generalising the flow to curved spacetimes and generic Hadamard states.

The derivation works by a straightforward computation of the generating functionals' scale dependence, and it is analogous to the Euclidean case [217, 246].

The first step is computing the k−derivative of Z_k: from its definition we have

$$\partial_k Z_k(j) = \frac{i}{\hbar}\omega\left(S(V)^{-1} \star S(V+Q_k+J) \cdot_T \partial_k Q_k\right) . \tag{5.1}$$

The k-derivative of W_k follows immediately,

$$\partial_k W_k(j) = \langle \partial_k Q_k \rangle . \tag{5.2}$$

This last equation computes the scale derivative of W_k in terms of the mean value of the regulator term, which is proportional to the Wick-square.

To obtain a closed expression, the derivation of the RG equations for W_k proceeds expressing the mean value of the regulator Q_k in terms of the interacting propagators $W_k^{(2)}$. First of all, notice that $\langle \partial_k Q_k \rangle$ is the sum of two contributions,

$$\langle \partial_k Q_k \rangle = -\frac{1}{2}\int_x \partial_k q_k(x)\langle T\mathcal{A}^2(x)\rangle + \partial_k \tilde{q}_k(x)\left(\langle T\bar{c}(x)c(x)\rangle - \langle Tc(x)\bar{c}(x)\rangle\right) .$$

Both in the gauge and ghost sector, the expectation value of $T\varphi^2$ can be written as

$$T\varphi^2 = \lim_{y\to x} \varphi(x) \cdot_T \varphi(y) - H_F = \lim_{y\to x} \varphi(x) \star \varphi(y) - H . \tag{5.3}$$

Recall that $H_F = H + i\Delta_A$. Despite the compact notation, the parametrix H_F is actually a block-diagonal matrix of functions, written in terms of the collection of parametrices $\{H_F^g, H_F^b, H_F^{gh}, H_F^{gh}\}$.

The scale derivative of W_k then becomes

$$\begin{aligned}\partial_k W_k = &\\ -\frac{1}{2}\int_x \partial_k q_k \langle \lim_{y\to x} \mathcal{A}(x) \cdot_T \mathcal{A}(y) - H_F^g(x,y)\rangle &+ 2\partial_k q_k^b(x)\langle \lim_{y\to x} \mathcal{A}(x) \cdot_T b(y) - H_F^b\rangle \\ &+ 2\partial_k \tilde{q}_k \langle \lim_{y\to x} \bar{c}(x) \cdot_T c(y) - H_F^{gh}(x,y)\rangle . \end{aligned} \tag{5.4}$$

The Wick square is normal-ordered by subtraction of the Hadamard parametrix, which in the language of renormalization is the counterterm associated with the free propagator in the coincidence limit (a 1-loop divergence). The connection between the interacting propagator $W_k^{(2)}(x, y)$ and the interacting Wick square $W^{(2)}(x, x)$ requires the implicit definition of a new, interacting parametrix $\tilde{H}_F$, arising from the commutation of the Bogoliubov map with the coincidence limit.

Definition 5.1 (Counterterms $\tilde{H}_F$)**.** The *interacting Hadamard counterterms* $\tilde{H}_F$ are implicitly defined by the commutation of the limit operation with the expectation value in the following formula:

$$\langle \lim_{y \to x} \varphi(x) \cdot_T \varphi(y) - H_F(x, y) \rangle := \lim_{y \to x} \langle \varphi(x) \cdot_T \varphi(y) \rangle - i\hbar \tilde{H}_F(x, y) \,.$$

Thanks to $\tilde{H}_F$, we can write the coincidence limit of the time-ordered expectation value $T\chi^2$ as

$$\begin{aligned} &\langle \lim_{y \to x} \varphi(x) \cdot_T \varphi(y) - H_F(x, y) \rangle \\ &\qquad = \lim_{y \to x} \left(-i\hbar W_k^{(2)}(x, y) - i\hbar \tilde{H}_F(x, y) \right) + W_k^{(1)}(x) W_k^{(1)}(y) \,. \end{aligned}$$

By implicitly defining the counterterms $\tilde{H}_F(x, y)$ as a block diagonal matrix, the scale derivative of W_k gives the RG flow equation for W_k, known as the *Polchinski equation* [208]. To write the Polchinski equation in compact form, the matrix $W_k^{(2)}(x, y)$ in Eq. (4.37) can be written as

$$W_k^{(2)}(x, y) = \begin{pmatrix} \frac{\delta^2 W(j)}{\delta j_A(x) \delta j_A(y)} & \frac{\delta^2 W(j)}{\delta j_A(x) \delta j_b(y)} & 0 & 0 \\ \frac{\delta^2 W(j)}{\delta j_b(x) \delta j_A(y)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta^2 W(j)}{\delta j_{\bar{c}}(x) \delta j_c(y)} \\ 0 & 0 & \frac{\delta^2 W(j)}{\delta j_c(x) \delta j_{\bar{c}}(y)} & 0 \end{pmatrix} .$$

Using the normal-ordering notation $: A :_H = A - H$, the Polchinski equation in matrix notation then is

$$\partial_k W_k = \frac{i\hbar}{2} \lim_{y \to x} \int_x \mathrm{Tr} \Big\{ \partial_k q_k(x) \Big[: W_k^{(2)} :_{\tilde{H}_F} (x, y) + W_k^{(1)}(x) W_k^{(1)}(y) \Big] \Big\} \,. \tag{5.5}$$

The last equation is a generalisation of Polchinski's equation to curved spacetimes and generic Hadamard states ω. The trace is performed over internal and Lorentz indices, as well.

The RG flow for Γ_k follows from the RG flow for W_k. Since it is one of the central results of this thesis, we summarise the equations in the following theorem.

Theorem 5.1 (RG flow equations)**.** *The RG flow equations are given by the flow equation for* Γ_k *with a consistency relation:*

$$\begin{aligned} \partial_k \Gamma_k &= \frac{i\hbar}{2} \lim_{y \to x} \int_x \mathrm{Tr} \big\{ \partial_k q_k(x) : G_k :_{\tilde{H}_F} (x, y) \big\} \\ (\Gamma_k^{(2)} &- q_k) G_k = -1 \,. \end{aligned} \tag{5.6}$$

Proof. From the Definition 4.5 of the effective average action as the modified Legendre transform of W_k, we have

$$\begin{aligned}\partial_k \Gamma_k &= \partial_k W_k(j_\phi) + W^{(1)}(j_\phi)\partial_k j_\phi - \phi \partial_k j_\phi - \partial_k Q_k(\phi) \\ &= \partial_k W_k(j_\phi) - \partial_k Q_k(\phi) \,.\end{aligned}$$

The cancellation of the scale derivative of j_ϕ follows as usual by the identification $W^{(1)}(j_\phi) = \phi$. By the same identification, the RG flow for W_k, when written with ϕ as independent variable, is

$$\partial_k W_k = \frac{i\hbar}{2} \lim_{y \to x} \int_x \mathrm{Tr}\Big\{ \partial_k q_k(x) \Big[: W_k^{(2)}(j_\phi) :_{\tilde{H}_F} (x,y) \Big] \Big\} + \partial_k Q_k(\phi) \,.$$

The classical term $\partial_k Q_k(\phi)$ arises because

$$-\frac{1}{2} \int_x \partial_k q_k(x) W_k^{(1)}(x)|_{j_\phi} W_k^{(1)}(x)|_{j_\phi} = \partial_k Q_k(\phi) \,.$$

From the last two relations the scale derivative of the effective average action is

$$\partial_k \Gamma_k = \frac{i\hbar}{2} \lim_{y \to x} \int_x \mathrm{Tr}\left\{ \partial_k q_k(x) : W_k^{(2)}\Big|_{j_\phi} :_{\tilde{H}_F} (x,y) \right\} .$$

The last step is the evaluation of $W_k^{(2)}(x,y)|_{j_\phi}$ in terms of the effective average action. In fact, Lemma (4.4) shows that $W_k^{(2)}|_{j_\phi}$ is the interacting propagator for the quantum wave operator $(\Gamma_k^{(2)} - q_k)$, so that it can formally be written as the inverse $W_k^{(2)}|_{j_\phi} = (\Gamma_k^{(2)} - q_k)^{-1}$. However, since the inverse is not unique this last relation does not uniquely fixes the interacting propagator, and thus cannot provide a well-defined RG flow for the effective average action. Instead, denoting

$$W_k^{(2)}(x,y)\Big|_{j_\phi} := G_k(x,y) \,,$$

to highlight that G_k is a function of ϕ and a fundamental solution of the QEOM, the RG flows equations for the average effective follow, and are given in the statement of the theorem. □

In the case of the scalar field, every matrix in field space reduces to a scalar, and there are no ghost terms; otherwise, the derivation follows the same steps [83].

In the r.h.s of Eq. (5.6), the derivative of the regulator term includes an infra-red cut-off function f with compact support, that limits the integral to a finite region of spacetime so that it is finite. The cut-off function f thus regularises the IR divergences. The counterterms $\tilde{H}_F$ instead regularises the UV divergences associated with the coincidence limit of the interacting propagator, so that the flow is finite in the UV as well.

The coincidence limit of the interacting propagator $G_k(x,x)$ arises from the application of the Bogoliubov map to the Wick square $T\chi^2$. This corresponds in perturbation theory to a summation of all Feynman diagrams, traced over each spacetime point. Graphically, it amounts to the sum of loops, with n internal points, that are closed by the Wick square $T\chi^2(x)$.

However, thanks to normal-ordering, the Wick square $T\chi^2$ is smooth by definition. It follows that in the loops, each propagator is connected to another at each point, but the coincidence limit is taken on a smooth function; there is no propagator multiplied at the same spacetime point, and so the loops are actually finite.

In the following Section we will gain more insight in the r.h.s of the RG flow equation.

There are two main differences from the Euclidean case in the RG flow equation. The first difference is the dependence of the RG flow on the interacting propagator: since the quantum wave equations are hyperbolic, the interacting propagator is simply *an* inverse of the second derivative of the effective average action, or, more precisely, a fundamental solution for the quantum wave operator. Therefore, the RG flow equations are actually a system of equations: the scale derivative of the effective average action is given in terms of the interacting propagator, and in turn the interacting propagator must be a fundamental solution of the quantum wave equations.

The second difference is the UV regularisation. Since we choose to work with a mass-like regulator, the flow equations depend on the coincidence limit of the interacting propagator. Nevertheless, the RG flow equations are finite, because the regulator term is normal-ordered. In fact, the r.h.s. of the RG flow equations is the Bogoliubov map applied to the coincidence limit of a time-ordered quantity, which provides a finite result.

Therefore, the RG flow equations (5.6) are finite, and they compute the renormalization flow of the effective average action. However, they are not closed, since the interacting propagator is not uniquely fixed by the effective average action, but it is *a* propagator among the infinite family of inverses of the QEOM.

We solve this issue in Section 5.2.

The RG flow equations are not closed for two reasons: the first is the ambiguity in the choice of the inverse G_k; the second is the definition of the counterterms $\tilde{H}_F$ only in perturbation theory. In the next Section we solve both issues, applying perturbation theory not on the construction of the Bogoliubov map, and therefore to the solution of the classical equations of motion, but rather to the construction of the interacting propagator as a fundamental solution of the QEOM.

We first rewrite Γ_k as the sum of the quadratic, classical action and quantum corrections, which we encode in the *effective potential*. The effective potential arises from the re-summation of all Feynman diagrams in perturbation theory, and therefore can contain any number of derivatives and non-local terms. Moreover, it is defined non-perturbatively in the coupling constant. The decomposition allows to rewrite the interacting propagator from the free propagator and the effective potential. The freedom in the choice of the interacting propagator is fixed by the requirement that, in the limit of vanishing interactions, it reduces to the propagator for the free theory.

We thus re-write G_k as a Neumann series in the quantum corrections to the effective average action. Since these corrections come from summing over all Feynman diagrams, the interacting propagator retains its non-perturbative nature in the coupling constant. The ambiguity in the choice of G_k is fixed by the requirement that, in the free and unregularised limits $V \to 0$ and $k \to 0$, the interacting propagator must coincide with the free Feynman propagator. The choice of a reference state for the free theory thus uniquely fixes the interacting propagator as a function of the effective average action.

In a similar way, regarding the QEOM as perturbative wave equations, the counterterms $\tilde{H}_F$ admit a perturbative construction in terms of the free Hadamard parametrix. In particular, we will find a closed expression for the counterterm $\tilde{H}_F$ as a power series defined from the effective average action itself.

Thus, the only information coming from the bare theory is the smooth part of Δ_+, the two-point function for a state that must be chosen *a priori*. Such state dependence seems unavoidable, since a Green hyperbolic operator has an infinite family of inverses.

5.2 RG FLOW FOR THE EFFECTIVE POTENTIAL

In this Section we write the inverse G_k explicitly in terms of the effective average action and the free Feynman propagator Δ_F.

We start with some general considerations on the connection between the free and the quantum wave operators in terms of the Møller operators, introduced in Section 2.9. First of all, the DSE (4.29) suggests to decompose the effective average action into

$$\Gamma_k(\phi) := I_0(\phi) + U_k(\phi)\ , \tag{5.7}$$

where P_0 is the Green hyperbolic operator defined by the free action, while the *effective potential* $U_k(\phi)$ is defined by relation

$$U_k^{(1)}(x) := \langle V^{(1)}(x)\rangle\ . \tag{5.8}$$

The effective potential includes all quantum corrections to the interaction V, and it can be seen as a non-perturbative definition for the sum of perturbative Feynman diagrams. In its perturbative expansion, the effective potential contains non-localities and possibly higher-derivative terms.

In terms of the effective potential, in operator notation the interacting propagator is defined by equation

$$(P_0 - q_k + U_k^{(2)})G_k = -1\ . \tag{5.9}$$

In the following, we will assume that, despite quantum corrections, the quantum wave operator $P_0 - q_k + U_k^{(2)}$ remains Green hyperbolic; that is, it admits advanced and retarded propagators such that

$$(P_0 - q_k + U_k^{(2)})\Delta^U_{A,R}(f) = f\ , \text{and } \operatorname{supp}\Delta^U_{A,R}(f) \subset J^{\pm}(\operatorname{supp} f)\ , \tag{5.10}$$

where the future light-cone corresponds to the retarded propagator Δ_R.

Thanks to this assumption, the advanced and retarded propagators $\Delta^U_{A,R}$ have an expression in terms of the propagators $\Delta_{A,R}$ of the free theory, through the classical Møller operators.

5.2.1 *Møller operators*

There is a standard procedure to intertwine the free and the quantum wave operators $(P_0 - q_k)$ and $P_0 - q_k + U_k^{(2)}$. Consider the operator $(1 - \Delta^U_A U_k^{(2)})$ applied to any function f; we have

$$(P_0 - q_k + U_k^{(2)})(1 - \Delta^U_A U_k^{(2)})f = (P_0 - q_k)f\ , \tag{5.11}$$

with a similar relation for the operator $r^R_U := 1 - U_k^{(2)}\Delta^U_R$. It follows that the operators $(1 - \Delta^U_{A,R}U_k^{(2)})$ intertwine between the free and quantum wave operators.

The operators $(1 - \Delta^U_{A,R} U_k^{(2)})$ are the Møller operators r_U that we introduced in Section 2.9, for the potential U_k.

We can now rewrite the Møller operators in terms of the propagators for the free theory and the effective potential. We start from the defining property of $\Delta^U_{A,R}$, that they are fundamental solutions of the QEOM:

$$(P_0 - q_k + U_k^{(2)})\Delta^U_{A,R} = 1 \,.$$

It follows that

$$(P_0 - q_k)(1 + \Delta_{R,k} U_k^{(2)})\Delta^U_{A,R} = 1 \,, \tag{5.12}$$

and so the following recursive formula for $\Delta^U_{A,R}$ holds

$$\Delta^U_{A,R} = \Delta_{A,R,k}(1 - U_k^{(2)} \Delta^U_{A,R}) \,. \tag{5.13}$$

$\Delta_{R,k}$ is the retarded propagator associated with $P_0 - q_k$, which is normally hyperbolic because q_k does not change the principal symbol of P_0.

Acting with $(1 + \Delta_{R,k} U_k^{(2)})^{-1} \Delta_{R,k}$ from the left in Eq. (5.12) gives

$$\Delta^U_{A,R} = (1 + \Delta_{R,k} U_k^{(2)})^{-1} \Delta_{R,k} \,. \tag{5.14}$$

From this expression, we can rewrite the Møller operators using the Neumann series for the operator inverse, so that

$$\begin{aligned} 1 - \Delta^U_A U_k^{(2)} = 1 - \sum_{n=0} (-\Delta_{R,k} U_k^{(2)})^n \Delta_{R,k} U_k^{(2)} &= 1 - \sum_{n=1} (-\Delta_{R,k} U_k^{(2)})^n \\ &= \sum_{n=0} (-\Delta_{R,k} U_k^{(2)})^n = (1 + \Delta_{R,k} U_k^{(2)})^{-1} \,. \end{aligned} \tag{5.15}$$

It is possible to derive a similar expression for the Feynman propagator. Recall that, by definition, $\Delta_{F,k} = \Delta_{+,k} + i\Delta_{A,k}$, where $\Delta_{+,k}$ is a Hadamard bisolution of $P_0 - q_k$ and $\Delta_{A,k}$ is the advanced propagator, so that $(P_0 - q_k)\Delta_{F,k} = i$. Now, starting from the observation

$$(P_0 - q_k + U_k^{(2)})\Delta^U_F = i \,,$$

it follows

$$(P_0 - q_k)(1 - i\Delta_{F,k} U_k^{(2)})\Delta^U_F = i \,,$$

and so

$$\Delta^U_F = (1 - i\Delta_{F,k} U_k^{(2)})^{-1} \Delta_{F,k} \,. \tag{5.16}$$

Clearly, the above formulas hold only if the operators $(1 + \Delta_{A,R} U_k^{(2)})$ and $(1 - i\Delta_{F,k} U_k^{(2)})$ admit an inverse. In the important examples of the No-Derivative and Local Potential Approximations, that we discuss later in Chapter 6 and in Chapter 8, this holds true and the inverse admits an exact expression thanks to the principle of perturbative agreement [95]. In general, the interacting propagator Δ^U_R and the inverse $(1 + \Delta_{R,k} U_k^{(2)})^{-1}$ have a definition only as a formal power series in the space of operators, with a formal parameter implicitly defined in $U_k^{(2)}$. This parameter can be the coupling constant or $\hbar$, but also a formal coupling, counting, for example, the number of derivatives; in this case, the perturbative construction of $(1 + \Delta_{A,R} U_k^{(2)})^{-1}$ corresponds to a Derivative Expansion (DE), one of the most used approximations in the fRG. This is how perturbation theory in U_k retains the non-perturbative structure of the RG flow.

The definition of Δ_F^U in Eq. (5.16) is a perturbative construction of the Feynman propagator for the wave operator $P_0 - q_k + U_k^{(2)}$, in terms of the Feynman propagator of $P_0 - q_k$. However, the usual definition of the Feynman propagator is in terms of the 2-point function and of the advanced propagator, and should read

$$\tilde{\Delta}_{F,U} = \Delta_{+,U} + i\Delta_{A,U} \,.$$

In the following Lemma, we prove that the two definitions coincide in the case of mass perturbations.

Lemma 5.2. *Consider the two wave operators P_0 and $P_0 - M^{(2)}$, where $M^{(2)}$ is a mass perturbation that does not contain derivatives. Then we have*

$$(1 + i\Delta_F M)^{-1}\Delta_F = \Delta_{+,M} + i\Delta_{A,M} \,.$$

Proof. According to Lemma 3.1 and Proposition 3.11 in Ref. [95], and Definition 2.19 of the Møller operators, we have

$$\Delta_{+,M} = r_M \circ \Delta_+ \circ r_M^* \,, \tag{5.17}$$

where r_M is the Møller operator,

$$r_M = (1 - \Delta_{R,M} M) \,,$$

and $\Delta_{R,M}$ is the unique retarded propagator of $P_0 - M^{(2)}$.

Now, recalling that $\Delta_F = \Delta_+ + i\Delta_A$, it holds that

$$\Delta_{F,M} = \sum_{n\geq 0} (\Delta_+ + i\Delta_A)[-iM(\Delta_+ + i\Delta_A)]^n \,. \tag{5.18}$$

According to Lemma 3.10 in Ref. [95], we have that

$$\Delta_{A,M} = \Delta_A \sum_{n\geq 0} (M\Delta_A)^n = \Delta_A(1 + M\Delta_{A,M}) = \Delta_A r_M^* \,.$$

Rearranging the sum, Eq. (5.18) becomes

$$\begin{aligned}
\Delta_{F,M} &= i\Delta_{A,M} + [1 + \Delta_{A,M} M]\Delta_{+,M} r_{Q_k}^* + \sum_{n\geq 1} \mathsf{p}(\Delta_{+,M} r_M^*)(-M\Delta_+ r_M^*)^n \\
&= i\Delta_{A,M} + r_M \Delta_+ r_M^* - \Delta_M M \Delta_+ r_M^* + \sum_{n\geq 1} \mathsf{p}(\Delta_+ r_M^*)(-M\Delta_+ r_M^*)^n \,,
\end{aligned}$$

where $\mathsf{p} = (1 + \Delta_{A,M} M) = (1 + \Delta_{R,M} M - \Delta_M M) = r_M - \Delta_M M$. Notice that

$$-\Delta_M M \Delta_+ = \Delta_M((P_0 - M^{(2)}) - P_0)\Delta_+ = 0 \,,$$

where in the last step we used the fact that Δ_M is a weak solution of $P_0 - M^{(2)}$ in both entries, and Δ_+ is a weak solution of P_0. Similarly, for every $n \geq 2$, $(\Delta_+ r_M^*)(M\Delta_+ r_M^*)^{n-1} = 0$ because $\Delta_+ r_M^*(P_0 - M^{(2)}) = 0$. Finally, we have that

$$\Delta_{F,M} = i\Delta_A r_M^* + r_M \Delta_+ r_M^* \,. \tag{5.19}$$

□

Thanks to the Møller operators, we can write solutions and propagators of the quantum wave operator $P_0 - q_k + U_k^{(2)}$ in terms of the solutions and propagators of the free theory and of the effective potential $U_k^{(2)}$.

In what follows, we want to compute the regularised, interacting propagator $:G_k:_{\tilde{H}_F}$ in terms of the propagators of the free theory and the effective potential U_k, in order to write the RG flow equations as a closed differential equation for U_k.

5.2.2 *Regularised propagator and RG flow of the effective potential*

Recall that the regularised propagator $:G_k:_{\tilde{H}_F}$ is a solution for the quantum wave operator $(\Gamma_k^{(2)} - q_k) : G_k :_{\tilde{H}_F} = 0$ up to known smooth terms. Furthermore, denoting by $\mathcal{O} \subset \mathcal{M}$ the support of V and of q_k, which is a compact set because of the cut-off functions used in their construction, it holds by causality that

$$-i : G_k(x,y) := \Delta_F(x,y) - H_F(x,y) = w(x,y)\,, \qquad \forall x,y \in \mathcal{M} \setminus J^+(\mathcal{O}).$$

To prove this result, we use the next Lemma.

Lemma 5.3. *The interacting propagator in the limit of vanishing interactions, $V = 0$, and vanishing regulator, $k = 0$, $G_0^{(0)} := \lim_{k\to 0} \lim_{V\to 0} G_k$ corresponds to the Feynman propagator for the free theory, with free wave operator P_0:*

$$-iG_0^{(0)} = \Delta_F\,.$$

Proof. The proof works by computing the free, unregularised propagator $G_0^{(0)}$ from its definition in terms of the Bogoliubov map applied to $\varphi(x) \cdot_T \varphi(y)$:

$$-i\hbar G_k(x,y) = -i\hbar\, W_k^{(2)}(x,y)\Big|_{j_\phi} = \langle \varphi(x) \cdot_T \varphi(y)\rangle - \phi(x)\phi(y)\,.$$

It follows that, in the limit $k \to 0$ and with vanishing interactions, the normal-ordered interacting propagator is given by the expression

$$-i\hbar : G_0^{(0)} : (x,y) = \langle \varphi(x) \cdot_T \varphi(y)\rangle_{\substack{k=0\\V=0}} - \phi(x)_{\substack{k=0\\V=0}}\phi(y)_{\substack{k=0\\V=0}} - H_F(x,y)\,. \tag{5.20}$$

We can now explicitly compute the mean value in the last expression. By definition, the mean value is

$$\langle \varphi(x) \cdot_T \varphi(y)\rangle_{\substack{k=0\\V=0}} = \frac{1}{Z_{V=0}}\omega(S(J_\phi) \cdot_T \varphi(x) \cdot_T \varphi(y))\,,$$

where $Z_{V=0} = \omega(S(J_\phi))$, and we recall that $Z := Z_{k=0}$. Using Eq. (2.51), the time-ordered product with $S(J)$ at the numerator produces a shift in the fields times a phase, which cancels out with the denominator. It follows that

$$\langle \varphi(x) \cdot_T \varphi(y)\rangle_{\substack{k=0\\V=0}} = \omega((\varphi + i\Delta_F j) \cdot_T (\varphi + i\Delta_F j))\,.$$

Moreover, the DSE (4.29) in the $k = 0$ and $V = 0$ limits reduces to

$$\frac{\delta\Gamma}{\delta\phi}\bigg|_{V=0} = \frac{\delta I_0}{\delta\phi} = P_0\phi\,,$$

where again $\Gamma = \Gamma_{k=0}$, while the QEOM (4.25) gives $-j_\phi = \frac{\delta\Gamma}{\delta\phi}\big|_{V=0}$. The source term j_ϕ thus depends on ϕ through the free equations of motion, $j_\phi = -P_0\phi$, and so the mean value simplifies into

$$\begin{aligned}\langle \varphi(x) \cdot_T \varphi(y)\rangle_{\substack{k=0\\V=0}} &= \omega\left((\varphi - \phi)(x) \cdot_T (\varphi - \phi)(y)\right) \\ &= \omega(\varphi(x)\varphi(y)) + \hbar\Delta_F \\ &\quad + \phi(x)\phi(y) - \omega(\varphi(x))\phi(y) - \omega(\varphi(y))\phi(x)\,.\end{aligned} \tag{5.21}$$

However, we have, again by definition,

$$\phi_{\substack{k=0\\V=0}} = \frac{1}{Z_k}\omega\left(S(J_\phi)\cdot_T \varphi(x)\right) .$$

Reasoning as before results in

$$\phi_{\substack{k=0\\V=0}} = \frac{1}{Z_k}\omega\left(S(J_\phi)\cdot_T \varphi(x)\right) = \omega(\varphi) - \phi_{\substack{k=0\\V=0}} .$$

Therefore, in the presence of sources, the mean field ϕ and the one-point function are proportional to each other:

$$\phi_{\substack{k=0\\V=0}} = \frac{1}{2}\omega(\varphi) . \tag{5.22}$$

Equations (5.20), (5.21), and (5.22) imply that the propagator $G_0^{(0)}$ is proportional to the free Feynman propagator

$$-i : G_0^{(0)} := \Delta_F(x,y) - H_{F,k} .$$

Notice that the $\hbar$ factor cancels out with the one coming from the T−product in Eq. (5.21). □

The normal-ordered free propagator $: G_k^{(0)} := \lim_{V\to 0} : G_k :$ can now be computed from $G_0^{(0)}$ using the Møller operators. Since we know that $G_k^{(0)}$ is a fundamental solution of Eq. (5.9) for vanishing U_k, it follows that $-i\tilde{H}_F$ reduces to H_F in the free, unregularised limit. Therefore, using the Møller operators $1-\Delta_{R,k}q_k$ and $1 - q_k\Delta_{A,k}$ intertwining between P_0 and $P_0 - q_k$, we get

$$-i : G_k^{(0)} := (1-\Delta_{R,k}q_k)(\Delta_F - H_F)(1-q_k\Delta_{A,k}) = \Delta_{F,k} - H_{F,k} .$$

Reasoning in the same way for the normal-ordered interacting propagator $: G_k :$, intertwining between $P_0 - q_k + U_k^{(2)}$ and $P_0 - q_k$, we get

$$\begin{aligned} -i : G_k :_{\tilde{H}_F} &= (1-\Delta_R^U U_k^{(2)})(\Delta_{F,k} - H_{F,k})(1 - U_k^{(2)}\Delta_A^U) \\ &= \sum_{n=0,m=0} (-\Delta_{R,k}U_k^{(2)})^n(\Delta_{F,k} - H_{F,k})(-U_k^{(2)}\Delta_{A,k})^m , \end{aligned} \tag{5.23}$$

where with the subscript k we denote the propagators for the operator $P_0 - q_k$.

Therefore, thanks to the last equation, we conclude that the RG flow equation can be rewritten as a differential equation for the effective potential U_k

$$\begin{aligned} \partial_k U_k &= \\ &= -\frac{\hbar}{2}\int_{x,y,z} \mathrm{Tr}\left\{\partial_k q_k (1-\Delta_{R,k}U_k^{(2)})^{-1}(\Delta_{F,k} - H_{F,k})(1-U_k^{(2)}\Delta_{A,k})^{-1}\right\} , \end{aligned} \tag{5.24}$$

The last equation can be written in two more, equivalent ways. First, expanding the Møller operators in their perturbative series gives

$$\partial_k U_k = -\frac{\hbar}{2}\int_x \mathrm{Tr}\left\{\partial_k q_k \sum_{n=0,m=0} (-\Delta_{R,k}U_k^{(2)})^n(\Delta_{F,k} - H_{F,k})(-U_k^{(2)}\Delta_{A,k})^m\right\} . \tag{5.25}$$

The last equation can also be written with the representation of the Møller operators with the Feynman propagator only, as

$$\partial_k U_k = -\frac{\hbar}{2} \lim_{y \to x} \int_x \mathrm{Tr}\, \partial_k q_k \left[\sum_{n \geq 0} (i\Delta_{F,k} U_k^{(2)})^n (\Delta_{F,k} - H_{F,k}) \right] . \tag{5.26}$$

As anticipated, the RG flow equation is a closed partial differential equation for the effective potential U_k. To solve the RG flow equations, we need a set of initial data: the regulator term, an initial value $U_{k=\Lambda}$, and a choice of a reference state for the free theory, with respect to the operator P_0.

Notice, however, that at least in some applications there is no need to specify a bare free action from the outset. In fact, most applications of the fRG start with an ansatz for the effective average action Γ_k as some expression in terms of the fields ϕ and their derivatives. Starting from the ansatz, the quadratic part of the limit $k \to \infty$ provides the operator P_0. In other words, choosing an ansatz for Γ_k or a free wave operator P_0 are both ways to fix the starting point of the flow at $k \to \infty$.

Although the r.h.s. of Eq. (5.24) can be regarded as a series in $U_k^{(2)}$, it retains the non-perturbative nature of the RG flow equation for two reasons: first, such Dyson-type series can in some cases be summed to their exact result, e.g. in the case of a local potential.

Moreover, U_k is itself defined non-perturbatively from the interaction, via the relation (5.8). As it is possible to expand $\langle V^{(1)} \rangle$ perturbatively, writing the Møller operators in terms of Feynman diagrams, it is as well possible to choose non-perturbative expansions for U_k, such as the derivative or the vertex expansion, keeping non-perturbative information on the interaction.

Remark 5.1. It is possible to rewrite the RG flow for the effective potential (5.24) in an equivalent way, constructing the interacting propagator G_k from the Møller operators intertwining P_0 and $P_0 - q_k + U_k^{(2)} = P_0 + \tilde{U}_k^{(2)}$, where $\tilde{U}_k$ is defined by

$$\tilde{U}_k^{(1)}(x) = \langle V^{(1)}(x) + Q_k^{(1)}(x) \rangle = U_k^{(1)}(x) + \langle Q_k^{(1)}(x) \rangle , \tag{5.27}$$

as

$$\partial_k \tilde{U}_k = \\ = -\frac{\hbar}{2} \int_{x,y,z} \mathrm{Tr} \Big\{ \partial_k q_k (1 + \Delta_R \tilde{U}_k^{(2)})^{-1} (\Delta_+ - H)(1 - \tilde{U}_k^{(2)} \Delta_A)^{-1} \Big\} . \tag{5.28}$$

This expression will be the starting point for the proof of existence of solutions in Chapter 6. ■

5.2.3 *Hadamard regularisation*

In the last Section, we derived the RG flow equations for the effective potential as closed equations in terms of the free reference state. Equation (5.24) is on of the main results of this thesis and it will provide the starting point for applications to concrete physical models. We will also prove that, in the important No Derivative Approximation, Eq. (5.24) always admits local solutions.

We now want to clarify some points in the derivation of Eq. (5.24). The equation heavily relies on the use of Møller operators intertwining between partial differential operators, to derive the interacting propagator G_k and its normal-ordering

$: G_k :$. However, the counterterms $\tilde{H}_F$ were never explicitly defined from the quantum wave operator; instead, we gave explicit expressions only for the combination $: G_k :_{\tilde{H}_F} = G_k - \tilde{H}_F$. The main reason is that, while the combination $: G_k :_{\tilde{H}_F}$ is unambiguous, it can be defined by different subtractions schemes, as we will discuss later. We then provide two different normal-ordering prescriptions to define $\tilde{H}_F$.

Moreover, the objects $: G_k :_{\tilde{H}_F}$ and $\tilde{H}_F$ were first defined via the Bogoliubov map from interacting observables in the quantum algebra. In this Section, we provide the connection between their definitions using the Møller operators and their perturbative expansion in Feynman diagrams.

We solve the issue on the definition of $\tilde{H}_F$ by providing a Hadamard-type subtraction for the quantum wave operator $P_0 - q_k + U_k^{(2)}$ and its inverse.

In the derivation of the RG flow equation, we implicitly defined the counterterms $\tilde{H}_F(x, y)$ by the relation

$$\langle \lim_{y \to x} \varphi(x) \cdot_T \varphi(y) - H_F(x, y) \rangle = \lim_{y \to x} \langle \varphi(x) \cdot_T \varphi(y) \rangle - i\hbar \tilde{H}_F(x, y) . \tag{5.29}$$

In fact, in the coincidence limit $\langle \varphi(x) \cdot_T \varphi(y) \rangle$ diverges more badly than $\varphi(x) \cdot_T \varphi(y)$, due to the loop diagrams coming from the Bogoliubov map in the mean value. On formal terms, the two limits have different singular parts because one is exchanging the limit with the convolutions in the expectation value, and the two operations in general do not commute.

In this Section, we provide a practical tool to compute $\tilde{H}_F$. From the double series in Eq. (5.23) we have

$$-i : G_k :_{\tilde{H}_F} = (1 - \Delta_R^U U_k^{(2)})(\Delta_F - H_F)(1 - U_k^{(2)} \Delta_A^U) . \tag{5.30}$$

Since $: G_k :_{\tilde{H}_F} = G_k - \tilde{H}_F$, it is natural to identify

$$-iG_k = (1 - \Delta_R^U U_k^{(2)}) \Delta_{F,k} (1 - U_k^{(2)} \Delta_A^U) \text{ , and} \tag{5.31}$$

$$-i\tilde{H}_F = (1 - \Delta_R^U U_k^{(2)}) H_{F,k} (1 - U_k^{(2)} \Delta_A^U) . \tag{5.32}$$

We already showed that G_k is the Feynman propagator for the quantum wave operator. In the same way, $\tilde{H}_F$ is a Feynman-Hadamard parametrix, that is, a propagator of the QEOM up to smooth terms,

$$(P_0 - q_k + U_k^{(2)}) \tilde{H}_F = i\delta + \lambda .$$

Remark 5.2. Using the fact that the Møller operators applied to the Feynman propagator can be written in terms of the Feynman propagator only, we have a more compact expression for G_k, equivalent to the above:

$$-iG_k = (1 + i\Delta_F^U U_k^{(2)}) \Delta_{F,k} = \sum_n (\Delta_{F,k} U_k^{(2)})^n \Delta_{F,k} = (1 - i\Delta_{F,k} U_k^{(2)})^{-1} \Delta_{F,k} . \tag{5.33}$$

However, the expression of G_k as a double series (5.31) has better convergence properties, especially in curved spacetimes, than the expression with the Feynman propagator. For this reason, in the following we prefer this formula, in particular when we need to prove important results on the convergence of the series. ■

There is an equivalent way to write the regularised $: G_k :$. Since we have the equivalence $\Delta_+ - H = \Delta_F - H_F$, from the first of these two expressions, we get

$$-i : G_k :_{\tilde{H}_F} (1 - \Delta_R^U U_k^{(2)})(\Delta_{+,k} - H_k)(1 - U_k^{(2)} \Delta_A) , \tag{5.34}$$

where $\Delta_{+,k}$ is a two-point function for $P_0 - q_k$. This leads to

$$-iG_k^+ = (1 - \Delta_R^U U_k^{(2)})\Delta_{+,k}(1 - \Delta_A^U U_k^{(2)})\text{ , and} \tag{5.35}$$
$$-i\tilde{H}_k = (1 - \Delta_R^U U_k^{(2)})H_k(1 - U_k^{(2)}\Delta_A^U)\ . \tag{5.36}$$

Since $(P_0 - q_k)\Delta_{+,k} = 0$, by the properties of the Møller operators we have

$$(P_0 - q_k + U_k^{(2)})G_k^+ = (P_0 - q_k)\Delta_{+,k} = 0\ ,$$

that is, G_k^+ is a solution of the QEOM, rather than a propagator, while $\tilde{H}_k$ is a solution of the QEOM with smooth source, $(P_0 - q_k + U_k^{(2)})\tilde{H}_k = \lambda$.

We therefore have two equivalent expressions for $:G_k:$, Eq.(5.33) and Eq. (5.34), analogous to the two equivalent expressions for $w = \Delta_+ - H = \Delta_F - H_F$. Notice that the normal ordered $:G_k:$ appearing in the flow equations is unambiguous; only the split in an unregularised term and counterterms introduces fictitious ambiguities in the definition. For this reason, and in order to highlight that the normal-ordered object is unique, we will write $:G_k:$ without specifying the normal-ordering prescription, unless necessary.

The connection between $:G_k:$ and its expansion in Feynman diagrams is

$$:G_k(x,x) := \langle T\varphi^2(x)\rangle\ , \tag{5.37}$$

as it comes from the expectation value of the mass term Q_k. From the expression above, it seems slightly preferable to consider the unregularised propagator $G_k(x,y)$ as the Feynman propagator for the interacting theory, rather than a solution G_k^+ of the equations. However, when dealing with the regularised object entering physical quantities, both choices are valid.

As a sanity check, we can explicitly see that G_k^F is actually proportional to the inverse of the quantum wave operator. In fact, the Møller operators appearing in the RG flow are equivalent to [95]

$$\begin{aligned} -iG_k^F &= (1 + \Delta_{R,k}U_k^{(2)})^{-1}\Delta_{F,k}(1 + U_k^{(2)}\Delta_{A,k})^{-1} \\ &= (1 - i\Delta_{F,k}U_k^{(2)})^{-1}\Delta_{F,k}\ . \end{aligned} \tag{5.38}$$

Now, we can formally write

$$\begin{aligned} (1 - i\Delta_{F,k}U_k^{(2)})^{-1} &= \\ &= -[i\Delta_{F,k}(P_0 - q_k + U_k^{(2)})]^{-1} \\ &= -(P_0 - q_k + U_k^{(2)})^{-1}(P_0 - q_k)\ . \end{aligned}$$

From the above expressions, the double series for G_k in Eq. (5.31) can be written as

$$-iG_k^F = (1 - i\Delta_{F,k}U_k^{(2)})^{-1}\Delta_{F,k} = -i(P_0 - q_k + U_k^{(2)})^{-1}\ . \tag{5.39}$$

Since $\Gamma_k^{(2)} - q_k = P_0 - q_k + U_k^{(2)}$ by definition, we see explicitly that $-iG_k^F$ is the Feynman propagator for the quantum wave operator.

Finally, notice that the RG flow can be written in a way more suggestive of its Euclidean counterpart. First of all, since $\Gamma_k = I_0(\phi) + U_k$, we can substitute in the l.h.s the scale derivative of the effective average action, $\partial_k\Gamma_k = \partial_k U_k$. Now, the inverse $(P_0 + U_k^{(2)})^{-1}$ is nothing but the inverse of the quantum wave operator,

$(P_0 + U_k^{(2)})^{-1} = (\Gamma_k^{(2)} - q_k)^{-1}$; it follows that the RG flow equations can formally be written as

$$\partial_k \Gamma_k = -\frac{\hbar}{2} \lim_{y \to x} \int_x \mathrm{Tr}\left[\partial_k q_k (\Gamma_k^{(2)} - q_k)^{-1} (P_0 - q_k)(\Delta_{+,k} - H_k)\right] . \tag{5.40}$$

This equation closely mimics the Wetterich equation in Euclidean spaces [217, 246], with the important difference that the Feynman propagator for the free theory appears, providing an explicit state dependence. However, the equation above hides the choice in the inverse $(\Gamma_k^{(2)} - q_k)^{-1}$; from the discussion above, Eq. (5.40) must be supplemented with the prescription that $(\Gamma_k^{(2)} - q_k)^{-1}$ is the Feynman propagator for the interacting theory, which reduces to the Feynman propagator of the free theory in the limit of vanishing mass and interactions.

