Advanced Estimation Algorithms for Nonlinear Systems: Design Methods and Applications

Thèse

Présentée et soutenue publiquement pour l'obtention du titre de

DOCTEUR DE L'UNIVERSITÉ DE GÊNES ET DE L'UNIVERSITÉ DE LORRAINE

(Mention : Dott. Ric. in Ingegneria Matematica e Simulazione (XXXVII Ciclo) Mention : Automatique)

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List of publications

International Journal Papers

[1] **H. Arezki**, A. Zemouche, P. Bagnerini. *Observer Design for Nonlinear Systems with Delayed Output Measurement* . Automatica, *(major revision)*

[2] **H. Arezki**, A. Zemouche, A. Alessandri. *Robust Moving Horizon Estimation Schemes for Nonlinear Systems Through Advanced Prediction Strategies*. IEEE Transactions on Automatic Control, *(major revision)*

[3] **H. Arezki**, A. Zemouche, A. Alessandri, P. Bagnerini. *LMI Design Procedure for Incremental Input/Output-to-State Stability in Nonlinear Systems*. IEEE Control Systems Letters, *vol. 7, pp. 3403–3408, 2023.*

[4] **H. Arezki**, A. Zemouche. *Simple but Useful Contributions to High-Gain Observer for Non-Triangular Systems*. IEEE Control Systems Letters, *vol. 7, pp. 3343–3348, 2023.*

International Conference Papers

[1] **H. Arezki**, A. Zemouche Observer Design for Nonlinear Systems with Delayed Nonlinear Output Measurement. *European Control Conference, ECC 2024* , June 2024, Stockholm, Sweden.

[2] **H. Arezki**, A. Zemouche, A. Alessandri, P. Bagnerini. LMI Design Procedure for Incremental Input/Output-to-State Stability in Nonlinear Systems. *IEEE American Control Conference, ACC 2024*, Toronto, Canada, July 2024.

[3] **H. Arezki**, A. Zemouche. Simple but Useful Contributions to High-Gain Observer for Non-Triangular Systems. *IEEE American Control Conference, ACC 2024*, Toronto, Canada, July 2024.

[4] **H. Arezki**, A. Zemouche, P. Bagnerini, S. Djennoune. LMI Feasibility Analysis in Observer Design for Some Families of Nonlinear Systems. 62^{th} *IEEE Conference on Decision and Control, CDC 2023*, pp. 2645-2650, Singapore, December 2023.

[5] **H. Arezki**, A. Alessandri, A. Zemouche. Robust Moving-Horizon Estimation for Quasi-LPV Discrete-Time Systems *The 22nd IFAC World Congress, IFAC WC'23*, vol. 56 (2) pp. 6771-6776, Yokohama, Japan, July 2023.

[6] **H. Arezki**, A. Alessandri, A. Zemouche. Robust Stability Analysis of Moving-Horizon Estimator for LPV Discrete-Time Systems *The 22nd IFAC World Congress, IFAC WC'23* , vol. 56 (2) pp. 11578-11583, Yokohama, Japan, July 2023.

[7] **H. Arezki**, A. Alessandri, A. Zemouche. Robust Moving Horizon Estimation for Lateral Vehicle Dynamics *21st European Control Conference, ECC 2023*, pp. 1-6, Bucharest, Romania, June 2023.

[8] **H. Arezki**, F. Zhang, A. Zemouche. A HG/LMI-Based Observer for a Tumor Growth Model *European Control Conference, ECC 2022*, pp. 259-263, London, UK, July 2022.

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Introduction

In mathematics, chemistry, and physics, a dynamical system consists of a set of data and a law that governs how these data evolve over time, such as the swinging of a clock pendulum, the movement of planets in the solar system, or the evolution of a computer's memory.

One of the key points in the study of dynamical systems is their stability. Stability analysis ensures that the system behaves predictably under conditions and disturbances and can determine whether a system remains in equilibrium or exhibits chaotic behavior. For example, in engineering, the stability of control systems is critical to prevent failures. Stability can be understood particularly through, Lyapunovs' methods [83], which provide an approach to determining whether a system will converge to a desired state. Furthermore, Input-to-State Stability (ISS) [111–115], provides a powerful set of tools for determining whether a system with external disturbances will converge to a desired state or if it will exhibit undesirable behavior over time.

In parallel, state estimation is another critical challenge, especially in systems where direct measurement of all states is not possible. In practice, all the state variables are not always available for feedback. Possible reasons include expensive sensors; available sensors not acceptable (due to high noise, high power consumption, etc.); and non-availability of sensors. Estimation techniques allow for the reconstruction of internal states from available measurements, which is essential for feedback control and monitoring in real-world applications. Estimators and observers are both used to estimate the internal states of a system, but they differ in approach. A state observer is a specific type of estimator, it's a system that provides an estimate of the internal state of a given real system, from measurements of the input and output of the real system. It is typically computer-implemented. An observer focuses on real-time state estimation using deterministic models, making it more computationally efficient but less robust to noise. Several observer methods have been developed, including the sliding mode observer [118], dissipative type observers [26,27], the high-gain observer approach [2,13,24,41,141], sliding mode observer [37,108] and the LMI-based observer techniques [1, 71, 131, 136]. Estimators like the EKF and MHE incorporate statistical or optimization methods to handle noise and uncertainties, making them suitable for systems with significant disturbances. Among these methods, without being exhaustive, we mention the extended Kalman filter [66, 81, 110, 117], the moving horizon estimators [5–7, 10, 15, 105]. Every observer is an estimator, but Not Vice Versa. The choice between them depends on the application's requirements for accuracy, computational complexity, and real-time performance.

Thesis Contributions

This thesis proposes several contributions to the study of dynamical systems, particularly in the areas of stability analysis and state estimation:

1. **Analysis of robust moving horizon estimation (MHE) schemes using incremental exponential input/output-to-state stability (i-EIOSS) property of the system : Chapter 2** explores the application of i-EIOSS properties in robust MHE schemes. It provides both theoretical analysis and practical implementation strategies, including methods for bounding estimation errors and improving prediction accuracy.

    - Two numerical design procedures are proposed to ensure that a nonlinear system achieves the i-EIOSS property and to compute the associated i-EIOSS parameters. The first method is a new Linear Matrix Inequality (LMI) condition guaranteeing the computation of the (i-EIOSS) coefficients. The proposed design method is easily tractable by numerical software and may be used for several real-world applications. The second method is an innovative LMI-based method for synthesizing the i-EIOSS coefficients.
    - The robust stability of moving horizon estimation (MHE) is proven for a class of nonlinear systems satisfying the i-EIOSS property. Based on some new mathematical tools, novel design conditions to tune the parameters of the cost function of the MHE scheme are proposed. These conditions are linked to the size of the MHE window and coefficients associated with the system's incremental exponential input/output-to-state stability (i-EIOSS). To enhance the MHE's robust stability while minimizing the window size, various prediction techniques. Furthermore, innovative LMI-based methods for synthesizing the i-EIOSS coefficients and prediction gains are established. The performances resulting from adopting the proposed prediction methods are compared through numerical tests.

2. **Contributions to LMI-based and high-gain-based observers**: the thesis presents new Linear Matrix Inequality (LMI) based methods for the synthesis of stability criteria and observer designs, providing practical tools for the design and implementation of robust control systems. **Chapter 3** can be divided into two parts:

    - The first part deals with observer design for nonlinear systems via LMIs. The main goal consists of showing that for some families of nonlinear systems, the LMI-based observer design techniques always provide exponential convergent observer. Indeed, until now, this advantageous feature is unique to some types of observers/estimators, such as the high-gain observer, the sliding mode observer, and the moving horizon estimator, under certain conditions of detectability or observability. More specifically, the proposed LMI conditions always provide solutions to both systems in companion form and feedforward structure. An extension to a general class of nonlinear triangular systems without linear components is provided, which renders the applicability of LMI-based methods possible for a wide class of nonlinear systems without the need for nonlinear diffeomorphism-based transformations.
    - The second part deals with nonlinear observer design for a class of non-triangular systems satisfying the sector condition. Under such a condition on the nonlinearity of the system, contributions to the design of high-gain observers are proposed for systems with arbitrary nonlinear structures contrary to the standard results on high-gain observer methodology developed only for triangular nonlinearities. First, based on the use of a convenient decomposition of the nonlinear function, a general design method is provided. Compared to the standard high-gain observer, the proposed method requires an

extra constraint on the tuning parameter to dominate the non-triangular components of the nonlinearity of the system. To reduce the conservatism of the extra-condition, further results are established by exploiting the LMI-based approach. A numerical design algorithm is provided to build all the observer parameters. Finally, an illustrative example is presented to show the effectiveness and validity of the proposed technique.

3. **Contributions to observer design for nonlinear systems with delayed measurements: Chapter 4** deals with nonlinear observer design for systems with delayed nonlinear outputs. The main idea behind consists of using a dynamic extension technique to transform a system with delayed nonlinear outputs into a system with linear outputs and a delay-dependent integral term in the dynamic process. First, a general result for arbitrary nonlinear structures is proposed, and then further contributions are provided for specific families of systems, namely systems in companion form and feedforward systems. For systems in companion form, we obtain novel high-gain observer synthesis conditions, while new low-gain results are obtained in the case of feedforward systems. To relax the necessary conditions related to the observer design procedure, a different state observer structure is proposed, and analytical comparisons are provided. Moreover, to relax the design conditions, we propose an alternative design approach using a specific Lyapunov–Krasovskii functional. This method can handle high values of the maximum allowable delay, however, it guarantees only asymptotic convergence of the error rather than exponential convergence.

Thesis Organization

The thesis is organized into four chapters, each building on the foundational concepts introduced previously:

- **Chapter** 1 introduces foundational ideas in stability analysis, starting with Introduction. Then, Section 1.2, introduces the key concepts in stability analysis, including Lyapunov's methods and Input-to-State Stability (ISS). Section 1.3, follows with an overview of estimation in dynamical systems, covering observers, LMI-based observers, and estimators.

- **Chapter** 2 begins with Section 2.2, which introduces the class of systems under consideration and the associated assumptions. It also presents the mathematical tools necessary for the main design methods. Section 2.3 outlines two methods for proving that a nonlinear system satisfies the i-EIOSS property, providing expressions for computing the related parameters. Section 2.4 demonstrates that MHE schemes are robust for nonlinear systems with the i-EIOSS property, addressing error bounds, predictions, and gain synthesis. Section 2.5 presents numerical results, featuring examples such as chaotic systems and tumor growth models. The chapter concludes with Section 2.6.

- **Chapter** 3 is organized as follows: Section 3.2 focus of this section is on proving an "Always LMI Feasibility Proof" for observer design in certain families of nonlinear systems including those in canonical form, systems with feedforward structures, and extensions to a broader class of systems. Next, Section 3.3 focuses on contributions to high-gain observer design for non-triangular systems. This section starts by outlining the motivation and formulation and presenting results specifically tailored to address non-triangular nonlinearities. Finally, it discusses additional refinements to the observer design, emphasizing the innovations and improvements made to overcome the challenges posed by non-triangular systems.

- **Chapter** 4 deals with observer design for nonlinear systems with delayed output Measurement. Section 4.2 describes the problem formulation and presents the motivations of this work compared to the available methods in the literature. Section 4.3 provides the main idea of this work, starting with a preliminary result as a main tool. Section 4.4 proposes specific new results for particular families of systems, namely systems in companion form and feedforward systems. Section 4.5 gives a relaxation technique to avoid certain required assumptions on the system state. Section 4.6 introduces some constructive comments and analytical comparisons. Section 4.7 presents an alternative method relaxing the existing results in the literature. Section 4.8 introduces an illustrative example to show the validity and efficiency of the proposed methods. Finally, Section 4.9 concludes the work and discusses future endeavors.

- **Conclusion:** The thesis concludes with a summary of the contributions and suggestions for future research directions, highlighting the potential for further advancements in stability analysis and estimation techniques.

CHAPTER

1

Brief Overview on State Estimation

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1.1 Introduction

This chapter gives some fundamental concepts of stability and estimation in dynamical systems. It begins with Section 1.2, introducing some notions of stability, including Lyapunov's methods and the concept of Input-to-State Stability (ISS), both crucial for analyzing and ensuring system robustness. Then, in Section 1.3 a brief recall of some estimation techniques is given, where key topics such as observability, observer design, and estimation methods. These foundational tools and methods are necessary to understand further contributions in designing reliable estimation schemes.

1.2 Notions of Stability

Stability is a fundamental concept in the study of dynamical systems. It is crucial to determine whether a system, when subjected to small disturbances, will return to its equilibrium state or diverge away from it. This section introduces the foundational concepts of stability, particularly focusing on Lyapunov's methods, which provide powerful tools for analyzing the stability of nonlinear systems. Let us consider the following autonomous system described by the differential equation:

$$\dot{x} = f(x), \tag{1.1}$$

where $x \in D \subset \mathbb{R}^n$, and D is a domain containing the origin. We assume that the function f governing the system's dynamics is piecewise continuous for time t and locally Lipschitz with respect to the state x. This implies that there exists a constant $k > 0$ such that:

$$\|f(x_1) - f(x_2)\| \leq k\|x_1 - x_2\|, \tag{1.2}$$

for all $x_1, x_2 \in D$. Additionally, we assume that $f(x_0)$ is bounded, where x_0 represents the initial condition at $t = 0$.

Definition 1.2.1 ([61, 109, 127]). *An equilibrium point x^* of the system (1.3) is a state where the system remains at rest if it starts from that state. Mathematically, this is expressed as:*

$$x(0) = x^* \Rightarrow x(t) = x^* \quad \forall t \geq 0.$$

Equivalently, x^* is an equilibrium point if and only if:

$$f(x^*) = 0, \quad \forall t \geq 0.$$

Without loss of generality, we will focus on the stability of the origin ($x = 0$) for the autonomous system (1.3). This simplifies the analysis while still capturing the essential behavior of the system.

1.2.1 Basic Concept of Stability

The concept of stability can be categorized into different types depending on how the system behaves when it is perturbed from its equilibrium state. These categories include stability, asymptotic stability, and exponential stability.

Definition 1.2.2 ([69]). *The origin is said to exhibit different types of stability based on the following criteria:*

1. ***Stability:*** *The origin is a stable equilibrium point if, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that:*

   $$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0.$$

   If this condition is not met, the origin is considered unstable.

2. ***Asymptotic Stability:*** *The origin is asymptotically stable if it is stable, and in addition, δ can be chosen such that:*

   $$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0.$$

   This means that solutions not only remain close to the origin but also converge to the origin as time goes to infinity.

3. ***Exponential Stability:*** *The origin is locally exponentially stable if there exist constants $\alpha \geq 1$ and $\beta > 0$ such that:*

   $$\|x(t)\| \leq \alpha\|x(0)\| \exp(-\beta t), \quad \forall t \geq 0, \ \forall x(0) \in \mathcal{B}_r.$$

   If $\mathcal{B}_r = \mathbb{R}^n$, we say that the origin is globally exponentially stable. This indicates that solutions decay to the origin at an exponential rate.

In summary, an equilibrium point is stable if small deviations from the point do not cause the system to diverge significantly from it. It is asymptotically stable if such deviations not only stay small but also tend to zero over time. Exponential stability implies a stronger form of asymptotic stability, where the rate of convergence is exponential.
These definitions highlight the importance of understanding the system's behavior in response to initial perturbations. However, directly applying these definitions to prove stability requires explicit solutions of the system, which can be challenging or even impossible to obtain. Alternative methods, such as Lyapunov's methods, are employed to overcome this difficulty.

1.2.2 Lyapunov's Methods for Stability Analysis

Lyapunov's methods provide powerful tools for determining the stability of equilibrium points without requiring explicit solutions of the system's differential equations. The two main approaches are Lyapunov's indirect method and Lyapunov's direct method.

1.2.2.1 Lyapunov's Indirect Method

Lyapunov's indirect method relates the stability of the origin for the nonlinear system (1.3) to the stability of the system's linear approximation near the origin. This method is particularly useful because linear systems are generally easier to analyze.

Theorem 1.2.3. *[57, 109] Let $x = 0$ be an equilibrium point for the nonlinear system (1.3), where $f : D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. The linearized system around the origin is given by:*

$$\dot{x} = f(x) \Rightarrow \dot{x} = Ax,$$

where $A = \frac{\partial f}{\partial x}\Big|_{x=0}$ is the Jacobian matrix evaluated at the origin.
The stability of the origin can be classified as follows:

1. *The origin is locally asymptotically stable if all eigenvalues of A have negative real parts Re$(\lambda_i) < 0$.*

2. *The origin is unstable if at least one eigenvalue of A has a positive real part Re$(\lambda_i) > 0$.*

3. *If some eigenvalues have zero real parts Re$(\lambda_i) = 0$ and the rest have non-positive real parts, the system's behavior is more complex:*

   - *If Re$(\lambda_i) \leq 0$ for all i, the origin may be marginally stable.*
   - *Otherwise, the origin is unstable.*

This theorem provides a criterion for local stability based on the linearized system, making it a practical tool for stability analysis in many cases. However, when the linear approximation is insufficient to determine stability, Lyapunov's direct method can be used.

1.2.2.2 Lyapunov's Direct Method

The direct method of Lyapunov is based on the concept of energy dissipation. In physical systems, if the total energy of the system continuously decreases over time, the system will eventually settle into an equilibrium state. This observation can be formalized using a scalar function $V(x)$, often interpreted as the system's total energy.

Theorem 1.2.4 (Lyapunov Function and Lyapunov Stability [69]). *Let $x = 0$ be an equilibrium point for the nonlinear system (1.3), and let D be a neighborhood of the origin. Consider a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions:*

- *$V(x) = 0$ if and only if $x = 0$.*
- *$V(x) > 0$ if $x \neq 0$ (i.e., $V(x)$ is positive definite).*
- *The time derivative of V along the trajectories of the system is non-positive:*

$$\dot{V}(x(t)) = \frac{d}{dt}V(x(t)) = \nabla V(x) \cdot f(x) \leq 0, \quad \forall x \neq 0.$$

Then $V(x)$ is called a Lyapunov function, and the system is stable.

- *The system is asymptotically stable if $\dot{V}(x) < 0$ for all $x \neq 0$.*
- *The system is globally asymptotically stable if, in addition, $V(x) \to +\infty$ as $\|x\| \to +\infty$ (this condition is known as "radial unboundedness").*

Lyapunov's direct method provides a powerful way to prove stability without solving the differential equations explicitly. By constructing an appropriate Lyapunov function, one can conclude the stability of the system's equilibrium.

1.2.3 Input-to-State Stability (ISS) and Its Extensions

In practical applications, control systems are frequently influenced by noise, perturbations in controls, and observational errors. Therefore, it is desirable for a system to be not only stable but also input-to-state stable (ISS). ISS was first introduced in the pioneering work of E. D. Sontag in 1989in his work [111]. His research laid the foundation for understanding how external inputs or disturbances affect the stability of nonlinear systems. Let the following differential equation

$$\dot{x} = f(x, u), \tag{1.3}$$

where $x \in \mathbb{R}^n$, where $u : \mathbb{R}_+ \to \mathbb{R}^m$ is a Lebesgue measurable essentially bounded external input and f is a Lipschitz continuous function w.r.t. the first argument uniformly w.r.t. the second one. This ensures that there exists a unique absolutely continuous solution of the system. with input $u : [0, \infty) \to \mathbb{R}^n$ (also called "controls" or "disturbances" depending on the context) being measurable locally essentially bounded maps.

1.2.3.1 Comparison Functions Formalism

In the context of stability analysis, comparison functions are essential mathematical tools used to describe the behavior of dynamical systems over time. These functions help formalize notions of stability, such as global asymptotic stability (GAS) and input-to-state stability (ISS), using sets and function properties. Below, we define three important classes of comparison functions: $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$.

$$\begin{aligned}
\mathcal{K} &:= \{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \text{continuous, strictly increasing, and } \alpha(0) = 0\} \\
\mathcal{K}_\infty &:= \{\alpha \in \mathcal{K} \mid \alpha \text{ is unbounded}\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \text{continuous, } \beta(\cdot, r) \in \mathcal{K}\ \forall r \geq 0, \\
&\qquad \text{and } \beta(s, t) \text{ converges strictly to } 0 \text{ as } t \to \infty \text{ for all fixed } s \in \mathbb{R}_0^+\}.
\end{aligned} \tag{1.4}$$

The class $\mathcal{K}$ represents continuous, strictly increasing functions that start at zero. The class $\mathcal{K}_\infty$ is a subset of $\mathcal{K}$ that includes functions that are also unbounded, meaning they grow without bound as their argument increases. Lastly, the class $\mathcal{KL}$ contains functions that depend on two variables, where the first variable behaves as a $\mathcal{K}$-class function, and the second variable ensures strict convergence to zero over time. These comparison functions are useful for defining stability concepts in dynamical systems. For more information about the comparison functions formalism one can refer to the works [40, 63, 70, 111, 113, 120]. For instance, a system is said to be globally asymptotically stable (GAS) if there exists a $\beta \in \mathcal{KL}$ such that the following inequality holds:

$$|x(t, x_0)| \leq \beta(|x_0|, t), \ \forall x_0 \in \mathbb{R}^n, t \geq 0 \tag{1.5}$$

$|x(t, x_0)|$ denotes the Euclidean norm of the system state at time t starting from the initial condition x_0. The function β describes how the system's state evolves over time, ensuring that the state decays to zero as t increases, thereby demonstrating stability.

1.2.3.2 The Necessity of ISS

A linear system is 0-GAS if and only if its system matrix A is Hurwitz, meaning all eigenvalues of A have negative real parts. A 0-GAS linear system satisfies reasonable input/output stability properties: bounded inputs lead to bounded state trajectories and outputs, and inputs converging to zero ensure the system's solutions (and outputs) also converge to zero. In other words, for linear systems, zero-detectability is equivalent to detectability: two trajectories that produce the same output must approach each other. However, for nonlinear systems, 0-GAS alone does not guarantee desirable behavior concerning inputs.

Example 1.2.5 ([112]). *Consider the following one-dimensional system, with scalar inputs:*

$$\dot{x} = -x + (x^2 + 1)u$$

- *This system is clearly 0-GAS, since it reduces to $\dot{x} = -x$ when $u \equiv 0$. On the other hand, solutions diverge even for some inputs that converge to zero. For example, take the control $u(t) = (2t+2)^{-\frac{1}{2}}$ and $x_0 = \sqrt{2}$, there results the unbounded trajectory $x(t) = (2t+2)^{\frac{1}{2}}$ as shown in Figure 1.1.*

- *This is in spite of the fact that the unforced system is GAS. Thus, the converging-input converging-state property does not hold.*

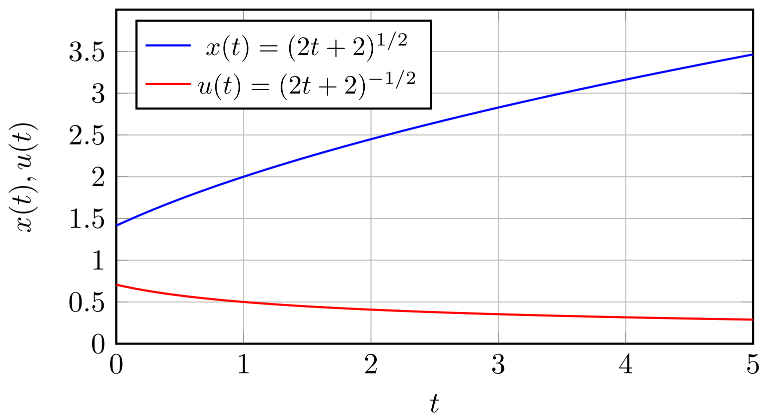

Figure 1.1: Diverging state for converging input

ISS extends the GAS property to nonlinear systems. Intuitively, ISS means that the system's behavior remains bounded when the inputs are bounded and tends towards an equilibrium point when the inputs approach zero [113]. ISS is closely related to the concept of stability under perturbations (total stability), which is well-studied in the classical dynamical systems literature. ISS requires that the system remains GAS up to an error term dependent on the magnitude of the input u, measured via the essential supremum norm:

$$\|u\|_\infty := \operatorname{ess}\sup_{t\geq 0} \|u(t)\|. \tag{1.6}$$

Definition 1.2.6 ([116]). *System is called ISS, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial values x_0, all perturbation functions u and all times $t \geq 0$, the following inequality holds:*

$$|x(t, x_0; u)| \leq \beta(|x_0|, t) + \gamma(\|u\|_\infty, t) \tag{1.7}$$

ISS can be characterized via ISS Lyapunov functions. The ISS Lyapunov function provides a powerful tool for analyzing the stability of nonlinear systems subject to input disturbances. It guarantees that the state trajectories of the system remain bounded, provided that the inputs are also bounded, and the Lyapunov gain ensures that the inputs do not cause the system state to grow faster than a certain rate.