Even if the regulator term q_k is only a local IR cut-off, the flow is finite thanks to the smoothing term $(P_0 - q_k)\Delta_{+,k} - H_k$. Recall that the smooth term $\Delta_{+,k} - H_k$ is a solution of $P_0 - q_k$ only up to smooth terms, so the equations always provide a non-trivial flow. The flow is non-trivial also for ground states, that usually have $w = 0$, since the difference $\Delta_{+,k} - H_k$ differs from the smooth part of the two-point function w by smooth terms. For example, in the case of Minkowski vacuum, while $w = 0$, $\Delta_{+,k} - H_k$ is proportional to a logarithm of the mass term.

6 RG flow and the Nash-Moser theorem

This Chapter discusses a proof of the existence of local solutions of the RG flow, in a particular approximation and for scalar field theories, on globally hyperbolic spacetimes and without assuming analyticity of the effective potential.

In order to provide the estimates required to prove the theorem, it is best to formulate th RG flow in terms of $\tilde{U}_k$, given in Equation (5.28). The definition of $\tilde{U}_k$ is recalled here,

$$\tilde{U}_k^{(1)}(x) = \langle V^{(1)}(x) + Q_k^{(1)}(x)\rangle = U_k^{(1)}(x) + \langle Q_k^{(1)}(x)\rangle \, .$$

In this Chapter, we will call $\tilde{U}_k$ the *effective potential*. Since we never refer to U_k in this Chapter, no confusion should arise.

Having derived the RG flow equations, Eq. (5.6), we are now interested in studying its solutions, provided by the effective average action Γ_k.

However, according to the RG philosophy, every possible interaction term is admitted in principle in the effective average action. The fRG flow reflects this behaviour in its mathematical structure; due to the appearance of the inverse of the second derivative of Γ_k on the r.h.s of Eq. (5.6), independently from the initial data for Γ_k, the flow will always produce additional interaction terms in the effective average action. A way of seeing it is to expand the inverse as a Neumann power series. A similar problem appears in the study of semi-classical gravity [186–188], where the back-reaction generates higher derivative terms. Only the truncation of Γ_k in a polynomial expression allows for the generation of finite terms.

Mathematically, the problem of the generation of every possible term along the flow is connected with the problem of *loss of derivatives*: intuitively, since the r.h.s of the RG flow (5.6) depends on the inverse of the second derivative of the effective average action, a Green operator (a fundamental solution) for the RG equation will also depend on the second derivative $\Gamma_k^{(2)}$. It follows that, if the RG equation is an operator acting on some space of C^n functions of the fields, its solutions will generally be only $C^{n-2}-$regular, losing two derivatives. Due to the loss of derivatives, standard iterative procedures to produce solutions in suitable Banach spaces fail to converge.

As we will see in Chapter 8 on concrete applications of the fRG, a very simple approximation to solve the RG equations (5.24) consists in a truncation of the effective potential in a finite sum of polynomials of the fields, $U_k^t = \sum_j^n \lambda_{j,k}\phi^j$, that reduces the functional RG equations (5.24) in a system of coupled differential equations for the dimensionless couplings, the beta-functions. Despite its simplicity, even in this setting the flow is non-trivial, and contains non-perturbative effects, as can be seen e.g. from the non-polynomial dependence of the beta-functions on the couplings in the last sections, equations (8.19), (8.22), and (8.32).

However, the β−functions describe only approximate solutions, that neglect higher-order polynomial terms, due to the truncation. Furthermore, little control on the quality of the approximation scheme, compared to the full theory space, is possible.

A step forward in this approximation would be an expansion of the effective potential in a series of field polynomials, $U_k^t = \sum_j^\infty \lambda_{j,k}\phi^j$, which would produce an infinite system of coupled differential equations. Systems of this kind have been recently studied in the context of semi-classical gravity [138], and it would be interesting to investigate a similar approach for the RG flow.

A more sophisticated approximation consists in considering the effective potential an unknown function of some variable, and solving the RG flow as a partial differential equation for this function. An example is the $f(R)$ approximation in quantum gravity, where the effective average action is assumed to be a function of the Ricci scalar R [193].

Almost the entirety of the fRG literature presents scheme-dependent results on a case-by-case basis. While the literature on the fRG is by now comprehensive of a large number of applications (see e.g. Ref. [102] for a recent review), the mathematical results on the fRG flows based on the Wetterich equation are more sparse. The main prejudice on the quality of fRG results comes from the lack of control on the truncation scheme. Precisely because the fRG is a non-perturbative approach, its approximation schemes do not allow to quantitatively estimate the error in the truncation on general grounds. The quality of the results obtained from the fRG have been verified only on a case-by-case basis.

In this Chapter, we take a step further in clarifying the mathematical structure of the RG flow equations, and we prove that, assuming that there are no derivatives of the fields in $\tilde{U}_k$, with a possibly non-polynomial effective potential $\tilde{U}_k$, the equations admit local solutions. This is a first step in establishing mathematically rigorous results on the RG flow derived from the fRG, without reference to particular applications. From the analysis of the local solutions, it will be possible to deduce if such solutions can be extended to global ones, if they have non-trivial fixed points, and their stability.

In order to prove the existence of local solutions for the RG flow (5.24), we need to choose an appropriate approximation. Inspired by Euclidean fRG approaches, as a first step towards more general results we choose to approximate $\tilde{U}_k$ with the *No Derivative Approximation* (NDA), as a local function of the field ϕ with no derivatives, given in Eq. (6.3),

$$\tilde{U}_k(\phi) = \int_{\mathcal{M}} u(\phi(x), k) f(x) \mathrm{d}\mu_x \, , \qquad \tilde{U}_k^{(2)}(\phi)(x,y) = \partial_\phi^2 u(\phi, k) f(x) \delta(x,y) \, ,$$

where f is a compactly supported smooth function which is equal to 1 on large regions of the studied spacetime. We further assume that the field ϕ is constant over the whole space, so that $\partial_\phi^2 u$ is function of k and ϕ only. However, notice that $\tilde{U}_k$ can be any smooth non-polynomial function of the field ϕ.

Within this approximation, the r.h.s. of the RG flow equation can be written in terms of the map given in Eq. (6.5), which we recall here

$$G_k(\partial_\phi^2 u) := -\frac{1}{2\,|||f|||_1} \int_{\mathcal{M}} \partial_k q_k(x) \left\{ (1 - \partial_\phi^2 u \Delta_R^u f) \otimes (1 - \partial_\phi^2 u \Delta_R^u f)(w)(x,x) \right\} \mathrm{d}\mu_x \, .$$

The RG flow equation reduces to an equation for $u(\phi, k)$. We are thus interested in studying the existence of solutions for the following problem,

$$\begin{cases} \partial_k u = G_k(\partial_\phi^2 u) \, , \\ u(\phi, a) = \psi \, , \\ u|_{\partial X \times [a,b]} = \beta \end{cases} \tag{6.1}$$

where ψ and β are known functions which characterise, respectively, the initial and boundary conditions of the problem.

The NDA greatly simplifies the RG flow equation, which now is an equation for u, as a function of k and ϕ. However, the NDA does not simplify the problem of the loss of derivatives.

The main result of this Chapter is the proof of local existence of solutions of this problem, which is given below in Theorem 6.14. The proof is an application of the renown Nash-Moser Theorem.

Nash provided a beautiful theorem to prove local existence of solutions of non-linear partial differential equations in spaces of smooth functions, which are particularly suited to deal with the problem of loss of derivatives [197]. The theory was first developed in the context of isometric embeddings of Riemannian manifolds by Nash, and then further generalised by Moser [195, 196]. Hamilton [144] provided a particularly natural setting for the theorem in the space of tame Fréchet spaces. We review Hamilton's formulation of Nash-Moser theorem in Section 6.1.

6.1 HAMILTON'S FORMULATION OF THE NASH-MOSER THEOREM

In the formulation of Hamilton, the Nash-Moser theorem is given for elements in a suitable tame Fréchet space. Mainly to fix notation, we recall here some basic definitions and the statement Nash-Moser theorem.

Definition 6.1. A *seminorm* on a vector space F is a function $\|\cdot\| : F \to \mathbb{R}$ such that, $\forall f, g \in F$ and $\forall c \in \mathbb{R}$, the following hold: (i) $\|f\| \geq 0$; (ii) $\|f + g\| \leq \|f\| + \|g\|$; (iii) $\|cf\| = |c|\|f\|$. A collection of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ defines a unique topology such that a sequence $f_i \to f \Leftrightarrow \|f_i - f\|_n \to 0 \; \forall n \in \mathbb{N}$. A *locally convex topological vector space* is a vector space with a topology arising from a collection of seminorms. The topology is called *Hausdorff* if $f = 0$ when $\|f\|_n = 0 \; \forall n$. The topology is called *metrizable* if the family $\{\|\cdot\|_n\}_n$ is countable, and the space F is *complete* if every Cauchy sequence converges. A *Fréchet space* is a complete Hausdorff metrizable locally convex topological vector space, and a graded Fréchet space has a collection of seminorms that are increasing in strength, so that $\|f\|_n \leq \|f\|_{n+1} \; \forall n$.

Definition 6.2. A graded space F is *tame* if, given the space $\Sigma(B)$ of exponentially decreasing sequences in some Banach space B, it is possible to find two linear maps $L : F \to \Sigma(B)$, $M : \Sigma(B) \to F$, such that $ML : F \to F$ is the identity

$$F \to^L \Sigma(B) \to^M F \, . \tag{6.2}$$

Consider two graded spaces F and G, and a map $P : \mathcal{U} \subset F \to G$ from an open subset $\mathcal{U}$ of F to G. The map P is *tame* of degree r and base b if it is continuous and satisfies

$$\|P(f)\|_n \leq C(1 + \|f\|_{n+r})$$

for all f in the neighbourhood of each $f_0 \in \mathcal{U}$, for all $n \geq b$, and with a constant C that may depend on n.

In this setting, Hamilton's formulation of the classic Nash-Moser theorem on the inverse function problem can be stated as follows.

Theorem 6.1 (Nash-Moser theorem in Hamilton's formulation)**.** *Consider a smooth tame map $P : \mathcal{U} \subset F \to G$ between two tame Fréchet spaces F and G. Suppose that*

i. the linear map $DP(u)v = f$ obtained as the first functional derivative of P has unique inverse $E(u)f = v \;\forall\, u \in \mathcal{U}$ and all $f \in G$, and

ii. the inverse map $E : \mathcal{U} \times G \to F$ is smooth tame.

Then P is locally invertible and P^{-1} is a smooth tame map.

6.2 NO DERIVATIVE EXPANSION

We restrict our attention to the *No Derivative Expansion*: in this approximation, the effective potential is a local functional which does not contain derivatives of the fields. Furthermore, we consider the case in which the classical field ϕ is constant throughout spacetime.

More precisely, the *No Derivative Expansion* (NDA) assumes that the effective potential and its second functional derivative are

$$\tilde{U}_k(\phi) = \int_{\mathcal{M}} u(\phi(x), k) f(x) \mathrm{d}\mu_x \,, \quad \tilde{U}_k^{(2)}(\phi)(x,y) = \partial_\phi^2 u(\phi(x), k) f(x) \delta(x,y) \,, \tag{6.3}$$

where $f \in C_0^\infty(\mathcal{O})$ is an adiabatic cut-off ($f \geq 1$ and $f = 1$ on the relevant part of the spacetime we are working with), which is inserted to keep the theory infrared finite, and $\mathcal{O} \in \mathcal{M}$ is a compact region in the space-time containing the support of f.

The arbitrary function $u(\phi, k)$ and its second field derivative $\partial_\phi^2 u$ thus determine the effective potential, and so the effective average action.

Notice that what we call here NDA slightly differs from the usual approximation found in the physics literature under the name of *Local Potential Approximation*. In fact, it is standard practice to expand the effective potential around an arbitrary background $\phi = \bar{\phi} + \hat{\varphi}$, and then retaining in the effective potential only terms that are quadratic in the fluctuation field $\hat{\varphi}$. This greatly simplifies the structure of the quantum wave operator, which in this way is approximated by an operator that is linear in the fluctuation field. On the contrary, even though we assume that the effective potential does not contain derivatives of the field, we are retaining its full non-linear dependence on the field ϕ, without expanding on a fixed background.

We further assume that the background spacetime $\mathcal{M}$ is ultra-static. This assumption simplifies the explicit form of the retarded and advanced propagators for the free theory $\Delta_{A,R}$, and it allows for simple estimates of their norms. However, these estimates can be easily generalised to static spacetimes, and are known to holds in some special cases, such as de Sitter space.

Finally, in the simplest approximation, we choose the field ϕ to be a constant throughout the spacetime, so that also $u(\phi, k)$ and $\partial_\phi^2 u(\phi, k)$ are constants in spacetime.

In the limit where $V \to 0$, the effective potential reduces to Q_k and u reduces to $-q_k\phi^2/2$. We shall take this into account in fixing the initial conditions for u.

Thanks to this approximation, the second derivative of the effective potential $\tilde{U}_k^{(2)}$ appearing in the QEOM reduces to a perturbation of the free wave operator P_0 with a smooth external potential that has compact support, and in the limit where $f \to 1$ on $\mathcal{M}$ the potential reduces to a mass perturbation, that is, to a term without derivatives. It follows that many techniques of the generalised principle of perturbative agreement [95] become readily available.

In particular, it is known that the interacting advanced and retarded propagators $\Delta^U_{A,R}$ are given by the free propagators $\Delta_{A,R}$ associated to P_0, with a mass modified by the external potential. The recursive relations given in Eq. (5.13) permit to analyse analytically how G_k depends on u.

By the NDA, the RG flow equation (5.24) becomes a partial differential equation for $u(\phi, k)$. Thus, we are interested in studying the existence and uniqueness of solutions of the problem associated with the RG flow equation (5.24), supplemented with suitable boundary conditions and a set of initial values explicitly given in terms of the functions ψ and β as:

$$\begin{cases} \partial_k u = G_k(\partial^2_\phi u) \, , \\ u(\phi, a) = \psi \, , \\ u|_{\partial X \times [a,b]} = \beta \, . \end{cases} \tag{6.4}$$

where the function G_k is defined as

$$G_k(\partial^2_\phi u) := -\frac{1}{2 \, |||f|||_1} \int_{\mathcal{M}} \mathrm{d}\mu_x \partial_k q_k(x) \left\{ (1 - \partial^2_\phi u \Delta^u_R f) \otimes (1 - \partial^2_\phi u \Delta^u_R f)(w)(x,x) \right\} \, . \tag{6.5}$$

$w \in C^\infty(\mathcal{M}^2)$ is a given symmetric smooth function (the smooth part of the chosen background state); $f \in C^\infty_0(\mathcal{M})$ is the positive cut-off function used in U and $|||f|||_1$ is the L^1 norm of f computed with respect to the standard measure on $\mathcal{M}$; q_k is the integral kernel of the adiabatic regulator Q_k, which is assumed to be smooth and with compact support in x.

$\Delta^u_R : C^\infty_0(\mathcal{M}) \to C^\infty(\mathcal{M})$ is the retarded fundamental solution of $(P_0 + f\partial^2_\phi u)g = 0$; it coincides with Δ^U_R, and it exists and it is unique because $P_0 + f\partial^2_\phi u$ is a Green-hyperbolic operator [18]. Furthermore, in the integrand in Eq. (6.5), f is a multiplicative operator, which maps $C^\infty(\mathcal{M}) \to C^\infty_0(\mathcal{M})$, and 1 is the identity map in $C^\infty(\mathcal{M})$. Notice that $\partial^2_\phi u$ is constant with respect to P_0.

Furthermore, thanks to the support properties of f, we have that $O := (1 - \partial^2_\phi u \Delta^u_R f) \otimes (1 - \partial^2_\phi u \Delta^u_R f)$ is a linear operator on $C^\infty(\mathcal{M} \times \mathcal{M})$ to itself. Since w is smooth on $\mathcal{M}$, the evaluation of Ow on (x,x) can be easily taken and the integral over $\mathcal{M}$ is finite because q_k is of compact support.

To keep the analysis of this part as simple as possible, we shall assume

$$q_k(x) := (k_0 + \varepsilon k) f(x) \tag{6.6}$$

where f is the same spacetime cut-off function used in U and where k is assumed to have the dimension of a mass squared. With this choice, $\partial_k q_k = f(x)$ and it is independent on k. We furthermore observe that the contribution proportional to k_0 is constant in k and it can always be reabsorbed in a redefinition of the mass of the free theory. Many other choices, like the more usual $q_k(x) = k^2 f(x)$ can be brought to the same case using k^2 in the equation in place of k.

The function u in Eq. (6.5) is a smooth function on compact spaces, and therefore the tame Fréchet space we are working with is $F = C^\infty(X \times [a,b])$, where X is a compact space in $\mathbb{R}$ containing all possible values of ϕ and k is in the positive interval $[a,b] \subset \mathbb{R}^+$, because the sign of k is always assumed to be positive.

This space is Fréchet with seminorms

$$\|u\|_n = \sum_j^n \sum_{|\alpha|=j} \sup_{\phi,k} |D^\alpha u(\phi,k)| \,, \tag{6.7}$$

where $\alpha \in \mathbb{N}\times\mathbb{N}$ is a multi-index: the derivatives D^α are both in ϕ and k.

The space F is tame because it is the space of smooth functions over a compact space [144].

To have uniqueness of the solution of Eq. (6.4) we need to provide suitable boundary conditions and to prescribe initial values. We thus assume that

$$u(\phi,a) = \psi, \qquad u|_{\partial X\times[a,b]} = \beta \,, \tag{6.8}$$

where ψ is a given smooth function on X and β is a given smooth function on $\partial X \times [a,b]$ compatible with ψ. To impose the initial values and the boundary conditions we introduce the tame Fréchet subspace of F

$$F_0 := \{u \in F \,|\, u(\phi,a) = 0,\ u|_{\partial X\times[a,b]} = 0\}.$$

The solution $\tilde{u}$ of Eq. (6.4) we are looking for is then of the form

$$\tilde{u} = u_b + u, \qquad u \in F_0 \tag{6.9}$$

where u_b is a given element of F selected in such a way that it satisfies the boundary conditions and respects the initial values given in Eq. (6.8).

We also further assume that $\partial^2_\phi u$ and its second derivatives lie in a suitably small neighbourhood of 0, that is, $\|u\|_4 \leq A$ for some positive constant A.

6.3 STRATEGY OF THE PROOF

In order to prove the main theorem of this Chapter, using Hamilton's formulation of Nash-Moser theorem, the RG flow equation needs to satisfy a number of assumptions. First of all, it must be cast in the form of a suitable map acting on a tame Fréchet space. Requiring that ϕ and k are limited in some compact interval, and that $\tilde{U}_k$ is a smooth function, is sufficient for $\tilde{U}_k$ to be an element of a tame Fréchet space.

We already remarked that u lives in a suitable tame Fréchet space F_0.

Secondly, the RG flow equations, acting on u, determine a RG operator $\mathcal{RG} : \mathcal{U} \subset F_0 \to F$, given below in Definition 6.11. The operator $\mathcal{RG} : u \in F_0 \to F$ defining the RG flow equation must be a smooth tame map between tame Fréchet spaces. In order to be tame, the RG operator must satisfy some estimates on its seminorms. Assuming that u lies in some neighbourhood of 0 (by requiring that a suitable seminorm of u, given below in Eq. (6.7), is $\|u\|_4 < A$ for sufficiently small A), it is possible to prove these estimates using the Grönwall lemma, since the normal-ordered interacting propagator $G_k(\partial^2_\phi u)$ satisfies a recursive integral inequality.

Then, the linearisation of the RG operator must be an invertible smooth tame operator, and its inverse must be tame smooth. In the NDA, the linearisation $L = D\mathcal{RG}$ takes the form of a parabolic equation, analogous to a heat equation with a $k,\phi-$dependent heat conductivity σ. The inverse of linear parabolic equations is known [120] (see also Refs. [87, 133]), and the inverse of the linearised RG operator can be constructed from the heat kernel. Once the inverse of the linearised RG operator is known, it is possible to prove that it is tame smooth.

All these results are presented and proved in Propositions 6.3, 6.8, 6.9, and 6.10. These are used to prove our main result, Theorem 6.14, on the existence of local solutions of the RG flow.

Since we can prove the assumptions of Nash-Moser theorem, it follows that the RG operator admits a local inverse. The solution of the RG flow equations is then determined as the unique solution of the equation [144]

$$\frac{d}{dt}u_t = -cD\mathcal{RG}^{-1}(S_t u_t)S_t\left(\mathcal{RG}(u_t)\right) \tag{6.10}$$

with a given $u_0 = 0$. In this equation c is a positive fixed arbitrary constant and S_t is a smoothing operator [144].

If $\mathcal{RG}$ is a smooth tame map, if $D\mathcal{RG}(u)$ admits a unique inverse for every u in a suitable subset of F_0, and if the inverse $D\mathcal{RG}^{-1}$ is also tame, a unique solution of Eq. (6.10) exists for all t such that the limit of the sequence of approximated solutions converges to a solution of the RG flow equations: $\lim_{t\to\infty} u_t = u_\infty$ is such that $\mathcal{RG}(u_\infty) = 0$ [144].

6.4 PROOF OF THE EXISTENCE OF SOLUTIONS

6.4.1 *The RG operator is tame smooth*

Following the strategy presented in the last Section, we start with a formal definition.

Definition 6.3 (RG operator)**.** Let $u_b \in F$ be such that it satisfies the initial values and the boundary conditions given in Eq. (6.8). The *RG operator* $\mathcal{RG} : \mathcal{U} \subset F_0 \to F$ is defined as

$$\mathcal{RG} : u \mapsto \mathcal{RG}(u) := \partial_k(u + u_b) - G_k(\partial_\phi^2(u + u_b)) \tag{6.11}$$

where G_k is given in Eq. (6.5).

As a first step in the proof of existence of local solutions, we want to prove that the RG operator is of the right class to apply the Nash-Moser theorem, i.e., it is tame smooth. In order to prove it, we start considering G_k in Eq. (6.5).

G_k depends on (ϕ, k) only through $\partial_\phi^2\tilde{u}$, where we recall that $\tilde{u} = u + u_b$, since with our choice of q_k, given in Eq. (6.6), $\partial_k q_k$ is constant in (ϕ, k).

Consider now G_k, written as

$$G_k(\partial_\phi^2\tilde{u}) = -\frac{1}{2\,|||f|||_1}\int_{\mathcal{M}} \mathrm{d}\mu_x \partial_k q_k(x) : G_k : (x,x)\,,$$

for $\tilde{u} \in F$. We analyse how $: G_k : (x,x)$ depends on $\partial_\phi^2\tilde{u}$. Notice that $: G_k : (x,y)$ can be given explicitly as

$$: G_k : (x,y) := \int \mathrm{d}\mu_{z_1}\mathrm{d}\mu_{z_2}(\delta - \Delta_R^U \tilde{U}_k^{(2)})(x,z_1)w(z_1,z_2)(\delta - \tilde{U}_k^{(2)}\Delta_A^U)(z_2,y)\,,$$

where δ is the Dirac delta function (the integral kernel of the identity). Recalling that $(1 - \Delta_R^U\tilde{U}_k^{(2)}) \circ (1 + \Delta_R\tilde{U}_k^{(2)}) = 1$, using the recursive relations given in Eq. (5.13), we obtain a recursive formula for $: G_k : (x,y)$:

$$: G_k : (x,y) = \tilde{w}(x,y) - \int \mathrm{d}\mu_z \Delta_R \tilde{U}_k^{(2)}(x,z) : G_k : (z,y)\,, \tag{6.12}$$

where

$$\tilde{w}(x,y) := \int \mathrm{d}\mu_z w(x,z)(\delta - \tilde{U}_k^{(2)}\Delta_A^U)(z,y)\ .$$

This recursive relation can be used to get estimates of $: G_k(x,y) :$, for x,y contained in some compact region of the spacetime $\mathcal{M}$. First of all, we can prove the following Lemma, providing estimates of the retarded propagator $\Delta_R^U g$ acting on some compactly supported smooth function g.

Lemma 6.2. *Let $\mathcal{M}$ be a ultra-static spacetime and let t be a time function. Let $\tilde{u} \in F$, and consider*

$$h = \Delta_R^U \chi\ ,$$

where g is a compactly supported smooth function on $\mathcal{M}$. It then holds that h is a past-compact smooth function with compact support on every Cauchy surface Σ. Moreover, recalling Eq. (5.13), writing h as

$$h = (1 - \Delta_R^U U^{(2)})\chi\ ,$$

where $\chi = \Delta_R g$, the following estimates hold:

$$|||h|||_\infty^t \leq c|||h|||_{2,2}^t \leq c|||\chi|||_{2,2}^t e^{C|\partial_\phi^2 \tilde{u}|} \leq c e^{C\|\tilde{u}\|_2}|||\chi|||_{2,2}^t \tag{6.13}$$

and

$$|||h|||_\infty^t \leq c|||h|||_{2,2}^t \leq c e^{C\|\tilde{u}\|_2}\int_{-\infty}^t \mathrm{d}\tau(t-\tau)|||g|||_{2,2}^\tau \leq \tilde{C}e^{C\|\tilde{u}\|_2}\sup_{\tau\leq t}|||g|||_{2,2}^\tau\ . \tag{6.14}$$

In the above inequalities, $C > 0$ is a positive constant, which depends on the support of f in U but not on $\tilde{u}$; similarly, $\tilde{C} > 0$ depends only on the support of g and c is positive and does not depend on U. Furthermore, $|||\cdot|||_{2,2}^t$ is the norm on the Sobolev space $W_{2,2}(\Sigma_t)$ and $|||\cdot|||_\alpha^t$ is the norm on $L^\alpha(\Sigma_t)$ where $\Sigma_t = \{x \in \mathcal{M} | t(x) = t\}$ is the Cauchy surface at fixed time t.

Proof. We recall that both Δ_R and Δ_R^U map past-compact smooth functions to past-compact smooth functions, hence both $\chi = \Delta_R g$ and $h = \Delta_R^U$ are smooth and past-compact. We also recall that

$$\Delta_R^U = \Delta_R(1 - \tilde{U}_k^{(2)}\Delta_R^U) = (1 - \Delta_R^U\tilde{U}_k^{(2)})\Delta_R\ .$$

Since $\tilde{U}_k^{(2)} = f\partial_\phi^2\tilde{u}$, where f is a smooth compactly supported function and $\partial_\phi^2\tilde{u}$ is constant on $\mathcal{M}$, the following recursive relation holds

$$h = \Delta_R\chi - \Delta_R\tilde{U}_k^{(2)}h = \chi - \Delta_R\tilde{U}_k^{(2)}h.$$

Now, let D be the (positive) Laplace operator on Σ_t constructed with the induced metric on Σ_t, and define $\omega = \sqrt{D + m^2}$ as the square root of the positive operator $D + m^2$. Hence

$$h(t,\mathbf{x}) = \chi(t,\mathbf{x}) - \partial_\phi^2\tilde{u}\int_{-\infty}^t \mathrm{d}\tau\frac{\sin(\omega(t-\tau))}{\omega}(fh)(\tau,\mathbf{x})\ .$$

We thus have

$$|||h|||_2^t \leq |||\chi|||_2^t + |\partial_\phi^2\tilde{u}|\int_{-\infty}^t \mathrm{d}\tau(t-\tau)\|f\|_\infty^\tau\, |||h|||_2^\tau\ ,$$

or, passing to the Sobolev norm $|||h|||^t_{2,2} = |||h|||^t_2 + |||Dh|||^t_2$, we have

$$\begin{aligned}|||h|||^t_{2,2} &\leq |||\chi|||^t_{2,2} + |\partial^2_\phi \tilde{u}| \int_{-\infty}^t \mathrm{d}\tau (t-\tau)\, |||fh|||^\tau_{2,2} \\ &\leq |||\chi|||^t_{2,2} + |\partial^2_\phi \tilde{u}| \int_{-\infty}^t \mathrm{d}\tau (t-\tau)(\|f\|^\tau_\infty + \sup_i 2\|\partial_i f\|^\tau_\infty + \|Df\|^\tau_\infty)\, |||h|||^\tau_{2,2} \\ &\leq |||\chi|||^t_{2,2} + C|\partial^2_\phi \tilde{u}| \int_a^t \mathrm{d}\tau\, |||h|||^\tau_{2,2}\end{aligned}$$

where $a = \inf_{x \in \operatorname{supp} f}\{t(x)\}$ and for a suitable positive constant C independent on $\partial^2_\phi \tilde{u}$. C is in fact finite because f is smooth and with compact support on $\mathcal{M}$.

Applying the Grönwall Lemma in integrated form to the previous inequality we obtain

$$|||h|||^t_{2,2} \leq |||\chi|||^t_{2,2}\, e^{C|\partial^2_\phi \tilde{u}|}.$$

To conclude the proof of the first inequality (6.13), we observe that Σ_t is a three dimensional space, and so by standard arguments we have

$$|||h|||^t_\infty \leq \left|\left|\left|\hat{h}\right|\right|\right|^t_1 \leq |||(1+D)h|||_2 \left|\left|\left|(1+D)^{-1}\right|\right|\right|_2 \leq c\, |||h|||_{2,2}\,.$$

where the $\left|\left|\left|(1+D)^{-1}\right|\right|\right|_2$ is the L-2 norm of $(1+D)^{-1}$. To prove Eq. (6.14) we use Eq. (6.13) for $\chi = \Delta_R g$. Recalling that

$$\chi(t, \mathbf{x}) = \partial^2_\phi \tilde{u} \int_{-\infty}^t \mathrm{d}\tau \frac{\sin(\omega(t-\tau))}{\omega} g(\tau, \mathbf{x}).$$

and taking the Sobolev norms we have

$$|||\Delta_R g|||^t_{2,2} \leq \int_{-\infty}^t \mathrm{d}\tau (t-\tau)\, |||g|||^\tau_{2,2}\,.$$

□

Starting from the above analysis and the previous Lemma, we can prove that the RG operator is tame smooth.

Proposition 6.3. *Assume that $\mathcal{U} \subset F_0$ is a small neighbourhood of 0, so that for $u \in \mathcal{U}$, $\|u\|_2 < A$ for some constant A. Then the RG operator is a smooth tame map.*

Proof. We start considering $\tilde{u} = u_b + u$ for $u \in F_0$ and for a given u_b satisfying Eq. (6.8), so that $\tilde{u} \in F$ and it satisfies the prescribed initial values and boundary conditions. We recall that, from Eq. (6.11),

$$\mathcal{RG}(u) = \partial_k(u + u_b) - G_k(\partial^2_\phi(u + u_b))\,,$$

where G_k is given in Eq. (6.5). To prove that $\mathcal{RG}$ is tame smooth we just need to prove that G_k is tame smooth for $\tilde{u} \in u_b + \mathcal{U}$. We start proving the following Lemma.

Lemma 6.4. *The functional G_k is a smooth function of $\partial^2_\phi \tilde{u}$. Furthermore, it is tame smooth for $\tilde{u} \in u_b + \mathcal{U}$.*

Proof. We observe that $\partial_k q_k$ is constant on $X \times [a,b]$; hence, recalling the definition of G_k given in Eq. (6.5), we have that $G_k(\partial^2_\phi \tilde{u})$, as a function on $X \times [a,b]$, depends on (ϕ, k) only through $\partial^2_\phi \tilde{u}$; that is, $G_k(\partial^2_\phi \tilde{u})(\phi, k) = G_k(\partial^2_\phi \tilde{u}(\phi, k))$. We also observe that $G_k(\partial^2_\phi \tilde{u})$ depends smoothly on $\tilde{u} \in F$. Actually, the n−th order functional derivative of $\tilde{G}_k(\tilde{u}) = G_k(\partial^2_\phi \tilde{u})$ with respect to $\tilde{u}$ can be explicitly computed, and it is well-defined for every n; in fact, it is given by

$$\tilde{G}_k^{(n)}(v_1, \dots, v_n) = \frac{(-1)^{n+1}}{|||f|||_1} n! \sum_{l=0}^{n} \int_{\mathcal{M}} \mathrm{d}\mu_x \partial_k q_k(x) \cdot$$
$$\left\{ (\Delta^U_R f)^l \otimes (\Delta^U_R f)^{n-l} \circ (1 - \partial^2_\phi \tilde{u} \Delta^U_R f) \otimes (1 - \partial^2_\phi \tilde{u} \Delta^U_R f)(w)(x,x) \right\} \prod_{j=1}^{n} \partial^2_\phi v_j \qquad (6.15)$$
$$=: A_n(\tilde{u}) \prod_{j=1}^{n} \partial^2_\phi v_j \, .$$

In the last formula, f in $\Delta^U_R f$ is a multiplicative operator, and $A_n(\tilde{u})$ are suitable functionals of $\tilde{u}$. Notice that both cut-off functions f and q_k have compact support, and w is a smooth function on $\mathcal{M}^2$. Hence, for every $\tilde{u} \in F$, the integral defining A_n always gives a finite bounded result. We thus have that G_k is a smooth function of $\partial^2_\phi \tilde{u}$.

We now prove that $G_k(\partial^2_\phi \tilde{u})$ is also tame.

Recall that $\|u\|_2 < \|u\|_4 < A$, and that G_k depends on ϕ and k only through $\partial^2_\phi \tilde{u}$, because $\partial_k q_k = f$. By direct inspection, we have that

$$\|G_k\|_n < \|A_0(\tilde{u})\|_0 + \sum_{p=1}^{n} \sum_{l=1}^{p} \|A_l(\tilde{u})\|_0 \|(\partial^2_\phi \tilde{u})^l\|_{p-l} \, . \qquad (6.16)$$

To estimate $\|(\partial^2_\phi \tilde{u})^l\|_{p-l}$, we use Leibniz rule together with an interpolating argument (See Corollary 2.2.2 in Ref. [144]), stating that, for every $f, g \in F$,

$$\|f\|_n \|g\|_m \leq C(\|f\|_{n+m}\|g\|_0 + \|f\|_0 \|g\|_{n+m}) \, .$$

Hence, by Leibniz rule, we have that

$$\|\partial^2_\phi \tilde{u}^l\|_r \leq C \sum_{R=(r_1,\dots,r_l),|R|=r} \prod_{i=1}^{l} \|\partial^2_\phi \tilde{u}\|_{r_i} \leq C' \|\partial^2_\phi \tilde{u}\|_r \|\partial^2_\phi \tilde{u}\|_0^{l-1} \, .$$

Using this in Eq. (6.16) we get

$$\begin{aligned}
\|G_k\|_n \; &< C \left(\|A_0(\tilde{u})\|_0 + \sum_{p=1}^{n} \sum_{l=1}^{p} \|A_l(\tilde{u})\|_0 \|(\partial^2_\phi \tilde{u})\|_0^{l-1} \|\tilde{u}\|_{p+2} \right) \\
&< C \left(\|A_0(\tilde{u})\|_0 + \sum_{p=1}^{n} \sum_{l=1}^{p} \|A_l(\tilde{u})\|_0 \|\tilde{u}\|_2^{l-1} \|\tilde{u}\|_{p+2} \right) \\
&< C\left(1 + \|\tilde{u}\|_{n+2}\right) \; ,
\end{aligned}$$

where in the last step we used the fact that $\|A_l(\tilde{u})\|_0 \leq C(1 + \|\tilde{u}\|_2)$. This last inequality is proved in the following Lemma 6.5. □

Lemma 6.5. *Consider the functionals $A_l(\tilde{u})$ for $\tilde{u} \in F$ given in Eq. (6.15). If $\|\tilde{u}\|_2 < A$, it holds that*

$$\|A_l(\tilde{u})\|_0 \leq C(1 + \|\tilde{u}\|_2).$$

Proof. To prove this result we observe that both $\partial_k q_k$ and f are smooth compactly supported functions on $\mathcal{M}$. The integral present in Eq. (6.15) is thus taken on a compact region, even if w is a smooth function supported in general everywhere on $\mathcal{M}^2$. Now, we need to estimate the action of each $\Delta_R^U f$ and of $(1 - \partial_\phi^2 \tilde{u} \Delta_R^U f)$ by means of Lemma 6.2.

Actually, Lemma 6.2 implies that if g is a smooth past-compact function, the following estimates hold:

$$\left|\left|\left|(\Delta_R^U f)^n g\right|\right|\right|_{2,2}^t \leq \sup_{\tau<t} \left|\left|\left|(\Delta_R^U f)^{n-1} g\right|\right|\right|_{2,2}^\tau \tilde{C} e^{C\|\tilde{u}\|_2} \leq \sup_{\tau<t} |||g|||_{2,2}^\tau \, \tilde{C}^n e^{nC\|\tilde{u}\|_2} \,,$$

where the constant $\tilde{C}$ depends on f. Similarly,

$$\left|\left|\left|(1 - \partial_\phi^2 \tilde{u} \Delta_R^U f) g\right|\right|\right|_{2,2}^t \leq c e^{C\|\tilde{u}\|_2} \,.$$

We now use these estimates in

$$a_{l_1,l_2}(x,y) := \left\{(\Delta_R^U f)^{l_1} \otimes (\Delta_R^U f)^{l_2} \circ (1 - \partial_\phi^2 \tilde{u} \Delta_R^U f) \otimes (1 - \partial_\phi^2 \tilde{u} \Delta_R^U f)(\chi w \chi)(x,y)\right\} \,,$$

for $l_1 + l_2 = n$, and where θ is a smooth compactly supported function, equal to 1 in a region containing the support of q_k and f. Recall also that $\chi = \Delta_R g$. Thanks to this choice, we can replace w in G_k with $\theta w \theta$, getting

$$\sup_{x \in \operatorname{supp} f} |a_{l_1,l_2}(x,x)| \leq \sup_{x,y \in \operatorname{supp} f} |a_{l_1,l_2}(x,y)| \leq \sup_{t_x,t_y \in \operatorname{supp} f} |||\theta w \theta|||_{4,2}^{(t_x,t_y)} c^2 \tilde{C}^n e^{(n+2)C\|\tilde{u}\|_2}$$

where $|||\cdot|||_{4,2}^{(t_x,t_y)}$ is the Sobolev norm for functions defined on $\Sigma_{t_x} \times \Sigma_{t_y}$. Using this estimate sufficiently many times in A_l, and recalling that $e^{C\|\tilde{u}\|_2} \leq C_1(1 + \|\tilde{u}\|_2)$ for a sufficiently large C_1 because $\|\tilde{u}\|_2 < A$, we have the thesis. □

With this results, we can conclude our proof, recalling that the linear combinations of smooth tame functionals is tame smooth. □

6.4.2 *The linearisation of the RG operator is tame smooth*

The first derivative of the RG operator defines the linearised RG operator $L(u)v = D\mathcal{RG}(u)v$, which by direct inspection is given by the linear operator

$$D\mathcal{RG}(u)v = \partial_k v - \sigma \partial_\phi^2 v \,,$$

where

$$\sigma(u) := \frac{1}{|||f|||_1} \int_{\mathcal{M}^2} \mathrm{d}\mu_x \mathrm{d}\mu_y \, \partial_k q_k(x) \Delta_R^U(x,y) f(y) \left\{(1 - \partial_\phi^2 u \Delta_R^u f) \otimes (1 - \partial_\phi^2 u \Delta_R^u f)(w)(y,x)\right\} . \tag{6.17}$$

The function σ as a function on $X \times [a,b]$ depends on ϕ through $\partial_\phi^2 u$ and on k through $\partial_k q_k$ and $\partial_\phi^2 u$. Moreover, σ depends on (ϕ, k) only through u, since, the choice of q_k given in Eq. (6.6) implies that $\partial_k q_k$ is constant in k.

Definition 6.4. Let $u_b \in F$ be a function satisfying the initial values and boundary conditions given in Eq. (6.8), and let $\mathcal{U}$ be a neighbourhood of 0 in F_0. The linearised RG operator is defined as the map

$$L : (u_b + \mathcal{U}) \times F_0 \longrightarrow F$$
$$L(u)f := \partial_k g - \sigma(u)\partial_\phi^2 g.$$

where σ is the map defined in Eq. (6.17).

The following Proposition specifies some of the properties of σ that will be useful in the analysis of $L(u)$.

Proposition 6.6. *The function $\sigma(u)$ is tame smooth.*

Proof. The function σ is linear in q_k, and q_k is a smooth function of k: moreover, $\partial_k q_k$ is constant in (ϕ, k). Hence, σ depends on k and on ϕ only through u. Furthermore, the $n-$th order functional derivative σ with respect to $\partial_\phi^2 u$ is always well-defined because it equals the $n+1$ order functional derivative of G_k with respect to $\partial_\phi^2 u$, and we already proved in Lemma 6.4 that G_k is a smooth function of $\partial_\phi^2 u$. Furthermore, σ is a smooth function and it is tame with respect to u, because it is related to the functional derivative of G_k, which is tame smooth for Lemma 6.4. □

The next proposition shows that, by a suitable choice of smooth functions w (or, equivalently, by suitable choices of states), the assumptions that: i) σ is larger than some positive constant c, and ii) that $\|u\|_2 \leq A$ is in some small neighbourhood of 0, hold.

Proposition 6.7. *If the boundary conditions given in Eq. (6.8) are such that $\|\beta\|_2 + \|\psi\|_2 < \epsilon$ for a sufficiently small ϵ and if u_b in Eq. (6.9) is chosen to be such that $\|u_b\|_2 \leq \epsilon$, then, for certain choices of the function $w \in C^\infty(\mathcal{M}^2)$, it exists a neighbourhood $\mathcal{U} \subset F_0$ such that, for every $u \in \mathcal{U}$, $\sigma(u_b + u) \geq c > 0$ and $\|u\|_2 < A = \epsilon$.*

Proof. We recall that

$$\sigma(0) = \frac{1}{|||f|||_1} \int_{\mathcal{M}^2} \mathrm{d}\mu_x \mathrm{d}\mu_y \, \partial_k q_k(x) f(y) \Delta_R(x,y)(w)(y,x) \, .$$

$\sigma(0)$ is linear in w and it cannot be identically 0 for every w, and so it is possible to choose a w such that $\sigma(0) \geq (2\epsilon C + c) > 0$, where $C > \sup_{\lambda \in [0,1]} \|\sigma^{(1)}(\lambda(u+u_b))\|_0$. Moreover, σ depends smoothly on u. We can choose u_b so that $\|u_b\|_2 \leq (\|\beta\|_2 + \|\psi\|_2) < \epsilon$, and we can choose a sufficiently small $\mathcal{U} \subset F_0$ such that every $u \in \mathcal{U}$ satisfies $\|u\|_2 < \epsilon$. Therefore, the smoothness of $\sigma(u)$ implies that

$$\begin{aligned} \sigma(u) &= \sigma(0) + \int_0^1 \mathrm{d}\lambda \, \frac{d}{d\lambda}\sigma(\lambda(u+u_b)) \\ &\geq \sigma(0) - \sup_\lambda \|\sigma^{(1)}(\lambda(u+u_b))(u+u_b)\|_0 \, . \end{aligned} \tag{6.18}$$

The functional derivative $\sigma^{(1)}$ is related to $G_k^{(2)}$, and it can be given explicitly in terms of the functions A_n with $n = 2$ defined in Eq. (6.15), as

$$\begin{aligned} \sigma^{(1)}(\tilde{u})(v) &= \frac{(-1)^3}{|||f|||_1} 2 \sum_{l=0}^{2} \int_{\mathcal{M}} \mathrm{d}\mu_x \partial_k q_k(x) \cdot \\ &\left\{ (\Delta_R^U f)^l \otimes (\Delta_R^U f)^{2-l} \circ (1 - \partial_\phi^2 \tilde{u} \Delta_R^U f) \otimes (1 - \partial_\phi^2 \tilde{u} \Delta_R^U f)(w)(x,x) \right\} \partial_\phi^2 v \\ &= A_2(\tilde{u}) \partial_\phi^2 v. \end{aligned} \tag{6.19}$$

Thanks to the estimate given in Lemma 6.5, we have that

$$\begin{aligned}\|\sigma^{(1)}(\lambda(u_b+u))(u_b+u)\|_0 &\le \|A_2\|_0\|(u_b+u)\|_0\\ &\le C'(1+\|u_b+u\|_2)\|(u_b+u)\|_0\\ &\le C''\|u_b+u\|_2\end{aligned}$$

for suitable constants C' and C'', depending on A. Using this estimate in Eq. (6.18), and recalling the choices we made for w in $\sigma(0)$, we obtain that for a suitable c'

$$\sigma(u) \ge \sigma(0) - \sup_{\lambda} \|\sigma^{(1)}(\lambda(u+u_b))(u+u_b)\|_0 \ge c' > 0\,.$$

□

Remark 6.1. Thanks to Proposition 6.7, σ can be chosen to be positive. In applications to physics, when w is obtained as the smooth part of the 2-point function of a quantum state, it is not obvious *a priori* that the choices necessary to have σ positive can be made. However, this is the case in many physically sensible states [83], also thanks to the freedom in the split of the smooth and singular parts present in any Hadamard 2-point function. The freedom in the split is related to the ordinary renormalization freedom in the coincidence limit of 2-point functions in Eq. (2.24), and can be exploited to make σ positive. ■

Now we can prove the following Proposition.

Proposition 6.8. *The linearisation of the RG operator*

$$L(u)v = \partial_k v - \sigma\partial_\phi^2 v$$

is tame smooth.

Proof. Since L acts as a second order linear differential operator, its $n-$th order seminorm is controlled by the $n+2-$th order seminorm of v. Using the Lebiniz rule and an interpolating argument (see e.g. in Corollary 2.2.2 in Ref. [144])

$$\|L(u)v\|_n \le \|v\|_{n+1} + C\left(\|\sigma\|_0\|v\|_{n+2} + \|\sigma\|_{n+2}\|v\|_0\right)$$

where C is a constant. σ is tame smooth and the composition of tame smooth maps is tame smooth, and thus $L : (\mathcal{U}) \times F_0 \to F$ is tame smooth. □

6.4.3 *The linearisation of the RG operator is invertible, and the inverse is tame smooth*

If $\sigma \ge c > 0$ on $X \times [a,b]$, the linearised RG operator $L(u)$ on $X \times [a,b]$ has the form of a parabolic equation. The existence and uniqueness of an inverse, satisfying the chosen boundary conditions

$$E(g)(\phi,a) = 0, \qquad E(g)|_{\partial X\times[a,b]} = 0\,, \qquad g \in C_0^\infty(X\times[a,b])\,,$$

is known [120]. Furthermore, by an application of the maximum principle, it is possible to prove that E is continuous with respect to the uniform norm; see e.g. Section 3 of Chapter 2 in Ref. [120]. We collect these results in the following Proposition.

Proposition 6.9. *Consider the linearised RG operator L. Assume that $\sigma(u)$ is positive for every $u \in \mathcal{U} \subset F$. Then, it exists an unique inverse $E : F \to F_0$ which is compatible with the initial and boundary conditions, thus satisfying*

$$E(L(g)) = L(E(g)) = g\,, \qquad g \in F_0\,.$$

Moreover, the inverse is continuous with respect to the uniform norm. More precisely, it exists a positive constant $C > 0$ such that

$$\|E(g)\|_0 < C\|g\|_0\,.$$

We now pass to analyse the regularity of E, which is a necessary condition to apply the Nash-Moser Theorem.

Proposition 6.10. *Consider the case where $\sigma \geq c > 0$, let $u \in \mathcal{U} \subset F_0$ such that $\|u\|_4 \leq A$, and assume that $\sup_{i\in\{\phi,k\}} |D_i \log\sigma(u)| < \epsilon$ with a sufficiently small ϵ. The inverse E of the linearised RG operator L is tame smooth.*

Proof. We first observe that $L(u)$ depends on u only through σ. Furthermore, σ is a tame map of u. The composition of tame maps is tame, and so, to prove the statement, it suffices to study how L depends on σ. To this end, with a little abuse of notation in this proof we shall denote $L(u)$ by $L(\sigma(u))$ and we estimate how L depends on σ. Consider $L(\sigma)(v) = g$. We look for an estimate which permits to control the higher derivative of v with those of g. We start with two Lemmas.

Lemma 6.11. *Under the hypothesis of Proposition 6.10, the following estimate holds:*

$$\|v\|_1 < C\left(\|g\|_1 + \|\sigma\|_1\|g\|_0\right)\,.$$

Proof. The uniform continuity of E stated in Proposition 6.9 implies that if $L(\sigma)v = g$,

$$\|v\|_0 < C\|g\|_0\,.$$

The same continuity result applies to Dv, where $D \in \{\partial_\phi, \partial_k\}$:

$$\|Dv\|_0 < C\|L(\sigma)Dv\|_0\,.$$

Now, it holds that

$$\begin{aligned} L(\sigma)Dv &= DL(\sigma)v - D(\sigma)\partial_\phi^2 v \\ &= DL(\sigma)v + \frac{D(\sigma)}{\sigma}\left(L(\sigma)v - \partial_k v\right)\,. \end{aligned} \tag{6.20}$$

Therefore, the uniform continuity of E and the fact that $\sigma \geq c > 0$ imply that

$$\begin{aligned} \|Dv\|_0 &< C\left(\|DL(\sigma)v\|_0 + \|D\log(\sigma)\|_0(\|L(\sigma)v\|_0 + \|v\|_1)\right) \\ &< C\left(\|Dg\|_0 + \|D\log(\sigma)\|_0(\|g\|_0 + \|v\|_1)\right)\,. \end{aligned}$$

Considering all possible D, using the uniform continuity of E and the fact that $1/\sigma > 1/c'$, we obtain

$$\begin{aligned} \|v\|_1 &\leq \left(\|v\|_0 + \sum_{D\in\{\partial_\phi,\partial_k\}} \|Dv\|_0\right) \\ &< C\left(\|g\|_1 + \|\sigma\|_1\|g\|_0 + \sup_D \|D\log(\sigma)\|_0\|v\|_1\right)\,, \end{aligned}$$

and so

$$(1 - C \sup_D \|D\log(\sigma)\|_0)\|v\|_1 < C\left(\|g\|_1 + \|\sigma\|_1\|g\|_0\right) \,.$$

Notice that, by hypothesis, $\sup_D \|D\log(\sigma)\|_0 \leq \epsilon$; therefore, if ϵ is chosen sufficiently small, it holds that

$$(1 - C \sup_D \|D\log(\sigma)\|_0) \geq c' > 0 \,, \tag{6.21}$$

and

$$\|v\|_1 < \frac{1}{(1 - C\sup_D \|D\sigma\|_0)} C\left(\|g\|_1 + \|\sigma\|_1\|g\|_0\right) \,,$$

from which the thesis follows. □

Lemma 6.12. *Under the hypothesis of Proposition 6.10, it holds that for every n*

$$\|v\|_n < C\left(\|g\|_n + \|g\|_0\|\sigma\|_{n+1}\right) \,. \tag{6.22}$$

Proof. We prove it by induction. The case $n = 1$ follows from Lemma 6.11 and the standard property $\|\sigma\|_1 \leq \|\sigma\|_2$. We assume now that inequality Eq. (6.22) holds up n. To prove that it holds also for the case $n+1$, consider the estimate for Dv. We have

$$\|Dv\|_n < C\left(\|L(\sigma)Dv\|_n + \|L(\sigma)Dv\|_0\|\sigma\|_{n+1}\right). \tag{6.23}$$

Recalling Eq. (6.20), and using the Leibniz rule, the interpolating argument (Corollary 2.2.2 in Ref. [144]) and the fact that $\sigma \geq c' > 0$, we have

$$\begin{aligned}
\|L(\sigma)Dv\|_n &< \|g\|_{n+1} + \|D(\log\sigma)g\|_n + \|D(\log\sigma)\partial_k v\|_n \\
&< \|g\|_{n+1} + C\Bigg(\|\log\sigma\|_0\|g\|_{n+1} + \|\log\sigma\|_{n+1}\|g\|_0 + \|\frac{D(\sigma)}{\sigma}\|_0\|\partial_k v\|_n \\
&\qquad + \|\frac{D(\sigma)}{\sigma}\|_n\|\partial_k v\|_0\Bigg) \\
&< C\Bigg(\Big(1 + \|\log\sigma\|_0\Big)\|g\|_{n+1} + \|\log\sigma\|_{n+1}\|g\|_0 + \|D(\log\sigma)\|_0\|v\|_{n+1} \\
&\qquad + \|\frac{D(\sigma)}{\sigma}\|_n\|\partial_k v\|_0\Bigg) \,.
\end{aligned}$$

From the last inequality, using the results of Lemma 6.11, it thus follows that

$$\begin{aligned}
\|L(\sigma)Dv\|_n &< (1 + \|\log\sigma\|_0)\|g\|_{n+1} + \|\log\sigma\|_{n+1}\|g\|_0 \\
&\quad + \|D(\log\sigma)\|_0\|v\|_{n+1} + \|\frac{D(\sigma)}{\sigma}\|_n(\|g\|_1 + \|\sigma\|_1\|g\|_0) \,.
\end{aligned} \tag{6.24}$$

Furthermore, from Eq. (6.20) and Lemma 6.11, we can prove that

$$\begin{aligned}
\|L(\sigma)Dv\|_0 &< \|DL(\sigma)v\|_0 + \|D\sigma\|_0(\|L(\sigma)v\|_0 + \|v\|_1) \\
&< \|Dg\|_0 + \|D\sigma\|_0(\|g\|_0 + \|v\|_1) \\
&< \|g\|_1 + \|\sigma\|_1((1 + \|\sigma\|_1)\|g\|_0 + \|g\|_1) \,.
\end{aligned} \tag{6.25}$$

Therefore, combining the two inequalities Eq. (6.24) and Eq. (6.25) in Eq. (6.23), it holds that

$$(1 - C\|D(\log(\sigma))\|_0)\|v\|_{n+1} < C\,[(1 + \|\log\sigma\|_0)\,\|g\|_{n+1} + \|\log\sigma\|_{n+1}\|g\|_0 + \\ + \|D(\log\sigma)\|_n(\|g\|_1 + \|\sigma\|_1\|g\|_0) + \\ + (\|g\|_1 + \|\sigma\|_1(\|g\|_0 + \|\sigma\|_1\|g\|_0 + \|g\|_1))\,\|\sigma\|_{n+1}]\ .$$

Notice that, as stated in Eq. (6.21), $(1 - C\|D(\log(\sigma))\|_0) > 0$, and so

$$\|v\|_{n+1} < C\,[((1 + \|\log\sigma\|_0)\,\|g\|_{n+1} + \|\log\sigma\|_{n+1}\|g\|_0 + \\ + \|D(\log\sigma)\|_n(\|g\|_1 + \|\sigma\|_1\|g\|_0)) + \\ + (\|g\|_1 + \|\sigma\|_1((1 + \|\sigma\|_1)\|g\|_0 + \|g\|_1))\,\|\sigma\|_{n+1}]\ .$$

By the interpolating argument, it holds that $\|g\|_1\|h\|_n \leq C(\|g\|_0\|h\|_{n+1} + \|g\|_{n+1}\|h\|_0)$. Moreover, we have $\|\sigma\|_1 < A$, from which $\|D\log\sigma\|_n \leq C\|\sigma\|_{n+1}$ follows. Thus, we obtain

$$\|v\|_{n+1} < C\,(\|g\|_{n+1} + \|g\|_0\|\sigma\|_{n+2})\ .$$

□

The estimates of Lemma 6.12 imply that Ef is a tame map of σ and f. The map $\sigma(u)$ is a smooth tame function of u. The composition of tame maps is tame, and so we have the result. □

To prove that E is tame smooth we made two assumptions for σ: first, that $\sigma > c$, and second, that $\partial_i \log(\sigma) < \epsilon'$ for small ϵ'. We have already seen in Proposition 6.7 that u_b and $\mathcal{U}$ can be chosen in such a way that, for every $u \in \mathcal{U}$, $\sigma(u_b + u) > c$. We now want to prove that the second requirement also holds.

Proposition 6.13. *Let $\epsilon' > 0$, u_b in Eq. (6.9) be such that $\|u_b\|_3 \leq A$, and consider the initial conditions given in Eq. (6.8). If $[a, b]$ is such that $b - a$ together with A are sufficiently small, it holds that*

$$|\partial_i \log(\sigma)| < \epsilon'\ ,$$

for every $u \in \mathcal{U}$, recalling that $\|u\|_4 < A$.

Proof. Let D be either ∂_ϕ or ∂_k, and notice that $D\log\sigma = D\sigma/\sigma$. In Proposition 6.7 we have shown that there are choices of w for which $1/\sigma < 1/c$. We now observe that

$$D\sigma(\phi, k) = D\sigma(\phi, a) + \int_a^k \partial_x D\sigma(\phi, x)\mathrm{d}x\ .$$

Therefore, since both σ and $D(\sigma)$ are smooth, we have that

$$\|D\sigma(\phi, k)\|_0 \leq \|D\sigma(\phi, a)\|_0 + (b - a)\|\sigma\|_2 < C(A + (b - a)(1 + A))\ ,$$

where we used the fact that σ is tame, and in particular $\|\sigma\|_2 \leq C(1 + \|u\|_4) \leq C(1 + A)$. Furthermore, $\sigma(\phi, a)$ depends on ϕ and a through $u_b + u$; hence, in view of the continuity of σ,

$$|D\sigma(\phi, a)| \leq C|D\partial_\phi^2 u_b(\phi, a)| \leq \|u_b\|_3 \leq A\ ,$$

since $u = 0$ at $k = a$, and we can choose u_b so that $\|u_b\|_3 < A$. Therefore,

$$|D \log \sigma| = \frac{|D\sigma(\phi, k)|}{\sigma} \leq \frac{C}{c}(A + (b - a)(1 + A)) \leq \epsilon' \ ,$$

where we have chosen both A and $b-a$ sufficiently small to make the last inequality valid. □

Remark 6.2. Recall that in u_b there is a contribution $-q_k\phi^2$. For q_k given in Eq. (6.6) it is in general not possible to make the choice $\|q_k\phi^2\|_3 \leq A$ for small A, because of the constant contribution $k_0 f$ in Eq. (6.6), while the other corrections can be made small with judicious choices of the chosen parameters. However, such a contribution can always be reabsorbed in the mass of the free theory present in P_0. ■

Theorem 6.14. *Under the hypothesis of Proposition 6.10, the RG operator admits a unique family of tame smooth local inverses, and unique local solutions of the RG flow equations exist.*

Proof. The proof is a direct application of the Nash-Moser theorem [144], which can be applied thanks to the results of Propositions 6.3, 6.8, 6.9, and 6.10. Actually, it follows from the Nash-Moser theorem that the RG operator admits a unique family of tame smooth local inverses. This guarantees the existence of local solutions of the RG flow equations. □

7 Renormalization of local symmetries

In Chapter 3, we discussed the BV formalism on general grounds, as a tool to quantise any theory with local symmetries, both fundamental or effective. The keystones of the BV formalism are the Quantum Master Equation (3.35), imposing a sufficient condition on the bare action so that $S-$matrix elements are gauge-independent, and the requirement that interacting observables are in the zeroth cohomology of the interacting BV differential, Eq. (3.38).

The QME in particular controls the gauge dependence of the action and its renormalization [119]. It is a fundamental requirement for the bare action of any gauge theory, in order to have a well-defined quantum theory. Now, the QME will impose some structural constraints on the effective average action as well. In fact, in the classical limit the effective average action reduces to the bare action; it follows that the effective average action must satisfy some symmetry identity, that reduces to the CME in the $\hbar \to 0$ limit.

In this Chapter, we find the symmetry constraints to be imposed on the effective average action, to be consistent with BV invariance. The identity satisfied by the effective average action is usually called in the fRG context *modified Slavnov-Taylor identity*. In the pQFT approach based on a path integral formulation, the Slavnov-Taylor identity is derived from the gauge invariance and parametrization invariance of the generating functional Z. In the absence of a regulator term (or more precisely, assuming an implicit BV invariant regularisation), the Slavnov-Taylor identity assumes the form of the *Zinn-Justin equation* [256, 257]

$$\int_x \frac{\delta\Gamma_0}{\delta\phi(x)}\frac{\delta\Gamma_0}{\delta\sigma(x)} = 0\,. \tag{7.1}$$

The Zinn-Justin equation is usually interpreted as a symmetry constraint for Γ_0 and it plays a crucial role in the perturbative renormalization of gauge theories. For this reason, it is fundamental to have control on the possible terms that break the Zinn-Justin equation, known as *anomalies*. Thanks to its linearity in the first derivatives of the effective action, this anomalies can be captured by the cohomology of the BV differential. In fact, the Zinn-Justin equation can interpreted in a way formally identical to the CME,

$$(\Gamma_0, \Gamma_0) = 0\,,$$

where the brackets $(\cdot,\cdot)$ are defined declaring the classical BRST sources σ as conjugate variables to the fields ϕ. At lowest order in $\hbar$, the Zinn-Justin equation gives a consistency condition on candidate anomalies A in the form

$$\mathcal{S}A = 0\,,$$

where $\mathcal{S}$ is the *linearised Slavnov operator*. Terms in the form $A = \mathcal{S}\Theta$ can be reabsorbed through field redefinitions in the couplings, and so anomalies are constrained by the cohomology of the Slavnov operator [12, 22]. In the same way, expanding the effective action in loop order, the Zinn-Justin equation provides an infinite tower of equations that the effective action must satisfy, which can be written as cohomological constraints based on the Slavnov operator. Since it is possible to prove that the Slavnov operator and the BV differential have isomorphic cohomologies, cohomological problems reduce to the study of $H(s|\mathrm{d})$ and $H(s)$ at various form degrees and ghost numbers. Notice that cohomological methods are particularly relevant for perturbatively non-renormalizable theories, since they do not rely on power counting renormalizability.

In the presence of a non-invariant regulator term Q_k, the Zinn-Justin equation is modified by a symmetry breaking term that, in the context of fRG, is written as [105]

$$\int_x \frac{\delta \Gamma_k}{\delta \phi(x)} \frac{\delta \Gamma_k}{\delta \sigma(x)} = i\hbar \int_{x,y} q_k(x) G_k(x,y) \frac{\delta^2 \Gamma_k}{\delta \sigma(x) \delta \phi(y)} \,. \tag{7.2}$$

A detailed discussion of the mSTI can be found e.g. in Ref. [201].

This equation is called *modified Slavnov-Taylor identity*, and it is interpreted as the breaking of the symmetries of the effective action by the regulator-dependent term. In the $k \to 0$ limit, the mSTI reduces to the Zinn-Justin equation.

Since the mSTI are derived by the same functional as the flow equations, an exact solution of the Wetterich equation also satisfies the mSTI, governing how the symmetry are broken and restored along the flow.

Although conceptually satisfying, practical computations are based on approximation schemes for the RG flow equations, usually in the form of a truncation in parameter space. In this case the mSTI becomes a non-trivial constraint on the truncation, and solving the flow equation with an approximated effective average action that also satisfies the mSTI can become a difficult task. In fact, a standard strategy [214] is to find a suitable truncation scheme for the flow equation, which in general does not satisfy the mSTI, and measure the quality of the approximation by the order of magnitude of the mSTI breaking term.

Since the regulator dependent term is proportional to a second-order derivative of the effective average action, one cannot use cohomological methods [22] to discuss the renormalization of the effective action. For these reasons, many alternatives have been pursued in the literature, in particular in constructing manifestly gauge invariant flows (see e.g. the works by Morris, initiated in 1999 [192]). However, these attempts usually come at the price of rather involved computations.

In the following Section, we will modify the generating functional Z_k in order to obtain *extended Slavnov-Taylor identities*. These share the same linear structure as in the non-regularised case, and therefore are amenable to cohomological methods.

In order to obtain the extended Slavnov-Taylor identities, we first need to enlarge the space of fields and sources by two additional terms. The first one is a source term for the BRST variations of the fields, Σ. The classical BRST sources σ are conjugate variables to the mean fields ϕ; together, they control the symmetries of the effective average action just as antifields and field configurations $(\varphi^\ddagger, \varphi)$ control the symmetries of the bare action via the QME. The term Σ appears in the regularised generating functional Z_k, and can be understood as a further deformation of the interaction by an automorphism.

The second addition is of a source term H for the BRST variation of the regulator term. By including this term, we can extend the BV differential to a new, scale-dependent differential s_k, so that the combination $H + Q_k$ is s_k−exact. Thus, also the extended classical action $I + Q_k + H$ is s_k−exact, extending the BV invariance of the bare action I. The use of a local regulator is crucial to have an extended BV invariance, since in this case the regulator kernel q_k is a local field, and thus it is possible to introduce an auxiliary field η so that $s_k \eta = q_k$.

The extended BV invariance translates into an extended Slavnov-Taylor identity for the effective average action, sharing the same structure as in the unregularised case. Since this identity controls the symmetries of the effective average action, just as the QME controls the symmetries of the bare action, we call this extended Slavnov-Taylor identity *effective master equation* (EME), Eq. (7.21). The EME can be solved by cohomological methods, as the Zinn-Justin equation. Since the cohomology of s_k (or rather of its effective counterpart $\mathcal{S}_k$) is isomorphic to the cohomology of the BV differential s, the cohomology $H(s)$ controls the functional dependence of the effective average action on the mean fields and classical BRST sources.

Before entering the details of the discussion, we briefly comment on a difference in treatment with respect to the first formulation of the EME [85].

Here, we consider only the state-evaluated generating functionals. It follows that the EME that we prove in the next sections is slightly different from the one that holds for the "algebraic" effective average action [85]. In fact, the two equations differ by a term that vanish on-shell, i.e. the action of the Koszul operator on the effective average action. This does not modify the interpretation and the subsequent discussion of the results. The algebraic relations (the RG flow and the EME) may be seen as more fundamental, since evaluating them on a state gives the equations that we report here. However, state-evaluated generating functionals have a clearer physical interpretation as generating functionals of Green's functions. In any case, the derivation in both cases follow the same steps.

7.1 MODIFIED SLAVNOV-TAYLOR IDENTITIES

Before deriving the EME, we briefly show how to derive the mSTI in our context. This allows us to make contact with existing literature, and to explain the role of the classical BRST sources Σ. We derive the modified Slavnov-Taylor identities only in the case of Yang-Mills-type theories, such as Yang-Mills and gravity, that are linear in the antifields. However, the sources σ can be defined more generally. These are classical sources, coupled to the fields by

$$\Sigma(\varphi;\sigma) := I_{af}(\varphi, \varphi^{\ddagger} = \sigma) = \alpha_{\sigma\varphi}(I_{af})\Big|_{\varphi^{\ddagger}=0} . \tag{7.3}$$

The automorphism $\alpha_{\sigma\varphi}$ substitutes to the antifields the classical sources σ. Moreover, notice that

$$V + \Sigma = \alpha_{\sigma\varphi}(V) = V(\varphi, \varphi^{\ddagger} + \sigma) ,$$

so that Σ can be understood as a gauge-fixed interaction. In the case of theories that are linear in the antifields, as Yang-Mills and gravity, Σ is simply

$$\Sigma = \int_x \sigma(x)\{\varphi, I\} = \int_x \sigma(x) \frac{\delta V}{\delta \varphi^{\ddagger}(x)} .$$

The classical sources σ couple on-shell to the BRST variation only, $\omega(\Sigma) = \int_x \sigma\omega(\gamma\varphi)$, since $\omega(\delta\varphi) = 0$; for this reason, they have been historically called BRST sources.

Remark 7.1. The sources σ are called *classical BRST sources* because they are commutative (the $\star-$ and $T-$products do not act on them) and, on-shell, they only couple with the BRST variation of the fields: $\omega(R_V(s\varphi)) = \omega(R_V(\gamma\varphi))$, since $\omega(R_V(\delta\varphi)) = \omega(\delta_0\varphi) = 0$ by Lemma 2.1.

Historically, the cohomology of the BV operator is known as *local BRST cohomology* [22]. Here we follow the historical convention, although the main object of interest is the BV operator s. ■

Lemma 7.1. *If V is linear in the antifields, Σ is invariant under $\{\,\cdot\,, V\}$.*

Proof. We prove this by direct computation.

$$\begin{aligned}\{V,\Sigma\}_T = \int_x \sigma\left\{V, \frac{\delta V}{\delta\varphi^\ddagger}\right\}_T &= \int_x \sigma\frac{1}{2}\frac{\delta}{\delta\varphi^\ddagger}\{V,V\}_T \\ &= \int_x \sigma\frac{\delta}{\delta\varphi^\ddagger}(i\hbar\Delta V - \{V, I_0\}_T) = 0\,.\end{aligned}$$

We used the QME between the first and second line, and the linearity of V in the antifields in the final step. □

The regularised generating functional now is defined as

$$Z_k(j,\sigma) := \omega\left[S(V)^{-1}\star S(V+J+\Sigma+Q_k)\right]\,. \tag{7.4}$$

The definitions of W_k, Γ_k, and of the mean value operator, in terms of Z_k, are the same as before, (4.14), (4.5), and (4.18).

Usually, the Slavnov-Taylor identities are derived in the path integral approach starting from the invariance of the integral under re-parametrisations in the field space, and assuming invariance of the field measure, so that for an infinitesimal BRST transformation

$$0 = sZ(j) = \int \mathcal{D}\varphi\mathcal{D}\varphi^\ddagger s e^{\frac{i}{\hbar}(I(\varphi,\varphi^\ddagger)+J(\varphi)+\Sigma)}\,.$$

Following the same approach in the algebraic formalism, we start by computing the free BV variation of the "integrand" of Z_k, i.e., the Bogoliubov map before state evaluation. The free BV variation is given by the action of the free Koszul-Tate differential, because we decomposed the action so that there are no antifields in the quadratic term I_0; thanks to the QME (3.35), the free BV variation can be written as

$$\delta_0\left[S(V)^{-1}\star S(V+J+\Sigma+Q_k)\right] = S(V)^{-1}\star[\delta_0 S(V+J+Q_k+\Sigma)]\,. \tag{7.5}$$

Now, we can use relation (3.29) with $X = S(V)$ and $Y = S(J+Q_k+\Sigma)$. We observe that $\{S(Q_k+\Sigma), I_0\} = 0$ since neither terms contain antifields, while $\{S(V), I_0\}_\star$ vanishes thanks to the QME (3.35). So we can write

$$\delta_0\left[S(V)^{-1}\star S(V+J+\Sigma+Q_k)\right] = -i\hbar S(V)^{-1}\star\{S(Q_k+J+\Sigma), S(V)\}_T \tag{7.6}$$

Since the antibracket acts as a derivation, we arrive at

$$\delta_0\left[S(V)^{-1}\star S(V+J+\Sigma+Q_k)\right] = \frac{i}{\hbar}S(V)^{-1}\star[S(V+Q_k+J)\cdot_T\{Q_k+J,V\}_T] \tag{7.7}$$

Σ does not contribute to the antibracket thanks to Lemma 7.1.

Finally, since J is linear in the fields, we have

$$\begin{aligned}
&\frac{i}{\hbar}S(V)^{-1}\star[S(V+Q_k+J)\cdot_T\{J,V\}_T]\\
&\qquad=\frac{i}{\hbar}\int_x j(x)S(V)^{-1}\star\left[S(V+Q_k+J)\cdot_T\frac{\delta V}{\delta\varphi^\ddagger}\right]\\
&\qquad\qquad=\int_x j(x)\frac{\delta}{\delta\sigma(x)}\left[S(V)^{-1}\star S(V+Q_k+J)\right]\,,
\end{aligned}$$

by definition of Σ. Now, we can evaluate on any Hadamard state ω Eq. (7.7). Notice that the state evaluation does not act on classical sources (j,σ), so that the derivative with respect to the sources commute:

$$\omega\left(\frac{\delta}{\delta\sigma(x)}\left[S(V)^{-1}\star S(V+Q_k+J)\right]\right)=\frac{\delta}{\delta\sigma}Z_k\,.$$

On the other hand, the Koszul differential maps any functional into the space of functionals that vanish on-shell, $\mathcal{F}_0$. Since $\mathcal{F}_0\subseteq\operatorname{Ker}\omega$, the l.h.s. of (7.7) vanishes on-shell.

From Eq. (7.7), we write the action of δ_0 on Z_k as

$$0=\int_x j(x)\frac{\delta Z_k}{\delta\sigma(x)}+\frac{\delta}{\delta\sigma(x)}\omega\left[S(V)^{-1}\star\left(S(V+Q_k+J+\Sigma)\cdot_T\frac{\delta Q_k}{\delta\varphi(x)}\right)\right]\,.\tag{7.8}$$

The first term in the right-hand side is the standard term appearing in the Ward identity for the gauge symmetry of the generating functional $Z(j)$, while the second term is the contribution from the regulator.

From the identity for Z_k, the symmetry identities for W_k and the effective average action follow by substituting the definitions (4.14) and (4.5), and recalling the identities

$$\begin{aligned}
\frac{\delta Z_k}{\delta\sigma}&=\frac{i}{\hbar}e^{\frac{i}{\hbar}W_k}\frac{\delta W_k}{\delta\sigma}\\
\delta_0 Z_k&=\frac{i}{\hbar}e^{\frac{i}{\hbar}W_k}\delta_0 W_k\\
e^{\frac{i}{\hbar}W_k}\frac{\delta W_k}{\delta j(x)}&=S(V)^{-1}\star(S(V+Q_k+J+\Sigma)\cdot_T\varphi(x))\\
\frac{\delta Q_k}{\delta\varphi(x)}&=-q_k(x)\varphi(x)\,.
\end{aligned}$$

Using these relations, the Ward identity (7.8) becomes an identity for W_k,

$$0=\int_x e^{\frac{i}{\hbar}W_k}j(x)\frac{\delta W_k}{\delta\sigma(x)}+i\hbar q_k(x)\frac{\delta}{\delta\sigma(x)}\left[e^{\frac{i}{\hbar}W_k}\frac{\delta W_k}{\delta j(x)}\right]\,.$$

By direct computation the last expression gives

$$0=\int_x\left(j(x)-q_k(x)\frac{\delta W_k}{\delta j(x)}\right)\frac{\delta W_k}{\delta\sigma(x)}+i\hbar q_k(x)\frac{\delta^2 W_k}{\delta\sigma(x)\delta j(x)}\,.\tag{7.9}$$

Finally, changing variables from j to $\phi = \dfrac{\delta W_k}{\delta j_\phi}$ and making the substitutions

$$\frac{\delta W_k}{\delta \sigma} = \frac{\delta \Gamma_k}{\delta \sigma} \tag{7.10}$$

$$j - q_k \frac{\delta W_k}{\delta j_\phi} = -\frac{\delta \Gamma_k}{\delta \phi} \tag{7.11}$$

$$\frac{\delta^2 W_k}{\delta \sigma(y) \delta j(x)} = \int_z G_k(x,z) \frac{\delta^2 \Gamma_k}{\delta \phi(z) \delta \sigma(y)} , \tag{7.12}$$

we get the modified Slavnov-Taylor identity:

$$\int_x \frac{\delta \Gamma_k}{\delta \phi(x)} \frac{\delta \Gamma_k}{\delta \sigma(x)} = i\hbar \int_{x,y} q_k(x) G_k(x,y) \frac{\delta^2 \Gamma_k}{\delta \phi(x) \delta \sigma(y)} . \tag{7.13}$$

The main difference from the off-shell treatment of the generating functionals presented in Ref. [85] is the absence, in the last equation, of the Koszul map acting on the algebraic effective average action, $\delta_0 \Gamma_k$. If we defined the generating functionals without state evaluation, the term proportional to the Koszul map in the l.h.s of Eq. (7.7) would actually contribute to the identity.

Remark 7.2 (Gauge dependence of the effective average action)**.** The derivation of the mSTI can be compared to the direct evaluation of the gauge dependence of Γ_k; as we discussed in the Section 3.4, gauge dependence can be expressed evaluating the non-vanishing terms of $\frac{\mathrm{d}}{\mathrm{d}\lambda} \alpha_\psi \Gamma_k$. It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \alpha_\psi \Gamma_k = \frac{\mathrm{d}}{\mathrm{d}\lambda} \alpha_\psi W_k = -i\hbar e^{\frac{i}{\hbar} W_k} \frac{\mathrm{d}}{\mathrm{d}\lambda} \omega \left(R_{\tilde{V}}(S(J + Q_k)) \right) ,$$

where the last line is proportional to the term appearing in the condition for the gauge independence of interacting observables (3.34), with $\tilde{F} = S(J + Q_k)$. Then, the gauge dependence of the effective average action is given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \alpha_\psi \Gamma_k = -i\hbar e^{\frac{i}{\hbar} W_k} \omega \left[R_{\tilde{V}}(\psi \cdot_T \alpha_{\lambda\psi}(\hat{s}_V S(J + Q_k)) \right] . \tag{7.14}$$

It follows that Γ_k is gauge dependent since $S(J + Q_k)$ is not in the cohomology of $\hat{s}$. ■

7.2 EFFECTIVE MASTER EQUATION

As discussed, since a second-order derivative of the effective average action appears in the mSTI (7.13), cohomological methods to determine the structural form of Γ_k are not available. We thus derive a different symmetry identity, closely related to the mSTI but which lets us discuss the regulator term in cohomology. The idea is an adaptation of the treatment of symmetries broken by quadratic terms [257]. The key insight is in recognising that the regulator term is "half of a contractible pair": by the inclusion of an auxiliary field η, it is possible to extend the configuration space, so that the auxiliary field transforms into the regulator kernel q_k by the action of a scale-dependent BV differential s_k. Then, coupling η to the BV variation of φ^2 by a term H, the combination $Q_k + H$ becomes s_k–exact. The extended action $I_{ext} = I_0 + \Sigma + H + Q_k$ is invariant under this extended BV differential, which can be understood as an extended BV symmetry at any scale k. From the invariance of this extended action, a symmetry identity for the effective average action follows, taking the form of an extended Slavnov-Taylor identity.

We first discuss the case of V linear in the antifields, which includes the most physically relevant cases of Yang-Mills theories and gravity; we will comment later on how to proceed for a theory with V at order l in the antifields.

Definition 7.1 (Extended BV differential)**.** We enlarge the extended configuration space of fields $\mathscr{E}$ with a new field $\eta \in \mathfrak{g}'[-1]$, and we define

$$H := \frac{1}{2} \int_x \eta(x)\{V, \varphi^2(x)\} , \tag{7.15}$$

together with an extended BV differential

$$s_k := s + \int_x q_k(x) \frac{\delta}{\delta \eta} .$$

The variation of η under s_k given by

$$s_k \eta(x) = q_k(x) . \tag{7.16}$$

Since $s_k q_k = 0$, the pair (η, q_k) form a contractible pair in the cohomology of the extended BV differential s_k.

The pair (η, q_k) can be understood as an extension of the non-minimal sector. In fact, defining

$$\Psi := -\frac{1}{2} \int_x \eta(x)\varphi^2(x) ,$$

a short computation shows that

$$s_k \Psi = H + Q_k . \tag{7.17}$$

From its action on the fields, one can check that s_k is a differential, since $s_k^2 = 0$. Moreover, the BV symmetry of the original action I can now be extended to a larger symmetry, encoded in the operator s_k, for the extended action $I_{ext}(\varphi, \varphi^\ddagger, j, \sigma, \eta) := I_0 + V(\varphi, \varphi^\ddagger + \sigma) + Q_k + H = I + \Sigma + Q_k + H$:

$$s_k I_{ext} = 0 . \tag{7.18}$$

Proof. From its action on the fields, one can check that $s_k^2 = 0$, so s_k is a differential. In fact, first notice that s_k is defined as a sum of two, commuting terms, $[s, \int_x q_k(x)\frac{\delta}{\delta\eta}] = 0$; this follows immediately from the fact that the original BV differential and $\int_x q_k \frac{\delta}{\delta\eta}$ act on different fields. Since $s^2 = 0$, it remains to prove that $\int_x q_k \frac{\delta}{\delta\eta}$ is nilpotent; but this again is immediate to see, since η and q_k are a contractible pair, so that the action of $\int_x q_k \frac{\delta}{\delta\eta}$ on any functional of the field η produces a term which is proportional to q_k, and $s_k q_k = 0$ by definition.

The action of s_k on $I_0 + V$ is simply the action of the original BV differential s, and as such it vanishes for the CME; $s_k\Sigma = 0$ because Σ is both $\varphi^\ddagger-$ and $\eta-$independent; and finally, $s_k(Q_k + H) = s_k^2\Psi = 0$ because s_k is a differential. □

Remark 7.3 (Zinn-Justin equation for the extended action)**.** The functional derivative in H can be identified with the antifield of η, $\frac{\delta}{\delta\eta} = \eta^\ddagger$. Then, if we include the field η in the multiplet $\varphi = \{\mathcal{A}, c, \bar{c}, b, \eta\}$, and we define a new antifield contribution in the interactions $V \longrightarrow V + H_{af} = V + \int_x q_k \eta^\ddagger$, Eq. (7.18) can be written in the form of a CME,

$$(I_{ext}, I_{ext}) = 0\,. \tag{7.19}$$

Notice, however, that the addition of the term H_{af} to the interaction term V in the generating functionals would introduce a new scale dependence in the effective average action. The RG flow equations would then acquire new terms, proportional to the antifield $\eta^\ddagger$. Since state evaluation by ω sets the antifield to 0, these additional contributions would be removed by the state evaluation in the RG equations.

For simplicity, in the following we do not include the antifield contribution H_{af} in the generating functionals. In the same way, we do not introduce an effective BRST source σ_η for the BV variation of η; we will comment on its inclusion in Remark 7.4. ■

Now, we want to discuss the consequences of the extended classical symmetry (7.18) to the quantum correlation functions, deriving an equation for the effective average action.