Theorem 1.2.7 ([113]). *A system is ISS if and only if it admits an ISS Lyapunov function, i.e. a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\mathcal{K}_\infty$-functions α_1, α_2 and $\mathcal{K}$- functions α_3, γ such that*

$$\alpha_1|x| \leq V(x) \leq \alpha_2|x| \tag{1.8}$$

and

$$\dot{V}(x) \leq -\alpha_3|x| \tag{1.9}$$

hold for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ so that $|x| \geq \gamma(|u|)$; the function γ is called Lyapunov gain.

The ISS property of an observer ensures that its estimation error is bounded and decays over time, despite the presence of input and measurement disturbances. The observer is said to be ISS if the error between the estimated state and the true state of the system remains bounded in the presence of input and measurement disturbances. This is captured by equation (1.10), which states that the error $|x(t) - \hat{x}(t)|$ between the true state $x(t)$ and the estimated state $\hat{x}(t)$ is bounded by a function of the error and time, $\beta(|x(t) - \hat{x}(t)|, t)$, as well as the supremum of the magnitudes of the input disturbance $v(s)$ and the measurement disturbance $w(s)$ up to time t.

Definition 1.2.8 (ISS-observer [112]). *An observer is ISS if*

$$|x(t) - \hat{x}(t)| \leq \beta(|x(t) - \hat{x}(t)|, t) + \gamma_1\left(\sup_{s\in[0,t)} |v(s)|\right) + \gamma_2\left(\sup_{s\in[0,t)} |w(s)|\right) \tag{1.10}$$

for all $t \geq 0$, for some $\beta \in \mathcal{KL}$, $\gamma_1, \gamma_2 \in \mathcal{K}$.

1.2.3.3 Output State Stability (OSS)

Output to State Stability (OSS) is a property of dynamic systems that relates the system's output to its state. Specifically, a system is said to be OSS if, for any bounded output, there exists a bound on the initial state that ensures the state remains bounded for all time. In [113, 114] output-to-state stability (OSS) was proposed as a generalization of the notion of detectability to nonlinear systems without inputs. Consider the class of systems given by

$$\begin{cases} \dot{x} &= f(x) \\ y &= h(x) \end{cases} \tag{1.11}$$

where the map $f : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be locally Lipschitz with, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous with $h(0) = 0$.

Definition 1.2.9 ([114]). *The system (1.13) is output-to-state stable (OSS) if there exist some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$ such that*

$$|x(t, x_0)| \leq \beta(|x_0|, t) + \gamma(\|y\|_\infty, t) \tag{1.12}$$

1.2.3.4 Input to Output Stability (IOS)

Input-Output Stability (IOS) is a property that bridges the system's input to its output. Specifically, a system is said to be IOS if, for any bounded input, the system's output remains bounded as well. The notion of "input to output stability" (IOS) formalizes the idea that outputs depend in an "asymptotically stable" manner on inputs, while internal signals remain bounded. When the output equals the complete state, one recovers the property of input to state stability (IOS). Consider the class of systems given by

$$\begin{cases} \dot{x} &= f(x, u) \\ y &= h(x) \end{cases} \tag{1.13}$$

where the map $f : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be locally Lipschitz with, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous with $h(0) = 0$.

Definition 1.2.10 ([115,121]). *The system (1.13) is input-to-output stability (IOS): there exist some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$ such that*

$$|y(t)| \leq \beta(|x_0|, t) + \gamma\|u\|_\infty. \tag{1.14}$$

1.2.3.5 Input/Output to State Stability (IOSS)

Input/output-to-state stability (IOSS) is the combination of IOS and SS, it is the property of dynamic systems that relates the input-output behavior of the system to its internal state. An IOSS system is stable in the sense that its output remains bounded for any bounded input, and its internal state remains bounded as well. IOSS was introduced in [113] since it is not possible to deal with inputs and outputs separately in general. It combines the 'strong' observability with ISS.

Definition 1.2.11 ([112]). *The system (1.13) is input/output to state stable (IOSS) if there exist some $\beta \in \mathcal{KL}$ and some $\gamma_1, \gamma_2 \in \mathcal{K}$ such that*

$$|x(t)| \leq \beta(|x_0|, t) + \gamma_1(\|u_{[0,t]}\|_\infty) + \gamma_2(\|y_{[0,t]}\|_\infty) \tag{1.15}$$

The IOSS Lyapunov function is a useful tool for analyzing the IOSS property of a system, and for designing control strategies that ensure IOSS.

Theorem 1.2.12 ([76] (Lyapunov Characterisation)). *A system is IOSS if and only if it admits an IOSS Lyapunov function, i.e. a positive definite smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$ such that*

$$\dot{V}(x) \leq -\alpha_1|x| + \alpha_2|u| + \alpha_3|y| \tag{1.16}$$

hold for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$.

1.2.3.6 Incremental Input/Output-to-State Stability i-EIOSS

Consider the following nonlinear discrete-time system:

$$\begin{cases} x_{t+1} = f(x_t, w_t) \\ y_t = h(x_t, v_t) \end{cases} \tag{1.17}$$

where $x_t \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state of the system; $y_t \in \mathbb{R}^m$ is the output vector; $w_t \in \mathcal{W} \subseteq \mathbb{R}^p$ and $v_t \in \mathcal{V} \subseteq \mathbb{R}^q$ are unknown external disturbances. The functions $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are assumed to be continuous with respect to their arguments. For simplicity, we do not consider known external input, which however does not undermine the generality of what follows. Let us denote by $|x|$ the Euclidean norm $\sqrt{x^\top x}$, $x \in \mathbb{R}^n$.

Definition 1.2.13. *System (1.17) is incrementally exponentially input/output-to-state stable (i-EIOSS) if there exist constants $c_x, c_v, c_w > 0$ and $\varrho \in (0,1)$ such that for any pair of initial conditions $x_0, \tilde{x}_0 \in \mathcal{X}$ and any disturbance sequences $w_t, \tilde{w}_t \in \mathcal{W}$, the following holds:*

$$\begin{aligned} \left|x_t(x_0, w_0^{t-1}) - \tilde{x}_t(\tilde{x}_0, \tilde{w}_0^{t-1})\right|^2 &\leq c_x|x_0 - \tilde{x}_0|^2\varrho^t \\ &+ c_v \sum_{i=0}^{t-1} \varrho^{t-1-i} \left|y_i(x_0, w_0^{i-1}, v_0^i) - y_i(\tilde{x}_0, \tilde{w}_0^{i-1}, \tilde{v}_0^i)\right|^2 \\ &+ c_w \sum_{i=0}^{t-1} \varrho^{t-1-i} \left|w_i - \tilde{w}_i\right|^2 . \end{aligned} \tag{1.18}$$

In practice, this i-EIOSS property ensures that the system is capable of maintaining accurate estimation despite the presence of disturbances and uncertainties. This property guarantees that the difference between the actual state of the system and the estimation obtained by MHE decreases exponentially over time.

The i-EIOSS property is a stronger notion, ensuring that even when inputs and outputs vary incrementally, the estimation error converges exponentially, leading to more robust and reliable estimators. For more details on the above definition, we refer the reader to [85, 105] for a more general case. These stability notions are interconnected, each offering a different perspective on how inputs, outputs, and states interact in a dynamic system. Understanding these relationships is crucial for designing robust and stable control systems. The ISS focuses on the boundedness of the state with respect to the input, and OSS considers the boundedness of the state with respect to the output. The OSS is a generalization of ISS for systems where the output is more accessible. ISS ensures that the state remains bounded for bounded inputs, while IOS ensures that the output remains bounded. IOS is particularly useful in output-driven systems where the main concern is the behavior of the output. The IOSS combines the principles of both IOS and OSS, providing a comprehensive stability criterion that links the input and output to the system's state. The i-IOSS extends IOSS by ensuring that the system's behavior is robust to incremental changes in inputs and outputs, with exponential convergence properties.

1.3 Estimation in Dynamical Systems

Dynamical systems are important in various fields such as engineering, economics, and biology, where understanding the evolution of system states over time is crucial. However, in many practical situations, it is impossible to directly measure all internal states of a system. This leads to the need for state estimation, where the unmeasured states are inferred from available output measurements and known system dynamics. State estimation lies in the concept of observability, which determines whether the internal states of a system can be reconstructed from its output measurements.

In this section, we discuss the observability of both linear and nonlinear systems. We provide a recall of both geometric and rank conditions for observability. We also explore the design and role of observers and estimators.

1.3.1 Observability

The observability of a process is a fundamental concept in the field of state estimation. Indeed, to reconstruct the inaccessible states of a system, it is necessary to know, a priori whether the variables are observable or not. Observability stands for the possibility of reconstructing the full trajectory from the observed data, that is, from the output trajectory in the uncontrolled case, or from the couple (output trajectory, control trajectory) in the controlled case. This property is fundamental for both linear and nonlinear systems and dictates whether an estimator or observer can function effectively.

1.3.1.1 Observability of Nonlinear Systems

Let us consider the class of nonlinear systems described by :

$$\begin{cases} \dot{x} &= f(x,u), \\ y &= h(x,u), \end{cases} \tag{1.19}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$.

Definition 1.3.1. *Observability of nonlinear systems (Geometric Condition) [28]*

- *Indistinguishability: Let $y_u^0(t) \to x^0$ and $y_u^1(t) \to x^1$. Then x^0 and x^1 are indistinguishable if $y_u^0(t) = y_u^1(t),\ \forall\, t \geq 0,\ \forall\, u$. Otherwise, x^0 and x^1 are distinguishable.*
- *The system (1.19) is observable in x^0 if x^0 is distinguishable from any $x \in \mathbb{R}^n$. In addition, the system (1.19) is observable if $\forall\, x^0 \in \mathbb{R}^n$, x^0 is distinguishable.*

Definition 1.3.2 (Observability of nonlinear systems (rank condition))**.** *We say that system (1.19) is observable if the following rank condition holds*[1]*:*

$$\text{rank}\left(dh, dL_f h, \ldots, dL_f^{n-1} h\right)^\top = n$$

where the expression of $dL_f^k h$ is given by

$$dL_f^k h = \left(\frac{\partial L_f^k h}{\partial x_1}, \frac{\partial L_f^k h}{\partial x_2}, ..., \frac{\partial L_f^k h}{\partial x_n}\right).$$

For more details about the observability of nonlinear systems, one can refer to [28, 56, 56, 61, 93, 109].

1.3.1.2 Observability of Linear Systems

Now, let's consider the observability of linear systems, which are easier to analyze due to their linearity. Linear systems are described by the following equations:

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{cases} \tag{1.20}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control and $y \in \mathbb{R}^p$ is the measured output. The matrices A, B, C are constant matrices of appropriate dimensions. The Kalman observability criterion provides a straightforward test for the observability of linear systems. If the system is observable, then it is possible to reconstruct the entire state vector from the output data.

[1]Lie derivative: Let $h : D \to \mathbb{R},\ f : D \to \mathbb{R}^n$, the Lie derivative of h along f is

$$L_f h(x) = \sum_{i=1} f_i(x) \frac{\partial h}{\partial x_i}(x).$$

Definition 1.3.3 (Kalman observability criterion [54]). *the system (1.20) is said to be observable if and only if*

$$\operatorname{rank}\Big(\mathcal{O}(A,C)\Big) \triangleq \operatorname{rank}\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} = n \tag{1.21}$$

$\mathcal{O}(A,C)$ is called Kalman observability matrix (of size $np \times n$).

1.3.2 State Observers

A state observer is a system that provides an estimate of the internal state of a given real system, from measurements of the input and output of the real system. It is typically computer-implemented. There exist many types of observers: including the sliding mode observer [118], dissipative type observers [26, 27], the high-gain observer approach [2, 13, 24, 41, 141], sliding mode observer [37, 108, 118]. the LMI-based observer techniques [1, 71, 131, 136].
A state observer is a typically computer-implemented system that estimates the internal state of a given real system, from measurements of the input and output of the real system. Observers are needed since the full state can not be measured or is too expensive to measure in many applications. In addition, some variables in many applications have to be estimated and can not be measured due to the unavailability of sensors at any cost.

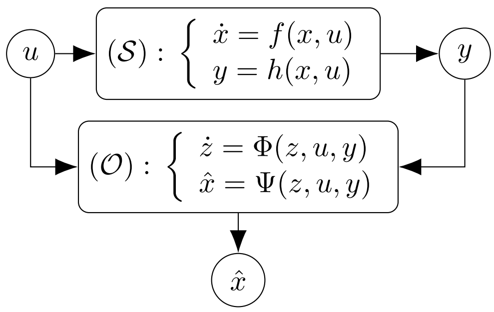

Figure 1.2: State observer principle

Definition 1.3.4 (State Observer [77], [130]). *Consider the following dynamical system:*

$$(\mathcal{S}) : \begin{cases} \dot{x} = f(x,u), \\ y = h(x,u) \end{cases} \tag{1.22}$$

The dynamical system

$$(\mathcal{O}) : \begin{cases} \dot{z} = \Phi(z,u,y), \\ \hat{x} = \Psi(z,u,y) \end{cases} \tag{1.23}$$

is a local asymptotic observer for system (1.22) if :

- $x(0) = \hat{x}(0) \quad \Rightarrow \quad x(t) = \hat{x}(t) \;\; \forall\, t \geq 0;$
- $\exists \Omega \subseteq \mathbb{R}^n \;: x(0) - \hat{x}(0) \in \Omega \quad \Rightarrow \quad \lim\limits_{t \to +\infty} \|x(t) - \hat{x}(t)\| \to 0.$

1. $\|x(t) - \hat{x}(t)\| \to 0$ *exponentially* $\Longrightarrow$ *Exponential observer.*
2. $\Omega = \mathbb{R}^n \Longrightarrow$ *Global observer.*

1.3.2.1 Luenberger Observer

The Luenberger observer is a classical observer widely used for state estimation in linear systems [64, 65, 82, 94]. It operates by estimating the internal states of a system through feedback from the system's output. The Luenberger Observer combines a model of the plant with a correction term. This correction, achieved through a feedback mechanism, adjusts the state estimates based on the difference between the actual system output and the observer's predicted output, using a designer-specified gain. This approach provides a simple yet effective and optimal solution for state estimation in linear systems. We consider the dynamic model of a linear system defined as follows:

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{cases} \tag{1.24}$$

where, at time t, $x(t)$ is the plant's state; $u(t)$ is its inputs; and $y(t)$ is its outputs. Luenberger's theory of observation is essentially based on pole placement techniques. The Luenberger observer for the system (1.24):

$$\dot{\hat{x}} = A\hat{x} + Bu(t) + K(y(t) - C\hat{x}).$$

The dynamics of the estimation error $e(t) = x(t) - \hat{x}$ has the expression

$$\dot{e}(t) = (A - KC)e(t).$$

Using a pole placement technique, it suffices to choose the gain K of the observer so that the eigenvalues of the matrix $A - KC$ are in the left complex half-plane.

1.3.2.2 Sliding Mode Observer

In the presence of unknown signals or system uncertainties, a Luenberger observer is generally unable to drive the output estimation error to zero, and the observer states fail to converge to the true system states. To address this limitation, a sliding mode observer offers an effective solution by feeding back the output estimation error through a nonlinear switching term. This approach enhances robustness against disturbances and uncertainties. For more details on sliding mode observers, see [37, 79, 108, 124]. When a bound on the magnitude of disturbances is known, the sliding mode observer can ensure finite-time convergence of the output estimation error to zero, while the observer states converge asymptotically to the system states. Additionally, this technique allows for the reconstruction of disturbances within the system.

In its simplest form, a sliding mode observer feeds back the output error through a discontinuous switched signal rather than through a linear feedback mechanism. Initially, consider a linear system as described by (1.24). We assume that C has full rank which means that each of the measured outputs is independent and that the pair (A, C) is observable. Consider the change of coordinates $x \mapsto T_c x$ whereby

$$T_c = \begin{bmatrix} N_C^T \\ C \end{bmatrix}$$

where the submatrix $N_c \in \mathbb{R}^{n\times(n-p)}$ spans the null-space of C. By construction $det(T_c) \neq 0$. Applying the change of coordinates $x \mapsto T_c x$, the triple (A, B, C) are rewritten in the new coordinates as follows:

$$T_c A T_c^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ T_c B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ C T_c^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix},\ T_c x = \begin{bmatrix} x_1 \\ y \end{bmatrix} \quad \begin{matrix} \Updownarrow n-p \\ \Updownarrow p \end{matrix}$$

where $A_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $B_1 \in \mathbb{R}^{(n-p)\times m}$. The system (1.24)can therefore be rewritten in the form:

$$\begin{cases} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}y(t) + B_1 u\nu, \\ \dot{y} &= A_{21}x_1(t) + A_{22}y(t) + B_2 u, \end{cases} \tag{1.25}$$

The observer

$$\begin{cases} \dot{\hat{x}}_1(t) &= A_{11}\hat{x}_1(t) + A_{12}\hat{y}(t) + B_1 u + L\nu, \\ \dot{\hat{y}} &= A_{21}\hat{x}_1(t) + A_{22}\hat{y}(t) + B_2 u - \nu, \end{cases} \tag{1.26}$$

where $(\hat{x}_1, \hat{y})$ the state estimates; $L \in \mathbb{R}^{(n-p)\times p}$ is a gain matrix; ν is a discontinuous injection term

$$\nu_i = \rho\,\mathrm{sign}(\hat{y}_i - y_i), \quad i = 1, 2, ..., p, \quad \rho \in \mathbb{R}_+,$$

The term ν is designed to be discontinuous with respect to the sliding surface $S = \{e : \ Ce = 0\}$ to force the trajectories of $e(t)$ onto S in finite time. Define $e_1(t) = \hat{x}(t) - x(t)$ and $e_y(t) = \hat{y}(t) - y(t)$ then, the error system is given by :

$$\dot{e}_1(t) = A_{11}\hat{e}_1(t) + A_{12}e_y(t) + L\nu, \tag{1.27a}$$

$$\dot{e}_y(t) = A_{21}e_1(t) + A_{22}e_y(t) - \nu, \tag{1.27b}$$

The gain L can be chosen to make the spectrum of $A_{11} + LA_{12}$ lies in $\mathbb{C}_-$. Now define a further change of coordinates by:

$$T_L = \begin{bmatrix} \mathbb{I}_{n-p} & L \\ 0 & \mathbb{I}_p \end{bmatrix}$$

The error system becomes

$$\begin{cases} \dot{\tilde{e}}_1(t) &= \tilde{A}_{11}\tilde{e}_1(t) + \tilde{A}_{12}e_y(t), \\ \dot{e}_y(t) &= A_{21}\tilde{e}_1(t) + \tilde{A}_{22}e_y(t) - \nu, \end{cases} \tag{1.28}$$

where $\tilde{e}_1 = e_1 + Ly$, $\tilde{A}_{11} = A_{11} + LA_{21}$, $\tilde{A}_{12} = A_{12} + LA_{22} - \tilde{A}_{11}L$ and $\tilde{A}_{22} = A_{22} - A_{21}L$.
In the domain

$$\Omega = \{(e_1, e_y) : \ \|A_{21}e_1\| + \frac{1}{2}\lambda_{\max}(\tilde{A}_{22} + \tilde{A}_{22}^T)\|e_y\| < \rho - \eta\}$$

where $\eta < \rho$ is some small positive scalar, the reachability condition $e_y^T\dot{e}_y < -\eta\|e_y\|$ is satisfied. After some finite time t_s, for all subsequent time, $e_y = 0$ and $\dot{e}_y = 0$. When every component of e_y has converged to zero, a sliding motion takes place on the surface

$$S_0 = \{(e_1, e_y) \ : \ e_y = 0\}$$

which, by choice of L, represents a stable system and so $\tilde{e} \to 0$ and consequently, $\hat{x} \to x$ as $t \to \infty$.

1.3.2.3 Standard High-Gain Observer

The high-gain observer is based on the concept of applying a high gain to the error between the observed output and the actual output. This approach forces the observer to converge quickly to the true state values, even if the initial estimation error is large. We recall the basic high gain observer as in [42]. Consider the class of nonlinear systems described by the equations:

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 + f_1(x_1) \\ x_3 + f_2(x_1, x_2) \\ \vdots \\ x_n + f_{n-1}(x_1, \dots, x_{n-1}) \\ f_n(x) \end{bmatrix} \\ y = x_1 \end{cases} \tag{1.29}$$

where $x(t) \in \mathbb{R}^n$ is the system state and $y(t) \in \mathbb{R}^p$ is the output measurement vector. For the sake of brevity, we consider the system (3.14) without control input. We assume that the functions $f_i : \mathbb{R}^i \longrightarrow \mathbb{R}, i = 1, \ldots, n$ are respectively γ_{f_i}–Lipschitz with respect to their arguments, and the Lipschitz constraint is assumed to be global. That is there exists $\gamma_{f_i} > 0, i = 1, \ldots, n$ such that

$$\Big| f_i(x_1 + \Delta_1, \ldots, x_i + \Delta_i) - f_i(x_1, \ldots, x_i) \Big| \leq \gamma_{f_i} \sum_{j=1}^{i} |\Delta_j| \,, \forall \Delta_i \in \mathbb{R} \,. \tag{1.30}$$

If the Lipschitz assumption is not global and the system (3.14) admits a positively invariant compact set on which f is Lipschitz, then we can apply the *Hilbert* projection theorem [90, 134] or the *Kirszbraun–Valentine* extension theorem [125, 133] to extend f to a globally Lipschitz function. The reader can also find details on this extension in [45]. To simplify the developments, we write system (3.14) under the form:

$$\begin{cases} \dot{x} = Ax + f(x) \\ y = Cx \end{cases} \tag{1.31}$$

where

$$C = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}, \; f(x) = \begin{bmatrix} f_1(x_1) & \ldots & f_n(x) \end{bmatrix}^\top \tag{1.32}$$

and the state matrix A is defined by

$$(A)_{i,j} = \begin{cases} 1 & \text{if} \quad j = i+1 \\ 0 & \text{if} \quad j \neq i+1 \end{cases} . \tag{1.33}$$

Let us introduce the following linear transformation

$$\zeta = \mathbb{T}_\tau x, \text{ where } \mathbb{T}_\tau \triangleq \operatorname{diag}\left(\frac{1}{\tau}, \ldots, \frac{1}{\tau^n}\right) \tag{1.34}$$

which transforms (3.14) into

$$\dot{\zeta} = \tau A \zeta + \mathbb{T}_\tau f(\mathbb{T}_{\frac{1}{\tau}} \zeta) \tag{1.35}$$

where the relation $\mathbb{T}_\tau^{-1} = \mathbb{T}_{\frac{1}{\tau}}$ is used. Now consider the following observer corresponding to (3.17):

$$\dot{\hat{\zeta}} = \tau A \hat{\zeta} + \mathbb{T}_\tau f\big(\mathbb{T}_{\frac{1}{\tau}} \hat{\zeta}\big) + L\left(y - C \mathbb{T}_{\frac{1}{\tau}} \hat{\zeta}\right) \tag{1.36}$$

where L, independent from τ, is the constant observer gain to be determined. Then the dynamics of the estimation error $e_\zeta = \zeta - \hat{\zeta}$ is expressed as

$$\dot{e}_\zeta = \tau \Big(A - LC\Big) e_\zeta + \Delta f \tag{1.37}$$

where

$$\Delta f \triangleq \mathbb{T}_\tau \left[f(\mathbb{T}_{\frac{1}{\tau}} \zeta) - f\big(\mathbb{T}_{\frac{1}{\tau}} \hat{\zeta}\big) \right]. \tag{1.38}$$

From the Lipschitz condition (1.30), the equivalence of norms in $\mathbb{R}^n$, the structure of $\mathbb{T}_\tau$ in (4.37), and the fact that $\tau \geq 1$, we can show as in [9] that there is a constant $k_f > 0$, independent of τ, such that

$$\|\Delta f\| \leq k_f \|e_\zeta\|. \tag{1.39}$$

Now, we can recall the following standard theorem from high-gain methodology.