First, we define the new, $\eta-$dependent generating functional Z_k as

$$Z_k(j,\sigma,\eta) := \omega\left(S(V)^{-1} \star S(V + J + Q_k + \Sigma + H)\right)\ . \tag{7.20}$$

From this new, $\eta-$dependent definition of Z_k, we define W_k and Γ_k, as in Eq. (4.14) and Definition (4.5). Since η is a classical field (the quantum and time ordered products do not act on it), we have $\frac{\delta W_k}{j_\eta} = \eta$ and $\frac{\delta \tilde{\Gamma}_k}{\delta \eta} = -j_\eta$.

We can now derive the symmetry constraint on Γ_k. This symmetry identity is a direct consequence of the QME, and as such it can be regarded as the translation of the gauge independence of physical observables on the level of the effective average action.

The identity is a main result of this thesis, and it is summarised in the following theorem:

Theorem 7.2 (Effective master equation for the effective average action)**.** *If the action I is linear in the antifields, the Quantum Master Equation (3.35) implies a symmetry constraint on the effective average action $\Gamma_k(\phi,\sigma,\eta)$, when the interaction terms V are regular functionals:*

$$\int_x \left[\frac{\delta\tilde{\Gamma}_k}{\delta\phi(x)}\frac{\delta\tilde{\Gamma}_k}{\delta\sigma(x)} + q_k(x)\frac{\delta\tilde{\Gamma}_k}{\delta\eta(x)}\right] = 0\,, \tag{7.21}$$

where we remember that $\tilde{\Gamma}_k = \Gamma_k + Q_k(\phi)$.

Proof. The proof works by showing that

$$-i\hbar\langle\mathcal{QME}\rangle = \int_x \left[\frac{\delta\tilde{\Gamma}_k}{\delta\phi(x)}\frac{\delta\tilde{\Gamma}_k}{\delta\sigma(x)} + q_k(x)\frac{\delta\tilde{\Gamma}_k}{\delta\eta(x)}\right]\,, \tag{7.22}$$

where $\mathcal{QME} := \{V, I_0\}_T + \frac{1}{2}\{V,V\}_T - i\hbar\Delta V$.

By definition, we have

$$0 = \langle\mathcal{QME}\rangle = e^{-\frac{i}{\hbar}W_k}\omega\left[S(V)^{-1} \star S(J + Q_k + \Sigma + H) \cdot_T S(V) \cdot_T \mathcal{QME}\right]\,, \tag{7.23}$$

where we recognise the alternative form of the QME (3.35):

$$0 = \omega \left[S(V)^{-1} \star S(J + Q_k + \Sigma + H) \cdot_T \{S(V), I_0\}_\star \right] . \tag{7.24}$$

Now, we can use Eq. (3.29): setting $X = S(J + Q_k + \Sigma + H)$ and $Y = S(V)$, since both $S(J + Q_k + \Sigma + H)$ and I_0 do not contain antifields, Eq. (3.29) becomes

$$\begin{aligned} &S(J + Q_k + \Sigma + H) \cdot_T \{S(V), I_0\}_\star \\ &\quad = \{S(V) \cdot_T S(J + Q_k + \Sigma + H), I_0\}_\star + i\hbar \{S(J + Q_k + \Sigma + H), S(V)\}_T . \end{aligned} \tag{7.25}$$

Substituting the last expression in Eq. (7.24), we get

$$\omega \left[S(V)^{-1} \star \{S(V + J + Q_k + \Sigma + H), I_0\}_\star + i\hbar \{S(V), S(J + Q_k + \Sigma + H)\}_T \right] = 0 . \tag{7.26}$$

The first term vanishes, since it is a combination of $\star-$products with a term proportional to the EOMs $\frac{\delta I_0}{\delta \varphi} = P_0 \varphi$; more explicitly, it can be written using the QME as

$$\begin{aligned} &\omega \left[S(V)^{-1} \star \{S(V + J + Q_k + \Sigma + H), I_0\}_\star \right] \\ &\qquad\qquad = \omega \left[\left\{ S(V)^{-1} \star S(V + J + Q_k + \Sigma + H), I_0\}_\star \right\} \right] = 0 . \end{aligned}$$

The second term can be computed recalling that the time-ordered antibracket acts as a derivation on S, so that Eq. (7.26) becomes

$$0 = \delta_0 Z_k - \frac{i}{\hbar} \omega \left[S(V)^{-1} \star S(V + J + Q_k + \Sigma + H) \cdot_T \{J + \Sigma + Q_k + H, V\}_T \right] . \tag{7.27}$$

Σ do not contribute in the antibracket, as we proved in Lemma 7.1. Since $Q_k + H$ is independent on the antifields, we have

$$\{Q_k + H, V\}_T = \{s_k \Psi, I\}_\star = \{\{\Psi, I\}, I\}_\star + \int_x q_k \{\frac{\delta \Psi}{\delta \eta}, I\}_\star .$$

The first term in the last step vanishes because $s^2 = 0$; the second term gives

$$\{Q_k + H, V\}_T = - \int_x q_k(x) \frac{\delta}{\delta \eta(x)} \{\Psi, I\}_\star = - \int_x q_k(x) \frac{\delta H}{\delta \eta(x)} .$$

Substituting back in (7.27), we obtain

$$0 = \int_x \left[j(x) \frac{\delta Z_k}{\delta \sigma(x)} - q_k(x) \frac{\delta Z_k}{\delta \eta(x)} \right] .$$

From the above expression, the result follows by noticing that

$$e^{-\frac{i}{\hbar} W_k} \frac{\delta Z_k}{\delta \sigma} = \frac{i}{\hbar} \frac{\delta W_k}{\delta \sigma} = \frac{i}{\hbar} \frac{\delta \tilde{\Gamma}_k}{\delta \sigma} ,$$

and similarly $\frac{\delta W_k}{\delta \eta} = \frac{\delta \tilde{\Gamma}_k}{\delta \eta}$. Recalling that $\dfrac{\delta \tilde{\Gamma}_k}{\delta \phi} = -j_\phi$ we obtain the result. □

7.2.1 *Renormalised effective master equation*

The above derivation strictly works for regular functionals only, since we made use of relation (3.29) between quantum and time-ordered antibrackets. When working with local functionals, we need to replace the $T-$products with the renormalized ones. For simplicity of notation, we denote the renormalized objects with the same symbol as the non-renormalized ones. However, in the renormalized case, some of the passages above are modified.

The QME is substituted by the rQME (3.41), which we can write as

$$\{S(V), I_0\}_\star = S(V) \cdot_T \left(\frac{1}{2}\{V,V\}_T + \{V, I_0\} - i\hbar \vartriangle (V) \right) = 0 \,.$$

The difference from the non-renormalized case is that the singular operator $\vartriangle$ gets replaced by the finite, but interaction-dependent term $\vartriangle(V)$. Denoting

$$\mathcal{RQME} = \frac{1}{2}\{V,V\}_T + \{V, I_0\} - i\hbar \vartriangle (V) \,,$$

we get the analogue of Eq. (7.24), i.e.:

$$\langle \mathcal{RQME} \rangle = 0 = e^{-\frac{i}{\hbar}W_k} S(V)^{-1} \star \left[S(J + Q_k + \Sigma + H) \cdot_T \{S(V), I_0\}_\star \right] \,. \tag{7.28}$$

The relation between the $\star-$bracket and the $T-$bracket in terms of the BV laplacian (3.28), however, does not hold any more; it is substituted by the master Ward identity (3.40).

We replace the argument used in the proof of Theorem (7.2) with the following Lemma.

Lemma 7.3. *The following relation holds in the renormalized theory:*

$$\begin{aligned} & i\hbar\{S(J + Q_k + \Sigma + H), S(V)\}_T \\ &= \{S(V) \cdot_T S(J + Q_k + \Sigma + H), I_0\}_\star - S(J + Q_k + \Sigma + H) \cdot_T \{S(V), I_0\}_\star \,. \end{aligned} \tag{7.29}$$

Proof. We apply the anomalous master Ward identity (3.40) in the following two cases:

$$\{S(V), I_0\}_\star = \frac{i}{\hbar} S(V) \cdot_T \left[\{V, I_0\}_T + \frac{1}{2}\{V,V\}_T + i\hbar \vartriangle (V) \right] \,, \tag{7.30}$$

and

$$\begin{aligned} & \{S(V + J + Q_k + \Sigma + H), I_0\}_\star \\ & \qquad = \frac{i}{\hbar} S(V + J + Q_k + \Sigma + H) \cdot_T \Big[\{V + J + Q_k + \Sigma + H, I_0\}_T \\ & + \frac{1}{2}\{V + J + Q_k + \Sigma + H, V + J + Q_k + \Sigma + H\}_T + i\hbar \vartriangle (V + J + Q_k + \Sigma + H) \Big] \,. \end{aligned} \tag{7.31}$$

Now, we use the fact that $J + Q_k + \Sigma + H$ does not depend on antifields, so its bracket with I_0 and with itself vanishes; therefore, Eq. (7.31) simplifies to

$$\begin{aligned} & \{S(V) \cdot_T S(J + Q_k + \Sigma + H), I_0\}_\star \\ & \qquad = \frac{i}{\hbar} S(V + J + Q_k + \Sigma + H) \cdot_T \Big[\{V, I_0\}_T \\ & + \frac{1}{2}\{V,V\}_T + \{V, J + Q_k + \Sigma + H\}_T + i\hbar \vartriangle (V + J + Q_k + \Sigma + H) \Big] \end{aligned} \tag{7.32}$$

Next, to obtain the r.h.s. of the statement (7.29), we subtract $S(J+Q_k+\Sigma+H)\cdot_T$(7.30) from (7.32) and obtain

$$\begin{aligned}&\{S(V)\cdot_T S(J+Q_k+\Sigma+H),I_0\}_\star - S(J+Q_k+\Sigma+H)\cdot_T \{S(V)\cdot_T, I_0\}_\star\\ &=\frac{i}{\hbar}S(V+J+Q_k+\Sigma+H)\cdot_T\Big[\{V,J+Q_k+\Sigma+H\}_T+i\hbar\triangle(V+J+Q_k+\Sigma+H)-i\hbar\triangle(V)\Big]\end{aligned} \tag{7.33}$$

To conclude the proof, we apply the fundamental theorem of calculus to obtain

$$\triangle(V+J+Q_k+\Sigma+H)=\int_0^1 \triangle_{V+\lambda(J+Q_k+\Sigma+H)}(J+Q_k+\Sigma+H)d\lambda+\triangle(V)\,.$$

Since $J+Q_k+\Sigma+H$ does not depend on the antifields, the integrand in the above expression is identically zero, so $\triangle(V+J+Q_k+\Sigma+H)-\triangle(V)=0$. Finally, we use the derivation property of the bracket to arrive at

$$\begin{aligned}&\{S(V)\cdot_T S(J+Q_k+\Sigma+H),I_0\}_\star - S(J+Q_k+\Sigma+H)\cdot_T \{S(V)\cdot_T, I_0\}_\star\\ &\qquad= -i\hbar\{S(V+J+Q_k+\Sigma+H),S(V)\}\,.\end{aligned} \tag{7.34}$$

□

We are now ready to prove the main theorem of this Section.

Theorem 7.4 (Effective master equation for the renormalized effective average action)**.** *If the action I is linear in the antifields, the Quantum Master Equation (3.35) implies a symmetry constraint on the effective average action $\Gamma_k(\varphi,\varphi^\ddagger;\phi,\sigma,\eta)$:*

$$\int_x\left[\frac{\delta\tilde\Gamma_k}{\delta\phi(x)}\frac{\delta\tilde\Gamma_k}{\delta\sigma(x)}+q_k(x)\frac{\delta\tilde\Gamma_k}{\delta\eta(x)}\right]=0\,. \tag{7.35}$$

Proof. We substitute (7.29) into (7.28) and obtain again (7.26). We then proceed exactly as in the proof of Theorem 7.2. □

Since we derive the EME from the QME, the EME is a necessary requirement on the effective average action to have a gauge-independent $S-$matrix. Requiring that the EME holds at each order in perturbation theory, the zeroth order of the EME reduces to the QME, so that the two identities are actually equivalent.

Notice that, due to the modification of the generating functional in Eq. (7.20), the scale dependence of Z_k is introduced by the extended BV differential s_k, through the term $H+Q_k=s_k\Psi$. Even though the field η enters the generating functional only through the trivial part of the cohomology, the same does not hold for the scale k. This implies that, although physical observables do not depend on η, they still depend non-trivially on k.

A way of seeing this is expressing the flow equations for Γ_k as

$$\partial_k\Gamma_k=\langle\partial_k s_k\Psi\rangle-\partial_k Q_k(\phi)\,. \tag{7.36}$$

Apart from the trivial contribution $\partial_k Q_k(\phi)$, the $k-$derivative of the effective average action is not a s_k-exact term, precisely because the scale dependence comes through the extended BV differential. This ensures that physical vertex functions derived from Γ_k depends non-trivially on k.

Equation (7.21) encodes the invariance of the effective average action under infinitesimal transformations generated by the *Slavnov operator* $\mathcal{S}_k := \int_x \frac{\delta \tilde{\Gamma}_k}{\delta \phi} + q_k \frac{\delta}{\delta \eta}$,

$$\mathcal{S}_k \tilde{\Gamma}_k = 0 \,.$$

The Slavnov operator is the natural translation of the extended BV differential to the space of mean fields and classical BRST sources (ϕ, σ). In this sense, the effective average action is invariant under the same symmetries of the extended action $I_{ext} = I_0 + V(\varphi, \varphi^\ddagger) + Q_k + H$,

$$s_k I_{ext} = 0 \,.$$

The infinitesimal symmetry transformations associated with the Slavnov operator are

$$\delta_\theta \phi = \frac{\delta \tilde{\Gamma}_k}{\delta \sigma} \theta \,, \; \delta_\theta \eta = q_k \theta \,. \tag{7.37}$$

In general these differ from $s\phi = \frac{\delta V(\phi)}{\delta \varphi^\ddagger}$. In fact, $\frac{\delta \tilde{\Gamma}_k}{\delta \sigma} = \frac{\delta \Gamma_k}{\delta \sigma}$, since $\tilde{\Gamma}_k$ and Γ_k differ by a term independent of σ. Next we compute

$$\frac{\delta \Gamma_k}{\delta \sigma} = \left\langle \frac{\delta V}{\delta \varphi^\ddagger} \right\rangle ,$$

and observe that in general

$$\left\langle \frac{\delta V}{\delta \varphi^\ddagger} \right\rangle \neq \frac{\delta V(\phi)}{\delta \varphi^\ddagger} \,. \tag{7.38}$$

The two terms coincide if V is linear both in fields and antifields. This agrees with the known result saying that if an action is invariant under linear symmetries, then the effective (average) action is also invariant under these same symmetries. In fact, the linear symmetry $s\eta = q_k \,, sq_k = 0$ is inherited by $\tilde{\Gamma}_k$.

Remark 7.4 (Zinn-Justin equation for the effective average action)**.** The effective master equation (7.21) is written to emphasise the contribution coming from the regulator term.

However, it is possible to rewrite Equations (7.18) and (7.21) to highlight the fact that both I_{ext} and $\tilde{\Gamma}_k$ are invariant under the same symmetry transformation, encoded in the Zinn-Justin equation. As we commented below Eq. (7.18), by including an antifield contribution associated with η in the form $H_{af} = \int_x q_k \eta^\ddagger$, the invariance of the extended action can be written as a CME, Eq. (7.19). In the same way, we can introduce a classical BRST source for the variation of η, by including a term $\Sigma_\eta = \int_x \sigma_\eta q_k$ in Z_k; by definition, it holds that $\frac{\delta \tilde{\Gamma}_k}{\delta \sigma_\eta} = q_k$. It follows that the EME can be written in Zinn-Justin form,

$$\int_x \frac{\delta \tilde{\Gamma}_k}{\delta \phi(x)} \frac{\delta \tilde{\Gamma}_k}{\delta \sigma(x)} = \frac{1}{2} (\tilde{\Gamma}_k, \tilde{\Gamma}_k) = 0 \,, \tag{7.39}$$

where the bracket is defined declaring (ϕ, σ) conjugate variables. The Zinn-Justin equation for the effective action is the starting point of an inductive technique to renormalization of gauge theories, both renormalizable and non-renormalizable ones [12, 24].

Just as for the H_{af} term, the source for the BRST variation of η introduces an additional $k-$dependence in the generating functional. There are two ways to deal with this additional term in the flow equation: i) one can simply evaluate the flow for $\sigma_\eta = 0$; or ii) since $\sigma_\eta q_k$ is a classical contribution, we can subtract it from the effective average action via a redefinition $\Gamma_k \longrightarrow \Gamma_k - \int_x \sigma_\eta q_k$. This term removes the additional $k-$dependence in the RG flow equation and we get the same flow, Eq. (5.6). ■

Example 7.1 (Yang-Mills theories)**.** In the case of Yang-Mills theories, we can write explicitly the new contributions Σ and H. Σ by definition is

$$\Sigma = I_{af}(\varphi^\ddagger = \sigma) = \int_x D_{\mathcal{A}} c\sigma_{\mathcal{A}} - \frac{i\lambda_{YM}}{2}[c, c]\sigma_c \, .$$

On the other hand, Ψ is

$$\Psi = -\frac{1}{2}\int_x \eta(|\mathcal{A}|^2 + \bar{c}c) \, ,$$

and so H is

$$H = \int_x \eta(x)\left[\mathcal{A}D_{\mathcal{A}}c + \frac{i}{2}(bc + \frac{\lambda_{YM}}{2}c[c, c])\right] \, .$$

■

7.2.2 *Effective master equation for general gauge theories*

If V is not linear in the antifields, the derivation above needs to be slightly modified. In order to derive an effective master equation for the effective average action also in the general case, we can further exploit the analogy between $\Sigma + H$ and a generalized gauge-fixing Fermion. Since for $\varphi^\ddagger-$linear theories we introduced Σ and H as BRST-exact terms, for general theories we introduce the functional

$$\Theta := \int_x \sigma(x)\varphi(x) + \frac{1}{2}\eta(x)\varphi^2(x) \, .$$

Then we have

$$\alpha_\Theta(V) = V\left(\varphi^\ddagger + \frac{\delta\Theta}{\delta\varphi}\right) = V\left(\varphi^\ddagger + \sigma + \eta\varphi\right) \, ,$$

and we define Z_k as

$$Z_k := S(V)^{-1} \star S(\alpha_\Theta(V) + Q_k + J) \, . \tag{7.40}$$

We now use the gauge-independence of the $S-$matrix (3.33), which is equivalent to

$$\frac{1}{2}\{\alpha_\Theta(V) + I_0, \alpha_\Theta(V) + I_0\}_T - i\hbar \bigtriangleup \alpha_\Theta(V) = 0 \, . \tag{7.41}$$

The above identity holds thanks to the QME. Then, just as in the linear case, we can write

$$0 = S(V)^{-1} \star [S(J + Q_k) \cdot_T \{S(\alpha_\Theta(V)), I_0\}_\star] \, .$$

Following the same steps as in the $\varphi^\ddagger-$linear case, we arrive at

$$0 = \frac{i}{\hbar}\omega\left\{S(V)^{-1} \star [S(\alpha_\Theta(V) + J + Q_k) \cdot_T \{\alpha_\Theta(V), J + Q_k\}_T]\right\} \, .$$

Finally, we have

$$\{J, \alpha_\Theta(V)\}_T = \int_x j(x) \frac{\delta}{\delta\varphi^\ddagger(x)} \alpha_\Theta(V) = \int_x j(x) \frac{\delta}{\delta\varphi^\ddagger(x)} V\left(\varphi^\ddagger + \frac{\delta\Theta}{\delta\varphi}\right)$$
$$= \int_x j(x) \frac{\delta}{\delta\sigma(x)} \alpha_\Theta(V)\,,$$

and

$$\{\alpha_\Theta(V), Q_k\}_T = -\frac{1}{2}\int_{x,y} q_k(y) \frac{\delta\varphi^2(y)}{\delta\varphi(x)} \frac{\delta}{\delta\varphi^\ddagger(y)} V\left(\varphi^\ddagger + \frac{\delta\Theta}{\delta\varphi}\right)$$
$$= -\int_x q_k(x) \frac{\delta\alpha_\Theta(V)}{\delta\eta}\,.$$

Therefore, we arrive at

$$0 = \int_x j(x) \frac{\delta Z_k}{\delta\sigma(x)} + q_k(x) \frac{\delta Z_k}{\delta\eta(x)}\,. \tag{7.42}$$

From here, the derivation of the effective master equation in Zinn-Justin form is identical to the linear case.

7.3 EFFECTIVE MASTER EQUATION AND BRST COHOMOLOGY

Since the EME (7.2) has the same algebraic structure of the Zinn-Justin equation, one can make use of standard methods to prove perturbative renormalizability, see e.g. [257]. In this Section, we are interested in how the symmetry constrains the functional dependence on the fields and the sources of $\tilde{\Gamma}_k$. The general form of the effective average action compatible with the symmetry will then be used as the input to solve the flow equation, i.e. to find the trajectories of the coupling constants in the parameter space under rescalings of k.

Here, we propose a non-perturbative method to solve the effective master equation (7.2). For simplicity, we consider only theories that are linear in antifields, such as Yang-Mills and gravity; the extension to general gauge theories is straightforward.

The starting point is the observation that, since Γ_k is defined from the action of the Bogliubov map on some functional, it is a formal power series in $\hbar$ [98, 149]. In the classical limit $\hbar \longrightarrow 0$, from its classical limit in Eq. (4.30) it follows that $\tilde{\Gamma}_k$ reduces to the classical action, with additional terms coming from Q_k and H:

$$\tilde{\Gamma}_k \longrightarrow_{\hbar\to 0} I_{ext}(\phi, \sigma) = I_0(\phi) + V(\phi, \sigma) + H(\phi, \eta) + Q_k(\phi)\,. \tag{7.43}$$

The dependence of V on the BRST sources σ comes from the Σ term, which is just a copy of the antifield dependence of V.

The above observation suggests to consider the decomposition

$$\tilde{\Gamma}_k = I_{ext} + \hbar\hat{\Gamma}_k\,. \tag{7.44}$$

Notice that we are not assuming an approximation at first loop order, but simply exploiting the fact the zeroth order of $\tilde{\Gamma}_k$ corresponds to the classical action to separate the $\hbar$–dependent contribution $\hat{\Gamma}_k$.

Substituting the decomposition (7.44) in Eq. (7.21), we get

$$\mathcal{S}_k I_{ext} + \hbar \mathcal{S}_k \hat{\Gamma}_k + \hbar^2 \frac{1}{2} (\hat{\Gamma}_k, \hat{\Gamma}_k) = 0 \,, \tag{7.45}$$

where we introduced the effective bracket

$$(A, B) = \int_x \frac{\delta A}{\delta \phi(x)} \frac{\delta B}{\delta \sigma(x)} + (-1)^{|A|} \frac{\delta A}{\delta \sigma(x)} \frac{\delta B}{\delta \phi(x)} \,, \tag{7.46}$$

defined by declaring the BRST sources to be conjugate to the respective effective fields, i.e., $(\phi^A(x), \sigma^B(y)) = \delta^{AB} \delta(x-y)$, where the indices A, B run on the field type. Moreover, the *linearised Slavnov operator* $\mathcal{S}_k$ is

$$\mathcal{S}_k A := \int_x \frac{\delta I_{ext}}{\delta \phi(x)} \frac{\delta A}{\delta \sigma} + (-1)^{|A|} \frac{\delta I_{ext}}{\delta \sigma(x)} \frac{\delta A}{\delta \phi(x)} + q_k(x) \frac{\delta A}{\delta \eta(x)} \,. \tag{7.47}$$

Since $\tilde{\Gamma}_k$ is a formal power series in $\hbar$, to solve the effective master equation, each term in the above decomposition must vanish independently.

The above decomposition suggests to extend the classical BV algebra of the functionals of the original fields and antifields $(\varphi, \varphi^\ddagger)$ to the space of functionals of (ϕ, σ) in the natural way. Thus, we complete the transition from the original field configurations $(\varphi, \varphi^\ddagger)$ to the space of effective fields and BRST sources (ϕ, σ), where a natural notion of BV algebra is inherited from the original structure.

The three conditions coming from Eq. (7.45) can now be interpreted as cohomological constraints.

7.3.1 *Effective BV invariance of I_{ext}*

$$\mathcal{S}_k I_{ext} = 0 \,. \tag{7.48}$$

This Equation is identically satisfied by the extended action, as we proved in Eq. (7.18). It encodes the effective BV invariance of I_{ext}, where the effective BV transformations are

$$\mathcal{S}_k \phi = \frac{\delta V}{\delta \sigma} \,, \ \mathcal{S}_k \eta = q_k \,, \ \mathcal{S}_k q_k = 0 \,, \tag{7.49}$$

The second part of the transformation shows that $(\eta,\ q_k)$ forms a contractible pair. This shows that I_{ext} plays the role of the proper solution to the CME in the space of functionals of (ϕ, σ).

7.3.2 *Cohomology condition*

$$\mathcal{S}_k \hat{\Gamma}_k = 0 \,. \tag{7.50}$$

This gives a non-trivial condition on the effective average action. As usual, $\mathcal{S}_k$-exact solutions can be re-absorbed by redefinitions of the fields [12, 22], and so $\hat{\Gamma}_k$ must be in the cohomology of $\mathcal{S}_k$. The operator can be decomposed into

$$\mathcal{S}_k = \mathcal{S} + \int_x q_k(x) \frac{\delta}{\delta \eta(x)} \,. \tag{7.51}$$

The second term acts only on the non-minimal sector and guarantees that q_k and η form a contractible pair [22]. Hence the non-trivial information about the cohomology of the effective BV operator is already encoded in $\mathcal{S}$, which acts as:

$$\mathcal{S} \phi = \mathcal{S}_k \phi \,, \quad \mathcal{S} \eta = 0 \,.$$

This is identical to the action of the BV operator s, but acting on (ϕ, σ) instead of $(\varphi, \varphi^{\ddagger})$. Hence, in particular, the effective antighost and the Nakanishi-Lautrup fields $(\phi_{\bar{c}}, \phi_b)$ form a contractible pair, since $\bar{c}$ and b form a contractible pair.

The quantum contribution $\hat{\Gamma}_k$ is then in the cohomology of the effective BV operator $\mathcal{S}$.

The cohomology of $\mathcal{S}$ is then determined by its minimal, regulator independent sector. By standard arguments, the cohomology of $\mathcal{S}$ on local functionals (that is, integrals of local functions) is characterised by the cohomology group of $\mathcal{S}$ modulo the exterior differential, $H^{g,n}(\mathcal{S}|d)$, where g is the effective-ghost number, the degree associated to the effective field corresponding to ghost fields, and n is the form degree. Here we are working on the space of local n-forms, rather than local functionals, as is standard in the literature [22].

Since the action of $\mathcal{S}$ in the space (ϕ, σ) is identical to the action of s in the space $(\varphi, \varphi^{\ddagger})$, the cohomology group $H^{g,n}(\mathcal{S}|d)$ can be characterised by the standard treatment of local BRST cohomology [22].

Since I_{ext} has effective-ghost number 0, in absence of anomalies $\hat{\Gamma}_k$ is determined by the BRST cohomology in ghost number 0, $H^{0,n}(\mathcal{S})$ in the space of functionals of (ϕ, σ).

In principle, the computation of the BRST cohomology provides the most general solution to the effective master equation. The effective average action is, in general, non-local, and therefore the cohomology of the BRST operator $H(\mathcal{S})$ provides the most general restriction on its functional dependence on the mean fields. However, one of the most used non-perturbative truncations for the effective average action is the *derivative expansion*,

$$\Gamma_k = \int_x f(\phi, \partial_\mu \phi, \dots, \partial_{(\mu_i} \dots \partial_{\mu_n)} \phi) \tag{7.52}$$

up to some finite order n in the order of the derivative. At each order in the truncation, the effective average action is local, and it is thus possible to apply the theorems on local BRST cohomology. The assumption behind this truncation is that non-local effects can be parametrised by $k-$dependent coefficients in front of the derivative expansion. The restriction of the effective average action to the functional form (7.52) implies that we can solve the effective master equation by standard local BRST cohomology techniques.

In the space of local functionals, the local BRST cohomology $H^{0,n}(\gamma|d)$ is completely determined by powerful theorems [20–23].

7.3.3 *Higher order cohomological restrictions*

$$(\hat{\Gamma}_k, \hat{\Gamma}_k) = 0 \,. \tag{7.53}$$

This condition tells us that, treating the regulator term as a generalised gauge-fixing, the correction $\hat{\Gamma}_k$ itself satisfies the same Zinn-Justin equation as the effective action without regulator.

This condition can be discussed once the cohomological constraint (7.50) is solved, as an identity or as an additional, non-trivial constraint. For example, in effective-ghost number 0, general theorems for Yang-Mills-type theories [22] guarantee that all representatives of $H^{0,d}(\mathcal{S}|d)$, where d the space-time dimension, can be chosen to be strictly gauge-invariant, except for Chern-Simons forms for d odd, and in particular the BRST sources σ can be removed in all counter-terms. In this case, the condition (7.53) reduces to a trivial identity, since $\frac{\delta \hat{\Gamma}_k}{\delta \sigma} = 0$. Similar considerations apply also for $H^{1,d}(\mathcal{S}|d)$, which controls gauge anomalies and where (7.53) can provide a non-trivial constraint on the anomaly term.

7.4 SYMMETRIES AND THE RG FLOW

Theorem (7.2) gives us powerful cohomological methods to constrain the structural form of the effective average action. However, Eq. (7.21) and its most important consequence, (7.50), are only useful if they are compatible with the RG flow equation (5.6), i.e., if solving the equation at fixed scale k_0 implies that the condition is satisfied at all scales.

To prove that the effective master equation is preserved along the RG flow, we consider the slightly more general problem of the flow for a composite, local operator $O_k(x; \varphi, \partial\varphi, \ldots)$, that is, an operator with arbitrary dependence on the field φ and its derivatives, which might also depend on the cut-off parameter k. A strategy to compute its flow equation is to further extend the definition of the generating functional Z_k, so that it also generates correlation functions between composite operators [200]. We thus introduce a new classical source υ, and we define

$$Z_k(j, \sigma, \eta, \upsilon) := \omega\left[S(V)^{-1} \star S\left(V + Q_k + J + \Sigma + H + \Upsilon_k\right)\right] , \tag{7.54}$$

where $\Upsilon_k := \int_x O_k(x)\upsilon(x)$. The definitions for the $\upsilon-$dependent $W_k(\upsilon, j)$ and effective average action are then the familiar ones from Section 4.2. Since $Z_k(j, \sigma, \eta) = Z_k(j, \sigma, \eta, \upsilon = 0)$, the derivative of Z_k with respect to υ, at $\upsilon = 0$, recovers the correlation functions for O_k in the original theory. We now compute the RG flow for composite operators.

Proposition 7.5. *Given a composite operator $O_k(x; \varphi, \partial\varphi, \ldots)$, its flow equation is given by*

$$\partial_k \langle O_k \rangle = -\frac{i\hbar}{2} \int_x \partial_k q_k(x) : G_k(x,x) : \frac{\delta^2 \langle O_k \rangle_k}{\delta\phi(x)\delta\phi(x)} : G_k(x,x) : + \langle \partial_k O_k \rangle \, . \tag{7.55}$$

Proof. The derivation of the RG flow equation for $Z_k(j, \upsilon)$ follows the same steps as in Section 5.1, with the addition of a contribution coming from the $k-$dependent O_k:

$$\partial_k Z_k(j, \upsilon) = \frac{i}{\hbar} S(V)^{-1} \star \left[S\left(V + Q_k + J + \Sigma + H + \int_x O_k \upsilon\right) \cdot_T \left(\partial_k Q_k + \partial_k \Upsilon_k\right)\right] . \tag{7.56}$$

Accordingly, the flow equation for the $\upsilon-$dependent effective average action gets modified into

$$\partial_k \Gamma_k = \frac{i\hbar}{2} \lim_{y \to x} \int_x \mathrm{Tr}\left[\partial_k q_k(x) : G_k : (x,y)\right] + \langle \partial_k \Upsilon_k \rangle_\upsilon \, . \tag{7.57}$$

Now, the derivative of $W_k(\upsilon)$ with respect to υ gives, as usual, the mean value of the corresponding operator O_k,

$$\frac{\delta W_k}{\delta \upsilon} = \langle O_k \rangle_\upsilon \, , \tag{7.58}$$

and since we do not take the Legendre transform with respect to υ, we have

$$\frac{\delta \Gamma_k}{\delta \upsilon} = \langle O_k \rangle_\upsilon \, , \tag{7.59}$$

where the mean value operator is taken for $j = j_\phi$.

Therefore, the derivative with respect to υ of the RG flow equation (7.57) gives the flow for the expectation value of the composite operator O_k:

$$
\begin{aligned}
\partial_k \frac{\delta \Gamma_k(\upsilon)}{\delta \upsilon} = \\
& - \frac{i\hbar}{2} \int_x \operatorname{Tr}\left\{ \partial_k q_k : G_k : \frac{\delta \Gamma_k^{(2)}}{\delta \upsilon} : G_k : \right\} \\
& + \frac{i}{\hbar} \left(\langle O_k \cdot_T \partial_k \Upsilon \rangle_\upsilon - \langle O_k \rangle_\upsilon \langle \partial_k \Upsilon \rangle_\upsilon \right) + \langle \partial_k O_k \rangle_\upsilon \,.
\end{aligned}
$$

The evaluation for vanishing source $\upsilon = 0$ annihilates all terms proportional to Υ, and so we arrive at the proposition. □

From the RG flow for composite operators, it is easy to check the compatibility of the effective master equation with the RG flow. In fact, the EME is equivalent to $\langle \mathcal{RQME} \rangle = 0$, thanks to Eq. (7.22). On the other hand, since $\mathcal{RQME}$ does not explicitly depend on k, its RG flow as a composite operator is

$$
\partial_k \langle \mathcal{RQME} \rangle = -\frac{i\hbar}{2} \int_x \operatorname{Tr}\left\{ \partial_k q_k : G_k : \frac{\delta^2 \langle \mathcal{RQME} \rangle}{\delta \phi \delta \phi} : G_k : \right\} . \tag{7.60}
$$

Since the flow of the operator is proportional to the operator itself, if the EME is satisfied at some scale $k = \Lambda$, $\langle \mathcal{RQME} \rangle|_{k=\Lambda} = 0$, it is automatically satisfied at all scales.

8 *Applications*

8.1 LOCAL POTENTIAL APPROXIMATION

In Chapters 5 and 7 we laid down the formalism to investigate the RG flow of scalar and gauge theories, in Lorentzian manifolds and for generic states, generalising the Wetterich equation [246]. The flow is non-perturbative, and can be applied to renormalizable as well as non-renormalizable theories.

In this Chapter we compute the approximate RG flow in some applications. In particular, we focus our attention on the existence of non-trivial fixed points in the flow, and their critical exponents.

In fact, the existence of an RG trajectory, depending on a finite number of parameters, connecting the well-defined limits for the effective average action in the IR ($k \to 0$) and the UV ($k \to \infty$) proves the renormalizability of a theory. Typically, the IR regime is controlled by the existence of a Gaussian fixed point, where the couplings vanish. Therefore, the existence of a non-trivial UV fixed point in the exact flow, connected to the Gaussian one, identifies a global RG trajectory, known as *separatrix*, where the effective average action remains finite at all k.

Naturally, the existence of global RG trajectories in the full parameter space of a theory is impossible to prove in general. We thus restrict our attention to a parameter subspace of the theory, characterised by a particular Ansatz for the effective average action known as *Local Potential Approximation*.

The applications we discuss have two main purposes: first, to show that the new formalism reproduces known results in special cases, and secondly, that it produces new interesting results in its wider field of application. We thus collect here a series of examples. We start discussing the textbook example of an interacting scalar field theory in flat spacetime. In Minkowski vacuum, and in 4 dimensions, we recover the known result that there is no non-trivial fixed point, the famous *triviality problem* of scalar field theory [4, 71]. Moreover, in 3 dimensions, the RG flow exhibits the Wilson-Fisher fixed point [251]. Even though the value of the fixed point is scheme-dependent, we recover standard results on the critical exponents.

To show that the RG flow equations can be applied to more general states, we then study the RG flow of a scalar field in Minkowski, choosing a KMS state (a thermal equilibrium state) at inverse temperature β for the free theory as reference state. In this case, we will see that the temperature introduces a new dimensionful parameter in the flow, allowing for the existence of a non-trivial fixed point even in 4 dimensions. The mechanism behind this fixed point is similar to a dimensional reduction, already discussed in the literature [177, 216, 236]. This effect demonstrates that different reference states dramatically change the RG flow of the same theory.

Finally, we apply the RG flow in curved spacetimes. We consider the simple example of a scalar field propagating over a de Sitter background. Similar to the thermal case, the curvature plays the role of an additional dimensionful parameter in the flow, other than k, and it is possible to identify again a non-trivial fixed point.

After the study of the scalar field, we study a theory of quantum gravity based on the metric tensor, known as *Quantum Einstein Gravity*. In fact, one of the main motivations to study the generalisation of the Wetterich equation to Lorentzian manifolds comes from the Asymptotic Safety scenario in quantum gravity [241, 242], in which gravity becomes non-perturbatively renormalizable thanks to a non-trivial fixed point in its RG flow, known as the *Reuter fixed point*.

First realised in $2+\epsilon$ dimensions [76, 129], the AS scenario in four dimensions has been explored through lattice simulations [11, 180], and, in the continuum, through functional Renormalization Group (fRG) techniques [41, 47, 198, 204, 214, 219, 226]. While lattice computations are based on a regularisation of the Lorentzian path integral, fRG approaches are mostly based on the Euclidean formulation of the Wetterich equation [191, 246], with few exceptions, based on foliated backgrounds in the ADM formalism [43, 184] or Minkowski space [108].

Here, we apply the Lorentzian RG flow equations (5.6) to quantum gravity. The flow is state dependent, as we discussed; while a state for the graviton in general spacetimes is not known, here we show that the universal terms that must contribute in the FRGE for any state, and in all backgrounds, already determine the existence of a Reuter-type fixed point for Lorentzian quantum gravity.

In order to apply the RG flow to all these cases, we need to approximate the effective average action. The standard approximations of the fRG literature work by truncating the functional dependence of the effective average action to some finite sum of monomials of the fields, with $k-$dependent coupling constants. The functional derivatives of the RG flow equations then provide the flow of these constant with k, and their dimensionless counterparts define the *$\beta-$functions.*

Here, we choose the simplest possible approximation, known in the literature as *Local Potential Approximation* (LPA). The LPA is based on two main assumptions. First, we assume that the effective potential U_k, defining the effective average action through equation (5.8), is a local functional of the fields that does not contain derivatives, and that the same holds for its second functional derivative; this is the approximation that we called *No-Derivative Approximation* in Chapter 6, and takes the form

$$U_k(\phi) = \int_x u(\phi(x), k) f(x)\,, \qquad U_k^{(2)}(\phi)(x,y) = \partial_\phi^2 u(\phi(x), k) f(x) \delta(x,y)\,. \tag{8.1}$$

In this approximation, f as usual denotes an IR cut-off function, so that the integral is IR convergent: we then have $f = 1$ on the relevant part of spacetime we are studying, and $f \in C_c^\infty$ with $\operatorname{supp} f \subseteq \mathcal{O}$ for some compact region $\mathcal{O} \subset \mathcal{M}$.

Moreover, after deriving $U_k^{(2)}$ we choose to set the field to a constant throughout the region $\mathcal{O}$, $\phi(x) = \phi$. Thanks to this approximation, the functions $u(\phi, k)$ and $\partial_\phi^2 u(\phi, k)$ are constant in space.

The key feature of the NDA is the absence of derivatives of the Dirac delta in $U_k^{(2)}$; it follows that the quantum wave operator $\Gamma_k^{(2)} - q_k = P_0 - q_k + U_k^{(2)}$ is a field-dependent, non-linear potential perturbation of the free wave operator P_0.

The LPA however proceeds further, approximating the effective potential to a mass perturbation of the bare action. The idea is the following.