Theorem 1.3.5 ([9]). *Let $P = P^\top > 0$ and $\mathcal{X}$ be matrices of appropriate dimensions, and $\tau > 0$ is a scalar, such that the following conditions hold:*

$$A^T P + PA - C^\top \mathcal{X} - \mathcal{X}^\top C + \lambda \mathbb{I}_n < 0, \tag{1.40}$$

$$\tau > \max\left(1, \frac{2k_f \lambda_{\max}(P)}{\lambda}\right). \tag{1.41}$$

Then the observer (3.18) corresponding to (3.17), with $L = \mathcal{P}^{-1}\mathcal{X}^\top$, converges exponentially towards zero. Moreover, the estimated state $\hat{x}(t) = \mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}(t)$ exponentially converges to the state $x(t)$ of the original system (3.14).

The key advantage of the high-gain approach is that it guarantees the existence of an exponentially convergent observer, achieved by adjusting a single parameter, which must be sufficiently large [43, 44]. However, the necessity for a large gain poses a significant drawback. Specifically, high-gain observers are highly sensitive to output measurement noise due to the large tuning parameter, which can become excessively high in systems with higher dimensions and nonlinearities with large Lipschitz constants. To address this issue, numerous studies have proposed various solutions : a time-varying high-gain observer can be designed where the gain starts high and gradually decreases over time [29]; a high-gain observer with limited gain power [25]; LPV/LMI-Based High-Gain Observer [138]. In [138], a new class of observers called the HG/LMI observer, was introduced by combining the standard high-gain methodology with the LPV/LMI technique [131]. This new observer offers the benefit of reducing the tuning parameters compared to previous high-gain observers, without resorting to saturation functions or filtering mechanisms. Our approach adopts the standard high-gain methodology, maintaining a state observer structure of dimension n. However, by utilizing the LPV/LMI technique developed in [131], the tuning parameter can be reduced, thereby decreasing the gain power. This is achieved by introducing a "compromise index" j_0, where $0 \leq j_0 \leq n$. This modification limits the gain power to j_0 but necessitates solving 2^{j_0} LMIs instead of just one, as is required with the standard high-gain observer.
Despite these improvements, research in this domain remains ongoing, as many challenges still need to be addressed to further enhance the high-gain observer's performance, particularly in the presence of measurement noise. The high-gain observer is typically developed for systems in triangular form or for any system that can be transformed into such a structure. However, finding the appropriate transformation, which is a diffeomorphism, can be challenging. This transformation is crucial for reformulating the system into a suitable triangular structure, but determining it is often complex and not always straightforward, especially for high-dimensional systems. To address this issue, we propose in this thesis an approach that eliminates the need for any transformation of the system into a triangular form.

1.3.3 LPV Approach

LMI (Linear Matrix Inequality)-based observers are designed using optimization techniques, where the observer gain is derived by solving a set of LMIs. These LMIs are typically derived from stability and performance conditions (e.g., Lyapunov stability). The LMI approach uses convex optimization to find the observer gains that satisfy certain criteria. We consider a class of nonlinear systems without a linear state part. Indeed, we will see later that the linear part is not necessary for the design methods we develop. For simplicity of presentation, we consider, without loss of generality, that the nonlinear function depends only on the system state, and the output is linear. The extension to nonlinear output is straightforward. The class of systems we treat in this section is described by the following equations:

$$\begin{cases} \dot{x} = \Psi(x) \\ y = Cx \end{cases} \tag{1.42}$$

where the nonlinear function $\Psi \; : \; \mathbb{R}^n \longrightarrow \; \mathbb{R}^n$ is assumed to be γ_Ψ-Lipschitz, i.e.:

$$\left\|\Psi(x) - \Psi(y)\right\| \leq \gamma_\Psi \left\|x - y\right\|, \;\; \forall \, x, y \in \mathbb{R}^n \tag{1.43}$$

This part is devoted to the reformulated Lipschitz property from [131]. This reformulation is necessary for our developed LPV approach. A useful definition and two lemmas are presented.

Definition 1.3.6. *Consider two vectors*

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\; \textit{and} \;\; Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n.$$

For all $i = 0, ..., n$, we define an auxiliary vector $X^{Y_i} \in \mathbb{R}^n$ corresponding to X and Y as follows:

$$\begin{cases} X^{Y_i} = \begin{pmatrix} y_1 \\ \vdots \\ y_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix} \;\; \textit{for } i = 1, ..., n \\ X^{Y_0} = X \end{cases} \tag{1.44}$$

Lemma 1.3.7 ([131])**.** *Consider a function $\Psi \; : \; \mathbb{R}^n \longrightarrow \; \mathbb{R}$. Then, for all*

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\; \textit{and} \;\; Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

there exist functions $\psi_j \; : \; \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \; \mathbb{R}$, $j = 1, ..., n$ so that

$$\Psi(X) - \Psi(Y) = \sum_{j=1}^{j=n} \psi_j\Big(X^{Y_{j-1}}, X^{Y_j}\Big) e_n^T(j) \Big(X - Y\Big) \tag{1.45}$$

Proof. The proof consists of rewriting $\Psi(X) - \Psi(Y)$ as

$$\Psi(X) - \Psi(Y) = \sum_{j=1}^{j=n} \Big[\Psi\Big(X^{Y_{j-1}}\Big) - \Psi\Big(X^{Y_j}\Big)\Big]$$

Now, defining the functions ψ_j by

$$\psi_j\Big(X^{Y_{j-1}}, X^{Y_j}\Big) = \begin{cases} 0 & \text{if} \;\; x_j = y_j \\ \frac{\Psi\left(X^{Y_{j-1}}\right) - \Psi\left(X^{Y_j}\right)}{x_j - y_j} & \text{if} \;\; x_j \neq y_j \end{cases} \tag{1.46}$$

we can write

$$\begin{aligned} \Psi(X) - \Psi(Y) &= \sum_{j=1}^{j=n} \Big[\psi_j\Big(X^{Y_{j-1}}, X^{Y_j}\Big)\Big] (x_j - y_j) \\ &= \sum_{j=1}^{j=n} \Big[\psi_j\Big(X^{Y_{j-1}}, X^{Y_j}\Big) e_n^T(j)\Big] \Big(X - Y\Big) \end{aligned} \tag{1.47}$$

□

Lemma 1.3.8 ([131]). *Considering the function $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the two following items are equivalent:*

- *Lipschitz property : Ψ is γ_Ψ-Lipschitz with respect to its argument, i.e.:*

$$\Big\|\Psi(X) - \Psi(Y)\Big\| \leq \gamma_\Psi \Big\|X - Y\Big\|, \ \ \forall\, X, Y \in \mathbb{R}^n \tag{1.48}$$

- *Reformulated Lipschitz property : for all $i, j = 1, ..., n$, there exist functions*

$$\psi_{ij} \ : \ \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \ \mathbb{R}$$

and constants $\underline{\gamma}_{\psi_{ij}}$ and $\bar{\gamma}_{\psi_{ij}}$, so that $\forall\, X, Y \in \mathbb{R}^n$,

$$\Psi(X) - \Psi(Y) = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \psi_{ij} H_{ij} \Big(X - Y\Big) \tag{1.49}$$

and

$$\underline{\gamma}_{\psi_{ij}} \leq \psi_{ij} \leq \bar{\gamma}_{\psi_{ij}} \tag{1.50}$$

where

$$\psi_{ij} \triangleq \psi_{ij}\Big(X^{Y_{j-1}}, X^{Y_j}\Big) \ \ \textit{and} \ \ H_{ij} = e_n(i) e_n^T(j)$$

In fact, Lemma (1.3.8) provides a best less conservative Lipschitz condition. Indeed, the reformulation (1.49)-(1.50) allows treatment of the nonlinearity with the best precision and exploits all the interesting properties of the nonlinearity of the investigated system. For instance, the Lipschitz condition (1.2) does not distinguish between the nonlinearities $\Psi(x) = \sin(x_2)$ and $\Psi(x) = \tanh(x_2)$. However, with the reformulation (1.50), we can see the difference. Indeed, for $\Psi(x) = \sin(x_2)$ we have $\underline{\gamma}_{\psi_{11}} = -1$ and for $\Psi(x) = \tanh(x_2)$, we have $\underline{\gamma}_{\psi_{11}} = 0$. Consider the following Luenberger observer:

$$\dot{\hat{x}} = \Psi(\hat{x}) + L\Big(y - C\hat{x}\Big) \tag{1.51}$$

The dynamic of the estimation error $e = x - \hat{x}$ is then given by:

$$\dot{e} = \Big[\Psi(x) - \Psi(\hat{x})\Big] - LCe \tag{1.52}$$

Since $\Psi(.)$ is γ_Ψ-Lipschitz, then following Lemma 1.3.8 there are functions

$$\psi_{ij} \ : \ \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \ \mathbb{R}$$

and constants $\underline{\gamma}_{\psi_{ij}}$ and $\bar{\gamma}_{\psi_{ij}}$, so that

$$\Psi(x) - \Psi(\hat{x}) = \Big[\sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \psi_{ij} H_{ij}\Big] e \tag{1.53}$$

and

$$\underline{\gamma}_{\psi_{ij}} \leq \psi_{ij} \leq \bar{\gamma}_{\psi_{ij}} \tag{1.54}$$

where

$$\psi_{ij} \triangleq \psi_{ij}\Big(x_t^{\hat{x}_{j-1}}, x^{\hat{x}_j}\Big)$$

is defined as in (1.46). By replacing Ψ by Ψ_i (the i^{th} component of Ψ). In the sequel, for simplicity of presentation, we use only ψ_{ij} instead of $\psi_{ij}\Big(x^{\hat{x}_{j-1}}, x^{\hat{x}_j}\Big)$.
Now, define the matrices

$$\Theta = \Big(\psi_{ij}\Big)_{ij} \tag{1.55}$$

and

$$\mathcal{A}\Big(\Theta\Big) = \sum_{i=1}^{i=n}\sum_{j=1}^{j=n} \psi_{ij} H_{ij} \tag{1.56}$$

Consequently, the dynamics (1.52) can be rewritten as

$$\dot{e} = \Big[\mathcal{A}\Big(\Theta\Big) - LC\Big]e \tag{1.57}$$

According to (1.54), the matrix parameter Θ belongs to a bounded convex set $\mathcal{H}_n$ for which the set of vertices is defined by:

$$\mathcal{V}_{\mathcal{H}_n} = \Big\{\Phi \in \mathbb{R}^{n\times n} \ : \ \Phi_{ij} \in \Big\{\underline{\gamma}_{\psi_{ij}}, \bar{\gamma}_{\psi_{ij}}\Big\}\Big\}. \tag{1.58}$$

The following theorem, which provides LMI conditions for observer design of Lipschitz systems.

Theorem 1.3.9 ([131]). *The observer (1.51) is asymptotically convergent if there exist a positive definite matrix $\mathcal{P}$, a matrix $\mathcal{R}$ of appropriate dimension so that the following LMI conditions hold:*

$$\mathcal{A}\Big(\Phi\Big)^T\mathcal{P} + \mathcal{P}\mathcal{A}\Big(\Phi\Big) - C^T\mathcal{R} - \mathcal{R}^T C < 0, \ \forall \ \Phi \in \mathcal{V}_{\mathcal{H}_n} \tag{1.59}$$

Hence, the observer gain is given by

$$L = \mathcal{P}^{-1}\mathcal{R}^T.$$

It is clear that from the computational complexity point of view, the LPV method is less interesting, viewing the cost of more demanding LMIs to be solved. In fact, in the LPV method, we have to solve 2^{n^2} LMIs for n-dimensional nonlinear vectors. But, despite that, the LPV method provides less restrictive LMI synthesis conditions. On the other hand, despite the cost of more demanding LMIs, this does not play an important role in the feasibility of the proposed LMIs. In general, computational complexity plays a role in online methods for real-time applications.

1.3.4 State Estimators

In many technological fields (telecommunication, remote sensing, geolocalisation, industrial control), useful information is not directly accessible as it is hidden in the observed signal; this issue requires the development of hidden information methods.

1.3.4.1 Kalman Filter

A simple and optimal solution to the problem of estimating the state of linear systems is the Kalman filter in the stochastic case [47,66,86]. The Kalman filter, based upon a linear state model puts into equation the evolution of the useful signal and its relationship to the signal measured from a series of incomplete or noisy measurements. Kalman filtering is an algorithm that provides estimates of some unknown variables given the measurements observed over time.
The Kalman filter is a recursive predictive filter that is based on the use of state space techniques and a recursive algorithm. It estimates the state of a dynamic system. This dynamic system can be distributed by some noise. To improve the estimated state, the Kalman filter uses measurements that are related to the state but distributed as well. Thus the Kalman filter consists of two steps, the prediction and the correction.

1.3.4.1.a Case of continuous Linear Time varying (LTV) systems:

For the LTV system of the form

$$\begin{cases} \dot{x} &= Ax + Bu + v(t), \\ y &= Cx + w(t), \end{cases} \tag{1.60}$$

where $v(t)$ and $w(t)$ are two Gaussian white noises of zero expectation, and non-correlated covariances Q and R, respectively.

$$\dot{\hat{x}} = A\hat{x} + Bu + PC^T R^{-1}(y - C\hat{x}), \tag{1.61}$$

$$\sum_{i=1}^{n} E(e_i(t)^2) \quad = \quad E(e^T(t)e(t))$$

$$\dot{\hat{x}} = A(t)\hat{x} + B(t)u + PC^T(t)R^{-1}(y - C(t)\hat{x}), \tag{1.62}$$

where P is a symmetric and definite positive solution of the following Riccati equation:

$$\dot{P} = AP + PA^T + Q - PC^T R^{-1} CP.$$

For the LTV system of the form

$$\begin{cases} \dot{x}_{x+1} &= A_t x_t + B_t u_t + v_t, \\ y_t &= C_t x_t + w_t, \end{cases} \tag{1.63}$$

where v and w are two Gaussian white noises of zero expectation, and non-correlated covariances Q and R, respectively. Consider the best possible predictor model and estimating that the value that the noise will take at the following instant will be zero, the best prediction that we can make is given by

$$\hat{x}_{t+1/t} = A\hat{x}_{t/t} + Bu_t.$$

As this prediction must be the same as:

$$\hat{x}_{t+1/t} = A\hat{x}_{t/t-1} + Bu_t + K_t(y_t - C\hat{x}_{t/t-1}).$$

We conclude that

$$\hat{x}_{t/t} = \hat{x}_{t/t-1} + A^{-1}K_t(y_t - C\hat{x}_{t/t-1})$$

1.3.4.1.b Extended Kalman Filter (EKF)

As we saw previously, the Kalman filter is frequently used to analyze the behavior of a linear system that operates under Gaussian noise conditions. In other words, it makes it possible to identify the state of a system over time from the current inputs and outputs and the Gaussian noise covariances that affect the system for the duration of the study. This method is very efficient, but since most physical systems are nonlinear, it is impossible to apply the Kalman filter directly to them. The Kalman filter for such nonlinear problems is the so-called Extended Kalman Filter (EKF). This Kalman filter linearizes the estimated state. The extended Kalman filter is a direct extension of the standard Kalman filter by replacing the state and output matrices, A, C of the linear system (1.60) or (1.63) by the Jacobians of the system non-linearities in question. Consider the following nonlinear system:

$$\begin{cases} \dot{x} &= f(x,u) + v(t), \\ y &= h(x,u) + w(t), \end{cases} \tag{1.64}$$

Algorithm 1 Standard Kalman Filter Algorithm

1. **Initialization:** Set the initial covariance matrix $P_0 = \mu \mathbb{I}_n > 0$.
2. **Set the correlation matrices:** $Q_t = E[v_t v_t^\top], R_{k+1} = E[w_{t+1} w_{t+1}^\top]$.
3. **One-step prediction of the state vector:** $\hat{x}_{t+1/t} = A_t \hat{x}_t + B_t u_t$.
4. **One-step prediction of the covariance error:** $P_{t+1/t} = A_t P_t A_t^\top + Q_t$.
5. **Compute the Kalman gain:** $K_{t+1} = P_{t+1/t} C_{t+1}^\top \left(C_{t+1} P_{t+1/t} C_{t+1}^\top + R_{t+1} \right)^{-1}$.
6. **Update the state estimate:** $\hat{x}_{t+1} = \hat{x}_{t+1/t} + K_{t+1} e_{t+1}$, where $e_{t+1} = y_{t+1} - C_{t+1} \hat{x}_{t+1/t}$ is the innovation or measurement residual.
7. **Update the covariance matrix of the estimation error:** $P_{t+1} = \left(P_{t+1/t}^{-1} + C_{t+1}^\top R_{t+1}^{-1} C_{t+1} \right)^{-1}$.

The EKF is expressed as follows:

$$\dot{x} = f(\hat{x}, u) + PH(\hat{x}, u) R^{-1} (y - h(\hat{x}, u)) \tag{1.65a}$$

$$\dot{P} = F(\hat{x}, u) P + PF(\hat{x}, u)^\top + Q - PH(\hat{x}, u)^\top R^{-1} H(\hat{x}, u) P, \tag{1.65b}$$

where

$$F(\hat{x}, u) = \frac{\partial f}{\partial x}(\hat{x}, u),$$

and

$$H(\hat{x}, u) = \frac{\partial f}{\partial x}(\hat{x}, u).$$

1.3.4.1.c EKF for discrete-time systems

The Extended Kalman Filter (EKF) is a nonlinear variant of the Kalman Filter, designed to handle systems where the process and observation models are nonlinear.
Unlike the standard Kalman Filter, which is optimal for linear systems, the EKF approximates the nonlinear system by linearizing the dynamics around the current estimate, making it suitable for a wide range of real-world applications, such as robotics, aerospace, and signal processing. In the case of a discrete-time system of the form:

$$\begin{cases} x_{t+1} &= f_t(x_t, u_t) + v_t, \\ y_t &= h_t(x_t, u_t) + w_t, \end{cases} \tag{1.66}$$

where the functions $f_t : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ and $h_t : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^m$ are differentiable functions; $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^p$, $y_t \in \mathbb{R}^m$, v_t and ω_t represent respectively the state, the input, the output and the two disturbances on the states and the output.
The standard form of the extended Kalman observer is

$$\hat{x}_{t+1} = \hat{x}_{t+1/t} + K_{t+1}(y_{t+1} - h(\hat{x}_{t+1/t}, u_{t+1})) \quad k = 1, ..., n.$$

We develop the error term $e_t = x_t - \hat{x}_t$ using the Taylor expansion for the nonlinear functions f_t and h_t. We seek the gain K_t minimizing:

$$\text{trace}(P_t) = E[e_t e_t^\top],$$

Algorithm 2 Kalman Filter Algorithm

1. **Initialization:** Set the initial covariance matrix $P_0 = \mu \mathbb{I}_n > 0$
2. **Computation of matrices:** $F_t = F(\hat{x}_t, u_t) = \frac{\partial f}{\partial x}(\hat{x}_t, u_t)$; $\quad H_t = H(\hat{x}_t, u_t) = \frac{\partial h}{\partial x}(\hat{x}_t, u_t)$
3. **Correlation matrices:** $Q_t = E[v_t v_t \top]$; $\quad R_{t+1} = E(w_{t+1} w_{t+1}^\top)$
4. **One-step prediction of the state vector:** $\hat{x}_{t+1/t} = f(\hat{x}_t, u_t)$
5. **One-step prediction of the covariance error:** $P_{t+1/t} = F_t P_t F_t \top + Q_t$
6. **Kalman gain:** $K_{t+1} = P_{t+1/t} H_{t+1}^\top \left(H_{t+1} P_{t+1/t} H_{t+1} + R_{t+1}\right)^{-1}$
7. **n+1-step estimation of the state vector:** $\hat{x}_{t+1} = \hat{x}_{t+1/t} + K_{t+1} e_{t+1}$, where $e_{t+1} = y_{t+1} - h(\hat{x}_{t+1/t}, u_{t+1})$
8. **Covariance matrix of the estimation error:** $P_{t+1} = \left(\mathbb{I}_n - K_{t+1} H_{t+1}\right) P_{t+1/t}$

where P_t is the estimation error covariance matrix.
Despite its effectiveness, the EKF has limitations, particularly when the system is highly nonlinear or when the linearization does not accurately capture the system dynamics. In such cases, more advanced techniques like the Unscented Kalman Filter (UKF) Filters may be preferred.

1.3.4.2 Moving Horizon Estimator

Moving Horizon Estimation (MHE) is a widely used technique for state estimation across diverse fields [5, 8]. The success of MHE lies in its ability to account for constraints on system variables [51, 91, 100, 101]. MHE infers the current system state by considering a recent batch of past inputs and measurements, using a moving window of the past N data points (Fig. 1.3.4.2). An optimization problem is solved to fit the available input/output data. By minimizing an objective function that quantifies the discrepancy between actual measurements and estimated states, MHE enables state estimation in the presence of noisy measurements. The MHE scheme is represented in Figure 1.3.4.2.
MHE is a state estimation method that is particularly useful for nonlinear or constrained dynamic systems. We consider the class of nonlinear systems given by (1.17). For simplicity, we do not consider known external input, which however does not undermine the generality of what follows. Let us denote by $|x|$ the Euclidean norm $\sqrt{x^\top x}$, $x \in \mathbb{R}^n$. MHE is based on the idea of minimizing a quadratic estimation cost function defined on a backward sliding window composed of a finite number of time stages, which will be denoted by the integer $N \geq 1$. For this, we define the classic quadratic objective function

$$J_t^N(\hat{x}_{t-N}) = \mu |\hat{x}_{t-N} - \bar{x}_{t-N}|^2 \eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i} \left|y_i - h(\hat{x}_{i|t}, 0)\right|^2 \tag{1.67}$$

where $\eta \in (0, 1)$ and $\mu, \nu > 0$
under the constraints

$$\hat{x}_{i+1|t} = f(\hat{x}_{i|t}, 0)\,,\; i = t-N, \dots, t-1 \tag{1.68}$$

and thus $\hat{x}_{t-N+1|t}, \dots, \hat{x}_{t|t}$ are generated by $\hat{x}_{t-N|t}$. We denote by MHE_N what results from the

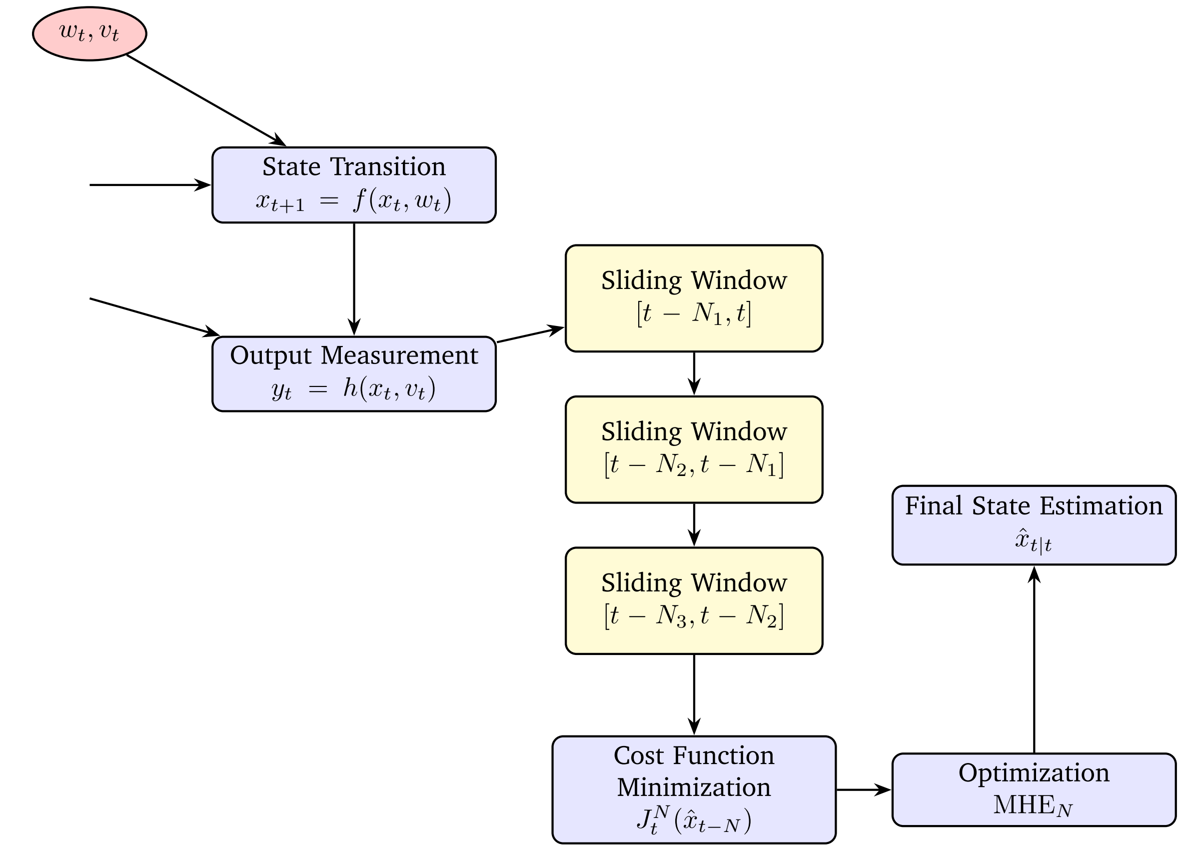

Figure 1.3: Moving horizon estimation technique (MHE) scheme

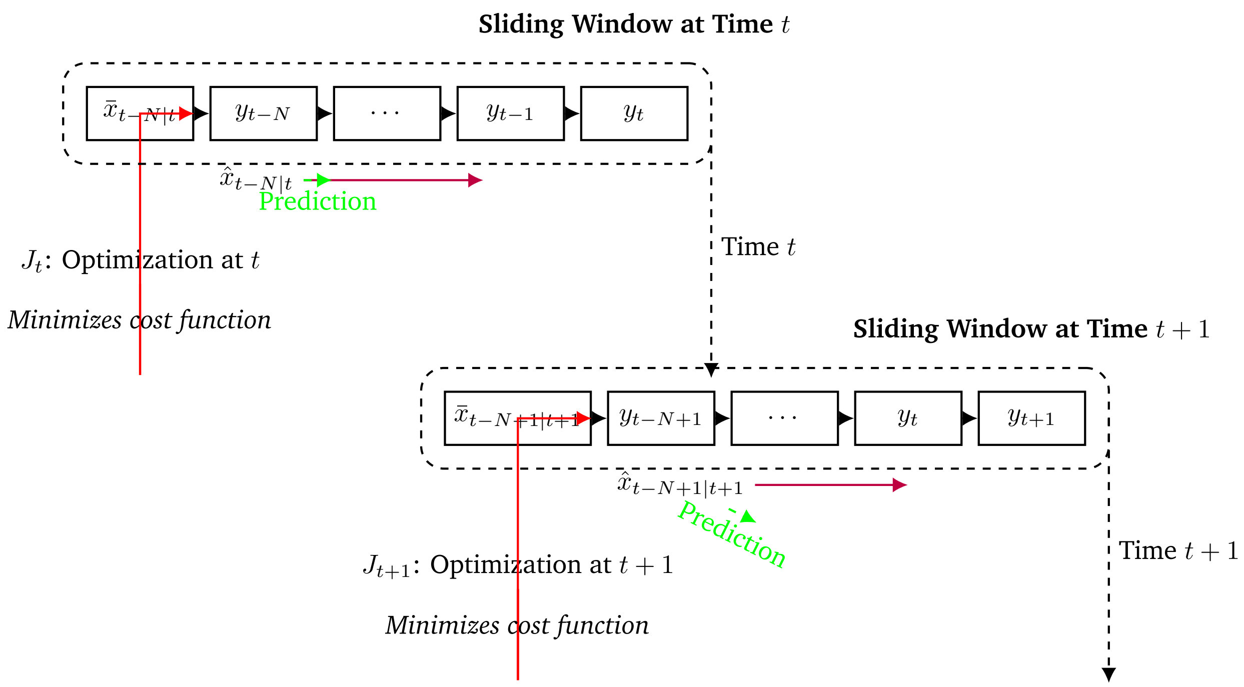

Figure 1.4: Sliding windows in (MHE) scheme.

minimization of the cost function (1.67) as follows

$$\begin{cases} \hat{x}_{0|t} \in \left\{ \underset{\hat{x}_0 \in \mathcal{X}}{\text{argmin}}\, J_t^t(\hat{x}_0) \text{ s.t. } (1.68) \text{ holds for } \quad t = 1, ..., N \right\} \\ \hat{x}_{t-N|t} \in \left\{ \underset{\hat{x}_{t-N} \in \mathcal{X}}{\text{argmin}}\, J_t^N(\hat{x}_{t-N}) \text{ s.t. } (1.68) \text{ holds for } \quad t = N+1, N+2, \ldots \right\} \end{cases}$$

together with (1.68), which provides $\hat{x}_{t|t}$. For further ease of presentation, note that the cost function

$$J_t^t(\hat{x}_0) = \mu|\hat{x}_0 - \bar{x}_0|^2\eta^t + \nu\sum_{i=0}^{t-1}\eta^{t-1-i}\left|y_i - h(\hat{x}_{i|t}, 0)\right|^2 \tag{1.69}$$

for all $t \leq N$.
Both the Extended Kalman Filter (EKF) and Moving Horizon Estimation (MHE) are advanced techniques used for state estimation in nonlinear systems. While the EKF is widely used due to its efficiency and simplicity, its robustness is not guaranteed. Moving Horizon Estimation (MHE), which is proven to be robust in Chapter 2, is more suitable for complex systems.

1.4 Conclusion

Estimators and observers are both used to estimate the internal states of a system, but they differ in approach. An observer is a specific type of estimator that focuses on real-time state estimation using deterministic models, making it more computationally efficient but less robust to noise. In contrast, estimators like the EKF and MHE incorporate statistical or optimization methods to handle noise and uncertainties, making them suitable for systems with significant disturbances. Every observer is an estimator, but Not Vice Versa. The choice between them depends on the application's requirements for accuracy, computational complexity, and real-time performance.

CHAPTER

2

Robust Moving Horizon Estimation Using Advanced Prediction Strategy

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2.1 Introduction

A rigorous theoretical framework for the robustness of control and estimation schemes has been developed in the last decades. In [114], the authors showed that systems admitting robust estimators must be i-IOSS. However, these results do not apply to moving-horizon estimators [11]. The use of i-IOSS, as defined in [12], allows for the synthesis conditions of the parameters of the MHE cost function to be less restrictive than previously done. In this part, we propose novel conditions for the robust stability of the MHE estimation error by considering the incremental exponential input/output-to-state stability (i-EIOSS) property. In more detail, the contributions of this chapter can be summarized as follows:

1. We propose novel numerical design procedures ensuring the computation of the i-EIOSS-related parameters which are necessary to tune the parameters of the robust estimators. We first introduce a general simple Lyapunov-based method, then we develop a new Linear Matrix Inequality (LMI) condition guaranteeing the computation of the i-EIOSS coefficients.

2. Based on the novel mathematical inequalities, we provide sufficient conditions to ensure the robust exponential stability (RES) of the MHE error by only assuming that the system is i-EIOSS. Such conditions involve the window size of the MHE, which needs to be appropriately selected to ensure RES.

3. Three prediction techniques are analyzed in detail, where each prediction equation demands different sufficient conditions for RES. To relax the conditions required by the two first prediction techniques on the size of the MHE, the third prediction method is proposed that involves an output correction term with additional design parameters.

4. The construction of the new proposed prediction strategy with a convenient selection of the design parameters is investigated in detail and suitable selection methods based on linear matrix inequalities (LMIs) are devised to the scope. A key innovation lies in the incorporation of an output-based dynamic extension approach. This technique not only simplifies but also enhances the outcomes compared to previous methodologies outlined in [19, 104].

Such contents allow to complement literature results previously established on RES for MHE (see, among others, [75, 85, 104, 105]). This chapter is organized as follows: Section 2.2 provides some preliminary results presented in lemmas that will be used to prove the main results in the rest of the chapter. In Section 2.3, we introduce two novel numerical design procedures for computing the parameters related to the i-EIOSS property. The first approach is based on Linear Matrix Inequalities (LMI), while the second leverages a Lyapunov-based stability criterion. Section 2.4 is dedicated to proving the robustness of Moving Horizon Estimation for nonlinear systems using advanced prediction strategies. A numerical case study is presented in Section 2.5. Finally, we end the chapter with conclusions in Section 2.6.

2.2 Preliminary Results

We start by introducing the class of systems and the related assumptions. Then, we will present some mathematical tools we need to use to establish the main LMI-based design method. The formulation of the problem we aim to solve and the motivations are given at the end. Consider the following nonlinear discrete-time system:
For convenience of calculation, and brevity as well and to avoid cumbersome notations, we consider the system (1.17) with the same disturbance input w_t in the output y_t. This is not a restriction. It is assumed without loss of generality since there are no constraints on the dependence of the functions $f(.)$ and $h(.)$ on w_t and v_t, respectively. To summarize, instead of system (1.17), we consider the following form:

$$\begin{cases} x_{t+1} = f(x_t, \boldsymbol{\omega}_t) \\ y_t = h(x_t, \boldsymbol{\omega}_t) \end{cases} \tag{2.1}$$

where $\boldsymbol{\omega}_t \triangleq \begin{bmatrix} w_t^\top & v_t^\top \end{bmatrix}^\top \in \mathbb{R}^q$. Then, all the next definitions and results are based on the system (2.1). This form is convenient in the LMI context as usual in the literature [131, 137].
Before introducing the main definitions, we need to make some assumptions, which are necessary for the developed design methodology.

Assumption 2.2.1. *The nonlinear functions $f(\cdot)$ and $h(\cdot)$ are differentiable with respect to their arguments and satisfy the following conditions:*

$$\sup_{\substack{x\in\mathbb{R}^n\\ \boldsymbol{\omega}\in\mathbb{R}^q}}\left|\frac{\partial f_i}{\partial x}(x,\boldsymbol{\omega})\right| < +\infty, \ \sup_{\substack{x\in\mathbb{R}^n,\\ w\in\mathbb{R}^q}}\left|\frac{\partial f_i}{\partial \boldsymbol{\omega}}(x,\boldsymbol{\omega})\right| < +\infty \tag{2.2}$$

$$\sup_{\substack{x\in\mathbb{R}^n\\ \boldsymbol{\omega}\in\mathbb{R}^q}}\left|\frac{\partial h_i}{\partial x}(x,\boldsymbol{\omega})\right| < +\infty, \ \sup_{\substack{x\in\mathbb{R}^n\\ \boldsymbol{\omega}\in\mathbb{R}^q}}\left|\frac{\partial h_i}{\partial \boldsymbol{\omega}}(x,\boldsymbol{\omega})\right| < +\infty. \tag{2.3}$$

where the functions $f_i, i = 1,\dots,n$ and $h_i, i = 1,\dots,p$ are the i^{th} component of the functions f and h, respectively.

The above assumption means that the Jacobians $\dfrac{\partial f(x,\boldsymbol{\omega})}{\partial x}$, $\dfrac{\partial f(x,\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}$, $\dfrac{\partial h(x,\boldsymbol{\omega})}{\partial x}$, and $\dfrac{\partial h(x,\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}$ are bounded, then they belong to convex polytopic sets defined respectively by:

$$\mathcal{V}_{f,x} \triangleq \left\{\sum_{j=1}^{n_x}\alpha_j\mathcal{F}_j^x, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{n_x}\alpha_j = 1\right\} \tag{2.4}$$

$$\mathcal{V}_{f,\boldsymbol{\omega}} \triangleq \left\{\sum_{j=1}^{n_w}\alpha_j\mathcal{F}_j^{\boldsymbol{\omega}}, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{n_{\boldsymbol{\omega}}}\alpha_j = 1\right\} \tag{2.5}$$

$$\mathcal{V}_{h,x} \triangleq \left\{\sum_{j=1}^{q_x}\alpha_j\mathcal{H}_j^x, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{q_x}\alpha_j = 1\right\} \tag{2.6}$$

$$\mathcal{V}_{h,\boldsymbol{\omega}} \triangleq \left\{\sum_{j=1}^{q_{\boldsymbol{\omega}}}\alpha_j\mathcal{H}_j^{\boldsymbol{\omega}}, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{q_{\boldsymbol{\omega}}}\alpha_j = 1\right\} \tag{2.7}$$

where $\mathcal{F}_j^x$, $\mathcal{F}_j^{\boldsymbol{\omega}}$, $\mathcal{H}_j^x$, and $\mathcal{H}_j^{\boldsymbol{\omega}}$ are known constant matrices of appropriate dimensions and the known integers $n_x, n_{\boldsymbol{\omega}}, q_x, q_{\boldsymbol{\omega}}$ are the number of vertices of each convex set, respectively.
The result is standard in the representation of elements in a convex set. Indeed, since the Jacobians are bounded, the partial derivatives admit lower and upper bounds from which we can construct a convex polytopic set containing the Jacobians. We refer the reader to the classic book [103] on the representation of elements in a convex set and the books [30, 106] for convex decomposition of nonlinear functions.
We will introduce the main definition concerned by this section, which is the incremental exponential input-to-state stability of the system (2.1).

Definition 2.2.2. *System (2.1) is incrementally Exponentially Input/Output-to-State Stable (i-EIOSS) if there exist constants $c_x, c_v, c_w > 0$ and $\varrho \in (0,1)$ such that for each pair of initial conditions $x_0, \tilde{x}_0 \in \mathcal{X}$ and each two disturbance sequences $\boldsymbol{\omega}_t, \tilde{\boldsymbol{\omega}}_t \in \Omega$, the following holds:*

$$\begin{aligned}|x_t(x_0,\boldsymbol{\omega}_0^{t-1}) - x_t(\tilde{x}_0,\tilde{\boldsymbol{\omega}}_0^{t-1})|^2 &\leq c_x|x_0-\tilde{x}_0|^2\varrho^t \\ &+ c_v\sum_{i=0}^{t-1}\varrho^{t-1-i}|y_i(x,\boldsymbol{\omega}_0^{i-1}) - y_i(\tilde{x},\tilde{\boldsymbol{\omega}}_0^{i-1})|^2 \\ &+ c_w\sum_{i=0}^{t-1}\varrho^{t-1-i}|\boldsymbol{\omega}_i - \tilde{\boldsymbol{\omega}}_i|^2\end{aligned} \tag{2.8}$$

where $x_t(x_0,\boldsymbol{\omega}_0^{t-1})$ means the solution of (2.1) generated from the initial state x_0 and $\boldsymbol{\omega}_0^{t-1} \triangleq \begin{bmatrix}\boldsymbol{\omega}_0 & \dots & \boldsymbol{\omega}_{t-1}\end{bmatrix}^\top$.

In the following, we present some basic results, presented in lemmas, that we will exploit after analyzing the i-EIOSS property of the system (2.1) by using quadratic Lyapunov functions. The Lemmas are presented in a general framework so that they can be exploited in different cases for different control design problems.

2.2.1 Tools for Stability Analysis

In this section, we present some key results, which we will exploit to analyze the robustness of the MHE for system (1.17). These results are presented in a general framework so that they can be exploited for other purposes as well.

Lemma 2.2.3 ([14]). *Let $(u_t)_{t\geq -\ell}$ be a sequence of nonnegative real numbers, and $\ell \geq 1$ is a natural number. Assume that*

$$u_t \leq \alpha u_{t-\ell} + \beta z_t, \, \forall t \geq \ell$$

where $\alpha \in (0,1)$, $\beta \geq 0$, and the sequence $(z_t)_{t\geq 0}$ is non negative. Then, the following inequality holds:

$$u_t \leq \alpha^{t/\ell} \max_{-\ell \leq j \leq 0} (u_j) + \frac{\beta}{1-\alpha} \max_{0 \leq j \leq t} (z_j) \, .$$

Furthermore, if $u_j = u_0, \forall j \in \{-\ell, \dots, 0\}$, then

$$u_t \leq u_0 \alpha^{t/\ell} + \frac{\beta}{1-\alpha} \max_{j \in [0,t]} (z_j) \, . \tag{2.9}$$

Proof. Since we work in *Archimedian* space, then for any $t \geq \ell$, there exists an integer $s \geq 1$ so that $t \in I_s = \{s\ell, s\ell+1, \dots, (s+1)\ell\}$. Then by backward substitution, we get

$$u_t \leq \alpha^{s+1} u_{t-(s+1)\ell} + \beta \sum_{j=0}^{s} \alpha^j z_{t-j\ell}. \tag{2.10}$$

It follows that

$$u_t \leq \alpha^{s+1} \max_{t \in I_s} \left(u_{t-(s+1)\ell}\right) + \beta \max_{0 \leq j \leq s} \left(z_{t-j\ell}\right) \sum_{j=0}^{s} \alpha^j. \tag{2.11}$$

Since $t \in I_s$, then

$$-\ell \leq t-(s+1)\ell \leq 0. \tag{2.12}$$

Also, since $0 \leq j \leq s$ and $t \in I_s$, then we have

$$0 \leq t - j\ell \leq t.$$

In addition, since $t \in I_s$, then

$$s+1 \geq \frac{t}{\ell}. \tag{2.13}$$

By using geometric sequence theory, it follows that

$$\sum_{j=0}^{s} \alpha^j = \frac{1-\alpha^{s+1}}{1-\alpha} \leq \frac{1}{1-\alpha}. \tag{2.14}$$

Consequently, by using(2.12)-(2.14), from (2.11) we obtain

$$u_t \leq \max_{-\ell \leq j \leq 0} (u_j) \, \alpha^{t/\ell} + \frac{\beta}{1-\alpha} \max_{0 \leq j \leq t} (z_j) \, . \tag{2.15}$$

If, in addition, $u_j = u_0, j = -\ell, \dots, 0$, then inequality (2.9) is inferred. This ends the proof.

□

Lemma 2.2.3 presented above can be beneficial for certain control design problems, such as time-delay systems [78, 123] and model-trajectory based approaches [84]. However, the results regarding MHE provide a conservative bound. To overcome this conservatism, we introduce two novel lemmas in the following sections, specifically tailored for the MHE context studied here. Lemma 2.2.4 presents a new and less conservative bound applicable to general control design schemes, not limited to the MHE problem in this chapter. Meanwhile, Lemma 2.2.5 provides a specific bound relevant to a particular sequence $(z_t)_{t\geq 0}$ concerning the exponential robustness of the MHE approach explored in this research.

Lemma 2.2.4 ([15]). *Let $(u_t)_{t\geq -\ell}$ be a sequence of non-negative real numbers and $\ell \geq 1$ such that*

$$u_t \leq \alpha u_{t-\ell} + \beta z_t, \forall t \geq \ell,$$

where $\alpha \in (0,1)$, $\beta \geq 0$. The sequence $(z_t)_{t\geq 0}$ is non-negative. Then the following inequality holds for any $\kappa \in \mathbb{N}, \kappa \geq 2$:

$$u_t \leq \alpha^{\frac{t}{\kappa\ell}} \max_{-\ell\leq j\leq 0} u_j + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa}}}\right) \max_{\substack{t-s\ell\leq j\leq t \\ \frac{t-j}{\ell}\in\mathbb{N}}} \left(\alpha^{t-j}\kappa\ell z_j\right). \tag{2.16}$$

Further, since $t - s\ell \geq 0$, the following strict inequality holds:

$$u_t \leq \alpha^{\frac{t}{\kappa\ell}} \max_{-\ell\leq j\leq 0} u_j + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa}}}\right) \max_{0\leq j\leq t} \left(\alpha^{\frac{t-j}{\kappa\ell}} z_j\right). \tag{2.17}$$

Proof. As in Lemma 2.2.3, we easily get (2.10). By relabeling j with $t - j\ell$, the second term in (2.10) becomes

$$\sum_{j=0}^{s} \alpha^j z_{t-j\ell} = \sum_{\substack{j=t-s\ell \\ \frac{t-j}{\ell}\in\mathbb{N}}}^{t} \alpha^{\frac{t-j}{\ell}} z_j\,.$$

On the other hand, for any $\kappa \in \mathbb{N}, \kappa \geq 2$, we can write

$$\begin{aligned}
\sum_{\substack{j=t-s\ell \\ \frac{t-j}{\ell}\in\mathbb{N}}}^{t} \alpha^{\frac{t-j}{\ell}} z_j &= \sum_{\substack{j=t-s\ell \\ \frac{t-j}{\ell}\in\mathbb{N}}}^{t} \left(\alpha^{\frac{\kappa-1}{\kappa\ell}}\right)^{t-j} \alpha^{\frac{t-j}{\kappa\ell}} z_j \\
&\leq \max_{\substack{t-s\ell\leq j\leq t \\ \frac{t-j}{\ell}\in\mathbb{N}}} \left(\alpha^{\frac{t-j}{\kappa\ell}} z_j\right) \sum_{\substack{j=t-s\ell \\ \frac{t-j}{\ell}\in\mathbb{N}}}^{t} \left(\alpha^{\frac{\kappa-1}{\kappa\ell}}\right)^{t-j}
\end{aligned}$$

From the properties of geometric series, after changing the index $i := (t-j)/\ell$, it follows that

$$\begin{aligned}
\sum_{\substack{j=t-s\ell \\ \frac{t-j}{\ell}\in\mathbb{N}}}^{t} \left(\alpha^{\frac{\kappa-1}{\kappa\ell}}\right)^{t-j} = \sum_{i=0}^{s} \left(\alpha^{\frac{\kappa-1}{\kappa}}\right)^{i} &= \frac{1-\left(\alpha^{\frac{\kappa-1}{\kappa}}\right)^{s+1}}{1-\alpha^{\frac{\kappa-1}{\kappa}}} \\
&\leq \frac{1}{1-\alpha^{\frac{\kappa-1}{\kappa}}}\,.
\end{aligned}$$

Consequently, we obtain

$$\sum_{j=0}^{s} \alpha^j z_{t-j\ell} \leq \left(\frac{1}{1-\alpha^{\frac{\kappa-1}{\kappa}}} \right) \max_{\substack{t-s\ell \leq j \leq t \\ \frac{t-j}{\ell} \in \mathbb{N}}} \left(\alpha^{\frac{t-j}{\kappa\ell}} z_j \right).$$

By using (2.12)-(2.13) as in Lemma 2.2.3 and since

$$\alpha^{\frac{t}{\ell}} \leq \alpha^{\frac{t}{\kappa\ell}}$$

for all integer $\kappa \geq 2$, from (2.10) we get

$$\begin{aligned} u_t &\leq \max_{-\ell \leq j \leq 0} (u_j)\, \alpha^{\frac{t}{\ell}} + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa}}} \right) \max_{\substack{t-s\ell \leq j \leq t \\ \frac{t-j}{\ell} \in \mathbb{N}}} \left(\alpha^{\frac{t-j}{\kappa\ell}} z_j \right) \\ &\leq \max_{\substack{t-s\ell \leq j \leq t \\ \frac{t-j}{\ell} \in \mathbb{N}}} (u_j)\, \alpha^{\frac{t}{\kappa\ell}} + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa}}} \right) \max_{\substack{t-s\ell \leq j \leq t \\ \frac{t-j}{\ell} \in \mathbb{N}}} \left(\alpha^{\frac{t-j}{\kappa\ell}} z_j \right), \end{aligned} \tag{2.18}$$

which coincides with (2.16). Hence, if $u_j = u_0, j \in \{-\ell, \ldots, 0\}$, then

$$u_t \leq u_0 \alpha^{\frac{t}{\kappa\ell}} + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa}}} \right) \max_{\substack{t-s\ell \leq j \leq t \\ \frac{t-j}{\ell} \in \mathbb{N}}} \left(\alpha^{\frac{t-j}{\kappa\ell}} z_j \right).$$

Further, since $t - s\ell \geq 0$ and $\frac{t-j}{\ell} \in \mathbb{N}$, then we have

$$\begin{aligned} \max_{\substack{t-s\ell \leq j \leq t \\ \frac{t-j}{\ell} \in \mathbb{N}}} \left(\alpha^{\frac{t-j}{\kappa\ell}} z_j \right) &\leq \max_{\substack{0 \leq j \leq t \\ \frac{t-j}{\ell} \in \mathbb{N}}} \left(\alpha^{\frac{t-j}{\kappa\ell}} z_j \right) \\ &\leq \max_{0 \leq j \leq t} \left(\alpha^{\frac{t-j}{\kappa\ell}} z_j \right). \end{aligned} \tag{2.19}$$

By substituting (2.19) in (2.18) the inequality (2.17) is inferred. This ends the proof.