We start expanding Γ_k close to a solution ϕ_{cl} of $\frac{\delta \Gamma_k}{\delta \phi}|_{\phi_{cl}} = 0$, so that

$$\Gamma_k(\phi) = \Gamma_k(\phi_{cl}) + \frac{1}{2}\Gamma_k^{(2)}(\phi_{cl})(\hat{\varphi} \otimes \hat{\varphi}) + \mathcal{O}(\hat{\varphi}^3) = \Gamma_k(\phi_{cl}) + \Gamma_k^t(\hat{\varphi}) + \mathcal{O}(\hat{\varphi}^3) ,$$

where we introduced the fluctuation field $\hat{\varphi} := \phi - \phi_{cl}$ and the *truncated effective action* Γ_k^t, disregarding the irrelevant constant $\Gamma_k(\phi_{cl})$. By a slight abuse of notation, we denoted

$$\Gamma_k^{(2)}(\phi_{cl})(\hat{\varphi} \otimes \hat{\varphi}) = \langle \Gamma_k^{(2)}(\phi_{cl}), \hat{\varphi} \otimes \hat{\varphi} \rangle .$$

In terms of the decomposition of the effective average action, Eq. (5.7), the Γ_k^t defines the truncated effective potential U_k^t as

$$\Gamma_k^t(\hat{\varphi}) = \frac{1}{2}\Gamma_k^{(2)}(\phi_{cl})(\hat{\varphi} \otimes \hat{\varphi}) = I_0(\hat{\varphi}) + \frac{1}{2}U_k^{(2)}(\phi_{cl})(\hat{\varphi} \otimes \hat{\varphi}) =: I_0(\hat{\varphi}) + U_k^t(\phi_{cl}, \hat{\varphi}) .$$

The LPA approximates $\Gamma_k(\phi_{cl} + \hat{\varphi})$ with $\Gamma_k^t(\phi_{cl}, \hat{\varphi})$, where ϕ_{cl} is a classical background and the dynamical mean field is the fluctuation field $\hat{\varphi}$. We now want to understand which bare action I_0^t produces Γ_k^t, just as Γ_k is the effective average action produced by $I = I_0 + V$. By Definition 5.8 of the effective potential, we have

$$U_k^{(1)}(\phi) = \langle TV^{(1)} \rangle = \langle TM^{(1)} \rangle + \mathcal{O}(\hat{\varphi}^2) ,$$

where M is the interaction term approximating V, so that

$$\frac{\delta}{\delta \hat{\varphi}} U_k^t(\phi_{cl}, \hat{\varphi}) = \langle T \frac{\delta M}{\delta \varphi} \rangle .$$

Now, since U_k^t is quadratic in $\hat{\varphi}$, it follows that

$$\frac{\delta}{\delta \hat{\varphi}} U_k^t(\phi_{cl}, \hat{\varphi}) = \frac{\delta}{\delta \phi} U_k^t(\phi_{cl}, \phi) = \langle \frac{\delta}{\delta \varphi} U_k^t(\phi_{cl}, \varphi) \rangle = \langle T \frac{\delta M}{\delta \varphi} \rangle ,$$

and so $TM^{(1)}(\varphi) = \dfrac{\delta}{\delta \varphi} U_k^t(\phi_{cl}, \varphi)$. Again discarding irrelevant constants it follows that the truncated bare action I_0^t producing the truncated effective average action is

$$I_0^t = I_0 + M = I_0(\varphi) + \frac{1}{2}U_k^{(2)}(\phi_{cl})(\varphi) ,$$

where $U_k^{(2)}(\phi_{cl})$ is equal to the contribution to the truncated Γ_k^t coming from the potential U_k. The truncated action I_0^t is the bare action producing the truncated effective average action of the LPA. Decomposing the full action $I = I_0 + V$ into $I = I_0^t + \mathcal{V} = I_0 + M + \mathcal{V}$, we see that the LPA consists in discarding the interacting terms $\mathcal{V} = V - M$. The term M contains the residual information about the interactions of the full Lagrangian. The action I_0^t then contains the usual kinetic term, plus a non-trivial, classical background, acting as a ϕ_{cl}-dependent mass for the fluctuation field $\hat{\varphi}$.

From the discussion in Section 5.2.2, it follows that the interacting propagator G_k is the Feynman propagator with a mass $M_k = m^2 + k^2 - U_k^{(2)}$. We prove the same result using the PPA in the following proposition.

Proposition 8.1 ([95])**.** *The perturbative expression of G_k in terms of the propagators for $P_0 - q_k$ converges, and it equals the Feynman propagator for the quantum wave operator $P_0 - q_k + U_k^{(2)}$*

$$G_k = -i\Delta_{F,k,U_k^{(2)}} . \tag{8.2}$$

Proof. To prove the statement, we analyse the perturbative construction of G_k through the Bogoliubov map, approximating $G_k = \langle T\varphi^2(x)\rangle_c$.

The truncated action I_0^t defining the effective average action in the LPA, and the free action I_0 and I_0^t differ only by the quadratic term M. We can then apply the PPA to construct the interacting algebra of I_0^t in the free algebra with propagators coming from I_0. In fact, the Bogoliubov map depends only on the interaction term M, which is a quadratic perturbation.

In fact, we want to use the Bogoliubov map $R_{\mathcal{V}}$ constructed around the new action I_0^t and consider only the zeroth order contribution.

In particular, making use of Theorem 4.1 in [95], we have that (*cf.* Eq. (2.47))

$$R_V = R_{\mathcal{V}+M} = r_M \circ R^M_{\rho\mathcal{V}} \circ \rho\,,$$

where r_M is the classical Møller map $r_M = 1 - \Delta_{R,M}M^{(2)}$ defined in definition (2.19), and ρ intertwines the time ordered product T constructed with Δ_F with the time-ordered product T_M for the free theory I_0^t; in fact, $\rho T = T_M$. The Feynman propagator $\Delta_{F,M}$ defining T_M is the one associated with the state ω_M, whose two-point function reads $r_M\Delta_+ r_M^*$. $R^M_{\rho V}$ is the Bogoliubov map constructed over the free theory I_0^t.

Using the formula for R_V in terms of $R_{\mathcal{V}+M}$, the Bogoliubov map in the mean value $\langle T\varphi^2\rangle$ defining the interacting propagator becomes

$$R_{\mathcal{V}+M}(S(J_\phi + Q_k)\cdot_T T\varphi^2) = r_M\left(R^M_{\rho\mathcal{V}}(S_M(\rho Q_k + \rho J_\phi)\cdot_T \rho T\varphi^2)\right)\,.$$

S_M is the $S-$matrix of the T_M time-ordered products. Now, discarding the contributions containing $\mathcal{V}$, the interacting propagator can be written in terms of the Bogoliubov map as

$$-iG_k = \langle T\varphi^2\rangle_c \simeq \frac{\omega_M(S_M(\rho Q_k + J_\phi)\cdot_{T_M} T_M\varphi^2)}{\omega_M(S_M(\rho Q_k + J_\phi))} - \phi^2\,.$$

where we used the fact that $\rho T\varphi^2 = T_M\varphi^2$. The last expression above corresponds to the computation of the interacting propagator in the free limit $G_k^{(0)}$, presented in Section 5.2.2, where the T_M-products substitute the $T-$products, and ρQ_k substitute Q_k. It follows that, by the same derivation, the regularised, interacting propagator in the LPA corresponds to the Hadamard-subtracted Feynman propagator for a theory with mass perturbed by $q_k - M^{(2)}$:

$$\begin{aligned}
&- i : G_k^{(0)} : (x,x) \\
&= \lim_{y\to x}\left(\frac{\omega_M(S_M(\rho Q_k + J_\phi)\cdot_{T_M}\varphi(x)\cdot_{T_M}\varphi(y))}{\omega_M(S_M(\rho Q_k + J_\phi))} - H_{F,M,k}(x,y) - \phi(x)\phi(y)\right) \\
&\qquad\qquad\qquad\qquad = \lim_{y\to x}\left(\Delta_{F,M,k}(y,x) - H_{F,M,k}(y,x)\right)\,, \qquad (8.3)
\end{aligned}$$

where $\Delta_{F,M,k}$ is a Feynman propagator for the theory $I_0^t + Q_k$ obtained from the two-point function Δ_+ of the state ω and where $H_{F,M,k}$ is the Hadamard parametrix of the theory $I_0^t + Q_k$. More precisely, arguing as in the proof of Lemma 5.3, we have that

$$\Delta_{F,M,k} = \sum_{n\geq 0}(-i)^n\Delta_{F,M}(q_k\Delta_{F,M})^n \qquad (8.4)$$

where $\Delta_{F,M} = \Delta_{+,M} + i\Delta_{A,M}$ is a Feynman propagator of the theory I_0^t, and it is computed from $\Delta_F = \Delta_+ + i\Delta_A$ using the Møller operators. □

The interacting propagator G_k in the LPA is thus the Feynman propagator for a state $\omega_{M,k}$, whose two-point function is $\Delta_{+,M,k}$. The r.h.s. of the RG flow equations (5.24) in the LPA then is nothing but the expectation value of $\partial_k Q_k$ in a quasifree state $\omega_{M,k}$ whose two-point function is

$$\Delta_{+,M,k} = r_{M+Q_k} \Delta_+ r^*_{M+Q_k} , \tag{8.5}$$

and the RG flow equations takes the simple form

$$\partial_k U_k(\phi) = \omega_{M,k}[\partial_k(Q_k)] . \tag{8.6}$$

According to the discussion in Section 5.2.3, the regularised propagator : G_k : can be written in two, equivalent ways, either by Hadamard subtraction of the Feynman propagator, or of the two-point function. Since the latter has better convergence properties, we can write the RG flow in the LPA in terms of the two-point function of the state $\omega_{M,k}$.

Actually, for $q_k = k^2 f$ in the region where the cut-off function f is 1, the RG flow equation in the LPA is

$$\partial_k U_k(\phi) = -\lim_{y \to x} \int_x k f(x) (\Delta_{S,M,k}(y,x) - H_{M,k}(y,x)) , \tag{8.7}$$

where $\Delta_{S,M,k}$ is the symmetric part of the two-point function $\Delta_{+,M,k}$ and $H_{M,k}$ is the Hadamard function related to the theory whose action is $I_0^t + Q_k = I_0 + M + Q_k$.

The regularisation provided by the point splitting procedure discussed here is compatible with the principles of local covariance [158, 162]. Many explicit computations of similar point-splitting regularisations are already present in the literature on flat and curved spacetimes (e.g. in de Sitter [90]).

Remark 8.1. We can adapt the discussion above in order to include a wavefunction renormalization. We discuss its inclusion in the case of a massless scalar field only for definiteness, but it can be generalised to Yang-Mills-type theories as well.

In this case, we start with an Ansatz for the effective average action in the form

$$\Gamma_k(\phi) = -\int \mathrm{d}^d x \left(\frac{z_k}{2} \nabla_a \phi \nabla^a \phi + U_k(\phi) \right) .$$

This approximation scheme is known as LPA'. It follows by an argument completely parallel to that for the LPA that the truncated effective action takes the expression

$$\Gamma_k^t = z_k I_0(\hat{\varphi}) + U_k(\phi_{cl}, \hat{\varphi}) ,$$

and, therefore, the truncated bare action I_0^t producing the truncated effective action is

$$I_0^t = z_k I_0(\varphi) + \frac{1}{2} U_k^{(2)}(\phi_{cl})(\varphi) .$$

Arguing as in Prop. 8.1, it follows that the interacting propagator in this case becomes

$$-iG_k = \Delta_{F,k,z_k,U_k^{(2)}} ,$$

where $\Delta_{F,k,z_k,U_k^{(2)}}$ is the Feynman propagator for the wave operator $z_k(P_0 - q_k) + U_k^{(2)}$, and it holds that

$$-iG_k = \Delta_{F,k,z_k,U_k^{(2)}} = z_k^{-1} \Delta_{F,k,z_k^{-1} U_k^{(2)}} .$$

For example, in the simple case of the Minkowski vacuum as the reference state ω, with $q_k = k^2$, taking the adiabatic limit and choosing a constant classical field ϕ_{cl}, the r.h.s. of the Wetterich equation in Fourier domain becomes

$$\partial_k \Gamma_k = \frac{1}{2(2\pi)^d} \int \mathrm{d}^d p \frac{\partial_k (z_k k^2)}{z_k(p^2 + k^2 + m_k^2) + U_k^{(2)}(\phi_{cl})} \,.$$

In the free limit $U_k = 0$, the wavefunction renormalization z_k correctly vanishes from the RG flow equations. ■

Remark 8.2. In the computation described above we have transformed the Wick square $T\varphi^2$ in $T_{M,k}\varphi^2$ with a two step procedure: (a) we moved $T\varphi^2$ in $T_M\varphi^2$ by invoking the principle of perturbative agreement; (b) we changed $T_M\varphi^2$ into $T_{M,k}\varphi^2$ roughly by computing $\omega(S(Q_k) \cdot_{T_M} T_M\varphi^2)/\omega(S(Q_k))$.

Although the net result in both cases is a change in the mass of the correlation functions, the procedures (a) and (b) are slightly different. The reason is that Q_k appears in the Bogoliubov map of the generating functional at the numerator only, so that $Z_k = \omega(S(V)^{-1} \star S(V + Q_k + J))$, but not in the inverse $S-$matrix. It follows that the PPA cannot be directly applied, because Q_k is not a perturbation of the action but rather a regulator on the correlation functions. The LPA, instead, approximates the interaction V with a quadratic term, and as such it is a deformation of the original action I.

Although this difference is not immediate when dealing with the effective average action, it is particularly relevant regarding the ambiguities in the choice of $T\varphi^2, T_M\varphi^2$ and $T_{M,k}\varphi^2$. Indeed, since point (a) respects the principle of local covariance [66], it follows that the ambiguities which define $T\varphi^2$ and $T_M\varphi^2$ *have to be the same*. The ambiguities of $T_{M,k}\varphi^2$ instead can be different, and even depend on k, since there is no reason why (b) should respect the principle of local covariance, because theories with different ks are in principle not deformable one into the other. We can thus make *different* choices for the ambiguities defining $T\varphi^2$ and $T_{M,k}\varphi^2$. ■

The LPA can be further simplified by taking successive functional derivatives with respect to ϕ_{cl} on both sides of (8.7), and evaluating at $\phi_{cl} = 0$. The flow for the $k-$dependent couplings gives scale evolution of the running coupling constants, defined as the coefficients of the Taylor series of $U_k^t(\phi_{cl})$ at $\phi_{cl} = 0$; in the simple case of a scalar field theory, the Taylor expansion is

$$U_k^t(\phi_{cl}) = U_{0,k} + m_k^2 \frac{\phi_{cl}^2}{2} + \lambda_k \frac{\phi_{cl}^4}{4!} + \mathcal{O}(\phi_{cl}^6) \,. \tag{8.8}$$

The β-functions are defined as the evolution equations for the dimensionless parameters $\tilde{m}_k^2$ and $\tilde{\lambda}_k$ with respect to the renormalization time $t = \log k/\Lambda$, where Λ is the scale at which the renormalization starts. In powers of k, the dimensions of the couplings are $[m_{\beta,k}^2] = 2$ and $[\lambda_k] = 4 - d$. Then we have

$$k\partial_k \tilde{m}_{k,\beta}^2 = k^{-1}\partial_k m_{k,\beta}^2 - 2\tilde{m}_{k,\beta}^2 \tag{8.9}$$

$$k\partial_k \tilde{\lambda}_k = k^{d-3}\partial_k \lambda_k + (d-4)\tilde{\lambda}_k \,. \tag{8.10}$$

8.2 VACUUM AND HIGH-TEMPERATURE STATES IN MINKOWSKI

Finally, we are ready to compute the β-functions in some example. We start from the simplest case of an interacting scalar field theory propagating in Minkowski; despite its simplicity, different states produce different β-functions, and we can see these effects in our unified formalism.

The action for a massive $\lambda\varphi^4$–theory is

$$I(\varphi) = -\int_x \frac{1}{2}\nabla_a\varphi\nabla^a\varphi\frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 ,$$

where $\varphi \in C^\infty(\mathcal{M})$. We choose to apply the formalism for the vacuum and in a KMS state, which are both quasifree states for the free theory, and we apply the LPA.

Of course, along the RG flow any type of coupling can arise, so the effective potential U_k is constrained by the symmetries of the effective average action only. Here we focus on the simple truncation of an effective potential with a mass and a ϕ^4 interaction term, thus retaining the first two terms in a polynomial expansion. Notice that, since the effective average action is in a quasifree state, it preserves the $\mathbb{Z}_2$ global parity of the action (*cf.* Section 4.3.3). Then, Taylor expanding the effective potential, we have

$$U_k(\phi) = U_{0,k,\beta} - \int_x \frac{1}{2}\bar{m}^2_{k,\beta}\phi^2 + \frac{1}{4!}\lambda_{k,\beta}\phi^4 , \tag{8.11}$$

The LPA now proceeds expanding $\phi = \phi_{cl} + \hat{\varphi}$, assuming a constant classical configuration ϕ_{cl}, and retaining the second-order fluctuations only, so that

$$\begin{aligned}\Gamma^t_k(\phi_{cl},\hat{\varphi}) &= I_0(\hat{\varphi}) + \frac{1}{2}\langle U^{(2)}_k(\phi_{cl}), \hat{\varphi}\otimes\hat{\varphi}\rangle \\ &= U_{0,k,\beta} - \int_x \frac{1}{2}\nabla^a\hat{\varphi}\nabla_a\hat{\varphi} + \frac{1}{2}\left(m^2_{k,\beta} + \lambda_{k,\beta}\rho\right)\hat{\varphi}^2 ,\end{aligned} \tag{8.12}$$

with $\rho = \frac{1}{2}\phi^2_{cl}$. The couplings $m^2_{k,\beta} := m^2 + \bar{m}^2_{k,\beta}$ and $\lambda_{\beta,k}$ can depend on the temperature as well as k, since, for example, there will be contributions coming from the one-loop renormalization of the thermal mass [95].

We now compute the interacting propagator in the LPA, the r.h.s. of the approximated RG flow equations (8.7). The reference state of the free theory is an equilibrium state with respect to Minkowski time evolution at inverse temperature β. This is a KMS quasifree state for the free theory $\lambda = 0$, and its 2-point function is invariant under translations. Thus, it can be written in momentum space, and has the expression [95]

$$\Delta^\beta_+(x^0,\mathbf{x};y^0,\mathbf{y}) = \int \frac{\mathrm{d}^{d-1}\mathbf{p}}{(2\pi)^{d-1}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{1}{2w}\left(\frac{e^{-iw(x^0-y^0)}}{1-e^{-\beta w}} + \frac{e^{iw(x^0-y^0)}}{e^{\beta w}-1}\right), \tag{8.13}$$

where $w = |\mathbf{p}|$. It follows that the two-point function appearing in the RG flow has the same expression, with the mass modified by the regulator and the effective potential. Taking the adiabatic limit, the 2-point function is [52, 93–95]

$$\begin{aligned}&\Delta^\beta_{+,M,k}(x^0,\mathbf{x};y^0,\mathbf{y}) = \\ &\quad = \int \frac{\mathrm{d}^{d-1}\mathbf{p}}{(2\pi)^{d-1}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{1}{2w_{M,k}}\left(\frac{e^{-iw_{M,k}(x^0-y^0)}}{1-e^{-\beta w}} + \frac{e^{iw_{M,k}(x^0-y^0)}}{e^{\beta w}-1}\right),\end{aligned} \tag{8.14}$$

where $w_{M,k} = \sqrt{w^2 + m^2_{k,\beta} + \lambda_{k,\beta}\rho + q_k}$. Notice that the w-factors associated with the modes have changed, while this is not the case for the Bose factors.

The above 2-point function differs from that of Ref. [179], since in our construction the Bose factors are k-independent. The reason is that the reference state for the free theory does not depend on the scale k, and we use this state to compute the inverse of the quantum wave operator. It would possible to compute the 2-point function of the thermal state in the interacting theory, since the perturbation is quadratic. However, choosing a $k-$dependent state would introduce additional contributions in the RG flow.

The RG flow in the LPA for a thermal reference state in Minkowski then is

$$\begin{aligned} -\partial_k U_k(\phi) &= \lim_{y\to x}(\Delta^{\beta}_{S,M,k}(y,x) - \Delta^{\infty}_{S,M,k}(y,x))\frac{\partial_k q_k(x)}{2} \\ &\quad + \lim_{y\to x}(\Delta^{\infty}_{S,M,k}(y,x) - H(y,x))\frac{\partial_k q_k(x)}{2}\,. \end{aligned} \tag{8.15}$$

The flow naturally splits in two contributions, a $\beta-$independent term plus a correction from thermal fluctuations. To get the β-functions for both $\lambda_{\beta,k}$ and $m_{\beta,k}$, we analyse these two contributions separately. The first term is the regularisation of the thermal 2-point function with respect to the vacuum, computed taking the $\beta \to \infty$ limit; from the expression (8.14), in the adiabatic limit and in four dimensions, the first contribution is

$$\begin{aligned} A &:= \lim_{y\to x}(\Delta^{\beta}_{S,M,k}(y,x) - \Delta^{\infty}_{S,M,k}(y,x)(y,x))\frac{\partial_k q_k(x)}{2} \\ &= \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{w_{M,k}}\left(\frac{1}{e^{\beta w}-1}\right)\frac{\partial_k q_k(x)}{2}\,. \end{aligned} \tag{8.16}$$

The second term instead is the regularisation of the vacuum 2-point function with respect to the Hadamard parametrix. For a massive theory in even-dimensional Minkowski space, the Hadamard distribution is known to depend on an additional arbitrary parameter μ, and it is given by [57]

$$H^{\mu}_{m}(x,y) = \Delta_{S,M,k}(x,y) + \frac{(-1)^{d/2}}{2(2\pi)^{d/2}}M^{d/2-1}\log\left(\frac{\mu^2}{M^2}\right)\sigma^{\frac{2-d}{4}}I_{d/2-1}(\sqrt{M^2\sigma})\,,$$

where $\Delta_{S,k}(x,y)$ is the symmetric contribution of the vacuum 2-point function, $I_\nu(x)$ is the modified Bessel function of the first kind and $\sigma = g_{ab}(x-y)^a(x-y)^b$ is the squared geodesic distance. The mass term for the truncated theory in the LPA is $M^2 = k^2 + m^2_{k,\beta} + \lambda_{k,\beta}\rho$. In the coincidence limit, $\sigma \to 0$ and

$$I_{d/2-1}(x,y) \simeq \frac{M^{d/2-1}\sigma^{\frac{d-2}{4}}}{2^{d/2-1}\Gamma(d/2)}\,.$$

Therefore, the σ dependence drops and we obtain

$$\begin{aligned} &\lim_{\sigma\to 0} H^{\mu}_{m} = \\ &= \Delta^{\infty}_{S,m,k} + \frac{(-1)^{d/2}}{\Gamma(d/2)(4\pi)^{d/2}}\left(k^2 + m^2_{k,\beta} + \lambda_{k,\beta}\rho\right)^{d/2-1}\log\left(\frac{k^2 + m^2_{k,\beta} + \lambda_{k,\beta}\rho}{\mu^2}\right)\,. \end{aligned} \tag{8.17}$$

While the state in the definition of Γ_k is $k-$independent, the Hadamard parametrix $\tilde{H}_F$ arises from the commutation of the coincidence limit with the mean value operator $\langle\cdot\rangle$, which depends on k.

Using this Hadamard parametrix, the second contribution in Eq. (8.15) is

$$\begin{aligned} B &:= \lim_{y\to x}(\Delta^{\infty}_{S,M,k}(y,x) - H(y,x))\frac{\partial_k q_k(x)}{2} \\ &= \frac{1}{8\pi^2}(k^2 + m^2_{k,\beta} + \lambda_{k,\beta}\rho)\log\left(\frac{k^2 + m^2_{k,\beta} + \lambda_{k,\beta}\rho}{\mu^2}\right)\frac{\partial_k q_k(x)}{2}\,. \end{aligned} \tag{8.18}$$

We now choose $q_k = k^2$, and we consider the different results in the vacuum or in the high-temperature limit.

8.2.1 Vacuum case

In the limit of vanishing temperature $\beta \to \infty$, the contribution (8.16) due to A vanishes, and in the four dimensional case the evolution equations for the running couplings are

$$\begin{aligned} k\partial_k U_{0,k} &= \frac{1}{8\pi^2}k^2(k^2+m_k^2)\log\left(\frac{k^2+m_k^2}{\mu^2}\right)\,, \\ k\partial_k m_k^2 &= \frac{1}{4\pi^2}k^2\left(1+\log\left(\frac{k^2+m_k^2}{\mu^2}\right)\right)\lambda_k\,, \\ k\partial_k\lambda_k &= \frac{3}{8\pi^2}\frac{k^2}{k^2+m_k^2}\lambda_k^2\,. \end{aligned}$$

In terms of the dimensionless couplings, $\tilde{U}_{0,k} = U_{0,k}/k^4$ $\tilde{m}_k = m_k/k$ and $\tilde{\lambda}_k = \lambda_k$ the β-functions then are

$$k\partial_k\tilde{U}_{0,k} = -4\tilde{U}_{0,k} + \frac{1}{8\pi^2}(1+\tilde{m}_k^2)\left[\log\left(1+\tilde{m}_k^2\right) + \log\left(\frac{k^2}{\mu^2}\right)\right]\,, \tag{8.19}$$

$$k\partial_k\tilde{m}_k^2 = -2\tilde{m}_k^2 + \frac{\tilde{\lambda}_k}{4\pi^2}\left[1+\log\left(1+\tilde{m}_k^2\right)+\log\left(\frac{k^2}{\mu^2}\right)\right]\,, \tag{8.20}$$

$$k\partial_k\tilde{\lambda}_k = \frac{3}{8\pi^2}\frac{\tilde{\lambda}_k^2}{1+\tilde{m}_k^2}\,. \tag{8.21}$$

The price to pay to have a local regularisation term is an additional freedom in the β-functions due to the ultraviolet renormalization scale μ, which is a residual freedom in the ultraviolet renormalization scheme we have adopted. However, the additional freedom may be safely removed setting $\mu = k$. This is equivalent to tune the renormalization ambiguities of $T_{M,k}\hat{\varphi}^2$ which, we recall, are not forced to be the same as the ones present in $T\hat{\varphi}^2$ (*cf.* Remark 8.2). This choice does not change the form of the Wetterich equation as it is made *after* deriving in k.

Setting $\mu = k$ the only fixed point is the Gaussian one, $\tilde{\lambda}_k = \tilde{m}_k = 0$, in agreement with the existing literature on the triviality of $\lambda\phi^4$ in four dimensions and in the vacuum case.

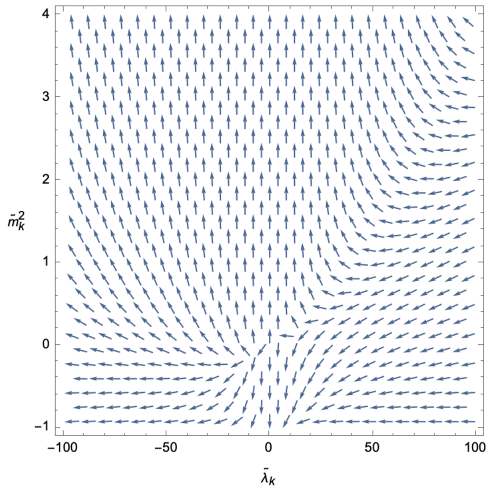

Figure 8.1: β−functions for the vacuum state for the free theory in 4-dimensional Minkowski spacetime. The flow is in the direction of decreasing k (towards the IR).

8.2.2 *High temperature limit*

The high temperature limit $\beta \longrightarrow 0$ of the RG flow (8.15) is very different from the vacuum case. The thermal contribution to the flow is

$$A = \frac{k^3}{2\pi^2}\int_0^\infty \mathrm{d}p \frac{p^2}{\sqrt{p^2 + \left(\frac{m_{k,\beta}}{k}\right)^2 + \frac{\lambda_{k,\beta}}{k^2}\rho + 1}} \frac{1}{e^{\beta k p} - 1}\,.$$

The expansion of A up to order 2 in powers of ρ gives

$$A \simeq \frac{k^3}{4\pi^2}\int_0^\infty \mathrm{d}p^2 \left[\left(p^2 + \left(\frac{m_{k,\beta}}{k}\right)^2 + 1\right)^{-\frac{1}{2}} - \frac{\lambda_{k,\beta}\rho}{2k^2}\left(p^2 + \left(\frac{m_{k,\beta}}{k}\right)^2 + 1\right)^{-\frac{3}{2}} \right.$$
$$\left. + \frac{3(\lambda_{k,\beta}\rho)^2}{8k^4}\left(p^2 + \left(\frac{m_{k,\beta}}{k}\right)^2 + 1\right)^{-\frac{5}{2}}\right] \frac{p}{e^{\beta k p} - 1}\,.$$

The contribution to the β-functions due to A diverges as $1/\beta$ in the limit $\beta \longrightarrow 0$, while B stays bounded. Therefore, B is negligible in the high-temperature limit, and the evolution equation for the running couplings are given by the functional derivatives of A with respect to ρ, evaluated at $\rho = 0$.

In order to derive the β-functions, we need to introduce dimensionless couplings. While in the vacuum case the only dimensionful scale is k, in the thermal case the inverse temperature β is an additional dimensionful parameter in the evolution equations; it is then possible to use different combinations of k and β to define dimensionless couplings. In order to have autonomous equations, we choose the dimensionless, rescaled constants $\tilde{U}_{0,k,\beta} = U_{0,k,\beta}\beta^2/k$, $\tilde{m}_{k,\beta} = m_{k,\beta}/k$

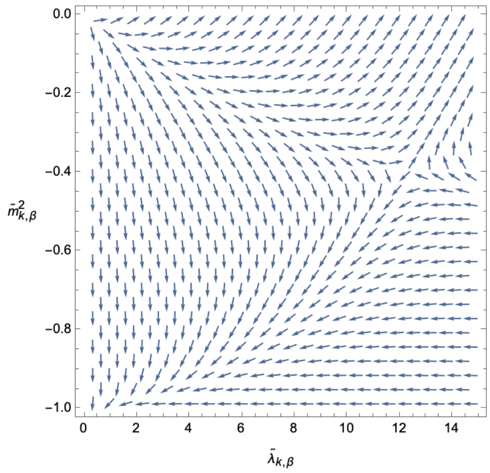

Figure 8.2: $\beta-$functions for the high-temperature regime of the scalar field in a KMS state for the free theory in 4-dimensional Minkowski spacetime. The flow is in the direction of decreasing k (towards the IR).

and $\tilde{\lambda}_{k,\beta} = \lambda_{k,\beta}/(\beta k)$. The β-functions then are

$$k\partial_k \tilde{U}_{0,k,\beta} = -\tilde{U}_{0,k,\beta} + \frac{\zeta(3)}{2\pi^2}\,, \tag{8.22}$$

$$k\partial_k \tilde{m}^2_{k,\beta} = -2(\tilde{m}_{k,\beta})^2 - \frac{1}{2\pi^2}\frac{\tilde{\lambda}_{k,\beta}}{\left(1+\tilde{m}^2_{k,\beta}\right)^{\frac{1}{2}}}\,, \tag{8.23}$$

$$k\partial_k \tilde{\lambda}_{k,\beta} = -\tilde{\lambda}_{k,\beta} + \frac{3}{8\pi^2}\frac{(\tilde{\lambda}_{k,\beta})^2}{\left(1+\tilde{m}^2_{k,\beta}\right)^{\frac{3}{2}}}\,, \tag{8.24}$$

where ζ is the Riemann zeta function.

The major difference from the vacuum case comes from the β-function for $\tilde{\lambda}_{k,\beta}$, which has an additional affine contribution. In d dimensions, this contribution for the dimensionless coupling $\lambda'_k = \lambda_k k^{d-4}$ is proportional to $(d-4)\lambda'_k$; the thermal state introduces a dimensional reduction in the fourth interaction coupling. Thanks to this dimensional reduction, also in $d = 4$, there is a non-trivial fixed point for $\tilde{U}_* = \zeta(3)/2\pi^2$, $\tilde{m}^2_* = -2/5$, $\tilde{\lambda}_* = (8/3)\pi^2(1+\tilde{m}^2_*)^{3/2}$. Notice the similarity between the phase diagram for the 4D, thermal case, Fig. 8.2 and the phase diagram in the 3D, vacuum case for the Wilson-Fisher fixed point, Fig. 8.3. We will see a similar mechanism in the Bunch-Davies case.

The minus sign in the mass fixed point indicates that the symmetry $\hat{\varphi} \to -\hat{\varphi}$ is spontaneously broken in the chosen state. To make it clearer, we can now repeat the above analysis, in which the effective potential takes the simple form [236]

$$U_k = \frac{\lambda_{k,\beta}}{2}(\rho - \rho_{0,k,\beta})^2 \qquad \rho = \frac{\phi^2}{2}\,,$$

which coincides with the effective potential written in Eq. (8.11) up to a constant. The new parameter $\rho_{0,k,\beta} = \phi^2_{0,k,\beta}/2$ is the minimum of the potential, located at $U^{(1)}_k(\phi_{0,k,\beta}) = 0$. The new parameters are then linked to the old coupling constants, in particular $m^2_{k,\beta} = -\lambda_{k,\beta}\rho_{0,k,\beta}$, while $\lambda_{k,\beta}$ is scaled by a factor 3.

8.2.3 *Wilson-Fisher fixed point*

We now recover another standard result, the Wilson-Fisher fixed point for the vacuum scalar field in 3 dimensions. Just as in the 4-dimensional case, we expand the effective potential up to second order in $\rho = \phi^2/2$, setting for simplicity the $\rho-$independent constant $U_{k,0}$ to zero:

$$U_k(\rho) = -\int_x m_k^2\rho + \lambda_k \frac{\rho^2}{6}\,.$$

The vacuum 2-point function in $d \geq 2$ dimensions and with a real mass is expressed in terms of modified Bessel functions as [57]

$$\Delta_+(x) = (2\pi)^{-\frac{d}{2}} M_k^{\frac{d}{2}-1} \sigma^{\frac{2-d}{4}} K_{\frac{d}{2}-1}(\sqrt{M_k^2\sigma})\,,$$

where the mass term M_k in the presence of regulator and in the LPA reads

$$M_k^2 = m_k^2 + k^2 + \lambda_k\rho\,.$$

σ as before is the squared geodesic distance $|x-y|^2$.

In odd dimensions, the unique Hadamard parametrix is

$$H = \frac{1}{4\sin\left(\frac{d}{2}-1\right)\pi}(2\pi)^{\frac{2-d}{2}} M_k^{\frac{d}{2}-1}\sigma^{\frac{2-d}{4}} I_{1-\frac{d}{2}}(\sqrt{m^2\sigma})\,.$$

In 3 dimensions, the 2-point function and the Hadamard parametrix reduce to

$$\Delta_+ = \frac{1}{4\pi}\frac{e^{-\sqrt{M_k^2\sigma}}}{\sqrt{\sigma}}\,,\qquad H = \frac{1}{4\pi}\frac{\cosh\sqrt{M_k^2\sigma}}{\sqrt{\sigma}}\,.$$

Taking the difference of the coincidence limit gives the smooth part of the 2-point function,

$$\lim_{\sigma\to 0}\Delta_+ - H = -\frac{1}{4\pi}\sqrt{m_k^2 + k^2 + \lambda_k\rho}\,.$$

The RG flow equations in the LPA (8.7) with regulator $q_k = k^2$ then give

$$\partial_k U_k = \frac{k}{4\pi}\int_x f(x)\sqrt{m_k^2 + k^2 + \lambda_k\rho}\,.$$

The first and second derivatives with respect to ρ, at vanishing ρ provide the evolution equations for the dimensionful parameters

$$k\partial_k m_k^2 = -\frac{k^2\lambda_k}{8\pi\sqrt{k^2+m_k^2}}$$

$$k\partial_k\lambda_k = \frac{3k^2}{16\pi}\frac{\lambda_k^2}{(k^2+m_k^2)^{\frac{3}{2}}}\,.$$

In terms of the dimensionless parameters $\tilde{m}_k^2 = k^{-2}m_k^2$ and $\tilde{\lambda}_k = k^{-1}\lambda_k$, the $\beta-$functions are

$$k\partial_k\tilde{m}_k^2 = -2\tilde{m}_k^2 - \frac{\tilde{\lambda}_k}{8\pi\sqrt{1+\tilde{m}_k^2}} \tag{8.25}$$

$$k\partial_k\tilde{\lambda}_k = -\tilde{\lambda}_k + \frac{3}{16\pi}\frac{\tilde{\lambda}_k^2}{(1+\tilde{m}_k^2)^{\frac{3}{2}}}\,. \tag{8.26}$$

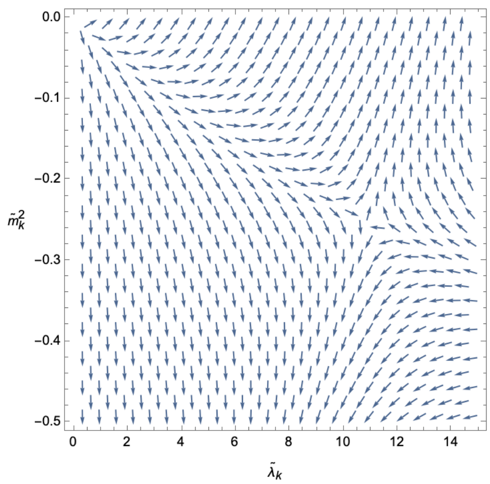

Figure 8.3: $\beta-$functions for the 3-dimensional Wilson-Fisher fixed point. The flow is in the direction of decreasing k (towards the IR).

Besides the Gaussian fixed point, the system admits a non-trivial solution for $\tilde{m}_k^{2,*} = -\frac{1}{4}$, $\tilde{\lambda}_k^* = 2\sqrt{3}\pi$. The fixed point is in the spontaneously broken phase, since the mass is negative, and qualitatively it coincides with known results. From the linearisation of the $\beta-$functions around the fixed point we can also characterise the two operators ϕ^2 and ϕ^4,

$$\left.\frac{\partial \beta_\lambda}{\partial \tilde{\lambda}_k}\right|_{\tilde{\lambda}_k^*} = 1\,, \qquad \left.\frac{\partial \beta_{m^2}}{\partial \tilde{m}_k^2}\right|_{\tilde{m}_k^{2,*}} = -\frac{5}{3}\,.$$

The critical exponents coincide with standard ones (see e.g. Ref. [206]). Since the linear coefficient in front of the mass $\beta-$function is negative, the operator ϕ^2 is *relevant*, while the operator ϕ^4 is *irrelevant* because the linearisation of the λ $\beta-$function is positive.

8.3 BUNCH-DAVIES STATE IN DE SITTER

In order to show that the same methods are applicable in the context of curved spacetimes, we study the $\lambda\varphi^4-$theory in de Sitter space. De Sitter spacetime is the four dimensional hyperboloid embedded in five dimensional flat space with the constraint

$$X^a X^b \eta_{ab} = H^{-2}\,,$$

where $H > 0$ is the Hubble constant and η is the five-dimensional flat space metric. We consider the free theory to be in the Bunch Davies state [70], which is the unique quasifree maximally symmetric state on the de Sitter spacetime; the symmetric part of its 2-point function is [8, 44]

$$\Delta_S^{\mathrm{BD},+}(x,y) = \frac{H^2}{16\pi}\frac{\left(\frac{1}{4}-\nu^2\right)}{\cos(\pi\nu)}\,{}_2F_1\left(\frac{3}{2}+\nu, \frac{3}{2}-\nu; 2; \frac{1+Z(x,y)}{2}\right)$$

where ${}_2F_1$ is the hypergeometric function, $Z(x,y) = H^2 X^a(x) X^b(y)\eta_{ab}$ is related to the geodesic distance $d(x,y) = H\cos(Z(x,y))$ between x and y, and

$$\nu = \sqrt{\frac{9}{4} - 12\xi + \frac{m^2}{H^2}}\,, \tag{8.27}$$

where m is the mass of the quantum field and ξ its coupling to the scalar curvature.

Reasoning as before in the LPA, the effective potential can be expanded in

$$U_k(\phi) = -\int_x f(x)\left[m_k^2\frac{\phi^2}{2} + \lambda_k\frac{\phi^4}{4!}\right], \qquad \rho = \frac{\phi^2}{2}\,.$$

For simplicity, ξ does not depend on k. Since we are not interested in back-reaction effects (which would cause the Hubble constant to flow as well), renormalization corrections to ξ can be included in the RG flow for the mass term.

As usual, to compute the interacting propagator in the LPA we expand up to second order the effective potential and we keep the quadratic terms, that modify the action of the bare theory. The 2-point function in the state with the mass modified by the regulator and the effective potential is given by the Møller operators,

$$r_{M+Q_k}\Delta_S^{\mathrm{BD}} r^*_{M+Q_k}\,.$$

However, instead of directly performing that computation, we observe that in the adiabatic limit the obtained states share the same symmetry properties as those of the original 2-point function, because the classical Møller map preserves the spacetime symmetry. The only maximally symmetric state in de Sitter is the Bunch Davies state, and the original state is maximally symmetric, and so the new state needs to be a Bunch Davies state too, with mass $m^2 = m_k^2 + \lambda_k\rho + k^2$.

For massive theories, with a general, non-minimal coupling ξ, the renormalized expectation value (via the Hadamard procedure) of the Wick square $\hat\varphi^2$ in this state is given by [42]

$$\omega(:\hat\varphi^2(x):) = \frac{1}{16\pi}\left\{-\frac{2H^2}{3}+\left[(m_k^2+k^2+\lambda_k\rho)+\left(\xi-\frac{1}{6}\right)12H^2\right]\left[\psi\left(\frac{3}{2}+\nu\right)+\psi\left(\frac{3}{2}-\nu\right)+\log\left(\frac{12H^2}{\mu^2}\right)\right]\right\}, \tag{8.28}$$

where $\psi(z) := \dfrac{\mathrm{d}\log\Gamma(z)}{\mathrm{d}z}$ is the digamma function, and μ is again an arbitrary mass parameter.