□

Lemma 2.2.4 may be viewed as a versatile tool for stability analysis, which is useful in many control design problems, including estimation and control of time-delay systems. It can also be used for the MHE problem we handle. It is more suitable than Lemma 2.2.3 since it provides a lower value of the coefficient of the second term in the upper bound. For the purpose of proving robust stability, it is suitable to state the following Lemma 2.2.5, derived from Lemma 2.2.4 for a specific case of $(z_t)_{t\geq 0}$ by judiciously using inequality (2.16) instead of (2.17).

Lemma 2.2.5 ([15]). *Let $(u_t)_{t\geq -\ell}$ be a nonnegative sequence of real numbers and $\ell \geq 1$ is a natural number such that*

$$u_t \leq \alpha\, u_{t-\ell} + \beta\, z_t,$$

for all $t \geq \ell$ with

$$z_t = \sum_{i=t-\ell}^{t-1} \eta^{t-1-i} |d_i|^2 \tag{2.20}$$

where $\alpha \in (0,1)$, $\beta \geq 0$, and $(d_j)_{j\geq -\ell}$ is any arbitrary bounded positive sequence. Then the following inequality holds for any $\kappa \in \mathbb{N}, \kappa \geq 2$:

$$u_t \leq \lambda^t \max_{-\ell \leq j \leq 0} u_j + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa}}} \right) \sum_{i=-\ell}^{t-1} \lambda^{t-1-i} |d_i|^2 \tag{2.21}$$

where

$$\lambda := \max\left(\eta, \alpha^{\frac{1}{\kappa\ell}}\right). \tag{2.22}$$

Further, if $u_j = u_0$ and $d_j = d_0$, for $-\ell \leq j \leq 0$, we get

$$u_t \leq u_0\,\lambda^t + \frac{\beta}{(1-\lambda)^2}\sum_{i=0}^{t-1}\lambda^{t-1-i}\,|d_i|^2\,. \tag{2.23}$$

Proof. From (2.16) in Lemma 2.2.4 and the definition of λ in (2.22), we get

$$u_t \leq \lambda^t \max_{-\ell\leq j\leq 0} u_j + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa}}}\right)\max_{\substack{t-s\ell\leq j\leq t\\ \frac{t-j}{\ell}\in\mathbb{N}}}\left(\lambda^{t-j}z_j\right).$$

On the other hand, by using (2.20) and (2.22) we obtain

$$\lambda^{t-j}z_j = \sum_{i=j-\ell}^{j-1}\lambda^{t-j}\,\eta^{j-1-i}\,|d_i|^2 \leq \sum_{i=j-\ell}^{j-1}\lambda^{t-1-i}\,|d_i|^2\,.$$

It follows that

$$\begin{aligned}\max_{\substack{t-s\ell\leq j\leq t\\ \frac{t-j}{\ell}\in\mathbb{N}}}\left(\lambda^{t-j}z_j\right) &\leq \max_{\substack{t-s\ell\leq j\leq t\\ \frac{t-j}{\ell}\in\mathbb{N}}}\left(\sum_{i=j-\ell}^{j-1}\lambda^{t-1-i}\,|d_i|^2\right)\max\left(\sum_{i=t-s\ell-\ell}^{t-s\ell-1}\lambda^{t-1-i}\,|d_i|^2,\right.\\ &\qquad\left.\sum_{i=t-s\ell}^{t-(s-1)\ell-1}\lambda^{t-1-i}\,|d_i|^2,\dots,\sum_{i=t-\ell=(t-s\ell)+s\ell-\ell}^{t-1=(t-s\ell)+s\ell-1}\lambda^{t-1-i}\,|d_i|^2\right)\\ &= \sum_{i=t-s\ell-\ell}^{t-1}\lambda^{t-1-i}\,|d_i|^2\,.\end{aligned} \tag{2.24}$$

Since $t-s\ell\geq 0$, then from (2.24) we have

$$\max_{\substack{t-s\ell\leq j\leq t\\ \frac{t-j}{\ell}\in\mathbb{N}}}\left(\lambda^{t-j}z_j\right) \leq \sum_{i=-\ell}^{t-1}\lambda^{t-1-i}\,|d_i|^2$$

which leads to (2.21).
Further, if $u_j = u_0$ and $d_j = d_0$, for $-\ell\leq j\leq 0$, then we have

$$\max_{-\ell\leq j\leq 0} u_j = u_0 \tag{2.25}$$

and

$$\begin{aligned}\sum_{i=-\ell}^{t-1}\lambda^{t-1-i}\,|d_i|^2 &= \sum_{i=-\ell}^{0}\lambda^{t-1-i}\,|d_i|^2 + \sum_{i=1}^{t-1}\lambda^{t-1-i}\,|d_i|^2\\ &= \sum_{i=1}^{t-1}\lambda^{t-1-i}\,|d_i|^2 + \lambda^{t-1}\,|d_0|^2\sum_{i=-\ell}^{0}\lambda^{-i}\\ &\leq \max\left(1,\sum_{i=-\ell}^{0}\lambda^{-i}\right)\sum_{i=0}^{t-1}\lambda^{t-1-i}\,|d_i|^2\\ &\leq \frac{1}{1-\lambda}\sum_{i=0}^{t-1}\lambda^{t-1-i}\,|d_i|^2\,.\end{aligned} \tag{2.26}$$

On the other hand, from the definition of λ in (2.22), since $\kappa \geq 2$, $\ell \geq 1$, which means that $\kappa - 1 \geq 1$, $1/\ell \leq 1$, then we have

$$\alpha^{\frac{\kappa-1}{\kappa}} \leq \left(\alpha^{\frac{\kappa-1}{\kappa}}\right)^{\frac{1}{\ell}} = \left(\alpha^{\frac{1}{\kappa\ell}}\right)^{\kappa-1} \leq \alpha^{\frac{1}{\kappa\ell}} \leq \lambda$$

which leads to

$$\frac{1}{1-\alpha^{\frac{\kappa-1}{\kappa}}} \leq \frac{1}{1-\lambda}. \tag{2.27}$$

Hence, by substituting inequality (2.25), (2.26), and (2.27) in (2.21), the relation (2.23) is inferred. This ends the proof of Lemma 2.2.5.

□

Lemma 2.2.5 can be used straightforwardly to show the estimation error of the MHE_N is RES according to Definition 2.4.1 for all $t \geq N$. The following lemma is well-known as a generalization of the differential mean value theorem, or the Taylor remainder exact formula, for vector-valued functions [132].

Lemma 2.2.6 (Differential Mean Value Theorem)**.** *Let $\Psi : \mathbb{R}^n \mapsto \mathbb{R}^q$ be a differentiable function and two vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Then, there exists*

$$\boldsymbol{z} \triangleq \begin{bmatrix} \boldsymbol{z}^1 \\ \boldsymbol{z}^2 \\ \vdots \\ \boldsymbol{z}^q \end{bmatrix} \in \mathbb{R}^{nq},\ \boldsymbol{z}^i \in \boldsymbol{Co}(x,y), i = 1, \dots, q \tag{2.28}$$

where $\boldsymbol{Co}(x,y)$ stands for the convex hull of convex combinations of x and y, such that

$$\Psi(x) - \Psi(y) = \nabla_x^{\Psi}(\boldsymbol{z})(x-y) \tag{2.29}$$

where

$$\nabla_x^{\Psi}(\boldsymbol{z}) \triangleq \begin{bmatrix} \frac{\partial \Psi_1(\boldsymbol{z}^1)}{\partial x} \\ \frac{\partial \Psi_2(\boldsymbol{z}^2)}{\partial x} \\ \vdots \\ \frac{\partial \Psi_q(\boldsymbol{z}^q)}{\partial x} \end{bmatrix}. \tag{2.30}$$

The notation (2.30) is employed to represent the Jacobian matrix.

2.3 LMIs for the Design of the i-EIOSS-related coefficients

Determining the i-EIOSS coefficients by using (2.8) in Definition 1.2.13 based on system trajectories is very challenging. The objective of consists of establishing design methods that can be easily exploited by numerical software and may be used for the design of the tuning parameters of any robust estimation scheme. The i-EIOSS notion has been investigated only recently in the discrete-time setting and, to our knowledge, up to now no method has been proposed to find the parameters involved in the i-EIOSS upper bound formulation.

2.3.1 Lyapunov–Based Stability Criterion for i-EIOSS

We propose a novel LMI-based technique, which ensures not only the i-EIOSS property of a system but more importantly allows the explicit computation of the i-EIOSS related coefficients while optimizing their values by solving a simple convex optimization problem. The result is based on the use of a convenient mathematical tool for stability analysis, which allowed us to develop a novel Lyapunov function-based criterion. Hence, the combination of a quadratic Lyapunov function and the convexity principle led to new LMI conditions.
The proposed LMI-based design procedure guaranteeing the i-EIOSS property plays an important role in designing robust estimators. The main motivation for developing this LMI method is the robust convergence analysis of the Moving Horizon Estimator (MHE). Indeed, the design of a robust MHE as developed in [14, 15] requires the i-EIOSS coefficients as tuning parameters. It is worth noticing that related results on ensuring the $\delta-$IOSS property by using matrix inequalities are established in [104], where the authors proposed general but not numerically tractable design conditions. We provide a general criterion based on Lyapunov theory to guarantee the i-EIOSS property of a given system. The result of this section is summarized in the following proposition.

Proposition 2.3.1. *Let $(x_t, \tilde{x}_t)$ be two arbitrary solutions of (1.17) generated from two initial conditions $x_0, \tilde{x}_0 \in \mathcal{X}$ and two disturbance sequences $w_t, \tilde{w}_t \in \Omega$, respectively. Let $\vartheta(x_t, \tilde{x}_t)$ be a Lyapunov function and*

$$\begin{aligned}\Delta^\theta \vartheta(x_t, \tilde{x}_t) &\triangleq \vartheta(x_{t+1}, \tilde{x}_{t+1}) - \theta\vartheta(x_t, \tilde{x}_t)\\ &\triangleq \Delta^\theta \vartheta_t = \vartheta_{t+1} - \theta\vartheta_t\end{aligned} \tag{2.31}$$

where $\theta > 0$. Define $\epsilon_t \triangleq x_t - \tilde{x}_t$ and assume that the following items hold:

(i) *There exist two positive scalars $\vartheta_{\min}$ and $\vartheta_{\max}$ with $\vartheta_{\min} < \vartheta_{\max}$ satisfying:*

$$\vartheta_{\min}|\epsilon_t|^2 \leq \vartheta(x_t, \tilde{x}_t) \leq \vartheta_{\max}|\epsilon_t|^2,\ \forall t \geq 0; \tag{2.32}$$

(ii) *There exist $\theta < 1$, $c_y > 0$, $\bar{c}_w > 0$ such that*

$$\Delta^\theta \vartheta_t \leq c_y|y_t - \tilde{y}_t|^2 + \bar{c}_w|w_t - \tilde{w}_t|^2, \forall t \geq 0, \tag{2.33}$$

where y_t and $\tilde{y}_t$ are the outputs generated by system (1.17) with x_t and $\tilde{x}_t$, respectively.

Then system (1.17) is i-EIOSS according to (2.8) in Definition 1.2.13 with the following coefficients $\forall\kappa \geq 2$:

$$\begin{cases} \varrho = \theta,\ c_\nu = \frac{c_y}{\vartheta_{\min}(1-\theta^{\kappa-1})} \\ c_x = \frac{\vartheta_{\max}}{\vartheta_{\min}},\ c_w = \frac{\bar{c}_w}{\vartheta_{\min}(1-\theta^{\kappa-1})} \end{cases} \tag{2.34}$$

Proof. From item (ii) and (2.31), we can write

$$\vartheta_t \leq \theta\vartheta_{t-1} + z_t, \forall t \geq 1, \tag{2.35}$$

where $z_t = c_y|y_{t-1} - \tilde{y}_{t-1}|^2 + \bar{c}_w|w_{t-1} - \tilde{w}_{t-1}|^2$. Therefore, by applying (2.16) of Lemma 2.2.4 with the parameters

$$\ell = 1, \beta = 1, s = t-1, \alpha = \theta^\kappa, \kappa \geq 2 \tag{2.36}$$

and from $\vartheta_j = \vartheta_0, \forall j \leq 0$, by convention and construction, we get

$$
\begin{aligned}
\vartheta_t &\leq \vartheta_0 \theta^t + \left(\frac{1}{1-\theta^{\kappa-1}}\right) \max_{1\leq j \leq t} \left(\theta^{t-j} z_j\right) \\
&\leq \vartheta_0 \theta^t + \frac{1}{(1-\theta^{\kappa-1})} \sum_{j=1}^{t} \theta^{t-j} z_j \\
&\underbrace{=}_{i:=j-1} \vartheta_0 \theta^t + \frac{1}{(1-\theta^{\kappa-1})} \sum_{i=0}^{t-1} \theta^{t-1-i} z_{i+1} \\
&= \vartheta_0 \theta^t + \frac{c_y}{(1-\theta^{\kappa-1})} \sum_{i=0}^{t-1} \theta^{t-1-i} |y_i - \tilde{y}_i|^2 \\
&\quad + \frac{\bar{c}_w}{(1-\theta^{\kappa-1})} \sum_{i=0}^{t-1} \theta^{t-1-i} |w_i - \tilde{w}_i|^2. \qquad (2.37)
\end{aligned}
$$

On the other hand, from item (i), we have $|\epsilon_t|^2 \leq \frac{1}{\vartheta_{\min}} \vartheta_t$ and $\vartheta_0 \leq \vartheta_{\max} |\epsilon_0|^2$. Hence, we deduce the following inequality for all $t \geq 1$:

$$
\begin{aligned}
|\epsilon_t|^2 \leq \frac{\vartheta_{\max}}{\vartheta_{\min}} |\epsilon_0|^2 \theta^t &+ \frac{c_y}{\vartheta_{\min}(1-\theta^{\kappa-1})} \sum_{i=0}^{t-1} \theta^{t-1-i} |y_i - \tilde{y}_i|^2 \\
&+ \frac{\bar{c}_w}{\vartheta_{\min}(1-\theta^{\kappa-1})} \sum_{i=0}^{t-1} \theta^{t-1-i} |w_i - \tilde{w}_i|^2 \qquad (2.38)
\end{aligned}
$$

which means that system (1.17) is i-EIOSS according to (2.8) with the coefficients given in (2.34).

□

Proposition 2.3.1 provides a criterion to guarantee i-EIOSS of a given system, which is in general difficult to characterize. That is, without this Lyapunov-based characterization, computing the values of the i-EIOSS coefficients c_x, c_ν, and c_w becomes a hard task. On the other hand, such coefficients are necessary to design the tuning parameters of any robust estimator of system (2.1). Thus the importance of establishing a quantitative synthesis procedure instead of a qualitative one, namely a Lyapunov-based procedure. More specifically, we will propose an LMI-based design procedure, which is easily tractable by numerical software.

2.3.1.1 New LMI-based i-EIOSS criterion

By considering a particular Lyapunov function, we will obtain sufficient conditions, expressed in terms of LMIs, ensuring the property of i-EIOSS of the system (2.1). To this end, let us consider the following quadratic Lyapunov function, usually used in the literature in the LMI context:

$$
\vartheta(x_t, \tilde{x}_t) \triangleq \Big(x_t - \tilde{x}_t\Big)^\top \mathbb{P} \Big(x_t - \tilde{x}_t\Big) \qquad (2.39)
$$

where $\mathbb{P} = \mathbb{P}^\top > 0$ and $(x_t, \tilde{x}_t)$ are two arbitrary solutions of (2.1) generated from two initial conditions $x_0, \tilde{x}_0 \in \mathbb{R}^n$ and two disturbance sequences $\boldsymbol{\omega}_t, \tilde{\boldsymbol{\omega}}_t \in \mathbb{R}^q$, respectively. Consider $\epsilon_t \triangleq x_t - \tilde{x}_t$, the error between the two trajectories, $\epsilon_{\boldsymbol{\omega}} \triangleq \boldsymbol{\omega}_t - \tilde{\boldsymbol{\omega}}_t$, $\epsilon_y \triangleq y_t - \tilde{y}_t$, and define ϑ_y as:

$$
\vartheta_y \triangleq \Delta^\theta \vartheta(x_t, \tilde{x}_t) - c_y |\epsilon_y|^2 - \bar{c}_w |\epsilon_{\boldsymbol{\omega}}|^2. \qquad (2.40)
$$

First, we have

$$\epsilon_{t+1} = \nabla_x^f(\boldsymbol{z}_f, \boldsymbol{\omega})\epsilon_t + \nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f)\epsilon_{\boldsymbol{\omega}} \tag{2.41}$$

$$\epsilon_y = \nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})\epsilon_t + \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\epsilon_{\boldsymbol{\omega}}. \tag{2.42}$$

After developing $\Delta^\theta \vartheta(x_t, \tilde{x}_t)$ and from Lemma 2.2.6, we get:

$$\begin{aligned}\vartheta_y = {} & \epsilon_t^\top \left[\left(\nabla_x^f(\boldsymbol{z}_f, \boldsymbol{\omega})\right)^\top \mathbb{P}\nabla_x^f(\boldsymbol{z}_f, \boldsymbol{\omega}) - c_y \left(\nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})\right)^\top \nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega}) - \theta\mathbb{P}\right]\epsilon_t \\ & + \epsilon_{\boldsymbol{\omega}}^\top \left[\left(\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f)\right)^\top \mathbb{P}\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f) - c_y \left(\nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) - \bar{c}_w\mathbb{I}_q\right]\epsilon_{\boldsymbol{\omega}} \\ & + 2\epsilon_t^\top \left[\left(\nabla_x^f(\boldsymbol{z}_f, \boldsymbol{\omega})\right)^\top \mathbb{P}\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f) - c_y \left(\nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\right]\epsilon_{\boldsymbol{\omega}}\end{aligned} \tag{2.43}$$

which can be written under the matrix form

$$\begin{bmatrix}\epsilon_t \\ \epsilon_{\boldsymbol{\omega}}\end{bmatrix}^\top \mathbb{M}\Big(\mathbb{P}, c_y, \bar{c}_w\Big)\begin{bmatrix}\epsilon_t \\ \epsilon_{\boldsymbol{\omega}}\end{bmatrix} \tag{2.44}$$

where $\mathbb{M}\Big(\mathbb{P}, c_y, \bar{c}_w\Big)$ is defined in (2.61). Then, we have $\vartheta_y < 0$ for all $\begin{bmatrix}\epsilon_t^\top & \epsilon_{\boldsymbol{\omega}}^\top\end{bmatrix}^\top \neq 0$ if the

$$\mathbb{M}\Big(\mathbb{P}, c_y, \bar{c}_w\Big) \triangleq \begin{bmatrix}\mathbb{N}(\mathbb{P}, c_y) - \theta\mathbb{P} & \left(\nabla_x^f(\boldsymbol{z}_f, \boldsymbol{\omega})\right)^\top \mathbb{P}\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f) - c_y \left(\nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) \\ (\star) & \left(\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f)\right)^\top \mathbb{P}\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f) - c_y \left(\nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) - \bar{c}_w\mathbb{I}_q\end{bmatrix} \tag{2.45}$$

$$\mathbb{N}(\mathbb{P}, c_y) \triangleq \left(\nabla_x^f(\boldsymbol{z}_f, \boldsymbol{\omega})\right)^\top \mathbb{P}\nabla_x^f(\boldsymbol{z}_f, \boldsymbol{\omega}) - c_y \left(\nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})\right)^\top \nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})$$

$$\begin{bmatrix}-c_y \left(\nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})\right)^\top \nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega}) - \theta\mathbb{P} & -c_y \left(\nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) & \left(\nabla_x^f(\boldsymbol{z}_f, \boldsymbol{\omega})\right)^\top \mathbb{P} \\ (\star) & -c_y \left(\nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) - \bar{c}_w\mathbb{I}_q & \left(\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f)\right)^\top \mathbb{P} \\ (\star) & (\star) & -\mathbb{P}\end{bmatrix} < 0 \tag{2.46}$$

inequality $\mathbb{M}\Big(\mathbb{P}, c_y, \bar{c}_w\Big) < 0$ is satisfied. Hence, from Schur Lemma, the previous inequality is equivalent to (2.62). Before stating the main Theorem, we need to introduce some convex polytopic sets. As in (4.10)-(2.7), from Assumption 2.4.10, the Jacobians $\left(\nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})\right)^\top \nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})$, $\left(\nabla_x^h(\boldsymbol{z}_h, \boldsymbol{\omega})\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)$, and $\left(\nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)$ are bounded. Therefore, by using the convex decomposition technique [103], they belong to the convex polytopic sets defined respectively as:

$$\mathcal{V}_h^\ell \triangleq \left\{\sum_{j=1}^{n_\ell} \alpha_j \mathcal{H}_j^\ell, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{n_\ell} \alpha_j = 1\right\} \tag{2.47}$$

for $\ell = 1, 2, 3$, respectively. The matrices $\mathcal{H}_j^\ell, j = 1, 2, 3$ are known and constant with appropriate dimensions. As for the known integers n_ℓ, they represent the number of vertices of $\mathcal{V}_h^\ell$, for $\ell = 1, 2, 3$. Now we are ready to state the main Theorem, which provides LMIs ensuring the i-EIOSS property of the system (2.1).

Theorem 2.3.2. *Assume that there exists a positive definite and symmetric matrix $\mathbb{P}$, positive scalars $c_y, \bar{c}_w$, and $\theta < 1$ such that the following matrix inequalities are satisfied:*

$$\begin{bmatrix} -c_y\mathcal{H}_i^1 - \theta\mathbb{P} & -c_y\mathcal{H}_j^2 & (\mathcal{F}_l^x)^\top\mathbb{P} \\ (\star) & -c_y\mathcal{H}_k^3 - \bar{c}_w\mathbb{I}_q & (\mathcal{F}_m^{\boldsymbol{\omega}})^\top\mathbb{P} \\ (\star) & (\star) & -\mathbb{P} \end{bmatrix} < 0 \tag{2.48}$$

for all $i \in \{1,\ldots,n_1\}$, $j \in \{1,\ldots,n_2\}$, $k \in \{1,\ldots,n_3\}$, $l \in \{1,\ldots,n_x\}$, and $m \in \{1,\ldots,n_{\boldsymbol{\omega}}\}$. Then the system (2.1) is i-EIOSS according to (2.8) in Definition 1.2.13 with the coefficients defined in (2.34) with $\vartheta_{\max} = \lambda_{\max}(\mathbb{P})$ and $\vartheta_{\min} = \lambda_{\min}(\mathbb{P})$.

Proof. From Schur Lemma we have equivalence between $\mathbb{M}\Big(\mathbb{P}, c_y, \bar{c}_w\Big) < 0$ and (2.62). On other hand, the left hand side of (2.62) is affine (then convex) with respect to all the Jacobian matrices

$$\frac{\partial f(x,\boldsymbol{\omega})}{\partial x}; \frac{\partial f(x,\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}; \frac{\partial h(x,\boldsymbol{\omega})}{\partial x}; \frac{\partial h(x,\boldsymbol{\omega})}{\partial \boldsymbol{\omega}};$$

$$\Big(\nabla_x^h(\boldsymbol{z}_h,\boldsymbol{\omega})\Big)^\top \nabla_x^h(\boldsymbol{z}_h,\boldsymbol{\omega}); \Big(\nabla_x^h(\boldsymbol{z}_h,\boldsymbol{\omega})\Big)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x},\boldsymbol{v}_h); \Big(\nabla_{\boldsymbol{\omega}}^h(\tilde{x},\boldsymbol{v}_h)\Big)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x},\boldsymbol{v}_h).$$

In addition, from (4.10)-(2.7) and (2.47), these Jacobians can be decomposed into a convex form by using the convex decomposition technique [103]. Hence, from the convexity principle [30], the inequality (2.62) is satisfied for any element on the convex sets defined by (4.10)-(2.7) and (2.47) if it is satisfied on the vertices $\mathcal{H}_i^1, \mathcal{H}_j^2, \mathcal{H}_k^3$, $\mathcal{F}_l^x$, and $\mathcal{F}_m^{\boldsymbol{\omega}}$. Since (2.48) are exactly (2.62) evaluated on the vertices, then from the convexity principle, we have $\mathbb{M}\Big(\mathbb{P}, c_y, \bar{c}_w\Big) < 0$, which leads to $\vartheta_y \leq 0$. Hence, from Proposition 2.3.1 it follows that the system (2.1) is i-EIOSS with the coefficients given in (2.34) with $\vartheta_{\max} = \lambda_{\max}(\mathbb{P})$ and $\vartheta_{\min} = \lambda_{\min}(\mathbb{P})$. This ends the proof.

□

Guaranteeing the i-EIOSS property of the system (2.1) with smaller values of the parameters ϱ, c_x, c_ν, and c_w may be useful in some applications. To optimize such values while solving the LMIs (2.48), we need to solve the following convex optimization problem:

$$\min_{c_y,\bar{c}_w} (c_y + \bar{c}_w) \text{ subject to } (2.48). \tag{2.49}$$

where the parameter θ is fixed by using the gridding method [21, *Algorithm 1, Fig. 1*]. However, an ill-conditioned $\mathbb{P}$ would make $\lambda_{\min}(\mathbb{P})$ small, which leads to large values of c_ν and c_ω even if c_y and $\bar{c}_\omega$ are small. To avoid this issue, one can resort to additional constraints on $\mathbb{P}$ such as

$$\mathbb{P} \geq \mathbb{I}_n. \tag{2.50}$$

by taking advantage of homogeneity. Nevertheless, the constraint (2.50) may increase the value of $c_x = \frac{\lambda_{\max}(\mathbb{P})}{\lambda_{\min}(\mathbb{P})} \leq \lambda_{\max}(\mathbb{P})$. To minimize c_x, we need the additional constraint:

$$\mathbb{P} \leq \alpha\mathbb{I}_n, \tag{2.51}$$

while minimizing α. To sum-up, to minimize the values of c_x, c_ν, and c_ω, we propose the optimization problem (2.52):

$$\min_{c_y,\bar{c}_w,\mathbb{P},\alpha} \Big(\gamma_1\alpha + \gamma_2 c_y + \gamma_3\bar{c}_\omega\Big) \text{subject to } (2.50), (2.51), (2.48) \tag{2.52}$$

where $\gamma_j, j = 1,2,3$ are constants to be fixed by the user. To give the same weight for α, c_y, and $\bar{c}_\omega$, we take $\gamma_j = 1$.

2.3.1.2 Case of a particular family of nonlinear systems

The class of systems (2.1) is general, thus the number of LMI conditions to be solved in (2.48). However, generally several real-world application models are simpler than (2.1), namely the following class of systems is often used in the literature, especially in the LMI context:

$$\begin{cases} x_{t+1} = f(x_t) + E\boldsymbol{\omega}_t \\ y_t = h(x_t) + D\boldsymbol{\omega}_t \end{cases} \tag{2.53}$$

In this case, the LMI (2.48) is reduced to the following one:

$$\begin{bmatrix} -c_y\mathcal{H}_i^1 - \theta\mathbb{P} & -c_y\mathcal{H}_j^x D & (\mathcal{F}_l^x)^\top\mathbb{P} \\ (\star) & -c_y D^\top D - \bar{c}_w\mathbb{I}_q & E^\top\mathbb{P} \\ (\star) & (\star) & -\mathbb{P} \end{bmatrix} < 0 \tag{2.54}$$

for all $i \in \{1,\dots,n_1\}$, $j \in \{1,\dots,q_x\}$, and $l \in \{1,\dots,n_x\}$.
In addition, if we consider systems with linear outputs, i.e.: $y_t = Cx_t + D\boldsymbol{\omega}_t$, then the LMI condition is much simplified as follows:

$$\begin{bmatrix} -c_y C^\top C - \theta\mathbb{P} & -c_y C^\top D & (\mathcal{F}_j^x)^\top\mathbb{P} \\ (\star) & -c_y D^\top D - \bar{c}_w\mathbb{I}_q & E^\top\mathbb{P} \\ (\star) & (\star) & -\mathbb{P} \end{bmatrix} < 0 \tag{2.55}$$

for all $j \in \{1,\dots,n_x\}$.

2.3.1.3 On the conservatism and feasibility of (2.48)

The conservatism related to the proposed approach lies, first, in converting (2.62) into (2.48) by using the convexity principle [30, 106]. Indeed, it is reported in [132] that using the polytopic approach based on the convexity principle always provides less conservative LMI conditions compared to other strong upper bounding techniques, namely the use of Lipschitz inequality or the Young's inequality instead of the convexity principle. Furthermore, the use of a constant Lyapunov matrix is conservative; however, it provides a systematic numerical procedure applicable to a wide class of nonlinear systems. A more general Lyapunov function and matrices instead of the scalars $c_y, \bar{c}_\omega$ may be used; however, we will lose getting a systematic synthesis procedure, or even the linearity of the synthesis conditions.
The decision variables in (2.48) are the matrix $\mathbb{P}$ and the positive scalars c_y and $\bar{c}_w$, while θ is fixed a priori. All these decision variables are free solutions returned by (2.48) and have not be fixed a priori by the gridding method. Indeed, the gridding method on $\theta \in]0,\ 1[$ consists in subdividing the interval $]0,\ 1[$ into ℓ subintervals and solving the LMI (2.48) for each value $\theta_j = \frac{1}{\ell}$ until a solution is returned. All the other matrices are known and specific to the system at hand. Especially, the matrices $\mathcal{H}_i^1, \mathcal{H}_j^2, \mathcal{H}_k^3$, $\mathcal{F}_l^x$, and $\mathcal{F}_m^{\boldsymbol{\omega}}$ are known and result from the convex decomposition of the Jacobian matrices of the nonlinear functions. These matrices implicitly depend on the Lipschitz constant and the structure of the functions f and h. Therefore, the feasibility of (2.48) depends strongly on the structure of those matrices [131].
the advantages of our proposed method here can be summarized as follows:

- The method in [104] is based on the differential dynamics of the system and its linearization at a given point. However, our technique uses the generalized version of the differential mean value Theorem for vector-valued functions to transform equivalently, without linearization, the nonlinear terms into a polytopic form.

- Our method is simpler than that of [104] due to the polytopic form of the error dynamics and the convexity principle to get a finite number of LMIs.

- Our method does not require any coordinate transformation to convert the original system into a new one with linear outputs as in [104]. Indeed, avoiding such a strong assumption while preserving LMIs is a significant advantage for real-world processes.

2.3.2 LMI-based Synthesis of i-EIOSS Coefficients

The primary objective of this section is to pioneer an advanced LMI-based methodology that ensures the i-EIOSS property of the system (1.17). This approach not only guarantees the desired property but also enables the explicit computation of key coefficients, specifically, ϱ, c_x, c_ν, and c_w.
Due to the presence of nonlinear outputs, the approach outlined in Subsection2.3.1, which relies on the polytopic approach and the convexity principle, requires the construction of an extended polytopic set with higher dimensionality. This, in turn, results in an exponential increase in the number of LMI conditions that need to be solved. To address this challenge, this part introduces a novel approach centered around a particular output-based dynamic extension technique.
To avoid cumbersome equations, we will use x_t and $\tilde{x}_t$ instead of $x_t(x_0, \boldsymbol{\omega}_0^{t-1})$ and $x_t(\tilde{x}_0, \tilde{\boldsymbol{\omega}}_0^{t-1})$, respectively. The same goes for the other variables.
As mentioned earlier, we aim to construct a novel nonlinear dynamical system with linear outputs. Subsequently, we utilize the Lyapunov-based criterion presented in Proposition 2.3.1 to establish the i-EIOSS property for the original system (1.17) with the appropriate coefficients. To achieve this, we define the following new system based on the dynamics of (2.1):

$$\begin{cases} z_{t+1} = h(x_t, \boldsymbol{\omega}_t) \\ x_{t+1} = f(x_t, \boldsymbol{\omega}_t) \\ y_t^\zeta = z_t \triangleq \underbrace{\begin{bmatrix} \mathbb{I}_p & 0 \end{bmatrix}}_{\mathcal{C}} \overbrace{\begin{bmatrix} z_t \\ x_t \end{bmatrix}}^{\zeta_t} \end{cases} \overset{\text{def}}{\iff} \begin{cases} \zeta_{t+1} = \chi(\zeta_t, \boldsymbol{\omega}_t) \\ y_t^\zeta = \mathcal{C}\zeta_t \end{cases} \tag{2.56}$$

which is a nonlinear system with linear output. Now we are ready to state the intermediate result summarized in the following proposition.

Proposition 2.3.3. *Assume that the system (2.56) is i-EIOSS. That is there exist constants $c_\zeta, c_v^\zeta, c_w^\zeta >$ 0 and $\varrho_\zeta \in (0,1)$ such that for any pair of initial conditions $\zeta_0, \tilde{\zeta}_0 \in \mathcal{X}_\zeta$ and any disturbance sequences $\boldsymbol{\omega}_t, \tilde{\boldsymbol{\omega}}_t \in \mathcal{W}_{\boldsymbol{\omega}}$, the following holds:*

$$\begin{aligned} \left|\zeta_t - \tilde{\zeta}_t\right|^2 \leq c_\zeta |\zeta_0 - \tilde{\zeta}_0|^2 \varrho_\zeta^t & \\ + c_v^\zeta \sum_{i=0}^{t-1} \varrho_\zeta^{t-1-i} \left|y_i^\zeta - \tilde{y}_i^\zeta\right|^2 & \\ + c_w^\zeta \sum_{i=0}^{t-1} \varrho_\zeta^{t-1-i} \left|\boldsymbol{\omega}_i - \tilde{\boldsymbol{\omega}}_i\right|^2 . & \end{aligned} \tag{2.57}$$

Then, the system (2.1) is i-EIOSS according to Definition 2.2.2 with the corresponding coefficients

$$\begin{cases} \varrho = \varrho_\zeta, \; c_\nu = \frac{c_v^\zeta}{\varrho_\zeta} \\ c_x = c_\zeta, \; c_w = c_w^\zeta. \end{cases} \tag{2.58}$$

Proof. The proof is intuitive. Let x_t and $\tilde{x}_t$ be two solutions of (2.1) generated by the initial states $x_0, \tilde{x}_0 \in \mathcal{X}$ and two disturbance sequences $\boldsymbol{\omega}_t, \tilde{\boldsymbol{\omega}}_t \in \mathcal{W}_{\boldsymbol{\omega}}$, respectively. Then, $\zeta_t = \begin{bmatrix} z_t & x_t \end{bmatrix}^\top$ and $\tilde{\zeta}_t = \begin{bmatrix} \tilde{z}_t & \tilde{x}_t \end{bmatrix}^\top$ are also solutions to (2.1) for any $z_0, \tilde{z}_0 \in \mathcal{Y}$. On the other hand, we have

$$\begin{cases} |x_t - \tilde{x}_t| \leq \left|\zeta_t - \tilde{\zeta}_t\right| \\ \left|\zeta_0 - \tilde{\zeta}_0\right|^2 = |x_0 - \tilde{x}_0|^2 + |z_0 - \tilde{z}_0|^2 \\ c_v^\zeta \sum_{i=0}^{t-1} \varrho_\zeta^{t-1-i} \left|y_i^\zeta - \tilde{y}_i^\zeta\right|^2 \leq c_v^\zeta \varrho_\zeta^{t-1} |z_0 - \tilde{z}_0|^2 \\ \qquad\qquad + \frac{c_v^\zeta}{\varrho_\zeta} \sum_{i=0}^{t-1} \varrho_\zeta^{t-1-i} |y_i - \tilde{y}_i|^2 \end{cases} \tag{2.59}$$

since $y_i^\zeta = y_{i-1}$ and $\tilde{y}_i^\zeta = \tilde{y}_{i-1}$. Consequently, if in particular we choose the solutions ζ_t and $\tilde{\zeta}_t$ generated by the same initial component $z_0 = \tilde{z}_0$, then from (2.57) and (2.59), the following final bound is inferred:

$$\begin{aligned} |x_t - \tilde{x}_t|^2 \leq\ & c_\zeta |x_0 - \tilde{x}_0|^2 \varrho_\zeta^t \\ & + \frac{c_v^\zeta}{\varrho_\zeta} \sum_{i=0}^{t-1} \varrho_\zeta^{t-1-i} |y_i - \tilde{y}_i|^2 \\ & + c_w^\zeta \sum_{i=0}^{t-1} \varrho_\zeta^{t-1-i} |\boldsymbol{\omega}_i - \tilde{\boldsymbol{\omega}}_i|^2 , \end{aligned}$$

which means that the system (2.1) is i-EIOSS with the corresponding coefficients in (2.58).

□

Instead of following the approach in [19], which utilizes an LMI-based method directly on the system (2.1) with nonlinear outputs, we will focus on the derived system (2.56) featuring linear output measurements. To start the development of the main LMI condition ensuring the i-EIOSS property of the system (2.56), we consider the Lyapunov function $\vartheta(\zeta_t, \tilde{\zeta}_t) \triangleq (\zeta_t - \tilde{\zeta}_t)^\top \mathcal{P} (\zeta_t - \tilde{\zeta}_t)$. Now define the error vectors $\epsilon_t \triangleq \zeta_t - \tilde{\zeta}_t$, $\epsilon_{\boldsymbol{\omega}} \triangleq \boldsymbol{\omega}_t - \tilde{\boldsymbol{\omega}}_t$, $\epsilon_y \triangleq y_t^\zeta - \tilde{y}_t^\zeta$ and the function

$$\vartheta_y \triangleq \Delta^\theta \vartheta(\zeta_t, \tilde{\zeta}_t) - \epsilon_y^\top \mathcal{S}_y \epsilon_y - \epsilon_{\boldsymbol{\omega}}^\top \mathcal{S}_{\boldsymbol{\omega}} \epsilon_{\boldsymbol{\omega}}.$$

where $\mathcal{S}_y$ and $\mathcal{S}_{\boldsymbol{\omega}}$ are two symmetric and positive definite matrices of appropriate dimensions. By applying Lemma 2.2.6 on the function χ, we get the following expression of the dynamic of the error ϵ_t:

$$\epsilon_{t+1} = \nabla_\zeta^\chi(\boldsymbol{z}_\chi, \boldsymbol{\omega})\epsilon_t + \nabla_{\boldsymbol{\omega}}^\chi(\tilde{\zeta}, \boldsymbol{v}_\chi)\epsilon_{\boldsymbol{\omega}}$$

where

$$\nabla_\zeta^\chi(\boldsymbol{z}_\chi, \boldsymbol{\omega}) \triangleq \left[\frac{\partial \chi_1(\boldsymbol{z}_\chi^1, \boldsymbol{\omega})}{\partial \zeta}\ \frac{\partial \chi_2(\boldsymbol{z}_\chi^2, \boldsymbol{\omega})}{\partial \zeta}\ \ldots\ \frac{\partial \chi_q(\boldsymbol{z}_\chi^q, \boldsymbol{\omega})}{\partial \zeta}\right]^\top$$

and

$$\nabla_{\boldsymbol{\omega}}^\chi(\tilde{\zeta}, \boldsymbol{v}_\chi) \triangleq \left[\frac{\partial \chi_1(\tilde{\zeta}, \boldsymbol{v}_\chi^1)}{\partial \boldsymbol{\omega}}\ \frac{\partial \chi_2(\tilde{\zeta}, \boldsymbol{v}_\chi^2)}{\partial \boldsymbol{\omega}}\ \ldots\ \frac{\partial \chi_q(\tilde{\zeta}, \boldsymbol{v}_\chi^q)}{\partial \boldsymbol{\omega}}\right]^\top . \tag{2.60}$$

By expanding the expression of $\Delta^\theta \vartheta(\zeta_t, \tilde{\zeta}_t)$, we obtain:

$$\begin{aligned} \vartheta_y = \epsilon_t^\top & \left[\left(\nabla_\zeta^\chi(\boldsymbol{z}_\chi, \boldsymbol{\omega})\right)^\top \mathcal{P} \nabla_\zeta^\chi(\boldsymbol{z}_\chi, \boldsymbol{\omega}) - \mathcal{C}^\top \mathcal{S}_y \mathcal{C} - \theta \mathcal{P}\right] \epsilon_t \\ + \epsilon_{\boldsymbol{\omega}}^\top & \left[\left(\nabla_{\boldsymbol{\omega}}^\chi(\tilde{\zeta}, \boldsymbol{v}_\chi)\right)^\top \mathcal{P} \nabla_{\boldsymbol{\omega}}^\chi(\tilde{\zeta}, \boldsymbol{v}_\chi) - \mathcal{S}_{\boldsymbol{\omega}}\right] \epsilon_{\boldsymbol{\omega}} \\ + 2\epsilon_t^\top & \left[\left(\nabla_\zeta^\chi(\boldsymbol{z}_\chi, \boldsymbol{\omega})\right)^\top \mathcal{P} \nabla_{\boldsymbol{\omega}}^\chi(\tilde{\zeta}, \boldsymbol{v}_\chi)\right] \epsilon_{\boldsymbol{\omega}} \end{aligned}$$

which we can write under the form

$$\begin{bmatrix}\epsilon_t\\ \epsilon_{\boldsymbol{\omega}}\end{bmatrix}^\top \mathbb{Q}\Big(\mathcal{P},\mathcal{S}_y,\mathcal{S}_{\boldsymbol{\omega}}\Big)\begin{bmatrix}\epsilon_t\\ \epsilon_{\boldsymbol{\omega}}\end{bmatrix}$$

where $\mathbb{Q}\Big(\mathcal{P},\mathcal{S}_y,\mathcal{S}_{\boldsymbol{\omega}}\Big)$ is expressed in detail in (2.61).

$$\mathbb{Q}\Big(\mathcal{P},\mathcal{S}_y,\mathcal{S}_{\boldsymbol{\omega}}\Big) \triangleq \begin{bmatrix}\left(\nabla_x^\chi(\boldsymbol{z}_\chi,\boldsymbol{\omega})\right)^\top \mathcal{P}\nabla_x^\chi(\boldsymbol{z}_\chi,\boldsymbol{\omega}) - \mathcal{C}^\top\mathcal{S}_y\mathcal{C} - \theta\mathcal{P} & \left(\nabla_x^\chi(\boldsymbol{z}_\chi,\boldsymbol{\omega})\right)^\top \mathcal{P}\nabla_{\boldsymbol{\omega}}^\chi(\tilde{x},\boldsymbol{v}_\chi)\\ (\star) & \left(\nabla_{\boldsymbol{\omega}}^\chi(\tilde{x},\boldsymbol{v}_\chi)\right)^\top \mathcal{P}\nabla_{\boldsymbol{\omega}}^\chi(\tilde{x},\boldsymbol{v}_\chi) - \mathcal{S}_{\boldsymbol{\omega}}\end{bmatrix} \tag{2.61}$$

Therefore, $\vartheta_y < 0$ for all $\begin{bmatrix}\epsilon_t^\top & \epsilon_{\boldsymbol{\omega}}^\top\end{bmatrix}^\top \neq 0$ if $\mathbb{Q}\Big(\mathcal{P},\mathcal{S}_y,\mathcal{S}_{\boldsymbol{\omega}}\Big) < 0$ holds, which is equivalent, by Schur Lemma, to the following inequality:

$$\begin{bmatrix}-\mathcal{C}^\top\mathcal{S}_y\mathcal{C} - \theta\mathcal{P} & 0 & \Big(\nabla_\zeta^\chi(\boldsymbol{z}_\chi,\boldsymbol{\omega})\Big)^\top \mathcal{P}\\ (\star) & -\mathcal{S}_{\boldsymbol{\omega}} & \Big(\nabla_{\boldsymbol{\omega}}^\chi(\tilde{\zeta},\boldsymbol{v}_\chi)\Big)^\top \mathcal{P}\\ (\star) & (\star) & -\mathcal{P}\end{bmatrix} < 0. \tag{2.62}$$

Since the Jacobians $\nabla_\zeta^\chi(\boldsymbol{z}_\chi,\boldsymbol{\omega})$ and $\nabla_{\boldsymbol{\omega}}^\chi(\tilde{\zeta},\boldsymbol{v}_\chi)$ are bounded from the previous assumptions, they belong to convex polytopic sets defined respectively by:

$$\mathcal{V}_\zeta^\chi \triangleq \left\{\sum_{j=1}^{n_\zeta}\beta_j F_j^\zeta,\ \text{such that } \beta_j \geq 0, \sum_{j=1}^{n_\zeta}\beta_j = 1\right\}$$

$$\mathcal{V}_{\boldsymbol{\omega}}^\chi \triangleq \left\{\sum_{j=1}^{n_{\boldsymbol{\omega}}}\beta_j F_j^{\boldsymbol{\omega}},\ \text{such that } \beta_j \geq 0, \sum_{j=1}^{n_{\boldsymbol{\omega}}}\beta_j = 1\right\}$$

where n_ζ and $n_{\boldsymbol{\omega}}$ represent the number of vertices of $\mathcal{V}_\zeta^\chi$ and $\mathcal{V}_{\boldsymbol{\omega}}^\chi$, respectively; F_j^ζ and $F_j^{\boldsymbol{\omega}}$ are constant matrices of appropriate dimensions. For more details on the representation of elements in a convex set and the convex decomposition of nonlinear functions, we refer the reader to [19, 30, 103, 106]. We are now ready to summarize the results of this section into the following main Theorem, which establishes LMI conditions guaranteeing the i-EIOSS property of the system represented by equation (2.56). Consequently, the Theorem provides the corresponding i-EIOSS coefficients for system (2.1).