From $\omega(T\hat\varphi^2)$ the evolution equations can be computed taking functional derivatives of the RG flow equation (8.7) and evaluating at vanishing fields, leading to

$$\begin{aligned} k\partial_k m_k^2 &= k^2\left.\frac{\delta\omega(:\hat\varphi^2:)_{U_k,k}}{\delta\rho}\right|_{\rho=0} \\ &= \frac{k^2\lambda_k}{16\pi}\left\{\left[\psi\left(\frac{3}{2}+\nu\right)+\psi\left(\frac{3}{2}-\nu\right)+\log\left(\frac{12H^2}{\mu^2}\right)\right]\right. \\ &\quad \left. -\frac{1}{2H^2\nu}\left(2H^2(6\xi-1)+k^2+m_k^2\right)\left[\psi^{(1)}\left(\frac{3}{2}-\nu\right)-\psi^{(1)}\left(\frac{3}{2}+\nu\right)\right]\right\}, \end{aligned} \tag{8.29}$$

and

$$k\partial_k\lambda_k = 3k^2 \left.\frac{\delta^2\omega(:\hat{\varphi}^2:)_{U_k,k}}{\delta\rho\delta\rho}\right|_{\rho=0} \tag{8.30}$$

$$\begin{aligned} &= \frac{3k^2\lambda^2}{16\pi H^2\nu}\left\{\left[\psi^{(1)}\left(\frac{3}{2}+\nu\right)-\psi^{(1)}\left(\frac{3}{2}-\nu\right)\right]\right. \\ &+\frac{1}{4H^2\nu}\left(2H^2(6\xi-1)+k^2+m_k^2\right)\left[\psi^{(1)}\left(\frac{3}{2}-\nu\right)-\psi^{(1)}\left(\frac{3}{2}+\nu\right)\right. \\ &\left.\left.+\nu\left(\psi^{(2)}\left(\frac{3}{2}+\nu\right)+\psi^{(2)}\left(\frac{3}{2}-\nu\right)\right)\right]\right\} . \end{aligned} \tag{8.31}$$

The IR cut-off function f has been taken equal to 1 on the support of ρ, and ρ is constant throughout space, so that the spacetime dependence of the functional derivatives vanishes.

Now, we would like to rewrite the evolution equations in terms of dimensionless couplings. However, due to the explicit appearance of an additional dimensional constant H, the naive substitution $\tilde{m}_k^2 := k^{-2}m_k^2$, $\tilde{\lambda}_k = \lambda_k$ would produce non-autonomous equations. We thus proceed as follows. We introduce new dimensionless couplings

$$\tilde{m}_k^2 := \frac{m_k^2}{H^2}, \quad \tilde{\lambda}_k = \frac{k^2}{H^2}\lambda_k .$$

Notice that, as in the thermal case, the appearance of a dimensionful parameter (H in this case) allows for a different scaling behaviour of the coupling constants. In terms of these couplings, the effective potential becomes

$$U_k(\rho) = -\int_x H^2\left(\tilde{m}_k^2\rho + k^{-2}\tilde{\lambda}_k\frac{\rho^2}{6}\right) .$$

We proceed rescaling the field to $\rho = k^2\tilde{\rho}$, so that the effective potential becomes

$$U_k(\tilde{\rho}) = -\int_x k^2H^2\left(\tilde{m}_k^2\tilde{\rho} + \tilde{\lambda}_k\frac{\tilde{\rho}^2}{6}\right) .$$

Finally, we see that a rescaling of the spacetime coordinates produces an effective potential equivalent to the original one, but in terms of dimensionless quantities only:

$$U_k(\tilde{\rho}) = -\int_{\sqrt{kH}x}\left(\tilde{m}_k^2\tilde{\rho} + \tilde{\lambda}_k\frac{\tilde{\rho}^2}{6}\right) .$$

The $\beta-$functions for the dimensionless couplings $\tilde{m}_k^2$ and $\tilde{\lambda}_k$ can now be read off the evolution equations of the dimensional couplings, since from their definitions

$$k\partial_k\tilde{m}_k^2 = H^{-2}k\partial_k m_k^2 , \quad k\partial_k\tilde{\lambda}_k = 2\tilde{\lambda}_k + \frac{k^2}{H^2}k\partial_k\lambda_k .$$

The substitution gives the $\beta-$functions for the dimensionless couplings:

$$\begin{aligned} k\partial_k\tilde{m}_k^2 &= \frac{\tilde{\lambda}_k}{16\pi}\left\{\log\frac{12H^2}{\mu^2}+\psi\left(\frac{3}{2}+\nu\right)+\psi\left(\frac{3}{2}-\nu\right)\right. \\ &\left.-\frac{1}{2\nu}\left(\tilde{m}_k^2+2(6\xi-1)+k^2/H^2\right)\left[\psi^{(1)}\left(\frac{3}{2}-\nu\right)-\psi^{(1)}\left(\frac{3}{2}+\nu\right)\right]\right\} . \end{aligned} \tag{8.32}$$

and

$$
\begin{aligned}
k\partial_k\tilde{\lambda}_k = & \qquad\qquad (8.33)\\
& 2\tilde{\lambda}_k + \frac{3\tilde{\lambda}_k^2}{16\pi\nu}\Bigg\{\psi^{(1)}\left(\frac{3}{2}+\nu\right) - \psi^{(1)}\left(\frac{3}{2}-\nu\right)\\
& + \frac{1}{4\nu^2}\left(\tilde{m}_k^2 + 2(6\xi-1) + k^2/H^2\right)\\
& \left[\psi^{(1)}\left(\frac{3}{2}-\nu\right) - \psi^{(1)}\left(\frac{3}{2}+\nu\right) + 2\nu\left(\psi^{(2)}\left(\frac{3}{2}+\nu\right) + \psi^{(2)}\left(\frac{3}{2}-\nu\right)\right)\right]\Bigg\}.
\end{aligned}
$$

By choosing $\mu^2 = 12H^2$, we can remove all the dependence on the additional parameter μ. This is possible in de Sitter since we have a new mass scale H^{-2} which enters the flow equations as an "external parameter", and does not depend on the scale, similar to the inverse temperature β in the flow equation for thermal theories we considered in the last Section. Comparing with the Minkowski equations, we see that the first term in the β-function for $\tilde{\lambda}_k$, corresponds to an effective dimension of 2 in the flow equation for $\tilde{\lambda}_k$.

However, the $\beta-$functions are still non-autonomous, due to terms k^2/H^2 in the r.h.s. of the $\beta-$functions. To remove this dependencies, we take the limit $k^2/H^2 \to 0$, corresponding to an inflationary regime in de Sitter. In this limit, the β-functions become autonomous and read

$$
\begin{aligned}
k\partial_k\tilde{m}_k^2 \xrightarrow{k^2/H^2\to 0} & \qquad\qquad (8.34)\\
& \frac{\tilde{\lambda}_k}{16\pi}\Bigg\{\psi^{(0)}\left(\frac{3}{2}+\sqrt{\tilde{m}_k^2 - 12\xi + \frac{9}{4}}\right) + \psi^{(0)}\left(\frac{3}{2}-\sqrt{\tilde{m}_k^2 - 12\xi + \frac{9}{4}}\right)\\
& - \frac{1}{2\sqrt{\tilde{m}_k^2 - 12\xi + \frac{9}{4}}}(\tilde{m}_k^2 + 12\xi - 2)\Bigg[\psi^{(1)}\left(\frac{3}{2}-\sqrt{\tilde{m}_k^2 - 12\xi + \frac{9}{4}}\right)\\
& - \psi^{(1)}\left(\frac{3}{2}+\sqrt{\tilde{m}_k^2 - 12\xi + \frac{9}{4}}\right)\Bigg]\Bigg\}.
\end{aligned}
$$

and

$$
\begin{aligned}
k\partial_k\tilde{\lambda}_k \xrightarrow{k^2/H^2\to 0} & \qquad\qquad (8.35)\\
& 2\tilde{\lambda}_k + \frac{3\tilde{\lambda}_k^2}{64\pi\nu^2}\Bigg\{\frac{(3\tilde{m}_k^2 - 60\xi + 11)}{\nu}\Bigg[\psi^{(1)}\left(\frac{1}{2}\left(\sqrt{4\tilde{m}_k^2 - 48\xi + 9} + 3\right)\right)\\
& - \psi^{(1)}\left(\frac{1}{2}\left(3 - \sqrt{4\tilde{m}_k^2 - 48\xi + 9}\right)\right)\Bigg]\\
& + (\tilde{m}_k^2 + 12\xi - 2)\Bigg[\psi^{(2)}\left(\frac{1}{2}\left(\sqrt{4\tilde{m}_k^2 - 48\xi + 9} + 3\right)\right)\\
& + \psi^{(2)}\left(\frac{1}{2}\left(3 - \sqrt{4\tilde{m}_k^2 - 48\xi + 9}\right)\right)\Bigg]\Bigg\}.
\end{aligned}
$$

The $\beta-$functions (8.34) and (8.35) can now be analysed at different non-minimal couplings ξ. For $\xi = 1/6$, they exhibit a non-trivial fixed point in the spontaneously broken phase. It can be approximately computed by expanding the $\beta-$functions up to third order, β_{m^2} in $\tilde{m}_k^2$ and β_λ in $\tilde{\lambda}_k$. The result is

$$\begin{aligned}\beta_{m^2} = & \\ & -\frac{(\tilde{m}_k^2)^3\tilde{\lambda}_k(3+\psi^{(2)}(1)+\psi^{(2)}(2))}{4\pi}+\frac{3(\tilde{m}_k^2)^2\tilde{\lambda}_k(2+\psi^{(2)}(1)+\psi^{(2)}(2))}{32\pi} \\ & -\frac{\tilde{m}_k^2\tilde{\lambda}_k}{8\pi}+\frac{(1-2\gamma)\tilde{\lambda}_k}{16\pi}+\mathcal{O}((\tilde{m}_k^2)^4)\,,\end{aligned} \tag{8.36}$$

and

$$\begin{aligned}\beta_\lambda = {} & 2\tilde{\lambda}_k+\frac{\tilde{\lambda}_k^2}{16\pi(4\tilde{m}_k^2+1)^{3/2}} \\ & \left\{6(3\tilde{m}_k^2+1)\left[\psi^{(1)}\left(\frac{1}{2}\left(\sqrt{4\tilde{m}_k^2+1}+3\right)\right)-\psi^{(1)}\left(\frac{1}{2}\left(3-\sqrt{4\tilde{m}_k^2+1}\right)\right)\right]\right. \\ & \left.+3\tilde{m}_k^2\sqrt{4\tilde{m}_k^2+1}\left[\psi^{(2)}\left(\frac{1}{2}\left(3-\sqrt{4\tilde{m}_k^2+1}\right)\right)+\psi^{(2)}\left(\frac{1}{2}\left(\sqrt{4\tilde{m}_k^2+1}+3\right)\right)\right]\right\}+O\left(\tilde{\lambda}_k^3\right)\,.\end{aligned} \tag{8.37}$$

These approximated $\beta-$functions exhibit two fixed points, the Gaussian one and a non-trivial fixed point at

$$\tilde{m}_k^2 \to -0.0809878\,, \quad \tilde{\lambda}_k \to 18.4348\,. \tag{8.38}$$

The non-trivial fixed point appears stable under the inclusion of higher orders, in a systematic expansion of the $\beta-$functions (8.34) and (8.35).

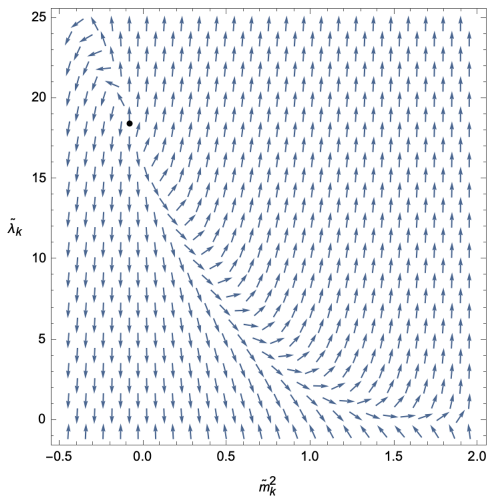

Figure 8.4: Flow diagram for the $\beta-$functions (8.34) and (8.35). The non-trivial fixed point is marked with a dot. The flow is in the direction of decreasing k (towards the IR).

8.4 ASYMPTOTIC SAFETY IN COVARIANT QUANTUM GRAVITY

As a final application, we study the RG flow of quantum gravity in a simple truncation, the *Einstein-Hilbert truncation*. To do so, we take as theoretical framework QG as a locally covariant QFT [64, 65]. In this context, gravity is quantised on a fixed, globally hyperbolic spacetime $(\mathcal{M}, \bar{g})$ with background metric $\bar{g}$; our computation is background independent in the sense that $\mathcal{M}$ is fixed, but arbitrary, thus studying the RG equations in all spacetimes at once [66].

Gravity as a locally covariant QFT can be treated with the standard BV formalism that we discussed in Chapter 3, and as such it is completely analogous to gauge theories such as Yang-Mills. Moreover, the proper solution of the CME (3.10) for gravity is linear in the antifields, i.e. it is of *Yang-Mills-type*. It follows that our formalism can handle the case of gravity with no difficulty. Here we review the main points, mainly to set the notation.

The space of off-shell configurations is $\mathscr{E}(\mathcal{M}) = \Gamma(T^*(\mathcal{M})^{\otimes 2}) \ni \hat{h}$, the space of symmetric bi-tensors. As usual, the configuration space must be extended to include the ghosts $\hat{c}$, the antighosts $\hat{\bar{c}}$, and the Nakanishi-Lautrup fields $\hat{b}$. We collect an element of the extended configuration space in the field multiplet $\varphi := \{\hat{h}, \hat{c}, \hat{b}, \hat{\bar{c}}\} \in \overline{\mathscr{E}}(\mathcal{M})$. As we discussed in Chapter 3, in the BV formalism [26–28] the configuration space is doubled to include the antifields, identified with the basis of the tangent space, $\varphi^\ddagger := \frac{\delta}{\delta\varphi}$. The classical BV algebra is thus the algebra of local functions on the odd cotangent bundle of the extended configuration space [116, 119].

The dynamics is governed by the Euler-Lagrange equations of the action

$$I := I_{EH} + I_{af} + \gamma\Psi = I_{EH} + I_{af} + I_{gh} + I_{gf} \,, \tag{8.39}$$

where $I_{EH} = 2\zeta^2 \int_{\mathcal{M}} \sqrt{-\det \hat{g}}(R(\hat{g}) - 2\Lambda)$ is the Einstein-Hilbert action in terms of the full metric $\hat{g} := \bar{g} + \hat{h}$, and $\zeta^2 = (32\pi G)^{-1}$ where G is Newton's constant. The antifield term is

$$I_{af} := \int_{\mathcal{M}} \sqrt{-\det \hat{g}} \mathcal{L}_{\hat{c}} \hat{g}^{ab} h^\ddagger_{ab} + c^b \partial_b c^a \, c^\ddagger_a + i\hat{b}^a \bar{c}^\ddagger_b \,,$$

where $\mathcal{L}_{\hat{c}}$ is the Lie derivative. The gauge-fixing Fermion Ψ in the De-Donder gauge is

$$\Psi = i \int_{\mathcal{M}} \sqrt{-\det \bar{g}} \; \hat{\bar{c}}^b (\nabla^a \hat{h}_{ab} - \frac{1}{2} \nabla_b \hat{h}_{ac} \bar{g}^{ac}) \,.$$

We recall that the BRST differential is defined as $\gamma := \{\cdot, I_{af}\}$ [29–31].

We then proceed with deformation quantization, as explained in Chapters 2 and 3. We split the action I into a term quadratic in the fields I_0 and a remaining, interacting term $V := I - I_0$. The free part I_0 is used to define the quantum products and the time-ordered products; the Epstein-Glaser renormalization procedure constructs the time-ordered products of local functions at coincidence points. Interacting observables are thus represented as formal power series in the $*$−algebra of free observables.

To define the generating functionals, we introduce the sources $J := \int_{\mathcal{M}} j_A \varphi^A$ and the classical BRST sources $\Sigma := \int_{\mathcal{M}} \sigma_A \gamma \varphi^A$. Finally, we need to introduce the regulator terms Q_k. These are chosen as local terms quadratic in the fields, acting as artificial masses in the correlation functions:

$$Q_k := -\frac{1}{2} \int_{\mathcal{M}} \sqrt{-\det \bar{g}(x)} \left[T(\hat{h}_{ab} {q_k}^{abcd} \hat{h}_{cd}) + 2T \hat{\bar{c}}_a {\tilde{q}_k}^{ab} \hat{c}_b \right] \,, \tag{8.40}$$

where T is the time-ordering operator. Together with the regulator term, we also introduce the source for its BRST variation,

$$H(\eta) := \frac{1}{2}\int_{\mathcal{M}} \sqrt{-\det \bar{g}(x)}\left[\eta(x)\,\gamma(\hat{h}_{ab}(x)\hat{h}^{ab}(x)) + \tilde{\eta}\gamma(\hat{\bar{c}}_a(x)\hat{c}^a(x))\right]\,. \tag{8.41}$$

The scale-dependent BV differential is $s_k := s + \int_{\mathcal{M}} q_{k\,A}\frac{\delta}{\delta \eta_A}$, where the BV differential is $s := \{\cdot, I\}$. The extended action $I_{ext} := I + \Sigma + Q_k + H$ then satisfies the symmetry identity (8.42), extending the BV invariance of the classical action I to

$$s_k I_{ext} = 0\,. \tag{8.42}$$

The regularised generating functional for time-ordered correlation functions is defined as in Def. 7.20,

$$Z_k(\bar{g}; j, \sigma, \eta) := \langle T \exp\{\Sigma + J + Q_k + H\}\rangle\,. \tag{8.43}$$

The Effective Average Action (EAA) $\Gamma_k(\bar{g}; \phi, \sigma, \eta)$ is defined in the standard way: it is the modified Legendre transform of the regularised generating functional of connected Green's functions $W_k = -i \log Z_k$ with respect to the sources. The EAA is thus a function of the classical fields $\phi := \{h, c, b, \bar{c}\}$, with $\phi = \langle \varphi \rangle$. Thanks to the extended symmetry (8.42), the Legendre EAA $\tilde{\Gamma}_k := \Gamma_k + Q_k(\phi)$ satisfies the extended Slavnov-Taylor identity, Eq. (7.35):

$$\int_{\mathcal{M}} \frac{1}{\sqrt{-\det \bar{g}(x)}}\left[\frac{\delta \tilde{\Gamma}_k}{\delta \sigma_A(x)}\frac{\delta \tilde{\Gamma}_k}{\delta \phi^A(x)} + q_k^A(x)\frac{\delta \tilde{\Gamma}_k}{\delta \eta^A(x)}\right] = 0\,. \tag{8.44}$$

The EAA is then constrained by the cohomology of the BRST operator γ in ghost number zero; it follows that the EAA must be a BRST-invariant functional of the full classical metric $g := \bar{g} + h$ thanks to Eq. (7.50).

8.4.1 *Renormalization Group flow equations*

The RG flow equations for gravity are derived in complete analogy with the gauge theory case and they are given by Eq. (5.6),

$$\partial_k \Gamma_k(\bar{g}; \phi) = \frac{i}{2}\int_{\mathcal{M}} \mathrm{Tr}\left\{\partial_k q_k(x) : G_k :_{\tilde{H}_F}(x, x)\right\}\,. \tag{8.45}$$

The trace is over Lorentz and field indices. The equations are written in terms of $\Gamma_k(\bar{g}, \phi) = \Gamma_k(\bar{g}, \phi, \sigma = 0, \eta = 0)$, with the field b integrated out, and the *interacting propagator*, satisfying

$$\frac{\delta^2}{\delta\phi(x)\delta\phi(z)}(\Gamma_k + Q_k)G_k(z, y) = -\delta(x, y)\mathbb{I}\,,$$

where $\mathbb{I}$ denotes an appropriate tensor identity. The normal-ordering prescription is given by a point-splitting procedure. Formally divergent quantities, such as $\langle \hat{h}^{ab}(x)\hat{h}_{cd}(x)\rangle = -iG_{k\ ab}^{hh\ \ cd}(x, x)$, are replaced by point-split expressions, such as $G_{k\ ab}^{hh\ \ a'b'}(x, y)$, for y in the vicinity of x and space-like separated. The singular terms in the coincidence limit are subtracted, obtaining the regularised corresponding quantity $: G_{k\ ab}^{hh\ \ ab} : (x, y)$.

We now assume that the operator $\Gamma_k^{(2)} - q_k$ is Green hyperbolic, with the kinetic term, apart from a possible wavefunction renormalization Z_k, given by the free part of the action: $\Gamma_k^{(2)} - q_k = Z_k D - q_k + U_k^{(2)}$, where $D = I_0^{(2)}$. In this approximation, the *effective potential* $U_k^{(2)}$ does not contain derivatives of the Dirac delta.

As we proved in Section 8.1, in this case the interacting propagator coincides with the propagator of the free theory, with a mass modified by U_k [83, 95]. Thus, for y in a normal convex neighbourhood of a given x, the interacting propagators must have the same Hadamard singularity structure of the free propagator:

$$G_k = \frac{i}{8\pi^2 \zeta_k^2} \left(H_k + W \right) \,, \tag{8.46}$$

written in terms of a smooth contribution W and the *Hadamard parametrix*, capturing its universal UV singularity structure:

$$H_k(x, y) = \frac{i}{8\pi^2 \zeta_k^2} \lim_{\epsilon \to 0^+} \left[\frac{\Delta^{1/2}}{\sigma_\epsilon(x, y)} \mathbb{I} + V \log \frac{\sigma_\epsilon(x, y)}{\mu} \right] .$$

In the last equations, $\zeta_k^2 := Z_k \zeta^2$, $\sigma(x, y)$ is the squared geodesic distance taken with sign between x and y and $\sigma_\epsilon(x, y) = \sigma(x, y) + i\epsilon$, Δ is the van-Vleck-Morette determinant.

The distributions V, W can be expanded in a covariant Taylor expansion as $V = \sum_{n=0} V_n \sigma^n$ and $W = \sum_{n=0} W_n \sigma^n$; the *Hadamard recursion relations* determine higher orders in the expansion from the zeroth order [91]. The zeroth term V_0 is completely determined by the quantum wave operator and the background geometry by the formula [143]

$$V_0 = -\frac{1}{2} \frac{\delta^2}{\delta \phi \delta \phi} (\Gamma_k + Q_k) \Delta^{1/2} \mathbb{I} \,, \tag{8.47}$$

and the coincidence values $\Delta^{1/2}(x, x) = 1$, $\nabla_a \nabla_b \Delta^{1/2}(x, x) = 1/6 \bar{R}_{ab}$ [75]. The smooth contribution W_0 remains arbitrary; once W_0 is fixed, it uniquely identify the state.

The subtraction of the Hadamard parametrix defines the normal-ordered quantity $: G_k :: = G_k - H_k$, smooth in the coincidence limit; the FRGE for Γ_k thus becomes

$$\partial_k \Gamma_k(\bar{g}; \phi) = -\frac{1}{16\pi^2 \zeta_k^2} \int_{\mathcal{M}} \mathrm{Tr} \left\{ \partial_k q_k(x) \left[S_0 + V_0 \log \frac{M^2}{\mu^2} \right] \right\} . \tag{8.48}$$

The logarithmic term $\log M^2$ is a smooth contribution coming from the arbitrary function W, and it is necessary to make the logarithm in (8.55) dimensionless; S_0 is the remaining smooth contribution in the coincidence limit.

8.4.2 *Einstein-Hilbert truncation*

The *Einstein-Hilbert truncation* assumes an Ansatz for the effective average action in the form

$$\Gamma_k(\bar{g}; \phi, \sigma, \eta) = \Gamma_k^{EH}(\bar{g}, g) + \Gamma_k^{gh}(\bar{g}, h, c, \bar{c}) + \Gamma_k^{gf}(\bar{g}, h, b, c, \bar{c}) + \Sigma(\bar{g}; \phi, \sigma) + H(\bar{g}; \phi, \eta) \,. \tag{8.49}$$

The Einstein-Hilbert contribution is

$$\Gamma_k^{EH} = 2\zeta_k^2 \int_{\mathcal{M}} \sqrt{-\det g}(R(g) - 2\Lambda_k) .$$

In terms of the fluctuation field $h := \langle \hat{h} \rangle = g - \bar{g}$, the ghost and gauge-fixing terms are

$$\Gamma_k^{gh} = \zeta_k^2 \int_{\mathcal{M}} \sqrt{-\det \bar{g}}\bar{c}_a(\bar{g}^{ab}\Box + \bar{R}^{ab}(\bar{g}))c_b ,$$
$$\Gamma_k^{gf} = -\zeta_k^2 \int_{\mathcal{M}} \sqrt{-\det \bar{g}}b^a(\nabla^b h_{ab} - \frac{1}{2}\nabla_a \bar{g}^{bc}h_{bc}) ,$$

and Σ and H correspond to the classical contributions.

The equations for the interacting propagators are derived expanding the effective average action up to second-order in a Taylor expansion in the fluctuation field h, $\Gamma_k(\bar{g}+h) = \Gamma_k(\bar{g}) + \mathcal{O}(h) + \Gamma_k^{\text{quad}}(h,\bar{g})$; the contribution $\Gamma_k^{\text{quad}}(h,\bar{g})$ gives raise to a h-independent, but background dependent quantum wave operator.

We can now specify the regulator terms q_k, $\tilde{q_k}$. They are chosen to act as artificial masses for the fields, dressing the d'Alembertians as $\Box \to \Box - k^2$:

$$q_k{}^{ab}{}_{cd} = \zeta_k^2 k^2 K^{ab}{}_{cd} , \quad \tilde{q}_{k\,ab} = \zeta_k^2 k^2 \bar{g}_{ab} , \tag{8.50}$$

where $K_{abcd} = 1/2(\bar{g}_{ac}\bar{g}_{bd} + \bar{g}_{bc}\bar{g}_{ad} - \bar{g}^{ab}\bar{g}_{cd})$. More explicitly, the regulator matrix is

$$k\partial_k q_k = (2\zeta_k^2 k^2 + k^2\, k\partial_k \zeta_k^2)\begin{pmatrix} I_{ab}{}^{cd} & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & \bar{g}_{ab} \\ 0 & 0 & -\bar{g}_{ab} & 0 \end{pmatrix} . \tag{8.51}$$

The graviton propagator may be decomposed in the sum of a tensor G_k^T and a scalar $G_k^S = \bar{g}^{ab}g_{c'd'}G_k^{hh}{}_{ab}{}^{cd}$ contribution [9]. The equations of motion then read

$$\zeta_k^2[\bar{g}_{ac}\bar{g}_{bd}\left(\Box - k^2 + 2\Lambda_k - \frac{1}{2}\bar{R}\right) - P_{abcd}]G_k^{T\,abc'd'} \tag{8.52}$$
$$= -\frac{1}{2}\left(\bar{g}_c{}^{c'}\bar{g}_d{}^{d'} + \bar{g}_d{}^{c'}\bar{g}_c{}^{d'} - \bar{g}_{cd}\bar{g}^{c'd'}\right)\delta(x,y)$$
$$-\frac{\zeta_k^2}{2}(\Box - k^2 + 2\Lambda_k)G_k^S = -\delta(x,y) , \tag{8.53}$$
$$\zeta_k^2\left[\bar{g}_{ab}(\Box - k^2) + \bar{R}_{ab}\right]\tilde{G}_k^{ab'} = -\bar{g}_b{}^{b'}\delta(x,y) . \tag{8.54}$$

The tensor $P_{ab}{}^{cd} := -2\bar{R}_{(a}{}^{c}{}_{b)}{}^{d} - 2\bar{g}^{(c}{}_{(a}\bar{R}^{d)}{}_{b)} + \bar{g}^{cd}\bar{R}_{ab} + \bar{g}_{ab}\bar{R}^{cd}$ is a potential term. In the last relation, all curvature quantities are constructed from the background metric $\bar{g}$; the d'Alembertian is $\Box = \bar{g}(\nabla,\nabla)$.

Each propagator has a corresponding Hadamard expansion:

$$G_k^S = -\frac{i}{4\pi^2\zeta_k^2}\left\{H_k^S + V_0^S \log M_S^2 + S_0^S\right\} \tag{8.55}$$

$$G_k^{T\,abc'd'} = \frac{i}{8\pi^2\zeta_k^2}\left\{H_k^{T\,abc'd'} + V_0^{T\,abc'd'}\log M_T^2 + S_0^{T\,abc'd'}\right\} \tag{8.56}$$

$$\tilde{G}_k^{ab'} = \frac{i}{8\pi^2\zeta_k^2}\left\{\tilde{H}_k^{ab'} + \tilde{V}_0{}^{ab'}\log\tilde{M}^2 + \tilde{S}_0{}^{ab'}\right\} , \tag{8.57}$$

The terms V_0^T, V_0^S, and $\tilde{V}_0$ arising from the equations (8.52)-(8.54) can be computed from Eq. (8.47); they are given by [9, 34]

$$V_0^S = \frac{1}{2}(k^2 - 2\Lambda_k) - \frac{1}{12}\bar{R} \tag{8.58}$$

$$V^T{}_{0ab}{}^{cd} = \frac{1}{2}\left(k^2 - 2\Lambda_k + \frac{1}{3}\bar{R}\right) K_{ab}{}^{cd} + \frac{1}{2}(P_{ab}{}^{cd} - \frac{1}{2}\bar{g}^{cd}P_{abe}{}^{e}) \tag{8.59}$$

$$\tilde{V}_0{}^{ab} = -\frac{1}{12}\bar{g}^{ab}\bar{R} + \frac{1}{2}(k^2\bar{g}^{ab} - \bar{R}^{ab}) \,. \tag{8.60}$$

8.4.3 *Universal terms and state dependence*

The RG equations (8.48) depends on the choice of a state. This is the main difficulty in applying the Lorentzian RG equations, in comparison with their Euclidean counterpart. In particular, Hadamard states for the graviton are not known in general spacetimes, but only in specific geometries [7, 10, 36, 111, 125, 130, 237]. The construction of a Hadamard vacuum state for the graviton is well beyond the scope of this short note. Thus, here we take into account only universal contributions to the evolution equations, that are present in any Hadamard state and in all backgrounds. The evaluation of state-dependent contributions is possible only selecting a class of backgrounds, and it will be addressed in future works.

To solve the FRGE (8.48), we need to evaluate S_0^S, M_S^2, S_0, M_T^2 and $\tilde{S}_0$, $\tilde{M}^2$. First of all, the smooth functions S_0^S, S_0, $\tilde{S}_0$ vanish in the flat space limit [10, 168]. Moreover, any $k-$independent term can be removed by a re-definition of the effective average action, while terms proportional to the scale k can be removed by an appropriate choice of the renormalization ambiguities [158, 160, 161]. Since the remaining contributions are completely state dependent, here we neglect S_0^S, S_0 and $\tilde{S}_0$.

On the other hand, while the specific expressions for the functions M_T^2, M_S^2, $\tilde{M}^2$ are state-dependent, they must be present in any Hadamard state. They are functions of mass dimension 2, analytic in the physical parameters. The only dimension-2 term in the Hadamard expansion for the interacting propagator is V_0; we thus choose

$$M_S^2 = V_0^S \,, \quad M_T^2 = V^T{}_{0ab}{}^{cd} I^{ab}{}_{cd} \,, \quad \tilde{M}^2 = \tilde{V}_0{}^{ab}\bar{g}_{ab} \,,$$

where $I_{abcd} = 1/2(\bar{g}_{ac}\bar{g}_{bd} + \bar{g}_{bc}\bar{g}_{ad})$ is the identity for symmetric four-tensors.

These choices completely fix W_0 , $\tilde{W}_0$ and thus they fix a vacuum-like state through the Hadamard recursion relations. In the case of the scalar field, this choice coincides with the Minkowski vacuum state [83].

The last term to be fixed is the arbitrary mass μ. Contrary to the mass terms M_T^2, M_S^2, and $\tilde{M}^2$, depending on the choice of the state, this term is actually an arbitrary mass contribution coming from the choice of the Hadamard parametrix. Thus, we are free to choose a running Hadamard mass $\mu = k^2$, adjusting the UV regularisation to the renormalization scale k.

With these choices, the FRGE (8.48) is written in terms of state-independent, universal quantities. Of course, state-dependent terms in specific backgrounds can significantly alter the FRGE.

8.4.4 *Phase diagram*

We can now compute the $\beta-$functions for the dimensionless constants g_k and λ_k, related to the dimensionful running Newton's and cosmological constants by

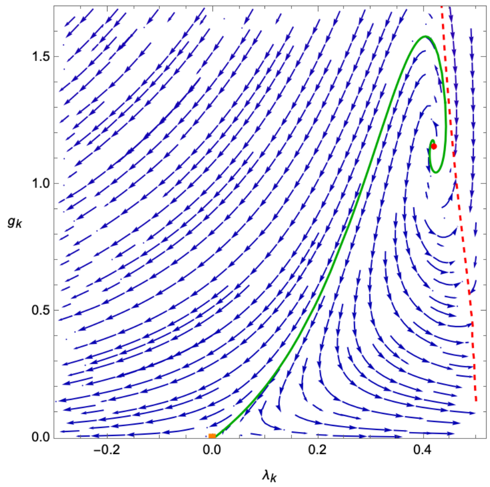

Figure 8.5: Phase diagram obtained by numerical integration of the β−functions (8.61)-(8.62), with the distinguished *Great Wave off Kawanaga* shape [154]. The solid line is the separatrix, connecting the non-Gaussian fixed point (circle) to the Gaussian one (square); the dashed line denotes the locus where η_N diverges.

canonical rescalings:

$$(32\pi\zeta_k^2)^{-1} = G_k = k^{-2}g_k\,, \quad \Lambda_k = k^2\lambda_k\,.$$

Expanding the r.h.s of (8.48) up to first order in the background Ricci scalar $\bar{R}$ then gives the β−functions for the dimensionless couplings g_k and λ_k,

$$k\partial_k g_k = (\eta_N + 2)g_k \tag{8.61}$$

$$k\partial_k\lambda_k = -(2-\eta_N)\lambda_k + \frac{g_k}{4\pi}\Bigg\{4(\log 16 - \eta_N\log 4) + (2-\eta_N) \tag{8.62}$$

$$(1-2\lambda_k)\left[8\log[4(1-2\lambda_k)] + \log[\frac{1}{2}(1-2\lambda_k)]\right]\Bigg\}\,,$$

in terms of the anomalous dimension $\eta_N := G_k^{-1}k\partial_k G_k$:

$$\eta_N(g_k,\lambda_k) = \frac{g_k}{6\pi}\frac{27\log(1-2\lambda_k) + 7 + 37\log 2}{1 + \frac{g_k}{12\pi}\left(37\log 2 + 27\log(1-2\lambda_k)\right)}\,. \tag{8.63}$$

The flow exhibits one non-trivial fixed point for positive g_k and λ_k, and it interrupts at $\lambda_k = 1/2$, where the logarithms diverge; the fixed point $g_* = 1.15$, $\lambda_* = 0.42$ realises the analogue of the Reuter fixed point in Lorentzian spacetimes. The product $g_*\lambda_* = 0.48$, known to be more stable under changes in the renormalization scheme, can be compared to the value found in the Euclidean case [215, 220], that is $g_*^E\lambda_*^E \approx 0.13$. The computation in the Lorentzian case with the ADM formalism gives $g_*^E\lambda_*^E \approx 0.06$ [184]. The critical coefficients for the Lorentzian fixed point are a pair of complex conjugate values, $\theta_{1,2} = 5.11 \pm 11.59i$; therefore λ_k and g_k are two relevant directions, agreeing again with Euclidean results. The fixed point $(g_*,\ \lambda_*)$ thus provides a realisation of the AS scenario in Lorentzian spacetimes.

The novel RG framework allows for an investigation of Lorentzian flows in a non-perturbative regime for gravity. In this short computation, we have seen that the contribution of universal, background independent terms in the flow of the

Einstein-Hilbert truncation supports the evidence that gravity is non-perturbatively renormalizable also in the Lorentzian case.

To preserve background independence, we have restricted our attention to contributions to the flow coming only from universal terms. The important question now is if the non-trivial fixed point persists when state-dependent terms are taken into account. The investigation of state-dependent terms, however, requires to select a background. The Lorentzian FRGE (8.48) then allows for a systematic investigation of these state-dependent contributions in specific background geometries.

On the other hand, the universal RG flow that we studied here allows for quick generalisations to more advanced truncations: in fact, in any given truncation, it is sufficient to compute the V_0 terms in the Hadamard expansion from Eq. (8.47) to investigate the universal contributions to the RG flow.

Finally, while the EAA is a gauge-dependent quantity, gauge-invariant relational observables have been already studied in the context of locally covariant QG [62, 121–123] and in Euclidean fRG flows [15]. In future works, we plan to investigate the RG flow of gauge-invariant observables in Lorentzian quantum gravity.

9 *Relative entropy and dynamical black holes*

9.1 A QUANTUM CUP OF TEA

In this final Chapter we leave the intricacies of interacting quantum field theories for the simpler model of a free scalar field, propagating on a curved spacetime. Despite its simplicity, even the toy model of a free scalar shows some remarkable properties, in particular in its interactions with the underlying geometry. In fact, if the background has been a non-dynamical spectator up until this point in the discussion, in this final Chapter we shall study the perturbations of the metric of a dynamical black hole under the influence of the scalar field. The interplay between black holes and scalar fields gives raise to an intriguing scenario, in which every area of physics plays a fundamental act: quantum physics and gravity, special relativity and thermodynamics, which was first discovered by the pioneering work of Hawking [146].

Nature enjoys the rich and the subtle, and gravity makes no exception: General Relativity is perhaps the best example of a theory hiding its best secrets *because of its essential loftiness, but not by means of ruse* [194]. In General Relativity, gravity and matter fields dance together a choreography that, giving shape to the Universe, determines the dynamical evolution of matter. The study of classical General Relativity already unravels spectacular phenomena, the most famous of which are without a doubt black holes. At a first glance, black holes are actually the simplest objects in the Universe: in fact, black hole solutions are characterised by three parameters only, their mass, their electric charge, and their angular momentum. However, the laws of black holes mechanics display a remarkable similarity with the principles of thermodynamics, suggesting that black holes, far from being the simple objects that they appear, are actually complicated thermodynamical actors interacting with continuous exchanges of energy with the surrounding matter. If black holes can have exchanges of energy, it is natural to wonder if they can also have thermodynamical properties, such as a temperature or entropy.

The idea to assign an entropy dates back to the beginning of the '70s, when professor John Wheeler and his then-graduate student, Jacob Bekenstein, were discussing over a cup of tea [199] the idea that black holes can be described by a handful of parameters only: in their own words, that black holes have "no hair" [189]. "What would happen", asked the professor, "if I throw this cup of tea into a black hole?"

General Relativity would say that the black hole would bear no trace of the cup of tea's microscopic properties, in striking contrast with the second law of Thermodynamics: if the black hole has no temperature nor entropy, the entropy of the Universe would decrease every time an object falls behind its horizon.

For this reason, Bekenstein proposed that black holes do indeed carry an entropy, which would compensate the loss of entropy due to matter falling behind the horizon. In fact, black hole entropy was already been introduced as a useful mathematical analogy between the laws of black hole mechanics and the laws of thermodynamics [19], but it did not have a true physical meaning. Thanks to a series of *gedanken experiments*, Bekenstein was able to predict that the entropy must be proportional to the black hole horizon area [32], with a proportionality factor of approximately $\eta = \frac{1}{2} \ln 2$ [33].

Bekenstein's theoretical insight was put on a firm ground by Hawking. He substituted the cup of tea with a *quantum cup of tea*; that is, he considered the effects of a black hole on the propagation of a quantum field, performing the computation in a semi-classical context. He showed that black holes do indeed emit a thermal radiation of particles with a black-body spectrum and a well-defined temperature [146, 147]; thanks to the analogy between the laws of black hole mechanics and the laws of thermodynamics [19], this in turn fixed the entropy of a black hole to one-fourth of the horizon area.

However, far from being resolved, the puzzle of black hole thermodynamics becomes just more mysterious after Hawking's discovery. In fact, thanks to Boltzmann law, we know that entropy counts the microscopic states of a macroscopic thermodynamical system; however, in General Relativity a black hole has no microstates, since it is described by three parameters only. It is then natural to look for the black hole microstates in terms of quantum degrees of freedom. The first proposal in this sense has been made by Bombelli and his collaborators [46]: their idea was that the black hole entropy comes from the entanglement entropy between the quantum degrees of freedom behind the event horizon, and those outside the black hole. Starting from a pure quantum state over the entire spacetime, defined assigning a free particle to each node on a lattice with spacing ϵ, the trace over degrees of freedom outside the black hole defines a *reduced density matrix* that accounts for the degrees of freedom inside the black hole only. The entanglement entropy is then computed as a von Neumann entropy for the reduced density matrix.

The entanglement entropy is indeed proportional to the area of the entangling surface, but it is divergent in the continuum limit $\epsilon \longrightarrow 0$, and it depends on the regularisation scheme. It also depends on the number of fields present in the model (see e.g. [157] for a review on the entanglement measures in the algebraic approach to QFT, and [228] for the applications to black hole entropy). For these limitations, it was argued (as Bekenstein originally stated [33]) that to correctly take into account the microscopic gravitational degrees of freedom, a quantum theory of gravity was needed: for example, string theory [234] or the more indirect approaches based on AdS/CFT, as in [107]. Indeed, both in the context of string theory [232] and loop quantum gravity [224], it has been possible to reproduce the Bekenstein-Hawking entropy from the counting of quantum-gravitational degrees of freedom. Both theories, however, apart from technical difficulties (in string theory, only the entropy of certain classes of supersymmetric black holes has been computed; in loop quantum gravity, the entropy is computed in terms of the Barbero-Immirzi parameter, which must be tuned to a peculiar value), make speculative assumptions on the nature of quantum gravity, which have not been experimentally confirmed yet.

In this Chapter, we prove an area law for the black hole entropy in a spirit close to Bekenstein's original proposal, without the mathematical analogy between the laws of black hole mechanics and the principles of thermodynamics, and without assumptions on the microscopic nature of quantum gravity. We will follow Hawking's original idea, of computing the effects of the entropy of a quantum field on the metric of a dynamical black hole, in the semi-classical context of a quantum field propagating on a curved spacetime.

In general QFTs, entropic measures are difficult to define and perhaps even harder to explicitly compute. In fact, the algebras arising in free QFT are generally von Neumann factors of type III, which do not admit a trace operation; it follows that the von Neumann entropy cannot be computed in this context. Here, we use results on the relative entropy between coherent states in flat spacetime developed in [73] and [183]. The Araki formula [13] defines the relative entropy as the expectation value of a particular operator, constructed using the Tomita-Takesaki theory of modular automorphisms of a von Neumann algebra [235], [140]. The Araki formula holds in general von Neumann algebras, so it is a suitable definition also in the context of QFTCS. In this regard, relative entropy completely avoids the divergences of entanglement entropy in the continuum limit, which arises because the continuum limit of Quantum Mechanics (generally described by a type I factor) is a QFT. Moreover, in Quantum Mechanics, the relative entropy reduces to the entanglement entropy formula, regularised subtracting a vacuum contribution.

We will also make extensive use of the geometric description of dynamical black hole backgrounds [3, 151].

The strength of this approach comes from the fact that it does not make use of unknown physics, since it is only based on QFTCS and General Relativity, and it does not rely on the mathematical analogy between the laws of black hole mechanics and the laws of Thermodynamics, which is based on a quasi-stationary approach; instead, we directly compute the black hole contribution to the relative entropy, that is, we compute the variation in the black hole geometrical quantities as the relative entropy of the matter fields coupled to the curvature varies in the black hole exterior.

Our starting point is the analysis made in [155] for a static black hole. In their work, Hollands and Ishibashi considered a free, massless, scalar field propagating over a Schwarzschild background, and they computed the relative entropy between a suitably defined ground state and a classical wave, treated as a coherent state which acted as a second order perturbation of the metric. The classical theory is defined as a solution of a Cauchy problem for the Klein-Gordon equation, for which one needs to give Cauchy data on some space-like Cauchy surface [239]. Hollands and Ishibashi chose as partial Cauchy surface a region of the event horizon and a region of the conformal future infinity, thus giving Cauchy data in the distant future. The ground state is given as the vacuum with respect to the affine parameters along the event horizon and the conformal future infinity, and they computed the relative entropy between the ground state and a coherent perturbation, using the Araki formula for coherent perturbations [73, 183]. The relative entropy is found to be the sum of two integrals, respectively over the event horizon and at future null infinity. Using the Raychaudhuri equation, to compute the evolution of the null geodesic generators of the event horizon [239], they showed that the event horizon integral can be rewritten as one-fourth of the event horizon area. They were able to find an evolution equation in the form of $\frac{\mathrm{d}}{\mathrm{d}t}\left(S + \frac{1}{4}\delta^2 A\right) = 2\pi F$, where S is the matter relative entropy, $\delta^2 A$ is the second order perturbation of the horizon area due to the scalar and gravitational perturbations, and F is the matter/gravitational radiation flux at infinity.

Here we follow the procedure outlined above, generalised to the case of a spher-

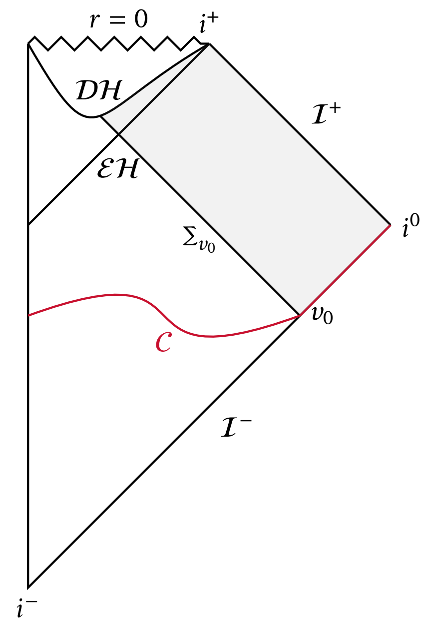

Figure 9.1: Penrose diagram of the dynamical black hole.

ically symmetric, asymptotically flat, dynamical black hole. We do not assume any energy condition on the source matter, so that the black hole can either grow due to infalling matter, or evaporate via Hawking radiation. Such a spacetime develops a dynamical horizon which does not coincide with the usual event horizon of a stationary black hole. Here, we consider the notion of dynamical horizon as the outer, future, trapping surface, in the sense of Hayward [151]; we will give more details on the definition in Section 9.2.

We then consider a free, massless, scalar field propagating over the background, giving initial data on a suitable Cauchy surface. Since the theory does not depend on the Cauchy surface, but it is influenced on its causal development only, we can consider the limit in which a part of the Cauchy surface coincides with a part of past null infinity. Therefore, we give initial data in the distant past, and we observe the evolution of the scalar field in time, as it is most natural from a physical point of view. The advantage of this approach is that we do not need to assume that the horizon is space-like or null, and thus our analysis applies both in the case of a black hole in formation or in evaporation.

A schematic Penrose diagram is given in figure 9.1. For simplicity, we consider a Schwarzschild black hole for $v < v_0$, which dynamically evolves from the instant v_0 on due to the presence of a non-vanishing energy-momentum tensor. The black hole develops a dynamical horizon $\mathcal{DH}$ inside the event horizon $\mathcal{EH}$, which evolves according to the presence of matter. We denoted conformal infinity with standard notation. The Cauchy surface C for the propagation of the quantum field is in red; we give non-vanishing initial data for the perturbation on the region lying at past null infinity. To prove our result, we will consider the flux of the Kodama current ([174], see Section 9.2.3) across the four boundaries of the shaded region.

We choose a ground state for the quantum field which is vacuum with respect to modes incoming from past infinity, and regular (in Hadamard sense [171], see Section 2.5.1 for the definition) otherwise. We will call such a state the ground state.

Our model is then independent on the details of the collapse of the black hole. We then consider a classical, massless wave, which acts as a perturbation on the geometry and as a coherent state with respect to the vacuum, which is switched on at the instant v_0; in other words, we give non-vanishing initial data, with spatially compact support, on the part of the Cauchy surface lying on past null infinity, $\mathcal{I}^-(v_0)$.

Assuming that the Klein-Gordon wave has initial data with spatially compact support, the second-order perturbation $\delta^2 g = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} g\Big|_{\lambda=0}$ goes to zero at space-like infinity i^0. Since the space is asymptotically a Schwarzschild background, we are able to use the same formula as in [155] for the relative entropy between the coherent state and the ground state, which is given in terms of integrals along the Cauchy surface. Furthermore, we can use the decaying properties of the scalar field at infinity for a Schwarzschild background [88]. Using the formulas for the relative entropy between coherent states [73, 183], we get an expression for the derivative of the relative entropy along a vector tangent to past null infinity, (9.2).

We then need to connect the variation of relative entropy with the variation of the dynamical horizon, as the classical wave $\hat{\varphi}$ perturbs the black hole. To this end, we make use of the Kodama vector [174]. The Kodama vector is a divergence-free vector, which in spherically symmetric spacetimes takes the same role as a time-like Killing vector in static spacetimes. It reduces to a time-like Killing vector at infinity, its integral parameter can be taken as a preferred time coordinate [3], and its contraction with a conserved energy-momentum tensor give raise to a conserved current. We consider the conserved current constructed from the energy momentum-tensor of the coherent wave. In the exterior of the black hole, we consider a region whose boundaries are given by the null hypersurface Σ_{v_0} at $v = v_0$, the future null, time-like, and space-like infinity $\mathcal{I}^+$, i^+, and i^0, the region of past null infinity with $v \geq v_0$, denoted by $\mathcal{I}^-(v_0)$, and the corresponding region of dynamical horizon $\mathcal{DH}(v_0)$; the region is coloured in grey in figure 9.1. Using the Stokes theorem [239], we can convert the conservation law for the Kodama current into an equation for its flux across the four boundaries. We then proceed to show that the flux term across past null infinity equals the variation of the relative entropy between the coherent wave and the ground state, with respect to a rigid translation of v_0 in the v direction. On the other hand, the flux term computed on the dynamical horizon can be written as the derivative, in the v direction, of one quarter of the perturbation of the horizon area caused by the coherent wave. Our main result is stated in Eq. (9.65): we find that, to a variation of the relative entropy of the quantum matter, we can associate a variation of one-fourth of the dynamical horizon area; it is therefore natural to interpret the area as the entropy of the black hole. The result thus generalises that of [155] to the case of dynamical, spherically symmetric, asymptotically flat black holes.

9.2 GEOMETRIC SETUP

9.2.1 *Warped-product spacetimes*

Our model for a dynamical black hole is a spherically symmetric Lorentzian spacetime $\mathcal{M}$, whose metric g can always be written as a warped product between the radial-temporal plane and the 2-spheres of symmetry, $\mathcal{M} = \mathcal{M}_2 \times S_2$. The metric admits a similar decomposition [3]

$$\mathrm{d}s^2 = x_{ab}\mathrm{d}x^a\mathrm{d}x^b = \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j + r^2(x)\sigma_{\alpha\beta}\mathrm{d}\theta^\alpha\mathrm{d}\theta^\beta\ , \tag{9.1}$$

where $\sigma_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta} = \mathrm{d}\theta^2 + \sin\theta\mathrm{d}\hat{\varphi}^2$ is the angular metric on the spheres of symmetry, and γ_{ij} is the metric on the radial-temporal plane; r is a function of the planar coordinates. Indices from the beginning of the Latin alphabet $(a, b, c, \ldots)$ denote the coordinates on the whole spacetime, indices from the middle of the Latin alphabet $(i, j, k, \ldots)$ are for the planar components, and indices from the Greek alphabet $(\alpha, \beta, \gamma, \ldots)$ denote the angular components. The metric highlights the warped product structure of a spherically symmetric spacetime, but the particular choice of planar and angular coordinates remains arbitrary; later, we will make a particular choice of coordinates to perform explicit computations, Eq. (9.6). Assuming asymptotic flatness the metric reduces to the standard Schwarzschild metric at infinity, and so it exhibits the same conformal structure at the boundary of a static black hole. By a conformal embedding we can identify the spacetime boundaries with past and future null infinities $\mathcal{I}^{\pm}$, time-like infinities $i^{\pm}$, and the space-like infinity i^0. An asymptotically flat dynamical black hole thus describes the formation or evaporation of a black hole influenced by some local distribution of matter, isolated from the rest of the Universe.

In spherical symmetry, it is possible to define the Israel-Hawking/Misner-Sharp quasi-local mass

$$m = \frac{r}{2}(1 - \nabla^a r \nabla_a r) \,. \tag{9.2}$$

The Misner-Sharp mass satisfies the correct Newtonian, small-sphere, large-sphere, special-relativistic, and test particle limits, and thus represents a good notion of gravitational mass in spherical symmetry [150].

Using the Misner-Sharp mass and the metric decomposition (9.1), the Einstein tensor takes the expression

$$G_{ij} = -\frac{2\nabla_i\nabla_j r}{r} + \left(\frac{2\Box r}{r} - \frac{2m}{r^3}\right)\gamma_{ij} \tag{9.3}$$

$$G_{i\alpha} = 0 \tag{9.4}$$

$$G_{\alpha\beta} = \left(-\frac{r^2}{2}R_{ij}\gamma^{ij} + r\Box r\right)\sigma_{\alpha\beta} \tag{9.5}$$

where R_{ij} is the planar component of the Ricci tensor.

In order to make explicit computations, we will choose coordinates so that the metric takes the form

$$\mathrm{d}s^2 = -e^{2\gamma(v,r)}f(v,r)\mathrm{d}v^2 + 2e^{\gamma(v,r)}\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2 \,. \tag{9.6}$$

$\gamma(v,r)$ is an arbitrary function of the coordinates, while $f(v,r)$ can be written in terms of the Misner-Sharp mass as

$$f(v,r) = 1 - \frac{2m(v,r)}{r} \,. \tag{9.7}$$

The assumption of asymptotic flatness implies that $\gamma \to 0$ and $f \to 1$ at infinity, so that the metric reduces to an asymptotically Schwarzschild spacetime.

Finally, we introduce the null vector field

$$l = e^{-\gamma(v,r)}\partial_v + \frac{f(v,r)}{2}\partial_r \,, \tag{9.8}$$

which describe the outgoing radial light-rays on the background metric.

9.2.2 Dynamical horizons

An event horizon is by definition a global feature of a spacetime [148]. To locate it, an observer should sit at infinite future, collect all light-rays coming from all regions of the Universe, and identify the event horizon as the boundary of the regions from which light-rays did not escape. While this definition provided dramatic insight in the causal structure of General Relativity, in dynamical settings, and more so for practical observations, the event horizon is not well-suited. Rather, in the literature it is possible to find different notions of horizons adapted to describes the dynamics of a black hole that evolve under in-falling matter [14, 49]. Here we follow the definition given by Hayward, based on marginal spheres [151].

A sphere is said to be *untrapped*, *trapped*, or *marginal* if $\nabla^a r$ is respectively spatial, temporal, or null. A hypersurface foliated by marginal spheres is called a *trapping horizon*, which can be outer, degenerate, or inner if $\Box r > 0$, $\Box r = 0$, or $\Box r < 0$ respectively, where $\Box = g^{ab}\nabla_a\nabla_b$. If $\nabla^a r$ is future- (past-) directed, the trapping horizon is said to be future (past). A *dynamical horizon* is then an outer, future trapping horizon, and defines the black hole boundary. Such a definition provides a local definition of a black hole, based on local measurements on the light-rays. The dynamical horizon is space-like if the black hole is growing, time-like if it is evaporating, and null if it is stationary; in this last case it coincides with the event horizon for stationary black holes. However, a dynamical horizon evolves according to the dynamics of local matter distributions, in contradistinction with the event horizon; their different behaviours can be analysed in detail in specific examples, as Vaidya backgrounds [50].

In the coordinates (9.6), the dynamical horizon is located at $f = 0$; the radial coordinate at the dynamical horizon thus satisfies the implicit equation $2m(v, r_{\mathcal{DH}}) = r_{\mathcal{DH}}$.

9.2.3 Kodama conservation law

In dynamical spacetimes, there is not an a priori preferred direction of time, since it is not possible to define an asymptotically time-like Killing vector field. However, in spherically symmetric spacetimes it is possible to introduce a vector that, while not being a Killing vector, shares many of the important properties of a time-like Killing vector field, and in particular has an associated conservation law [3, 174].

Definition 9.1 (Kodama vector)**.**

$$k^i = \epsilon^{ij}\nabla_j r\ ,$$

where ϵ^{ij} denotes the volume form (Levi-Civita tensor) associated with the planar metric γ_{ij}, $\epsilon^{ij} = \sqrt{-\det g}\,\tilde{\epsilon}^{ij}$, with $\tilde{\epsilon}^{ij}$ the Levi-Civita symbol.

In coordinates, the Kodama vector is

$$k = e^{-\gamma}\partial_v\ . \tag{9.9}$$

From the definition, it is possible to see that by symmetry

$$k_a\nabla^a r = 0\ . \tag{9.10}$$

The Kodama vector is divergence-free for symmetry, as can be seen by a short computation:

$$\nabla_i k^i = \nabla_i(\epsilon^{ij}\nabla_j r) = \epsilon^{ab}\nabla_a\nabla_b r + \nabla_a\epsilon^{ab}\nabla_b r = 0\ . \tag{9.11}$$

The first term vanishes because it is the product of a symmetric tensor and an antisymmetric tensor; the second term vanishes because the Levi-Civita symbol is constant, and the covariant derivative of the metric determinant vanishes, since it is proportional to the covariant derivative of the metric itself.

The conserved charge associated with the conservation law of the Kodama vector is just the areal volume $V = \frac{4\pi}{3}r^3$,

$$V := \int_\Sigma k_a \mathrm{d}\Sigma^a \,, \tag{9.12}$$

where $\mathrm{d}\Sigma^a$ is the oriented surface element of a 3-hypersurface, $\mathrm{d}\Sigma^a = n^a |\det h|^{1/2}\, \mathrm{d}^3 y$, with n^a the unit normal vector to the hypersurface, h the induced 3-metric, and y^a the induced coordinates on the hypersurface.

The miracle of the Kodama vector is that there is another conserved current, the *Kodama current*, defined by the contraction of the Kodama vector with the Einstein tensor:

$$J_a := \frac{1}{8\pi} G_{ab} k^b \,. \tag{9.13}$$

Naturally, the Kodama current can be written in terms of the stress-energy tensor via the Einstein's equations, $J_a := T_{ab}k^b$.

Lemma 9.1. *The conserved charge associated with the Kodama conservation law is then the Misner-Sharp mass [151],*

$$m(q) - m(p) = -\int_\Sigma J_a \mathrm{d}\Sigma^a \,, \tag{9.14}$$

where the integral is performed from the point p to the point q on a hypersurface Σ.

Proof. By definition, the contraction of the Einstein's tensor with the Kodama vector is

$$G_{ij}k^j = -\frac{2}{r}\nabla_i(\nabla_j r)\, \epsilon^{jk}\nabla_k r + \left(\frac{2}{r}\Box r - \frac{2m}{r^3}\right)k_i \,. \tag{9.15}$$

The result relies on the identity

$$\epsilon_{[jk}\nabla_{i]} r = 0 \,. \tag{9.16}$$

Since the radial-temporal covariant derivative of the 2-dimensional volume form vanishes, we can write

$$\begin{aligned}\nabla_j(\nabla_i r)\epsilon^{jk}\nabla_k r &= \nabla_j(\epsilon^{jk}\nabla_i r)\nabla_k r = \\ &= -\nabla^k r \nabla^j \left(\epsilon_{ki}\nabla_j r + \epsilon_{ij}\nabla_k r\right) = k_i \Box r - \frac{1}{2}\epsilon_{ij}\nabla^j(\nabla_k r \nabla^k r) \,.\end{aligned} \tag{9.17}$$

Using the definition of the Misner-Sharp mass (9.2), we find

$$G_{ij}k^j = -\frac{1}{r}\epsilon_{ij}\nabla^j(\frac{2m}{r}) - \frac{2m}{r^3}\epsilon_{ij}\nabla^j r = -\frac{2}{r^2}\epsilon_{ij}\nabla^j m \,, \tag{9.18}$$

Using the Einstein equations for the background energy-momentum tensor, and using the symmetries to extend the relation to the full spacetime, we obtain

$$J_a = -\epsilon_{ab}\frac{\nabla^b m}{4\pi r^2} \,. \tag{9.19}$$

□

Notice that the proof relies only on the geometrical properties of the Einstein's tensor, associated with the warped product structure of spherically symmetric spacetimes.

The Kodama vector reduces to the Killing field at infinity, where the spacetime is stationary, and its conserved current is naturally interpreted as the energy of the system, in analogy with the usual Minkowski case in which energy is the Noether charge associated to time translations. Moreover, from its definition we see immediately that the Kodama vector becomes null on the dynamical horizon, just as the Killing vector is null on the event horizon. It thus provides a notion analogous to the time-like Killing vector of Schwarzschild, and it defines a natural notion of time.

9.3 MATTER CONTENT AND COHERENT PERTURBATIONS

In the matter content sourcing the dynamics of the black hole, we consider, among other possible contributions, a free, minimally coupled, massless, scalar field φ. The scalar field is described by the Klein-Gordon (K-G) equation, Eq. (2.25), and the classical solutions, together with a symplectic form given in (9.29), forms the symplectic space of the classical theory. In particular, we give non-vanishing initial data on the region of null past infinity with $v \geq v_0$. The quantization of the scalar field goes along the lines presented in Section 2.4, and in particular, since the field is free, we can construct the Weyl C^*-algebra of observables. We consider a class of quasifree states ω that are of Hadamard type, and that are vacuum-like with respect to modes coming from past infinity; we give the explicit two-point function in Section 9.4.1.

From an arbitrary solution of the K-G equation $\lambda\hat{\varphi}$ with initial given on the interval $v \geq v_0$ on past infinity, it is possible to construct the corresponding Weyl unitary $W(f)$, where f are the initial data for $\hat{\varphi}$. Using the Hadamard lemma for the Baker-Campbell-Hausdorff formula,

$$e^X Y e^{-X} = \sum_{n\geq 0} \frac{1}{n!}[X, X, \underbrace{\dots}_{m} [X, Y] \dots] ,$$

and the Weyl relations it is possible to show that

$$\Phi := W(f)^* \varphi(x) W(f) = \varphi + \lambda\hat{\varphi}(x) ,$$

since $\lambda\hat{\varphi} = Ef$.

It follows that in the quasifree state ω, the expectation value of $W(f)^*\varphi(x)W(f)$ is given by

$$\omega\left(W(f)^* \varphi(x) W(f)\right) = \lambda\hat{\varphi} ,$$

since the state is quasifree, so that the observable Φ describes a coherent perturbation of the vacuum-like state ω. We call $\hat{\varphi}$ the *coherent wave*. We can then consider the combination $\Phi = \varphi + \lambda\hat{\varphi}$ as "the quantum field" φ plus a coherent perturbation, where we introduced the perturbative parameter λ. Φ has an associated stress-energy tensor $T(\Phi)$, defined by quantum counterpart of the familiar classical expression,

$$T(\Phi) = \partial_a \Phi \partial_b \Phi + \frac{1}{2} g_{ab} \partial_c \Phi \partial^c \Phi .$$

The expectation value of $T(\Phi)$ in the state ω represents a perturbation of the background by a classical perturbation $\hat{\varphi}$. Since the state is quasifree, the perturbative expansion of the expectation value of a term quadratic in the fields gives

$$\begin{aligned}\omega(\Phi(x)\Phi(y)) = \omega(\varphi(x)\varphi(y)) + \\ + \lambda\left(\omega(\varphi(x))\hat{\varphi}(y) + \omega(\varphi(y))\hat{\varphi}(x)\right) + \\ + \lambda^2\hat{\varphi}(x)\hat{\varphi}(y) + o(\lambda^2)\,.\end{aligned} \tag{9.20}$$

Since the state is quasifree, $\omega(\varphi) = 0$, the coherent wave represents only a second order perturbation of the stress-energy tensor, so that the expectation value of the stress-energy tensor associated with Φ gives a second-order correction,

$$\omega(T_{ab}(\Phi)) = \lambda^2 T_{ab}(\hat{\varphi}) = \lambda^2\left[\partial_a\hat{\varphi}\partial_b\hat{\varphi} + \frac{1}{2}g_{ab}\partial_c\hat{\varphi}\partial^c\hat{\varphi}\right]\,.$$

Remark 9.1 (Alternative description of the coherent wave)**.** There is an alternative description of the coherent wave. Above, we described the coherent wave as the operator $\Phi := W(f)^*\varphi W(f)$ in the C^*-algebra, whose expectation value in the vacuum-like state ω equals the classical wave solution $\omega(\Phi) = \hat{\varphi}$. Here, we discuss an equivalent description in terms of a coherent state $\omega_{\hat{\varphi}}$ in which we take expectation values of the stress-energy tensor associated with φ.

From an arbitrary solution of the K-G equation $\lambda\hat{\varphi}$ with initial data given on the interval $v \geq v_0$ on past infinity, it is possible to construct a coherent state of the vacuum, by acting with a Weyl unitary (see Section 2.7), as $\omega_{\hat{\varphi}}(\cdot) := \omega(W(f)\cdot W(f)^*)$. In the classical solution, we identified a perturbative parameter λ. The coherent state satisfies by definition $\omega_{\hat{\varphi}}(\varphi) = \hat{\varphi}$, that is, the expectation value of the quantum field corresponds to the classical solution.

The stress-energy tensor of the scalar field in the coherent state gives a classical contribution,

$$\omega_{\hat{\varphi}}(T_{ab}(\varphi)) = \lambda^2 T_{ab}(\hat{\varphi}) = \lambda^2\left[\partial_a\hat{\varphi}\partial_b\hat{\varphi} + \frac{1}{2}g_{ab}\partial_c\hat{\varphi}\partial^c\hat{\varphi}\right]\,. \tag{9.21}$$

Since the stress-energy tensor is quadratic in the field, the coherent wave represents a second-order correction in the energy. ■

The classical wave therefore acts as a second order correction to the stress-energy tensor. The expectation value of the stress-energy tensor then represents a perturbation of the black hole dynamics, and we have a one-parameter analytic family of stress-energy tensors $T(\lambda)$, which can be expanded in a perturbative series,

$$T(\lambda) = T(0) + \frac{1}{2}\lambda^2\left.\frac{\mathrm{d}^2T}{\mathrm{d}\lambda^2}\right|_{\lambda=0} + o(\lambda^2)\,. \tag{9.22}$$

$T^{(0)} := T(0)$ represents the background matter content sourcing the dynamical black hole, and $\frac{1}{2}\left.\frac{\mathrm{d}^2T}{\mathrm{d}\lambda^2}\right|_{\lambda=0} = T(\hat{\varphi})$ is the stress-energy tensor of the coherent wave.

The Einstein equations are then satisfied order by order: we can perturbatively expand the Einstein tensor,

$$G(\lambda) = G(0) + \frac{1}{2}\lambda^2\left.\frac{\mathrm{d}^2G}{\mathrm{d}\lambda^2}\right|_{\lambda=0} + o(\lambda^2)\,. \tag{9.23}$$

Denoting $G^{(0)} := G(0)$, the perturbed Einstein equations read

$$G^{(0)} = 8\pi T^{(0)} \tag{9.24}$$

$$\frac{1}{2}\left.\frac{\mathrm{d}^2G}{\mathrm{d}\lambda^2}\right|_{\lambda=0} = 8\pi\frac{1}{2}\left.\frac{\mathrm{d}^2T}{\mathrm{d}\lambda^2}\right|_{\lambda=0} := 8\pi T(\hat{\varphi}) := 8\pi T_{\hat{\varphi}}\,. \tag{9.25}$$

It follows from the perturbative expansion of Einstein's equations that there is a corresponding one-parameter family of metrics described by a function $g(\lambda)$ of the perturbative parameter λ; the background metric (9.1) (or, in explicit coordinates, Eq. (9.6)) equals $g(0)$. The Einstein tensor G thus depends through the metric on the perturbative parameter λ, $G(\lambda)$.

Since the coherent wave preserves spherical symmetry by assumption, the arbitrary functions $\gamma(v,r)$ and $m(v,r)$ (or, equivalently, $f(v,r) = 1-2m/r$) appearing in the background metric get corrected, due to the coherent wave, by a second order perturbation, which replaces in the definitions of the geometrical quantities $\gamma \longrightarrow \Gamma(v,r) = \Gamma^{(0)}(v,r) + \delta^2\gamma(v,r)$, $m \longrightarrow M(v,r) = M^{(0)}(v,r) + \delta^2 m(v,r)$, where $F^{(0)} = F(\lambda = 0)$ and $\delta^2 f = \frac{\lambda^2}{2}\left.\frac{\mathrm{d}^2F}{\mathrm{d}\lambda^2}\right|_{\lambda=0}$.

The location of the dynamical horizon is given by

$$r_{\mathcal{DH}} = 2M(v,r)\,, \tag{9.26}$$

that is, it is translated by $r_{\mathcal{DH}} - r^{(0)}_{\mathcal{DH}} = 2\delta^2 m$.

We can now introduce the geometric setup we discussed in the Introduction. To this end, we consider a region in the black hole exterior bounded by four hypersurfaces: future infinity, the null hypersurface at $v = v_0$, and the two segments of the dynamical horizon $r = 2M$ and of past null infinity with $v \geq v_0$. The resulting region, shaded in grey in figure 9.1, is a deformation of a double cone, in which one of the four null boundaries is substituted by the dynamical horizon.

We then construct the Kodama current associated with the stress-energy tensor of the scalar field, $j^a := T^{ab}_{\hat{\varphi}} k_b$. This current is conserved, and its integral along a segment between two points of any hypersurface Σ equals the variation of the second-order perturbation of the Misner-Sharp mass, (9.14).

Thanks to the Stokes theorem [239], the conservation law for the Kodama current can be converted into an equation for its flux F across the five boundaries of the shaded region. We denote with $\mathcal{DH}(v_0)$ (respectively $\mathcal{I}^-(v_0)$) the region of the perturbed dynamical horizon (resp. past null infinity) with $v \geq v_0$, and with Σ_{v_0} the null hypersurface at $v = v_0$. The remaining pieces of the boundary are space-like infinity i^0 and null ($\mathcal{I}^+$) and time-like (i^+) future infinity. Since the coherent perturbation is switched on at $v = v_0$, we can immediately see that $F_{v_0} = 0$ for causality. Since the coherent wave is a spatially compact solution of the Klein-Gordon equation, the flux at space-like infinity vanishes as well, $F_{i^0} = 0$. Finally, since the spacetime is asymptotically flat, we can apply the decaying properties for the scalar fields at infinity in a Schwarzschild background, studied in [88]; in particular, they showed that the flux at time-like future infinity vanishes as well, $F_{i^+} = 0$. Therefore, the flux is naturally composed by three terms:

$$F_{\mathcal{DH}} + F_{\mathcal{I}^+} = F_{\mathcal{I}^-}\,, \tag{9.27}$$

where each term is in the form $F_\Sigma = \int_\Sigma j_a \mathrm{d}\Sigma^a$, with the convention such that the directed surface element $\mathrm{d}\Sigma^a$ is always future-directed.

We will show in the next Sections that the horizon term gives the derivative along the outgoing light-rays of one-fourth of the horizon area, while the term at past infinity is the derivative of the relative entropy between the classical wave and the ground state. In order to relate the flux term at past infinity with an expression for the relative entropy, the next Section deals with introducing in some detail the algebraic formalism for the quantization of free fields on globally hyperbolic spacetimes, and the Tomita-Takesaki modular theory to define the relative entropy.

9.4 RELATIVE ENTROPY BETWEEN COHERENT STATES

9.4.1 *Quantization of the free, scalar, massless field on globally hyperbolic spacetimes*

On the black hole spacetime $\mathcal{M} = \mathcal{M}_2 \times S_2$, equipped with the metric g given in (9.1), we now consider a solution of the Klein-Gordon (K-G) equation (2.25),

$$\Box\varphi = 0\,,$$

where $\Box = g^{ab}\nabla_a\nabla_b$ is the K-G operator for a massless field with minimal coupling.

The scalar field can be quantised along the lines discussed in Chapter 2. In particular, we consider the symplectic space $(\mathscr{E}_{\text{os}}, \sigma)$ of smooth solutions of the K-G equation with spatially compact support, equipped with the symplectic form

$$\sigma(\varphi_1, \varphi_2) = \int_\Sigma (\varphi_2\nabla_\mu\varphi_1 - \varphi_1\nabla_\mu\varphi_2)n^\mu \,\mathrm{d}\Sigma\,, \tag{9.28}$$

where Σ is a Cauchy hypersurface (a surface whose causal development covers $\mathcal{M}$ [148]) and n is the unit, future-directed normal vector to Σ. As a Cauchy surface we choose the surface denoted by Σ in figure 9.1, and we give non-vanishing initial data for the coherent wave on $\mathcal{I}^-(v_0)$, i.e., the region of past null infinity with $v > v_0$. The resulting classical wave represents a perturbation coming from the distant past. The Cauchy surface Σ is a particular limit of a space-like Cauchy surface, in which a part of it corresponds with past null infinity. In this limit, the symplectic form becomes

$$\sigma(\varphi_1, \varphi_2) = \int_{\mathcal{I}^-(v_0)} (\tilde\varphi_2\nabla_\mu\tilde\varphi_1 - \tilde\varphi_1\nabla_\mu\tilde\varphi_2)n^\mu_{\mathcal{I}^-} \,\mathrm{d}v\mathrm{d}\Omega\,, \tag{9.29}$$

where $n_{\mathcal{I}^-}$ is the unit, future-directed vector normal to $\mathcal{I}^-$, and $\tilde\varphi = r\varphi$. It has been shown that, in the case of a Schwarzschild background, this is the correct limit to past infinity of a space-like Cauchy hypersurface [88]; since our spacetime approaches the Schwarzschild spacetime in the asymptotic past, we can assume the same formula for the symplectic form.

We then quantise the scalar field providing the Weyl C^*−algebra $\mathscr{W}$ of Weyl unitaries $W(f)$ presented in Section 2.7.

We now need to choose a quasifree, Hadamard state. We want a state that best represents a vacuum-like state, or more generally a ground state, for modes coming from past infinity. For a Schwarzschild black hole, the Unruh state [238] is the most physically sensible candidate for the vacuum [72] in both the interior and the exterior of the black hole, since it approaches a Minkowski vacuum at past infinity, exhibits Hawking radiation at late times, and is regular on the horizon. In particular, it has been proved [88] that it is of Hadamard form.

For our purposes, however, since we are not interested in the details of the black hole dynamics on the past horizon, we will consider a slightly more general class of states. In particular, we fix the form of the 2-point function at past infinity only, assuming that it is regular in the sense of Hadamard on the rest of the Cauchy surface. The 2-point function restricted to Σ then defines the quasifree state in $D(\Sigma)$, the causal development of Σ, via the K-G equation. Since our spacetime coincides with the Schwarzschild spacetime at past infinity, we choose the 2-point function of a Unruh state on $\mathcal{I}^-$, so that the state is a vacuum with respect to modes coming from past infinity. Denoting $\tilde\varphi_f = r\varphi_f\big|_{\mathcal{I}^-}$, and remembering that we have non-vanishing initial data on $\mathcal{I}^-(v_0)$ only, we choose the 2-point function to be

$$\Delta_+(f,g)\bigg|_{\mathcal{I}^-} = \frac{1}{\pi}\int_{\mathcal{I}^-(v_0)} \frac{\tilde{\varphi}_f(v_1)\tilde{\varphi}_g(v_2)}{(v_2-v_1-i\varepsilon)^2}\, dv_1 dv_2 d\Omega\,, \tag{9.30}$$

where $d\Omega = \sin\theta d\theta_1 d\theta_2$ is the volume form of a unit 2-sphere. The integral is computed only on the region of null past infinity in which we give initial data, $\mathcal{I}^-(v_0)$. The 2-point function (9.30) is given by the Unruh 2-point function on $\mathcal{I}^-$ of a Schwarzschild black hole. Our choice for a class of ground states, to which we will refer for simplicity as *the* ground state, is then defined by the requirement that the 2-point function is

$$\Delta_+(f,g) = \Delta_+(f,g)\bigg|_{\mathcal{I}^-} + \Delta_+(f,g)\bigg|_{\Sigma\,\setminus\,\mathcal{I}^-}\,. \tag{9.31}$$

We only assume that $\Delta_+(f,g)\big|_{\Sigma\,\setminus\,\mathcal{I}^-}$ is of Hadamard form.

This ground state can now be represented in the GNS reconstruction as the vacuum of a symmetrised Fock space, $|\Omega\rangle$. The coherent wave $\hat{\varphi}$, propagating over the spacetime with initial data on $\mathcal{I}^-(v_0)$ is then a coherent perturbation of the ground state, which can be represented as $|\Phi\rangle = W(f)\,|\Omega\rangle$, where $\hat{\varphi} = Ef$. The corresponding state functional and its GNS vector representative is $\omega_{\hat{\varphi}}(\cdot) = \langle\Phi|\cdot|\Phi\rangle$.

9.4.2 *Tomita-Takesaki modular theory*

Tomita-Takesaki modular theory provides a tool to describe the entropy of a free QFT as the expectation value of an operator in the corresponding von Neumann algebra, constructed from the GNS representation of the $*-$algebra on some Hilbert space. In recent years, the Araki formula [13] for the relative entropy between coherent states received some attention [73, 77, 78, 155], since it has been proven that it can be computed by a simple expression in terms of the symplectic structure of the classical theory and the modular flow only [182, 183]. Here we provide a brief review of Tomita-Takesaki modular theory (see e.g. [51, 140, 233, 252, 253] for reviews), in order to re-derive the formula for the relative entropy between coherent states in our setting.

We start very general, and we slowly reduce to the case of coherent states for a QFT. We will prove the properties of Tomita-Takesaki theory which are most useful for the characterization of the relative entropy of coherent states, with no attempt to be exhaustive.

Let $v\mathcal{N}(\mathcal{H})$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and let $|\Omega\rangle \in \mathcal{H}$ be a cyclic and separating vector, i.e., i) given a vector $|\Omega\rangle \in \mathcal{H}$ and an operator $\pi(A)$ over $\mathcal{H}$, $\pi(A)\,|\Omega\rangle$ is dense in $\mathcal{H}$, and ii) $\pi(A)\,|\Omega\rangle = 0 \Rightarrow \pi(A) = 0\ \forall\, A \in \mathscr{A}$. There exists a unique antilinear operator S_Ω, called the *modular involution* or *Tomita operator*, such that

$$S_\Omega A\,|\Omega\rangle = A^*\,|\Omega\rangle\ . \tag{9.32}$$

From the definition, it is clear that $S_\Omega^2 = 1$, and therefore it is invertible. Moreover, $S_\Omega\,|\Omega\rangle = |\Omega\rangle$.

An invertible, closed operator always admits a unique polar decomposition, $S_\Omega = J_\Omega \Delta_\Omega^{1/2}$, where the *modular conjugation* J_Ω is an anti-linear, unitary operator and the *modular operator* Δ_Ω is self-adjoint and non-negative. The Tomita operator as defined in (9.32) is not closed, but it is closable. We denote the closure of S_Ω with the same symbol.

As $S_\Omega |\Omega\rangle = S_\Omega^* |\Omega\rangle = |\Omega\rangle$, it is immediate to see that $\Delta_\Omega |\Omega\rangle = |\Omega\rangle$ and therefore $J_\Omega |\Omega\rangle = |\Omega\rangle$. Moreover, from $S_\Omega^2 = 1$, we have that

$$1 = J_\Omega \Delta_\Omega^{1/2} J_\Omega \Delta_\Omega^{1/2} \Rightarrow J_\Omega \Delta_\Omega^{1/2} J_\Omega = \Delta_\Omega^{-1/2} .$$

Expanding in Taylor series Δ^{is}, it is possible to show that $J_\Omega \Delta_\Omega^{is} J_\Omega = \Delta_\Omega^{is}$ holds. Using twice the above property,, and the invariance of $|\Omega\rangle$ under J_Ω, we have

$$S_\Omega = J_\Omega \Delta_\Omega^{1/2} = J_\Omega^2 \Delta_\Omega^{-1/2} J_\Omega = J_\Omega^2 J_\Omega \Delta^{1/2} .$$

By uniqueness of the polar decomposition it follows that $J_\Omega^2 = 1$.

The logarithm of Δ_Ω defines the *modular Hamiltonian*, in terms of which it is possible to compute the relative entropy between coherent states.

Definition 9.2 (Modular Hamiltonian)**.**

$$K_\Omega = -\log \Delta_\Omega .$$

K_Ω is a self-adjoint operator with generally unbounded spectrum, which is well-defined since Δ_Ω is non-negative. By Stone Theorem, K_Ω defines a 1-parameter group of unitary operators on the von Neumann algebra called *modular flow*,

Definition 9.3 (Modular flow)**.**

$$\alpha_s(A) = e^{-iK_\Omega s} A e^{iK_\Omega s} = \Delta_\Omega^{is} A \Delta_\Omega^{-is} .$$

The Tomita operator for the commutant of $v\mathcal{N}(\mathcal{H})$, that is, the set of bounded operators $v\mathcal{N}'(\mathcal{H}) = \{U' \mid [U, U'] = 0 \,\forall\, U \in v\mathcal{N}(\mathcal{H})\}$, is defined in an analogous way. The von Neumann algebra, the modular flow, and its commutant are related by [235]

$$J_\Omega v\mathcal{N}(\mathcal{H}) J_\Omega = v\mathcal{N}(\mathcal{H})' \tag{9.33}$$

$$\Delta_\Omega^{is} v\mathcal{N}(\mathcal{H}) \Delta_\Omega^{-is} = v\mathcal{N}(\mathcal{H}) \quad , \quad \Delta_\Omega^{is} v\mathcal{N}(\mathcal{H})' \Delta_\Omega^{-is} = v\mathcal{N}(\mathcal{H})' . \tag{9.34}$$

The first property says that the modular conjugation maps the algebra into its commutant. The second one states that the modular flow defines an automorphism of the algebra.