Theorem 2.3.4. *Assume that there exist symmetric and positive definite matrices $\mathcal{P},\mathcal{S}_y$, and $\mathcal{S}_{\boldsymbol{\omega}}$, of appropriate dimensions, and $\theta \in (0,1)$ such that the following matrix inequality is satisfied:*

$$\begin{bmatrix}-\mathcal{C}^\top\mathcal{S}_y\mathcal{C} - \theta\mathcal{P} & 0 & \Big(F_i^\zeta\Big)^\top \mathcal{P}\\ (\star) & -\mathcal{S}_{\boldsymbol{\omega}} & \big(F_j^{\boldsymbol{\omega}}\big)^\top \mathcal{P}\\ (\star) & (\star) & -\mathcal{P}\end{bmatrix} < 0 \tag{2.63}$$

for all $i \in \{1,\ldots,n_\zeta\}$, $j \in \{1,\ldots,n_{\boldsymbol{\omega}}\}$. Then the system (2.1) is i-EIOSS according to (2.8) in Definition 2.2.2 with the following coefficients $\forall \kappa \geq 2$:

$$\begin{cases}\varrho = \theta,\ c_\nu = \frac{\lambda_{\max}(\mathcal{S}_y)}{\theta\lambda_{\min}(\mathcal{P})(1-\theta^{\kappa-1})}\\ c_x = \frac{\lambda_{\max}(\mathcal{P})}{\lambda_{\min}(\mathcal{P})},\ c_w = \frac{\lambda_{\max}(\mathcal{S}_{\boldsymbol{\omega}})}{\lambda_{\min}(\mathcal{P})(1-\theta^{\kappa-1})}.\end{cases} \tag{2.64}$$

Proof. The proof is nearly complete. Given that the left-hand side of equation (2.62) is affine and, therefore, convex with respect to the generalized gradients $\nabla^{\chi}_{\zeta}(z\chi, \omega)$ and $\nabla^{\chi}_{\omega}(\tilde{\zeta}, v\chi)$, the convexity principle implies that if condition (2.63) is satisfied, then (2.62) is also fulfilled. Consequently, $\mathbb{Q}\Big(\mathcal{P}, \mathcal{S}y, \mathcal{S}\omega\Big) < 0$, leading to $\vartheta_y < 0$ for all non-zero vectors $\begin{bmatrix}\epsilon_t^\top & \epsilon_\omega^\top\end{bmatrix}^\top$. By invoking Proposition 2.3.1, we deduce that the system (2.56) is i-EIOSS as per (2.57), and this holds for all $\kappa \geq 2$ with the following coefficients:

$$\begin{cases} \varrho_\zeta = \theta, \ c_v^\zeta = \frac{\lambda_{\max}(\mathcal{S}_y)}{\lambda_{\min}(\mathcal{P})(1-\theta^{\kappa-1})} \\ c_\zeta = \frac{\lambda_{\max}(\mathcal{P})}{\lambda_{\min}(\mathcal{P})}, \ c_w^\zeta = \frac{\lambda_{\max}(\mathcal{S}_\omega)}{\lambda_{\min}(\mathcal{P})(1-\theta^{\kappa-1})}. \end{cases} \tag{2.65}$$

Finally, form Proposition 2.3.3 we conclude that the system (2.1) is i-EIOSS with the coefficients given in (2.64). This completes the proof.

□

Remark 2.3.5. *The optimization of the coefficients outlined in (2.65) can be accomplished by introducing supplementary constraints to be jointly solved with the LMIs (2.63). For a more detailed exploration of this aspect, we refer interested readers to [19], where valuable guidelines and insights on this matter have been presented.*

2.4 Moving Horizon Estimation Using Advanced Prediction Strategies

The use of the notion i-EIOSS in the concept of the MHE allows getting synthesis conditions of the parameters of the cost function that are numerically or analytically easy to verify. As shown in [114], systems admitting robust estimators must be i-IOSS. However, these results do not apply to moving-horizon estimators [11]. This objective is the main motivation for the work presented in this chapter, based on the i-EIOSS property of a given nonlinear system, we synthesize the parameters of the cost function guaranteeing the robust convergence of the MHE.
In the following, we present sufficient conditions for achieving the RES of the MHE_N, based on the assumption that the system (1.17) is i-EIOSS. Likewise in [75, 85, 104, 105], we study the RES of the estimation error but, as novelty w.r.t. such literature, with different prediction methods, for each of which sufficient conditions are established. Specifically, we will show that for a large enough size of the moving window, namely N, the error given by MHE_N is $N-$RES. We recall the class of system (1.17) nonlinear discrete-time given by:

$$\begin{cases} x_{t+1} = f(x_t, w_t) \\ y_t = h(x_t, v_t) \end{cases}$$

where $x_t \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state of the system; $y_t \in \mathbb{R}^m$ is the output vector; $w_t \in \mathcal{W} \subseteq \mathbb{R}^p$ and $v_t \in \mathcal{V} \subseteq \mathbb{R}^q$ are unknown external disturbances. The functions $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are assumed to be continuous with respect to their arguments. For simplicity, we do not consider known external input, which however does not undermine the generality of what follows. Let us denote by $|x|$ the Euclidean norm $\sqrt{x^\top x}$, $x \in \mathbb{R}^n$.
To establish robust stability of the estimation error given by MHE_N defined in (1.3.4.2), we first introduce the following definition of the robust exponential stability of an estimator. In practice, the property of robust exponential stability signifies the system's ability to maintain accurate estimation despite disturbances and uncertainties. This characteristic ensures that the disparity between the actual system state and the estimated state computed by MHE diminishes exponentially over time.

Definition 2.4.1. *An* MHE_N *is robustly exponentially stable (RES) if the following inequality holds:*

$$
\begin{aligned}
|x_t - \hat{x}_{t|t}| \leq \alpha_1 |x_0 - \bar{x}_0| \lambda^t + \alpha_2 \sum_{i=0}^{t-1} \lambda^{t-1-i} |v_i|^2 \\
+ \alpha_3 \sum_{i=0}^{t-1} \lambda^{t-1-i} |w_i|^2
\end{aligned}
\tag{2.66}
$$

for some $\lambda \in (0,1)$ *and* $\alpha_i > 0, i = 1,2,3$. *Further, if inequality (2.66) is satisfied for all* $t \geq \ell$*, where* $\ell \geq 1$ *is an integer, we say that the* MHE_N *is* ℓ*–RES.*

We propose novel robust stability conditions of the MHE for systems for which the property of incremental exponential input/output-to-state stability holds. The i-EIOSS property given in 2.8 characterizes the stability and robustness of a dynamic system with respect to inputs, outputs, and disturbances. This ensures that the difference between the system states, accounting for different initial conditions and disturbances, remains bounded and exponentially decreases over time, thereby ensuring reliable system behavior.

Remark 2.4.2. *The previous definition 2.8 can be applied not only to states at time* t *and* 0*, respectively, but also to account, for instance, for the exponential discount of the error on trajectories between* t *and* $t-\ell$*. Since especially for the MHE problems studied here, we will need to apply the definition for* $t \geq \ell$*, and between* t *and* $t-\ell$*, then we will use the following inequality:*

$$
\begin{aligned}
|x_t(x_{t-\ell}, w_{t-\ell}^{t-1}) - \tilde{x}_t(\tilde{x}_{t-\ell}, \tilde{w}_{t-\ell}^{t-1})|^2 \leq c_x |x_{t-\ell} - \tilde{x}_{t-\ell}|^2 \varrho^\ell \\
+ c_v \sum_{i=t-\ell}^{t-1} \varrho^{t-1-i} \Big| y_i(x_{t-\ell}, w_{t-\ell}^{i-1}, v_{t-\ell}^{i}) \\
- y_i(\tilde{x}_{t-\ell}, \tilde{w}_{t-\ell}^{i-1}, \tilde{v}_{t-\ell}^{i}) \Big|^2 + c_w \sum_{i=t-\ell}^{t-1} \varrho^{t-1-i} |w_i - \tilde{w}_i|^2
\end{aligned}
\tag{2.67}
$$

in line with (2.8). For more details on this inequality, we refer the reader to [105] and [85, Definition 2, and Lemma 7] for a more general case.

The above definitions will be utilized in the following to analyze robust stability analysis of the MHE_N. We provide sufficient conditions ensuring the RES of the MHE_N, according to the Definition 2.4.1 based solely on the assumption that system (1.17) is i-EIOSS. The observability condition is not necessary with the approach we propose in the paper, which is a significant improvement compared to previous results. Moreover, unlike [75,85,104,105], we focus on exponential stability of the MHE_N with quadratic cost functions and propose different prediction methods. PredictioR plays an important role in the stability conditions in terms of required assumptions to ensure the RES of the MHE_N.

2.4.1 Upper bound of the estimation error

Before introducing the different prediction equations, we start by providing an upper bound on the estimation error $e_t := x_t - \hat{x}_{t|t}$. For that, we will exploit the minimization of the cost function and the i-EIOSS property of the system (1.17). The upper bound on the error e_t depends on the prediction error $\bar{e}_t := x_t - \bar{x}_t$ (or $\bar{e}_{t-N} := x_{t-N} - \bar{x}_{t-N}$, at time $t-N$).
From the definition of minimizer $\hat{x}_{t-N|t}$, i.e.,

$$
J_t^N(\hat{x}_{t-N|t}) \leq J_t^N(x_{t-N}),
$$

we obtain

$$
\begin{aligned}
\mu|\hat{x}_{t-N|t} - \bar{x}_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|y_i - h(\hat{x}_{i|t},0)|^2 + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2 \\
\leq \mu|x_{t-N} - \bar{x}_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|h(x_i,v_i) \\
- h(x_i,0)|^2 + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2
\end{aligned}
\tag{2.68}
$$

for any $\omega \geq 0$.
To get a bound on the estimation error, we need to upper-bound the term $|h(x_i,v_i) - h(x_i,0)|$. From above, a specific assumption is required as follows.

Assumption 2.4.3. *The function $(x,v) \mapsto h(x,v)$ is γ_h Lipschitz with respect to its second argument, uniformly in $x \in \mathcal{X}$, namely*

$$
|h(x,v) - h(x,\tilde{v})| \leq \gamma_h \, |v - \tilde{v}| \, , \; \forall v, \tilde{v} \in \mathcal{V} \, .
$$

Assumption 2.4.3 is not conservative as it is easily satisfied by a large class of systems. For instance, in case of additive noise $h(x,v) := h(x) + v$, we have $\gamma_h = 1$. Generally speaking, Assumption 2.4.3 turns out to be not restrictive as compared with standard hypotheses adopted in the current literature on MHE. Using Assumption 2.4.3 and

$$
\frac{1}{2}\,|e_{t-N}|^2 \leq |\bar{e}_{t-N}|^2 + \left|\bar{x}_{t-N} - \hat{x}_{t-N|t}\right|^2 \, ,
$$

(2.68) yields

$$
\begin{aligned}
\frac{\mu}{2}\,|e_{t-N}|^2\,\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}\left|y_i - h(\hat{x}_{i|t},0)\right|^2 + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2 \\
\leq 2\mu\,|\bar{e}_{t-N}|^2\,\eta^N + \nu\,\gamma_h^2 \sum_{t=k-N}^{t-1} \eta^{t-i-i}\,|v_i|^2 + \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2.
\end{aligned}
$$

Since the system (1.17) is i-EIOSS according to Definition 1.2.13, then by applying inequality (2.67) with convenient parameters μ, ν, ω, and η such that

$$
\begin{cases}
\varrho \leq \eta < 1 \\
\mu \geq 2c_x, \; \nu \geq c_v, \; \omega = c_w
\end{cases}
$$

we obtain the following inequality:

$$
\begin{aligned}
|e_t|^2 \leq 2\mu\,|\bar{e}_{t-N}|^2\,\eta^N + \nu\,\gamma_h^2 \sum_{i=t-N}^{t-1} \eta^{t-1-i}\,|v_i|^2 \\
+ c_w \sum_{i=t-N}^{t-1} \eta^{t-1-i}\,|w_i|^2 \, .
\end{aligned}
\tag{2.69}
$$

As the parameter ω is not subject to tuning in the cost function, we have explicitly set its value to $\omega = c_w$.

The inequality (2.69) allows us to obtain an upper bound on the estimation error e_t; however, the term $\bar{e}_{t-N}$ prevents us from reaching a conclusion at this stage. The fate of this term depends on the prediction step, specifically how we choose $\bar{x}_{t-N}$. Consequently, we propose three distinct prediction strategies, each requiring specific conditions to ensure the robust stability of the resulting MHE_N. These strategies are denoted as follows:

1. ***Freeze-all prediction:*** In this prediction strategy, $\hat{x}_{t-N|t-N}$ is chosen as the prediction of x_{t-N} at time t. This means that we directly use the state estimate obtained from the MHE scheme for the prediction without any further adjustments. This prediction method is the most popular in the current literature [75, 85, 104, 105].

2. ***One-step-ahead prediction:*** Inspired by the past literature on MHE (see, among others, [5]). It hinges upon $\bar{x}_{t-N} = f(\hat{x}_{t-N-1|t-N-1}, 0)$ as a prediction of x_{t-N} at time t to capture the system dynamics more reliably by using the most recent estimate $\hat{x}_{t-N-1|t-N-1}$ of $\hat{x}_{t-N-1}$, obtained by the MHE algorithm at the previous time step $t-N-1$.

3. ***Output-driven one-step-ahead prediction:*** This new prediction strategy improves upon the one-step-ahead approach by taking into account y_{t-N-1}. In practice, it is based on

$$\begin{aligned}\bar{x}_{t-N} &= f(\hat{x}_{t-N-1|t-N-1}, 0) \\ &\quad + \psi\big(\hat{x}_{t-N-1|t-N-1}, y_{t-N-1}\big)\Big(y_{t-N-1} - h(\hat{x}_{t-N-1|t-N-1}, 0)\Big)\end{aligned} \tag{2.70}$$

   where $\psi(\cdot,\cdot)$ is a smooth "*prediction-gain*" function, which will be analyzed later.

Before proceeding, let us focus on the special case for $t \leq N$; from (1.69) it follows that

$$|e_t|^2 \leq 2\mu\,|\bar{e}_0|^2\,\eta^t + \nu\,\gamma_h^2 \sum_{i=0}^{t-1} \eta^{t-1-i}\,|v_i|^2 + c_w \sum_{i=0}^{t-1} \eta^{t-1-i}\,|w_i|^2\ , \tag{2.71}$$

independently of the specific prediction strategy we will adopt. It can be obtained by using the same arguments as in (2.68) and the i-EIOSS property. The objective here is to combine (2.69) and (2.71) to get a unified and general RES bound for the estimation error for all $t \geq N+1$.

2.4.2 Freeze-all prediction

The most straightforward solution consists of keeping and forcing the term $\bar{e}_{t-N}$ in (2.69) to be equal to e_{t-N}. To this end, the prediction $\bar{x}_{t-N}$ is determined from the estimate via the following scheme:

$$\bar{x}_{t-N} = \begin{cases} \bar{x}_0, & t = 1, ..., N \\ \hat{x}_{t-N|t-N}, & t \geq N+1. \end{cases} \tag{2.72}$$

In this case, the inequality (2.69) becomes as follows:

$$|e_t|^2 \leq 2\mu\,|e_{t-N}|^2\,\eta^N + \nu\,\gamma_h^2 \sum_{i=t-N}^{t-1} \eta^{t-1-i}\,|v_i|^2 + c_w \sum_{i=t-N}^{t-1} \eta^{t-1-i}\,|w_i|^2\ . \tag{2.73}$$

The above inequality (2.73) allows concluding on the RES of the MHE_N for N large enough. Now that the essential stability analysis tools are provided in Section 2.2.1, we only need to apply them with particular and convenient parameters. The result is summarized in the following main Theorem 3.2.1.

Theorem 2.4.4. *Assume that the system (1.17) is i-EIOSS according to (2.8) with exponential discount ϱ. Then, the MHE_N with prediction equation (2.72) is RES according to the following inequality:*

$$\left|x_t - \hat{x}_{t|t}\right|^2 \leq \max(2\mu, 1)\,|x_0 - \bar{x}_0|^2 \lambda^t + \frac{\nu\gamma_h^2}{(1-\lambda)^2} \sum_{i=0}^{t-1} \lambda^{t-1-i} |v_i|^2 + \frac{c_w}{(1-\lambda)^2} \sum_{i=0}^{t-1} \lambda^{t-1-i} |w_i|^2 \tag{2.74}$$

with exponential discount parameter

$$\lambda := \max\left(\eta, \left(2\mu\eta^N\right)^{\frac{1}{\kappa N}}\right)$$

and integer $\kappa \geq 2$ if μ, ν, η, and $N \geq 1$ satisfy the following conditions:

(i) $\varrho \leq \eta < 1$

(ii) $\mu \geq 2c_x$

(iii) $\nu \geq c_v$

(iv) $2\,\mu\,\eta^N < 1$.

Proof. First, notice that conditions (i)-(iv) guarantee the inequality (2.73). By applying Lemma 2.2.5 to (2.73) with

$$d_i = \begin{pmatrix} \gamma_h \sqrt{\nu}\; v_i \\ \sqrt{c_w}\; w_i \end{pmatrix} \quad \alpha = 2\mu\eta^N \quad \beta = 1, \ell = N\,,$$

and using (v), from (2.73) it follows that

$$\left|x_t - x_{t|t}\right|^2 \leq |x_0 - \bar{x}_0|^2 \lambda^t + \frac{\nu\gamma_h^2}{(1-\lambda)^2} \sum_{i=0}^{t-1} \lambda^{t-1-i} |v_i|^2 + \frac{c_w}{(1-\lambda)^2} \sum_{i=0}^{t-1} \lambda^{t-1-i} |w_i|^2$$

by means of Lemma 2.2.5 only for $t \geq N$. On the other hand, (2.74) holds for $t \leq N-1$ by considering (2.71) and the upper bounds

$$\nu\,\gamma_h^2 \leq \frac{\nu\,\gamma_h^2}{(1-\lambda)^2} \quad c_w \leq \frac{c_w}{(1-\lambda)^2}\,.$$

□

Remark 2.4.5. *From condition (iv) of Theorem 3.2.1, we obtain that the window size needs to be sufficiently high, namely*

$$N \geq 1 + \left\lfloor -\frac{\ln(2\mu)}{\ln \eta} \right\rfloor$$

where $\lfloor z \rfloor$ denotes the largest integer less than $z \in \mathbb{R}$. Thus, we deduce that the MHE_N is RES for any $N \geq 1$ if the parameter μ is chosen such that $\ln(2\mu) \leq 0$, i.e., $\mu \leq 1/2$. Taking into account the condition (ii) of Theorem 3.2.1, the necessary condition for the MHE_N to be RES for any $N \geq 1$ is

$$4c_x \leq 1.$$

Similarly, the RES of the MHE_N *is possible for any* $N \geq 2$ *if the necessary condition* $4c_x \leq \frac{1}{\eta}$ *is satisfied. Thus, in general, the necessary condition to design an RES* MHE_N *for* $N \geq \ell - 1, \ell \in \mathbb{N}, \ell \geq 2$*, is given as follows:*

$$\frac{1}{4c_x\eta^\ell} \geq 1. \tag{2.75}$$

Consequently, with the prediction equation (2.72), we cannot ensure exponential robustness of the MHE_N*, with* $N \leq \ell - 2$*, if (2.75) is not satisfied.*

It is clear that the stability of the MHE_N for any fixed window size, N, depends on the i-EIOSS-related parameters, namely c_x and η. On the other hand, the prediction equation (2.72) does not provide any additional tuning parameter to overcome the limitation related to the necessary condition (2.75). Hence we will analyze new prediction strategies in the next sections.

2.4.3 One-step-ahead prediction

To address this prediction strategy, we need to assume the following.

Assumption 2.4.6. *The function* $(x, w) \mapsto f(x, w)$ *is uniformly* (γ_x, γ_w) *Lipschitz with respect to its arguments, namely*

$$|f(x, w) - f(\tilde{x}, \tilde{w})| \leq \gamma_x\, |x - \tilde{x}| + \gamma_w\, |w - \tilde{w}|$$

for all $x, \tilde{x} \in \mathcal{X}, w, \tilde{w} \in \mathcal{W}$*.*

As previously pointed out, here we consider the prediction given by $\hat{x}_{t-N-1|t-N-1}$ instead of $\hat{x}_{t-N-1|t-1}$ as usually adopted in MHE. Therefore, the prediction of $\bar{x}_{t-N}$ is given by

$$\bar{x}_{t-N} = f(\hat{x}_{t-N-1|t-N-1}, 0) \tag{2.76}$$

and thus the term $\bar{e}_{t-N}$ in (2.69) becomes

$$\begin{aligned}\bar{e}_{t-N} &= f(x_{t-N-1}, w_{t-N-1}) - f(\hat{x}_{t-N-1|t-N-1}, 0)\\ &= f(x_{t-N-1}, w_{t-N-1}) - f(\hat{x}_{t-N-1|t-N-1}, w_{t-N-1})\\ &+ f(\hat{x}_{t-N-1|t-N-1}, w_{t-N-1}) - f(\hat{x}_{t-N-1|t-N-1}, 0).\end{aligned} \tag{2.77}$$

From (2.77) it follows that

$$|\bar{e}_{t-N}|^2 \leq 2\gamma_x^2 |e_{t-N-1}|^2 + 2\gamma_w^2 |w_{t-N-1}|^2 \tag{2.78}$$

by using Assumption 2.4.6 and Young's inequality[2] By substituting (2.78) in (2.69), we obtain

$$\begin{aligned}|e_t|^2 \leq 4\mu\gamma_x^2\, |e_{t-N-1}|^2\, \eta^N + \nu\gamma_h^2 \sum_{i=t-(N+1)}^{t-1} \eta^{t-1-i}\, |v_i|^2\\ + \max(4\mu\gamma_w^2, c_w) \sum_{i=t-N-1}^{t-1} \eta^{t-1-i}\, |w_i|^2\, .\end{aligned} \tag{2.79}$$

Thus, based on the aforementioned we can state what follows.

[2] The bound $2ab \leq a^2 + b^2$ holding for all $a, b \in \mathbb{R}$ will be referred to as Young's inequality.

Theorem 2.4.7. *Assume that system (1.17) is i-EIOSS according to (2.8) with exponential discount ϱ. Then, the MHE_N with prediction equation (2.76) is RES according to the following inequality:*

$$\begin{aligned}\left|x_t - \hat{x}_{t|t}\right|^2 &\le \max(2\mu, 1)\left|x_0 - \bar{x}_0\right|^2 \lambda^t \\ &+ \frac{\nu\gamma_h^2}{(1-\lambda)^2}\sum_{i=0}^{t-1}\lambda^{t-1-i}|v_i|^2 + \frac{\max(4\mu\gamma_w^2, c_w)}{(1-\lambda)^2}\sum_{i=0}^{t-1}\lambda^{t-1-i}|w_i|^2\end{aligned} \tag{2.80}$$

with exponential discount parameter

$$\lambda := \max\left(\eta, \left(4\mu\gamma_x^2\eta^N\right)^{\frac{1}{\kappa(N+1)}}\right)$$

and integer $\kappa \ge 2$ if μ, ν, η, and $N \ge 1$ satisfy the following conditions:

(i) $\varrho \le \eta < 1$

(ii) $\mu \ge 2c_x$

(iii) $\nu \ge c_v$

(iv) $4\mu\,\gamma_x^2\,\eta^N < 1\,.$

Proof. Again notice that conditions (ii)-(iv) guarantee that inequality (2.73) holds. Without expanding the computations, similarly to the proof of Theorem 2.4.7, by applying Lemma 2.2.5 with

$$d_i = \begin{pmatrix} \gamma_h\sqrt{\nu}v_i \\ \\ \sqrt{\max(4\mu, c_w)}w_i \end{pmatrix},$$

$\alpha = 4\mu\gamma_x^2\eta^N$, $\beta = 1$, $\ell = N+1$, we obtain (2.80) since condition (iv) allows applying Lemma 2.2.5. Again, as in the proof of Theorem 3.2.1, the bounds (2.71) for $t \le N$ need to be treated similarly.