The action of the modular operator on observables is in general very hard to compute, since from its definition the modular operator is a non-local operator. However, for free theories on flat spacetimes the *Bisognano-Wichmann Theorem* [45] links the modular flow to a geometric action.

In Minkowski $\mathcal{M}$, we consider the region $W = \{\mathbf{x} \in \mathbb{R}^{1,3} \mid x > |t|\}$, called *Rindler wedge*. This is the domain of dependence of the surface $\Sigma = \{\mathbf{x} \in \mathbb{R}^{1,3} \mid t = 0,\ x > 0\}$. Consider the vacuum vector $|\Omega\rangle$ for the theory defined on the whole Minkowski spacetime $v\mathcal{N}(\mathcal{M})$. By the Reeh-Schlieder theorem [252], this is a cyclic and separating vector for the algebra of observables restricted to the Rindler wedge $v\mathcal{N}(W)$. The Bisognano-Wichmann theorem states that [45]

$$J_\Omega = \Theta U(R_1(\pi)) \quad \Delta_\Omega = e^{-2\pi K_1} , \tag{9.35}$$

where Θ is the CPT operator, $U(R_1(\pi))$ is the unitary operator representing a space rotation of π degrees around the x axis and K_1 is the generator of the one-parameter group of boosts in the plane (t, x). The theorem admits a generalisation [67] to the algebras defined on a spacetime with a group of symmetries, in which a “wedge region” can be defined. In this perspective, past null infinity falls into this category: the group of symmetries are translations along v, and the region $v \geq v_0$ represents the wedge.

9.4.3 *Relative entropy*

Now, we consider the induced von Neumann algebra $v\mathcal{N}(\mathscr{A})$ as the GNS representation of a C*-algebra $\mathscr{A}$, defined by a state functional ω, and the cyclic and separating vector $|\Omega\rangle$ representative of ω. If we consider the so-called *natural cone*, that is, the set of vectors

$$\mathcal{P}_\Omega = \overline{\{Aj_\Omega(A)\,|\Omega\rangle \mid A \in \mathfrak{A}\}}\,, \tag{9.36}$$

where the bar means the closure and $j_\Omega(A) = J_\Omega A J_\Omega$, any state ω_Φ has a unique representative vector $|\Phi\rangle$ in $\mathcal{P}_\Omega$, represented in the usual way, $\omega_\Phi(A) = \langle\Phi|A|\Phi\rangle$.

Now, given two cyclic and separating vectors, $|\Omega\rangle \in \mathcal{H}$, $|\Phi\rangle \in \mathcal{P}_\Omega$, the relative entropy provides a notion of distance between the two. First, the *relative Tomita operator* (or *relative modular involution*) provides a generalisation of the Tomita operator, since

$$S_{\Omega,\Phi} A\,|\Phi\rangle = A^*\,|\Omega\rangle\;. \tag{9.37}$$

Just as the Tomita operator, the relative Tomita operator admits the unique polar decomposition $S_{\Omega,\Phi} = J_{\Omega,\Phi}\Delta^{1/2}_{\Omega,\Phi}$. Since $|\Phi\rangle$ is in the natural cone, one can show that $J_{\Omega,\Phi} = J_\Omega$ [252]. The *relative modular Hamiltonian* is defined as

$$K_{\Omega,\Phi} = -\log\Delta_{\Omega,\Phi}\,. \tag{9.38}$$

Finally, the Araki formula gives the relative entropy:

Definition 9.4 (Araki relative entropy)**.**

$$S(\omega_\Omega|\omega_\Phi) = \langle\Omega|\log\Delta_{\Omega,\Phi}|\Omega\rangle = -\langle\Omega|K_{\Omega,\Phi}|\Omega\rangle\;.$$

In the case of Quantum Mechanics, where states are realised as density matrices, $\omega(A) = \mathrm{Tr}\,\rho_\Omega A$, the relative entropy takes the familiar form $S(\Omega|\Phi) = -\mathrm{Tr}\,\rho_\Omega(\log\rho_\Phi - \log\rho_\Omega)$ [13], and represents a regularisation of the entanglement entropy by subtraction of the vacuum entanglement.

9.4.4 *Relative entropy for coherent states*

Now, we consider two cyclic and separating vectors $|\Omega\rangle$ and $|\Phi\rangle$ and some unitary operator $U \in v\mathcal{N}(\mathscr{A})$ which is an automorphism of the algebra, $U^* v\mathcal{N} U = v\mathcal{N}$. In this case, for $A \in v\mathcal{N}$, it holds that [73]

$$US_{\Omega,\Phi}U^*AU\,|\Phi\rangle = US_{\Omega,\Phi}(U^*AU)\,|\Phi\rangle = U(U^*AU)^*\,|\Omega\rangle = A^*U\,|\Omega\rangle\;, \tag{9.39}$$

which implies

$$US_{\Omega,\Phi}U^* = S_{U\Omega,U\Phi}\,. \tag{9.40}$$

We can now choose $|\Phi\rangle = j_\Omega(U)U\,|\Omega\rangle$, for some unitary operator $U \in v\mathcal{N}(\mathcal{H})$. The operator $\tilde{U} = j_\Omega(U)U$ is still a unitary operator, and the corresponding state functional is $\omega_\Phi(A) = \langle\Omega U|AU\Omega\rangle = \omega(U^*AU)$. The above property in this special case becomes

$$S_{\Omega,\Phi} = S_{\Omega,\tilde{U}\Omega} = S_{\tilde{U}\tilde{U}^*\Omega,\tilde{U}\Omega} = \tilde{U}S_{\tilde{U}^*\Omega,\Omega}\tilde{U}^* = Uj_\Omega(U)S_{\tilde{U}^*\Omega,\Omega}(Uj_\Omega(U))^*\,. \tag{9.41}$$

A similar property for the relative modular operator follows, $\Delta^{1/2}_{\Omega,U\Omega} = U\Delta^{1/2}_{U^*\Omega,\Omega}U^*$. These properties let us express the relative entropy between two coherent states in terms of the relative entropy of a coherent state with respect to the vacuum. In fact, suppose we have two different coherent vectors of the same state $|\Omega\rangle$, $|\Phi\rangle = W(f)\,|\Omega\rangle$ and $|\Psi\rangle = W(g)\,|\Omega\rangle$, in the natural cone of $|\Omega\rangle$, for some test functions f, g. Suppose we want to compute the relative entropy between the vectors $|\Phi\rangle$ and $|\Psi\rangle$. Then, one can show that [73]

$$S(\omega_\Psi|\omega_\Phi) = S(\omega_{W(f-g)\Omega}|\omega_\Omega)\,. \tag{9.42}$$

$W(g-f)\,|\Omega\rangle$ is still a coherent vector. Therefore, we can restrict our attention to the relative entropy of a coherent vector with respect to a reference vector $|\Omega\rangle$.

From property (9.33), acting with $US_\Omega j_\Omega(U^*)A$ on $|\Phi\rangle$ for some operator A, we can see that

$$(US_\Omega j_\Omega(U^*))\;A\,|\Phi\rangle = US_\Omega(AU)\,|\Omega\rangle = A^*\,|\Omega\rangle = S_{\Omega,\Phi}A\,|\Phi\rangle\;. \tag{9.43}$$

Therefore, the relative Tomita operator between $|\Omega\rangle$ and $|\Phi\rangle$ can be computed using the Tomita operator only:

$$S_{\Omega,\Phi} = US_\Omega j_\Omega(U^*)\,. \tag{9.44}$$

By polar decomposition, one can see that

$$\Delta^{1/2}_{\Omega,\Phi} = j_\Omega(U)\Delta^{1/2}_\Omega j_\Omega(U^*)\quad K_{\Omega,\Phi} = j_\Omega(U)K_\Omega j_\Omega(U^*)\,. \tag{9.45}$$

It follows that the relative entropy between $|\Omega\rangle$ and $|\Phi\rangle$ depends only on the modular operator, not the relative modular operator:

$$\begin{aligned} S(\omega_\Omega|\omega_\Phi) = {} & \langle\Omega|\log\Delta_{\Omega,U\Omega}|\Omega\rangle = \\ & = 2\,\langle\Omega|\log\Delta^{1/2}_{\Omega,\Phi}|\Omega\rangle = 2\,\langle\Omega|\log\Big(j_\Omega(U)\Delta^{1/2}_\Omega j_\Omega(U^*)\Big)|\Omega\rangle\;, \end{aligned} \tag{9.46}$$

Since $\log(j_\Omega(U)\Delta_\Omega j_\Omega(U^*)) = j_\Omega(U)\log\Delta_\Omega j_\Omega(U^*)$, using the invariance of $|\Omega\rangle$ under J_Ω, and the fact that $J_\Omega\Delta_\Omega J_\Omega = \Delta^{-1}_\Omega$, it is a straightforward computation to show that:

$$S(\omega_\Omega|\omega_{U\Omega}) = i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\langle U^*\Omega|e^{-iKt}|U^*\Omega\rangle\Big|_{t=0}\;. \tag{9.47}$$

Finally, we specialise to the case in which ω_Ω is the ground state functional, which will denote simply by ω, with vector representative $|\Omega\rangle$, and $\omega_{U\Omega}$ is a coherent state, for some test function f, which will be denoted by $\omega_f(A) = \langle\Phi|A|\Phi\rangle = \omega(W^*(f)AW(f))$. We use f as a label to remember that is a coherent state, and can be considered a classical perturbation $\hat\varphi_f = Ef$ of the ground state. The unique representative vector of ω_f in the natural cone $\mathcal{P}_\Omega$ is $|\Phi\rangle = j_\Omega(W(f))W(f)\,|\Omega\rangle$. The entropy for coherent states is indeed computed using the classical structure only, that is, from the symplectic form of $\mathscr{E}_{\mathrm{os}}$. In particular, it holds that

Proposition 9.2 (Relative entropy between coherent states)**.** *The relative entropy between two coherent states ω and ω_f, with GNS representatives, respectively, $|\Omega\rangle$ and $|\Phi\rangle = j_\Omega(W(f))W(f)\,|\Omega\rangle$, can be computed from the following formula:*

$$S(\omega|\omega_f) = \frac{1}{2}\sigma\Big(\frac{\mathrm{d}}{\mathrm{d}t}\alpha_t(\hat\varphi_f)\Big|_{t=0}, \hat\varphi_f\Big)\,.$$

Proof. Choosing $U = W(f)$ in Eq. (9.47), we have

$$S(\omega|\omega_f) = i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\Omega|W(f)^*\Delta^{it}W(f)|\Omega\rangle\Big|_{t=0}\,. \tag{9.48}$$

Using the invariance of the vacuum with respect to Δ_Ω it is possible to rewrite this expression as the action of the automorphism α_t introduced in (9.3) over the Weyl operator:

$$i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\Omega|W^*(f)\Delta^{it}W(f)\Delta^{-it}|\Omega\rangle\Big|_{t=0} = i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\Omega|W^*(f)\alpha_t W(f)|\Omega\rangle\Big|_{t=0}\,, \tag{9.49}$$

and using the Weyl relations,

$$W^*(f)W(\alpha_t(f)) = e^{-\frac{i}{2}\sigma(-\hat\varphi_f,\alpha_t(\hat\varphi_f))}W(\alpha_t(f)-f) \tag{9.50}$$

and

$$\langle\Omega|W(f)|\Omega\rangle = e^{-\frac{1}{2}\Delta_+(f,f)}\,. \tag{9.51}$$

The expectation value then becomes

$$\langle\Omega|W^*(f)W(\alpha_t(f))|\Omega\rangle = \exp\left\{-\frac{i\sigma(-\hat\varphi_f,\alpha_t(\hat\varphi_f))}{2} - \frac{1}{2}\Delta_+(\alpha_t(f)-f,\alpha_t(f)-f)\right\}. \tag{9.52}$$

Now, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Delta_+(\alpha_t(f)-f,\alpha_t(f)-f)\Big|_{t=0} &= \\ &= \Big(\Delta_+(\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_t(f)-f),\alpha_t(f)-f)+ \\ &\quad + \Delta_+(\alpha_t(f)-f,\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_t(f)-f)\Big)\Big|_{t=0} = 0\,,\end{aligned} \tag{9.53}$$

since $\alpha_t(f)\big|_{t=0} = f$. Therefore, the derivative with respect to t of (9.52), evaluated at $t=0$, gives

$$\begin{aligned}&i\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\Omega|W^*(f)W(\alpha_t(f))|\Omega\rangle\Big|_{t=0} = \\ &= \Big[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sigma(-\hat\varphi_f,\alpha_t(\hat\varphi_f)\Big)\times \\ &\times e^{-\frac{1}{2}(i\sigma(-\hat\varphi_f,\alpha_t(\hat\varphi_f))+\Delta_+(\alpha_t(f)-f,\alpha_t(f)-f)}\Big]_{t=0} = \\ &= \frac{1}{2}\sigma\left(\frac{\mathrm{d}\alpha_t(\hat\varphi_f)}{\mathrm{d}t}\Big|_{t=0},\hat\varphi_f\right).\end{aligned} \tag{9.54}$$

□

This conclude the derivation of the main result of this Section, which can be found (each in a slightly different derivation) in [155], [73], and [183]: the relative entropy between a coherent state and the vacuum is given by the derivative of the symplectic form of the associated classical solution.

We can now use proposition (9.2) to compute the relative entropy between the ground state defined by (9.30) and a coherent perturbation, with the symplectic form given in (9.29). First notice that, from their respective expressions, these equations actually defines an algebra on the half-infinite region of past null infinity $\mathcal{I}^-(v_0)$, which coincides with the restriction of the quantum theory in the bulk to past null infinity, $\mathscr{A}(\mathcal{I}^-(v_0))$. On such a spacetime, the generalisation of the Bisognano-Wichmann theorem by Brunetti, Guido, and Longo applies [67, 155], so that the modular action $\alpha_t(\hat{\varphi}(f))$ acts as a boost in the radial-temporal plane: in our coordinates,

$$\alpha_t(\hat{\varphi}(v,\theta,\hat{\varphi})) = \hat{\varphi}(v_0 + e^{-2\pi t}(v - v_0), \theta, \hat{\varphi}) \,. \tag{9.55}$$

Therefore, the relative entropy (9.2) in the region $v > v_0$ becomes

$$\begin{aligned} S(\omega|\omega_{\hat{\varphi}}) &= \\ &= -\pi \int_{\mathcal{I}^-(v_0)} \left[\tilde{\varphi}\partial_a[(v-v_0)\partial_v\tilde{\varphi}] - (v-v_0)\partial_v\tilde{\varphi}\partial_a\tilde{\varphi}\right] n^a_{\mathcal{I}^-} \mathrm{d}v\mathrm{d}\Omega \,. \end{aligned} \tag{9.56}$$

Integrating by parts the first term in the integral, it follows that

$$S(\omega|\omega_{\hat{\varphi}}) = 2\pi \int_{\mathcal{I}^-(v_0)} (v-v_0)\partial_a\tilde{\varphi}\partial_b\tilde{\varphi}\; n^a_{\mathcal{I}^-} k^a \;\mathrm{d}v\mathrm{d}\Omega \,, \tag{9.57}$$

where we used $k^a_{\mathcal{I}^-} = \partial_v$.

As argued in [155], using results on Schwarzschild background [88], this entropy formula remains finite, due to the decaying properties of $\hat{\varphi}$ at large radii.

In order to make the connection with the conservation law for the Kodama flux, (9.27), the above expression can be written in terms of the stress-energy tensor of the scalar field at infinity,

$$T_{ab} = \partial_a\tilde{\varphi}\partial_b\tilde{\varphi} - \frac{1}{2} g_{ab}\partial_i\tilde{\varphi}\partial^{\tilde{k}}\varphi \,. \tag{9.58}$$

Since the metric is anti-diagonal at past infinity, while the vectors k and $n_{\mathcal{I}^-}$ are parallel, we have $g_{ab}k^a n^b_{\mathcal{I}^-} = 0$. The relative entropy can thus be written as

$$S(\omega|\omega_{\hat{\varphi}}) = 2\pi \int_{\mathcal{I}^-(v_0)} (v-v_0) T_{ab}\; n^a_{\mathcal{I}^-} k^a \;\mathrm{d}v\mathrm{d}\Omega \tag{9.59}$$

We can now consider what happens if we rigidly translate the boundary of the region in which the field propagate by a finite amount, $v_0 \longrightarrow v_0 + \xi$. We can write the derivative of the relative entropy as

$$-\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}v_0} S(\omega|\omega_{\hat{\varphi}}) = \int_{\mathcal{I}^-(v_0)} T_{ab}k^b n^a_{\mathcal{I}^-} \;\mathrm{d}v\mathrm{d}\Omega \,. \tag{9.60}$$

Such a derivative can be interpreted as the derivative with respect to the instant in which we switched on the perturbation v, evaluated for $v = v_0$.

9.5 VARIATION OF GENERALISED ENTROPY

We now return to the original task of computing the various terms in the conservation law for the Kodama flux, (9.27). Of the three terms, we have just seen that $-\frac{1}{2\pi}l(S(\omega|\omega_{\hat{\varphi}}))(v_0) = F_{\mathcal{I}^-}$. On the other hand, on the dynamical horizon of the perturbed spacetime, we consider the flux term

$$F_{\mathcal{DH}} = \int_{\mathcal{DH}(v_0)} j_a \mathrm{d}\Sigma^a_{\mathcal{DH}} , \tag{9.61}$$

where the integral is extended over the region of dynamical horizon with $v > v_0$.

Using (9.14), and the fact that the perturbation goes to zero at future infinity, it is immediate to see that the flux term equals the mass contribution to the black hole by the scalar field,

$$F_{\mathcal{DH}} = \delta^2 m_{i^+} - \delta^2 m(v_0, r_{\mathcal{DH}}) , \tag{9.62}$$

where the minus sign in (9.14) is compensated by the minus sign coming from the evaluation of the integral on the lower extreme, and $\delta^2 m_{i^+} = \delta^2 m(v = \infty, r_{\mathcal{DH}})$.

On the other hand, we can compute the derivative of $A = 4\pi r^2$ along the *background* outgoing light-rays on the apparent horizon, $l(A)_{\mathcal{DH}}$. As we said, however, the apparent horizon is not at $r = 2m$, but gets translated to $r = 2M(v, r)$. This implies that $f = \frac{1}{2M}(2M - 2m)$, and so

$$l(A)\bigg|_{r=2M} = \frac{2M - 2m}{4M}\, \partial_r A\bigg|_{r=2M} = 8\pi(M - m) = 8\pi\delta^2 m . \tag{9.63}$$

Evaluating the above equation for $v = v_0$, we can conclude that

$$F_{\mathcal{DH}} = \frac{1}{8\pi}\, l(\delta^2 A)\bigg|_{\mathcal{DH}} (v_0) - \delta^2 m_{i^+} . \tag{9.64}$$

We can now conclude our computation of the Kodama flux. In the flux conservation law (9.27), we have shown that the dynamical horizon term can be rewritten as a variation of the area of the horizon, giving (9.64); on the other hand, the past infinity term has been rewritten as a variation of the relative entropy of the perturbation, via the Araki formula for coherent states (9.2), obtaining (9.60). Substituting in the flux conservation law (9.27) equations (9.64) and (9.60), gives

$$l(S(\omega|\omega_{\hat{\varphi}}))_{\mathcal{I}^-}(v_0) + \frac{1}{4} l(A)_{\mathcal{DH}}(v_0) = 2\pi(F_{\mathcal{I}^+} + \delta^2 m_{i^+}) . \tag{9.65}$$

Equation (9.65) generalises the result by Hollands and Ishibashi [155], from the event horizon of a static black hole to the case of dynamical horizons of dynamical, spherically symmetric black holes.

If we now consider a coherent wave such that $\hat{\varphi} \longrightarrow 0$ for large v, then $\delta^2 m \longrightarrow 0$ both at time-like and null future infinity, the right-hand side of the equation above vanishes identically. It is then straightforward to integrate the left-hand side along the geodesic congruence tangent to l. Choosing the surface $v = v_0$ as the surface on which the integral parameter τ of l vanishes, and integrating between 0 and $+\infty$, that is, from the surface $v = v_0$ to $\mathcal{I}^+$, we find

$$\Delta S_{gen} = c(\xi) , \tag{9.66}$$

where it is natural to introduce a generalised entropy $S_{gen} = S(\omega|\omega_{\hat{\varphi}}) + \frac{1}{4}A$, and $c(\xi)$ is an arbitrary function of the integral parameter of the ingoing light-rays.

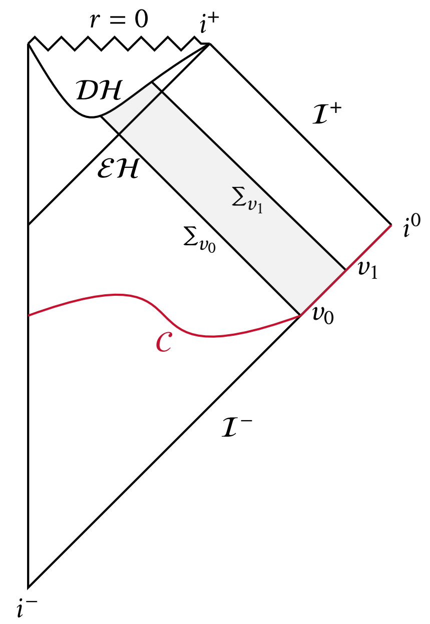

Figure 9.2: Kodama flux across the boundaries of a finite region.

9.6 THERMODYNAMIC INTERPRETATION

First of all, the result (9.65) immediately reduces to the Bekenstein-Hawking formula, and coincides with Hollands and Ishibashi's result, if we take the limit of a static black hole, $\Gamma \to 0$ and $M(v,r) \to M$. Moreover, we see that, at least for the case of coherent perturbations, the relative entropy approach solves two of the problems emerged using the entanglement entropy approach [46], [228]: the divergence in the continuum limit and the dependence on the number of fields. As proved in [155], the relative entropy formula (9.56) is finite. On the other hand, if we had more than one field, we could simply sum each contribution to get the relative entropy for the matter fields, while the sums of the energy-momentum tensors would give the total variation of the area.

We want to consider now what happens if, instead of performing the computation in the region extended up to infinity, we use the flux conservation law in a finite region. To this end, we take two null hypersurfaces, at $v = v_0$ and $v = v_1$, and the finite regions of past null infinity $\delta\mathcal{I}^-$ and $\delta\mathcal{DH}$ in the interval $v \in [v_0, v_1]$, see figure 9.2. We then consider the Kodama flux across the four boundaries of the shaded region, associated to a coherent wave with initial data on $\delta\mathcal{I}^-$,

$$F_{\delta\mathcal{DH}} + F_{\Sigma_{v_1}} = F_{\delta\mathcal{I}^-} \,. \tag{9.67}$$

This is the analogue of Eq. (9.27), where the flux across Σ_{v_0} can again be immediately put to zero for causality.

The term on the dynamical horizon can be computed following the same steps of Section 9.5; the only difference now is that, when we evaluate the integral (9.64), the boundary term does not vanish, yielding

$$F_{\delta\mathcal{DH}} = -\frac{1}{8\pi}\left(l(A)(v_1) - l(A)((v_0))\right)\Big|_{\mathcal{DH}} \,. \tag{9.68}$$

The construction of the relative entropy term is more delicate. We start considering a coherent wave with initial data on $\mathcal{I}^-(v_0)$, just as we did in 9.4.4. Following the same steps, we arrive at Eq. (9.60). We can now consider the same computation, in the case of a quantum theory with a Cauchy surface which coincides with past null infinity from the instant $v = v_1$, denoted with Σ_1 in figure 9.2, with non-vanishing initial data on $\mathcal{I}^-(v_1)$. In this case, the derivative of the relative entropy would give

$$-\frac{1}{2\pi} l(S(\omega|\omega_{\hat{\varphi}}))(v_1) = \int_{\mathcal{I}^-(v_1)} j_a n^a_{\mathcal{I}^-} \, \mathrm{d}v \mathrm{d}\Omega \,. \tag{9.69}$$

Taking the difference between the derivatives of the relative entropy, we get

$$\frac{1}{2\pi}\Big(l(S(\omega|\omega_{\hat{\varphi}})(v_1) - l(S(\omega|\omega_{\hat{\varphi}}))(v_0) \Big) = \Phi_{\delta\mathcal{I}^-} \,, \tag{9.70}$$

where the right-hand side is given by the difference of the integrals at past infinity. Then, substituting (9.68) and (9.70) in (9.67), we get

$$l\left(S(\omega|\omega_{\hat{\varphi}})_{\mathcal{I}^-} + \frac{1}{4}A_{DH}\right)(v_1) - l\big(S(\omega|\omega_{\hat{\varphi}})_{\mathcal{I}^-} + \frac{1}{4}A_{DH}\big)(v_0) = \\ = 2\pi F_{v_1} \,. \tag{9.71}$$

We see again the emergence of the generalised entropy, $S + \frac{1}{4}A$, as the entropy term associated to the gravitational and matter content of the model, equals to the work done on the black hole plus the energy flux emitted towards the future. The main difference, with the result (9.65), is that the difference of relative entropies cannot be immediately interpreted as the relative entropy of a coherent wave with non-vanishing initial data on $\delta\mathcal{I}^-$, and therefore the result can seem somewhat formal.

Finally, we want to make contact with the first law of dynamical black holes found by Hayward [151]. In his paper, he introduced two invariant quantities, the scalar

$$w = -\frac{1}{2}\gamma_{ij}T^{ij} \,, \tag{9.72}$$

and the vector

$$\psi = T_{ab}\nabla^b r + w k_a \,. \tag{9.73}$$

ψ plays the role of a localized Bondi energy flux, while w is interpreted as an energy density. Hayward showed that, along any vector t tangent to the dynamical horizon, it holds that

$$-F_{DH(v_0)} = \int_{DH} \mathrm{d}m = \int_{DH} \big(\frac{\varkappa}{8\pi}\nabla_a A + w\nabla_a V\big) t^a \mathrm{d}^3 y \,, \tag{9.74}$$

where $\varkappa$ is the surface gravity associated to the Kodama vector,

$$k^a \nabla_{[a} k_{b]} = \varkappa k_b \,. \tag{9.75}$$

Equation (9.74) resembles (and, in fact, coincides with, see [151] for the details) the first law of thermodynamics, $\mathrm{d}E = T\mathrm{d}S + p\mathrm{d}V$. Now, we want to show that the relative entropy term at past infinity can also be interpreted as a thermodynamic contribution to the system.

In fact, if we introduce a new coordinate $V = e^{\varkappa(v - v_0)}$, a boost acts in this case as

$$V \longrightarrow e^{\varkappa e^{-2\pi t}(v - v_0)} = V^{e^{-2\pi t}} \,. \tag{9.76}$$

The modular action on the scalar field restricted to $\mathcal{I}^-$ therefore is

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_t(\hat{\varphi}(V,\theta_1,\theta_2))\Big|_{t=0} = 2\pi V\partial_V\hat{\varphi}(V,\theta_1,\theta_2)\,. \tag{9.77}$$

On the other hand, the Kodama vector in this new coordinate is $k = \frac{\partial V}{\partial v}\partial_V = \varkappa V\partial_V$, and therefore, computing (9.2) in these coordinates, following the same passages we did in Section 9.4.4, we arrive at

$$S_{v_0}(\omega|\omega_{\hat{\varphi}}) = 2\pi\int_{\mathcal{I}^-(v_0)}\frac{1}{\varkappa}T_{ab}k^a\mathrm{d}\Sigma^b \tag{9.78}$$

Since $\varkappa$ is a constant at infinity, we get

$$\frac{\varkappa}{2\pi}S_{v_0}(\omega|\omega_{\hat{\varphi}}) = \Phi_{\mathcal{I}^-}\,. \tag{9.79}$$

We see then that the relative entropy has a clear thermodynamic interpretation as a true entropy contribution from the scalar field, where the temperature is given by $T = \frac{\varkappa}{2\pi}$, which is analogous to the Hawking temperature for the black hole, while the Kodama flux is the energy contribution of the scalar field. The role of $\varkappa/2\pi$ is that of temperature of a dynamical, spherically symmetric black holes, as the temperature of states of a scalar field near the dynamical horizon (in analogy with Hawking's characterization of a Schwarzschild black hole temperature) [175].

9.7 COSMOLOGICAL HORIZONS IN DE SITTER SPACE

The same approach can be used to test the entropy of cosmological horizons [82, 131]. In fact, using the Raychaudhuri equation for the null generators of the cosmological horizon, the back-reaction of relative entropy between coherent states can be put in relation with the area of the cosmological horizon in de Sitter (dS) space [86].

Here we only briefly recall the main steps in the computation, as the general construction closely follows the one given for dynamical black hole. The dS metric in null coordinates is

$$\mathrm{d}s^2 = \frac{4}{H^2(u+v)^2}\left[-\mathrm{d}u\mathrm{d}v + \frac{(v-u)^2}{4}\mathrm{d}\Omega^2\right]\,, \tag{9.80}$$

and the future cosmological horizon is the surface $v = 0$, $u \in [-2\ell, 0]$. As in the dynamical black hole case, the relative entropy between coherent states can be given as the surface integral of the stress-energy tensor for the coherent wave on some Cauchy surface Σ. Since the formula for the relative entropy is independent on the choice of the Cauchy surface [124], we can consider the limit in which Σ coincides with the cosmological horizon. In this limit, the relative entropy is given by [86]

$$S\left(\omega|\omega_{\hat{\varphi}}\right) = -\frac{2\pi}{\ell}\int_{-2\ell}^{0}\int T_{uu}|_{v=0}u(u+2\ell)\mathrm{d}\Omega\mathrm{d}u\,. \tag{9.81}$$

Here ℓ denotes a particular value for the affine parameter u along the horizon, similar to v_0 in the computation for dynamical black holes.

To determine the change of the horizon area to leading order in perturbations, we compute the back-reaction of the matter on the geometry analogously to the case of black holes [155]. The leading correction to the metric under the influence of the stress tensor of the coherent wave is again quadratic in the perturbation. Hence, the leading-order correction to the Raychaudhuri equation, in null coordinates, determines the back-reaction on the background geometry,

$$\frac{\mathrm{d}\delta^2\Theta}{\mathrm{d}u}|_{v=0} = -32\pi T_{uu}\,, \tag{9.82}$$

where $\delta\Theta$ denotes the geodesic expansion of the cosmological horizon due to the coherent perturbation. Both the shear and vorticity tensors, as well as the contribution quadratic in Θ, do not contribute to the right-hand side of the Raychaudhuri equation when evaluated at second order in perturbations [155].

Multiplying the Raychaudhuri equation with $u(u+2\ell)$ and integrating over u and Ω, Eq. (9.81) can be written as

$$S\left(\omega|\omega_{\hat{\varphi}}\right) = \int_{-2\ell}^{0}\frac{u(u+2\ell)}{16\ell}\int\frac{\mathrm{d}\delta^2\Theta}{\mathrm{d}u}|_{v=0}\mathrm{d}\Omega\mathrm{d}u\,. \tag{9.83}$$

Since on the background we have $\mathrm{d}\Theta/\mathrm{d}u = 0$, to leading order in perturbations we can replace $\delta^2\Theta$ by the complete geodesic expansion $\Theta+\delta^2\Theta = \tilde{\Theta}$. In the same way, $\mathrm{d}\Omega$ is substituted by the complete line element $\mathrm{d}\tilde{\Omega} = \sqrt{\gamma}\mathrm{d}\theta\mathrm{d}\phi$. Then we use the definition of the expansion as the logarithmic derivative of the cross-sectional area of the geodesic congruence [239], $\tilde{\Theta} = \mathrm{d}\left(\ln\sqrt{\gamma}\right)/\mathrm{d}u$, and integrate by parts. Since the fields are massless and propagate along null geodesics, the area perturbations at time-like and space-like infinity vanish. Thus, the computation results in

$$S\left(\omega|\omega_{\hat{\varphi}}\right) = \frac{1}{4}\langle\langle A_{\mathrm{H}}\rangle\rangle\,, \tag{9.84}$$

where the cross-sectional area of the perturbed cosmological horizon is given by

$$A_{\mathrm{H}}(u) = \int\sqrt{\gamma(u)}\mathrm{d}\theta\mathrm{d}\phi\,, \tag{9.85}$$

and we defined the null average

$$\langle\langle A\rangle\rangle = \frac{1}{2\ell}\int_{-2\ell}^{0}A(u)\mathrm{d}u\,. \tag{9.86}$$

It follows that to leading order in matter perturbations, the perturbation of generalized entropy $\delta S_{\mathrm{gen}} = -S\left(\omega|\omega_{\hat{\varphi}}\right) + \frac{1}{4G_{\mathrm{N}}}\langle\langle\delta A_{\mathrm{H}}\rangle\rangle$ vanishes for dS diamonds. We note that the minus sign in the generalized entropy, which gave raise to some subtle interpretations [17], naturally arises in this context. Hence the generalized entropy conjecture, according to which S_{gen} does not increase when perturbing dS, holds in our case, and we expect that it decreases when we take into account also the back-reaction of quantized metric perturbations.

9.8 FUTURE DIRECTIONS

Let us conclude with a review on the assumptions, and the possible future generalisations.

The method presented here relies on two essential geometrical assumptions: an asymptotic boundary and a conservation law. In the case of dynamical black holes, the asymptotic boundary is conformally flat, and this allows to directly use the Bisognano-Wichmann theorem, or its generalisation, to compute the relative entropy.

Any boundary with a group of symmetries in principle allows for such an analysis of the relative entropy. Moreover, a conservation law for the stress-energy tensor is necessary to compute the entropy flux on the horizon.

Our method can thus be generalised to a variety of situations, and it can be considered as a test to compute the reaction of the black hole to matter entropy. It can be in principle applied to asymptotically de Sitter or Anti de Sitter black holes, and in the presence of angular momentum and electric charge, both in a stationary and in a dynamical phase, either using a Killing field or the Kodama field.

Modifications on the matter content are more delicate. Usually, relative entropy is known when it can be reduced to a computation of the modular Hamiltonian, instead of the relative modular Hamiltonian; it thus can be computed whenever the modular Hamiltonian is known. By now, other than for coherent states of scalar fields, the relative entropy has been computed for unitarily excited states [176], in scalar field theories, and in certain conformal field theories. Recently, Galanda [126], and Galanda, Much and Verch [127] produced new results extending the computation of relative entropy to fermionic systems. All these cases can be further generalised to curved spacetimes: for example, Fröb computed the modular Hamiltonian for causal diamonds in de Sitter [124]. These recent advances in the computation of relative entropy open up exciting new directions to investigate the relation between entropy and geometry.

10 *Conclusions*

General Relativity was discovered in 1915, and Quantum Mechanics in 1926. The realisation that the gravitational field should have been quantised along the same lines of the electromagnetic field came almost immediately. Already in 1916, Einstein pointed out that quantum effects would modify the theory of General Relativity [104, 229]. The first technical paper on a quantum theory of gravity was published in 1930 by Rosenfeld [222]. For over a century, the search of quantum gravity has been fascinated generations of physicists. Despite enormous progress, and the development of many approaches [89], we still do not have a complete theory of quantum gravity.

Among many conceptual difficulties, the problem of quantum gravity faces two important technical considerations, one theoretical, the other experimental. The first one is the realisation that quantum gravity, when treated with the standard approaches of QFT, is perturbatively non-renormalizable [2, 136, 137]. The second issue is the difficulty of directly probing the energy scales at which quantum gravity should become relevant.

The problem of renormalizability leaves open the possibility of the Asymptotic Safety scenario [214, 241, 242], in which a QFT of the metric tensor provides a quantum theory of gravity, that remains finite at arbitrary high-energies thanks to the existence of a non-trivial fixed in its RG flow. In the absence of experimental verifications, theoretical approaches to quantum gravity must be based on internal consistency, conceptual clarity and mathematical rigour.

In this thesis, motivated by the asymptotic safety scenario, we took a small step forward in the search of quantum gravity, providing a tool to investigate the non-perturbative renormalization of gauge theories on curved spacetimes. Thanks to the algebraic approach, we were able to have good control on the assumptions behind our RG flow equations, so that that no divergences arise in any step of their derivation. A Hadamard-type regularisation, similar to the point-splitting procedure that have been extensively studied to construct the renormalized stress-energy tensor in curved spacetime, eliminates possible UV divergences in the RG equations, so that the flow remains finite. The RG flow equations can then be expanded in series, and it is possible to construct systematic truncation schemes based on the Hadamard expansion of propagators, similar to the ones that are studied in Euclidean signature based on the heat kernel expansion.

Thanks to the conceptual distinction between algebras and states at the heart of the algebraic approach, the RG flow equations in Lorentzian spacetime exhibits a distinctive dependence on the state. In order to write closed differential equations for the effective average action, we exploited this dependence fixing a reference Hadamard state for the free theory, as a new, initial datum that has to be provided in order to have a well-defined RG flow. We then showed that the choice of different states has important physical consequences in the phase structure of a given theory.

Thanks to the BV formalism, the gauge invariance of on-shell observables can be compactly summarised in the fundamental requirement that the bare action satisfies the Quantum Master Equation. This equation in turn provides a non-trivial symmetry identity for the effective average action, in the form of extended Slavnov-Taylor identities. The use of an extended regulator sector, supplemented by the BRST variation of the regulator term, allows a treatment of the extended Slavnov-Taylor identities with cohomological methods, providing a non-perturbative control on the symmetries of the effective average action.

Finally, we were able to prove a theorem on the existence of local solutions of the RG flow equations, based on the Nash-Moser theorem, with the assumptions that the effective average action is local in the fields and does not contain derivatives higher than second order of the fields. The proof of the Nash-Moser theorem is constructive, and it is based on an iteration procedure generalising Newton's method with the introduction of suitable smoothing operators. The method can then be used to construct approximate, explicit solutions of the RG flow and prove their convergences in specific examples.

The novel non-perturbative, Lorentzian RG flow developed in this thesis can now be applied to investigate QFT in curved spacetimes in strong coupling regimes, and in situations where classical and quantum gravity can play a major role. Indeed, we showed that the minimal ingredients of universal contributions provide a mechanism for Asymptotic Safety, the first background-independent and covariant result on the non-perturbative renormalizability of quantum gravity also in Lorentzian spacetimes.

Our journey started with a glass of wine, and it ended, as usual, with black holes. Perhaps the most important lesson is that there is still much to be learned about "good old Quantum Field Theory" [181]. QFT is, by now, our theoretical tool that makes the most accurate predictions on the observed Universe [145]. Despite this spectacular success, QFT is still a largely mysterious theory, whose mathematical foundations have yet to be understood completely. Going back to the minimal ingredients of a set of observables, their symmetries, and the causal structure of the underlying background, can provide valuable insight on what are the essential principles of the theory, and what are its most far-reaching consequences. And the applications of QFT to gravity, which are now living a reviving interest, can unravel a world of riches and subtle dynamics that shaped our Universe from its first instants to the cosmological scales around us, and it is all to be discovered. Looking at the starry night with new eyes, we can see that there still is so much beauty yet to be found.

Bibliography

PUBLICATIONS

The main results of this thesis appeared previously in the following publications:

80. D'Angelo, E. Entropy for spherically symmetric, dynamical black holes from the relative entropy between coherent states of a scalar quantum field. *Class. Quant. Grav.* **38,** 175001. arXiv: 2105.04303 [gr-qc] (2021) (cit. on p. 16).

81. D'Angelo, E. *Asymptotic Safety in Lorentzian quantum gravity* arXiv: 2310.20603 [hep-th] (cit. on p. 16).

83. D'Angelo, E., Drago, N., Pinamonti, N. & Rejzner, K. An algebraic QFT approach to the Wetterich equation on Lorentzian manifolds. *Ann. Henri Poincaré.* arXiv: 2202.07580 [math-ph] (July 2023) (cit. on pp. 16, 93, 116, 158, 160).

84. D'Angelo, E. & Pinamonti, N. *Local solutions of RG flow equations from the Nash-Moser theorem* arXiv: 2310.20596 [math-ph] (cit. on p. 16).

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COLOPHON

The typographical style of this document is inspired by the works of Edward R. Tufte and Robert Bringhurst. It is realised by a mixture of the package `classicthesis`, developed by André Miede and Ivo Pletikosić, and the class `tufte-latex`.
The text and math font is Cochineal, a fork of Sebastian Kosch's Crimson designed by Michael Sharpe, while the monospaced text uses Raph Levien's `Inconsolata`.
The image on the half title page is a stylistic rendition of *The Great Wave off Kanagawa*, by Hokusai (1831).

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