□

Likewise in what follows from Theorem 3.2.1, a sufficiently large window size N, i.e.,

$$N \ge 1 + \left\lfloor -\frac{\ln(4\mu\gamma_x^2)}{\ln\eta}\right\rfloor,$$

ensures the satisfaction of condition (iv), independently of all the system and other previously selected design parameters.

Remark 2.4.8. *With the prediction equation (2.76), the new necessary condition for the MHE_N to be RES for any $N \ge 1$ is $\ln(4\mu\gamma_x^2) \le 0$, which means that $4\mu\gamma_x^2 \le 1$. Taking into account the condition (ii) of Theorem 3.2.1, the necessary condition for the MHE_N to be RES for any $N \ge 1$ is*

$$8c_x\gamma_x^2 \le 1. \tag{2.81}$$

Then, the MHE_N is RES for $N \ge \ell - 1, \ell \in \mathbb{N}, \ell \ge 2$, if the following necessary condition holds:

$$\frac{1}{4c_x\eta^\ell} \ge 2\gamma_x^2. \tag{2.82}$$

Compared to (2.75), the robust stability of the MHE_N can be guaranteed for some systems having $2\gamma_x^2 < 1$, subject to (2.82), while with the prediction (2.72) it is not necessarily ensured. On the other hand, for systems having $2\gamma_x^2 > 1$, subject to (2.82), the prediction equation (2.72) is better than the modified standard prediction-based equation. Hence, both prediction techniques can be seen as alternative methods. It is for the user to fix which one is more appropriate for the model at hand while ensuring the RES property of the MHE_N. To overcome the limitation related to the necessary conditions, we propose an improvement in the modified standard prediction equation by introducing additional decision variables, which play the role of tuning parameters. This is the goal of the next section.

2.4.4 Output-driven one-step-ahead prediction

In this section, we present a novel prediction strategy that introduces an additional degree of freedom, which aims at a less conservative satisfaction of stability conditions. The key innovation lies in the incorporation of a correction term, linked to the output error. Specifically, the correction term depends on the output error, resembling the structure of a Luenberger observer. To achieve this novel approach, we rely on the following assumption.

Assumption 2.4.9. *$\mathcal{X}$, $\mathcal{W}$, and $\mathcal{V}$ are compact sets and the functions $f(\cdot,\cdot), h(\cdot,\cdot)$ are continuously differentiable with respect to their arguments.*

Therefore, $\mathcal{Y} := h(f(\mathcal{X},\mathcal{W}),\mathcal{V}) \subset \mathbb{R}^m$ is a compact set since $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are assumed to be continuous. We recall the prediction equation , namely,

$$\begin{aligned}\bar{x}_{t-N} = f(\hat{x}_{t-N-1|t-N-1},0) + &\psi\big(\hat{x}_{t-N-1|t-N-1},y_{t-N-1}\big)\\ &\Big(y_{t-N-1} - h(\hat{x}_{t-N-1|t-N-1},0)\Big)\end{aligned} \tag{2.83}$$

where $\psi(\cdot,\cdot)$ to be determined in such a way as to ensure the RES of the MHE_N. The aim is to select $\psi(\cdot,\cdot)$ to overcome the necessary condition (2.81). Instead of γ_x in (2.81), we rely on a parameter depending on $\psi(\cdot,\cdot)$ for which this condition can be satisfied with a convenient choice of $\psi(\cdot,\cdot)$. Under the prediction equation (2.83), the term $\bar{e}_{t-N}$ in (2.69) satisfies the following equation:

$$\begin{aligned}\bar{e}_{t-N} &= f(x_{t-N-1},w_{t-N-1}) - f(\hat{x}_{t-N-1|t-N-1},0)\\ &-\psi\big(\hat{x}_{t-N-1|t-N-1},y_{t-N-1}\big)\Big(h(x_{t-N-1},v_{t-N-1}) - h(\hat{x}_{t-N-1|t-N-1},0)\Big)\\ &= f(x_{t-N-1},w_{t-N-1}) - f(\hat{x}_{t-N-1|t-N-1},w_{t-N-1})\\ &+ f(\hat{x}_{t-N-1|t-N-1},w_{t-N-1}) - f(\hat{x}_{t-N-1|t-N-1},0)\\ &-\psi\big(\hat{x}_{t-N-1|t-N-1},y_{t-N-1}\big)\Big(h(x_{t-N-1},v_{t-N-1})\\ &- h(\hat{x}_{t-N-1|t-N-1},v_{t-N-1})\Big)\\ &-\psi\big(\hat{x}_{t-N-1|t-N-1},y_{t-N-1}\big)\Big(h(\hat{x}_{t-N-1|t-N-1},v_{t-N-1})\\ &- h(\hat{x}_{t-N-1|t-N-1},0)\Big).\end{aligned} \tag{2.84}$$

From Lemma 2.2.6, there exist vectors $\boldsymbol{z}_x, \bar{\boldsymbol{z}}_x, \boldsymbol{z}_w$, and $\boldsymbol{z}_v$ with $\boldsymbol{z}_x^i \in \mathbf{Co}\left(x_{t-N-1},\hat{x}_{t-N-1|t-N-1}\right)$, $\bar{\boldsymbol{z}}_x^i \in \mathbf{Co}\left(x_{t-N-1},\hat{x}_{t-N-1|t-N-1}\right)$, $\boldsymbol{z}_w^i \in \mathbf{Co}\left(w_{t-N-1},0\right)$, $\boldsymbol{z}_v^i \in \mathbf{Co}\left(v_{t-N-1},0\right)$, which transform (2.84) into the following form:

$$\begin{aligned}\bar{e}_{t-N} = &\left(\frac{\partial f}{\partial x}(\boldsymbol{z}_x,w_{t-N-1}) - \psi\big(\hat{x}_{t-N-1|t-N-1},y_{t-N-1}\big)\frac{\partial h}{\partial x}(\bar{\boldsymbol{z}}_x,v_{t-N-1})\right)e_{t-N-1}\\ &+\frac{\partial f}{\partial w}(\hat{x}_{t-N-1|t-N-1},\boldsymbol{z}_w)\,w_{t-N-1}\\ &-\psi\big(\hat{x}_{t-N-1|t-N-1},y_{t-N-1}\big)\frac{\partial h}{\partial v}(\hat{x}_{t-N-1|t-N-1},\boldsymbol{z}_v)v_{t-N-1}\end{aligned} \tag{2.85}$$

where

$$\boldsymbol{z}_x := \begin{bmatrix}\boldsymbol{z}_x^1\\ \boldsymbol{z}_x^2\\ \vdots\\ \boldsymbol{z}_x^n\end{bmatrix},\ \boldsymbol{z}_x^i \in \mathcal{X},\ \bar{\boldsymbol{z}}_x := \begin{bmatrix}\bar{\boldsymbol{z}}_x^1\\ \bar{\boldsymbol{z}}_x^2\\ \vdots\\ \bar{\boldsymbol{z}}_x^p\end{bmatrix},\ \bar{\boldsymbol{z}}_x^i \in \mathcal{X}, \boldsymbol{z}_w := \begin{bmatrix}\boldsymbol{z}_w^1\\ \boldsymbol{z}_w^2\\ \vdots\\ \boldsymbol{z}_w^n\end{bmatrix}, \boldsymbol{z}_w^i \in \mathcal{W}, \boldsymbol{z}_v := \begin{bmatrix}\boldsymbol{z}_v^1\\ \boldsymbol{z}_v^2\\ \vdots\\ \boldsymbol{z}_v^p\end{bmatrix}, \boldsymbol{z}_v^i \in \mathcal{V}.$$

From now on, we need some conditions on the nonlinearities of the system. More specifically, we introduce the following Lipschitz-like assumptions.

Assumption 2.4.10. *The following conditions hold:*

$$\gamma_w := \sup_{\substack{x\in\mathcal{X}\\ z\in\mathcal{W}}} \left| \frac{\partial f}{\partial w}(x,z) \right| < +\infty$$

$$\gamma_v := \sup_{\substack{x\in\mathcal{X}\\ y\in\mathcal{Y}\\ z\in\mathcal{V}}} \left| \psi(x,y)\frac{\partial h}{\partial v}(x,z) \right| < +\infty$$

$$\gamma_\psi := \sup_{\substack{z,\bar{z},\tilde{z}\in\mathcal{X},\\ w\in\mathcal{W},v\in\mathcal{V}}} \left| \frac{\partial f}{\partial x}(z,w) - \psi(\bar{z},y)\frac{\partial h}{\partial x}(\tilde{z},v) \right| < +\infty\,.$$

Under the Assumption 2.4.10, from (2.85) we obtain

$$|\bar{e}_{t-N}|^2 \le 3\gamma_\psi^2 |e_{t-N-1}|^2 + 3\gamma_w^2 |w_{t-N-1}|^2 + 3\gamma_v^2 |v_{t-N-1}|^2 \tag{2.86}$$

and, by using (2.86), from (2.69) it follows that

$$\begin{aligned} |e_t|^2 \le\ & 6\mu\gamma_\psi^2\, |e_{t-N-1}|^2\, \eta^N \\ & + \max\left(6\mu\gamma_v^2, \nu\gamma_h^2\right) \sum_{i=t-N-1}^{t-1} \eta^{t-1-i}\, |v_i|^2 \\ & + \max\left(6\mu\gamma_w^2, c_w\right) \sum_{i=t-N-1}^{t-1} \eta^{t-1-i}\, |w_i|^2\ . \end{aligned} \tag{2.87}$$

Now we are ready to state the general Theorem 2.4.11, which provides conditions for the MHE_N, under the prediction equation (2.83), to be RES. Such conditions depend on the choice of $\psi(\cdot,\cdot)$ (through the parameter γ_ψ), which can be selected appropriately together with the other conditions.

Theorem 2.4.11. *Assume that system (1.17) is i-EIOSS according to (2.8) with exponential discount ϱ. Then, the MHE_N with prediction equation (2.83) is RES according to the following inequality:*

$$\begin{aligned} \left|x_t - \hat{x}_{t|t}\right|^2 \le\ & \max(2\mu,1)\, |x_0 - \bar{x}_0|^2 \lambda^t \\ & + \frac{\max\left(6\mu\gamma_v^2, \nu\gamma_h^2\right)}{(1-\lambda)^2} \sum_{i=0}^{t-1} \lambda^{t-1-i}\, |v_i|^2 \\ & + \frac{\max\left(6\mu\gamma_w^2, c_w\right)}{(1-\lambda)^2} \sum_{i=0}^{t-1} \lambda^{t-1-i}\, |w_i|^2 \end{aligned}$$

with the exponential discount parameter

$$\boxed{\lambda := \max\left(\eta, \left(6\mu\gamma_\psi^2\eta^N\right)^{\frac{1}{\kappa(N+1)}}\right)}$$

and integer $\kappa \ge 2$ if μ, ν, η, and $N \ge 1$ satisfy the following conditions:

(i) $\varrho \le \eta < 1$;

(ii) $\mu \geq 2c_x$;

(iii) $\nu \geq c_v$;

(iv) $6\mu\,\gamma_\psi^2 \eta^N < 1$.

Proof. Based on Theorems 3.2.1 and 2.4.7, the proof of Theorem 2.4.11 becomes straightforward. In fact, we only have to apply Lemma 2.2.5 on the inequality (2.87) with

$$d_i = \begin{pmatrix} \sqrt{\max\left(6\mu\gamma_v^2, \gamma_h^2\right)} v_i \\ \sqrt{\max\left(6\mu\gamma_v^2, c_w\right)} w_i \end{pmatrix},$$

$\alpha = 6\mu\,\gamma_\psi^2 \eta^N$, $\beta = 1$, and $\ell = N+1$.

□

Remark 2.4.12. *Theorem 2.4.11 provides more general design conditions as compared to Theorem 3.2.1 and Theorem 2.4.7. The necessary condition for the* MHE_N *to be RES is now*

$$12 c_x \gamma_\psi^2 \leq 1.$$

Then, the MHE_N *is RES for any* $N \geq \ell - 1, \ell \in \mathbb{N}, \ell \geq 2$*, if the following necessary condition holds:*

$$\frac{1}{4c_x\eta^\ell} \geq 3\gamma_\psi^2. \tag{2.88}$$

The difference between (2.88) and the previous necessary conditions (2.75) and (2.82) is that in (2.88) we may exploit γ_ψ *by selecting* $\psi(\cdot,\cdot)$ *as to make inequality (2.88) feasible.*
Notice that the condition (2.88) is slightly more stronger than the usual detectability condition ($\gamma_\psi < 1$*). If the system (1.17) is observable, then it is possible to find* $\psi(\cdot,\cdot)$ *for which (2.88) is satisfied, contrarily to (2.75) and (2.82), where even observability cannot ensure the exponential robustness of the* MHE_N*, unless an analytical link between observability and the i-EIOSS property, namely the parameters* c_x *and* ϱ*, is developed. This issue is another research direction on the existence of robust estimators under the i-EIOSS property, which is not covered here and constitutes one of the future research problems we aim to solve.*

2.4.5 Synthesis of the Prediction Gain

This section is devoted to outlining a numerical procedure for designing the prediction gain. For this, we examine a specific prediction feedback term, precisely, a linear correction term featuring a constant matrix. This approach is borrowed from the methodology employed in the design of nonlinear observers, as elaborated in [131]. Then, first, we consider the specific output-driven one-step-ahead prediction equation:

$$\bar{x}_{j+1} = f(\hat{x}_{j|j}, 0) + \mathbb{K}\Big(y_j - h(\hat{x}_{j|j}, 0)\Big) \tag{2.89}$$

where $j := t-N-1$ is introduced from now on to avoid cumbersome equations and $\mathbb{K} \in \mathbb{R}^{n\times p}$ is the constant prediction gain to be determined later. From (2.85) and according to the notation (2.60), we have

$$\bar{e}_{j+1} = \Big(\nabla_x^f(\boldsymbol{z}_x, w_j) - \mathbb{K}\nabla_x^h(\bar{\boldsymbol{z}}_x, v_j)\Big)\, e_j + \Big[\nabla_w^f(\hat{x}_{j|j}, \boldsymbol{z}_w) \quad -\mathbb{K}\nabla_v^h(\hat{x}_{j|j}, \boldsymbol{z}_v)\Big]\, \boldsymbol{\omega}_j$$

where $\boldsymbol{\omega}_j \triangleq \begin{bmatrix} w_j^\top & v_j^\top \end{bmatrix}^\top \in \mathbb{R}^q$.
Before starting with the numerical design procedures, it is worthy noticing that from Assumptions 2.4.3-2.4.10, the Jacobians $\nabla_x^f(\boldsymbol{z}_x, w_j)$, $\nabla_w^f(\hat{x}_{j|j}, \boldsymbol{z}_w)$, $\nabla_x^h(\bar{\boldsymbol{z}}_x, v_j)$, and $\nabla_v^h(\hat{x}_{j|j}, \boldsymbol{z}_v)$ belong to convex polytopic sets defined respectively by:

$$\mathcal{V}_x^f \triangleq \left\{ \sum_{j=1}^{n_x} \beta_j \digamma_j^x, \text{ such that } \beta_j \geq 0, \sum_{j=1}^{n_x} \beta_j = 1 \right\}$$

$$\mathcal{V}_{\boldsymbol{\omega}}^f \triangleq \left\{ \sum_{j=1}^{n_{\boldsymbol{\omega}}} \beta_j \digamma_j^{\boldsymbol{\omega}}, \text{ such that } \beta_j \geq 0, \sum_{j=1}^{n_{\boldsymbol{\omega}}} \beta_j = 1 \right\}$$

$$\mathcal{V}_x^h \triangleq \left\{ \sum_{j=1}^{p_x} \beta_j \daleth_j^x, \text{ such that } \beta_j \geq 0, \sum_{j=1}^{p_x} \beta_j = 1 \right\}$$

$$\mathcal{V}_{\boldsymbol{v}}^h \triangleq \left\{ \sum_{j=1}^{p_{\boldsymbol{v}}} \beta_j \daleth_j^{\boldsymbol{v}}, \text{ such that } \beta_j \geq 0, \sum_{j=1}^{p_{\boldsymbol{v}}} \beta_j = 1 \right\}.$$

To design the prediction gain $\mathbb{K}$, we propose in this section two different LMI-based methods. The first method is borrowed from observer design techniques and the second one utilises conditions to tune the spectral norm of the matrix $\nabla_x^f(\boldsymbol{z}_x, w_j) - \mathbb{K}\nabla_x^h(\bar{\boldsymbol{z}}_x, v_j)$.

2.4.5.0.a $\mathcal{L}_2$ observer-based design method

The idea here is to adopt the observer-based technique to design a prediction gain, $\mathbb{K}$, which we can consider also as the observer gain of the state observer

$$\phi_{j+1} = f(\phi_j, 0) + \mathbb{K}\Big(y_j - h(\phi_j, 0)\Big)$$

which leads to an error dynamic of the form:

$$\tilde{\phi}_{j+1} = \Big(\nabla_x^f(\boldsymbol{z}_\phi, w_j) - \mathbb{K}\nabla_x^h(\bar{\boldsymbol{z}}_\phi, v_j)\Big)\tilde{\phi}_j + \Big[\nabla_w^f(\phi_j, \bar{\boldsymbol{z}}_w) \quad -\mathbb{K}\nabla_v^h(\phi_j, \bar{\boldsymbol{z}}_v)\Big]\boldsymbol{\omega}_j. \tag{2.90}$$

where $\tilde{\phi}_j \triangleq x_j - \phi_j$. The prediction $\bar{x}_j$ at time j is obtained by means of (2.89) with $\hat{x}_{j|j}$, where the gain $\mathbb{K}$ is selected by using (2.90). More specifically, we will analyze the error associated with $\bar{x}_j$ and exploit well-known observer design techniques for nonlinear systems [131, 135]. The LMI conditions ensuring an ISS bound on the estimation error $\tilde{\phi}_j$ are summarized in the following proposition.

Proposition 2.4.13. *Assume there exists a symmetric and positive definite matrix $\mathfrak{P}$, a matrix $\mathcal{X}$, of appropriate dimensions, and positive scalars α and λ, with $\alpha < 1$, such that the following LMI condition holds:*

$$\begin{bmatrix} -\alpha\mathfrak{P} + \mathbb{I}_n & 0 & (\digamma_i^x)^\top \mathfrak{P} - \left(\daleth_j^x\right)^\top \mathcal{X} \\ (\star) & -\lambda\mathbb{I}_q & \begin{bmatrix}\mathfrak{P}\digamma_k^{\boldsymbol{\omega}} & \mathcal{X}^\top \daleth_l^{\boldsymbol{\omega}}\end{bmatrix}^\top \\ (\star) & (\star) & -\mathfrak{P} \end{bmatrix} < 0 \tag{2.91}$$

for all $i \in \{1, \ldots, n_x\}$, $j \in \{1, \ldots, p_x\}$, $k \in \{1, \ldots, n_{\boldsymbol{\omega}}\}$, and $l \in \{1, \ldots, p_v\}$. Then with the observer gain $\mathbb{K} = \mathfrak{P}^{-1}\mathcal{X}^\top$, the estimation error $\tilde{\phi}_t$ satisfies the following $\mathcal{L}_2-$optimality criterion:

$$\|\tilde{\phi}\|_{\mathcal{L}_2} \leq \sqrt{\lambda\|\boldsymbol{\omega}\|_{\mathcal{L}_2} + \lambda_{\max}(\mathfrak{P})\,|\tilde{\phi}_0|^2}.$$

Proof. The proof is straightforward and relies on well-established arguments in LMI-based nonlinear observer design. Our approach involves utilizing the Lyapunov function defined as $V(\tilde{\phi}) \triangleq \tilde{\phi}^\top \mathfrak{P} \tilde{\phi}$ and demonstrating that, under the conditions (2.91), the inequality

$$\vartheta_j \triangleq \Delta V(\tilde{\phi}_j) + |\tilde{\phi}_j|^2 - \lambda |\boldsymbol{\omega}_j|^2 \leq 0.$$

holds, where $\Delta V(\tilde{\phi}_j) \triangleq V(\tilde{\phi}_{j+1}) - \alpha V(\tilde{\phi}_j)$. For a comprehensive understanding of this methodology, we recommend [131, 135] with details on the specific techniques of LMI-based nonlinear observer design, clarifying the steps involved in establishing the aforementioned inequality.

□

Proposition 2.4.13 provides a numerical procedure to design the prediction gain $\mathbb{K}$. Indeed, since (2.91) guarantees the $\mathcal{L}_2$ stability of $\tilde{\phi}_j$. On other hand, although the LMI conditions (2.91) imply that the eigenvalues of $\left(\nabla_x^f(\boldsymbol{z}_x, w_j) - \mathbb{K}\nabla_x^h(\bar{\boldsymbol{z}}_x, v_j)\right)$ are smaller than α; however, they cannot guarantee the norm

$$\gamma_{\mathbb{K}} \triangleq \sup_{\substack{\boldsymbol{z}_x, \bar{\boldsymbol{z}}_x \in \mathcal{X}, \\ w_j \in \mathcal{W}, v_j \in \mathcal{V}}} \left| \nabla_x^f(\boldsymbol{z}_x, w_j) - \mathbb{K}\nabla_x^h(\bar{\boldsymbol{z}}_x, v_j) \right|$$

to be upper bounded by α. This emphasizes the need for further exploration and refinement in ensuring a bound on $\gamma_{\mathbb{K}}$. These conditions do not provide the flexibility to select $\mathbb{K}$ in a way that minimizes $\gamma_{\mathbb{K}}$, thereby facilitating a reduction in the window size, N, of the MHE, while ensuring robust stability. Addressing this limitation is the primary motivation behind the second numerical design procedure outlined in the subsequent section.

2.4.5.1 Second method: Spectral norm-based design method

In this section, we provide an alternative LMI-based method allowing a bound on $\gamma_{\mathbb{K}}$. First, we need the following lemma.

Lemma 2.4.14. *Let $A \in \mathbb{R}^{n \times n}$ be an arbitrary matrix and $P \in \mathbb{R}^{n \times n}$ a symmetric and positive definite matrix. Let α be a positive scalar and $\kappa \in \mathbb{N}$ an integer with $\kappa \geq 2$. Assume that the following inequalities hold:*

$$A^\top P A - \frac{\alpha}{\kappa} P < 0 \tag{2.92}$$

$$P > \mathbb{I}_n \tag{2.93}$$

$$P < \kappa \mathbb{I}_n \tag{2.94}$$

Then, we have

$$|A| \triangleq \sqrt{\lambda_{\max}(A^\top A)} < \sqrt{\alpha} \tag{2.95}$$

where $\lambda_{\max}(A^\top A)$ is the maximum eigenvalue of the matrix $A^\top A$ and $|A|$ is called the spectral norm of A.

Proof. From (2.93), we get $A^\top P A > A^\top A$. On the other hand, from (2.94), we obtain

$$-\frac{\alpha}{\kappa} P > -\alpha \mathbb{I}_n.$$

It follows from (2.92) that

$$A^\top A - \alpha \mathbb{I}_n < A^\top P A - \frac{\alpha}{\kappa} P < 0$$

which implies that $|A| < \sqrt{\alpha}$. The last inequality uses the fact that $A^\top A < \alpha \mathbb{I}_n$ implies $\lambda_{A^\top A} < \alpha$.

□

By applying Lemma 2.4.14, we obtain the following theorem, which provides a bound on $\gamma_{\mathbb{K}}$.

Theorem 2.4.15. *Assume that there exists a symmetric and positive definite matrix $\mathfrak{P}$, a matrix $\mathcal{X}$, of appropriate dimensions, a positive scalar λ and an integer $\kappa \geq 2$ such that the following LMI condition holds:*

$$\begin{bmatrix} -\frac{\lambda}{\kappa}\mathfrak{P} & \left(\digamma_i^x\right)^\top \mathfrak{P} - \left(\daleth_j^x\right)^\top \mathcal{X} \\ (\star) & -\mathfrak{P} \end{bmatrix} < 0 \tag{2.96}$$

$$\mathbb{I}_n < \mathfrak{P} < \kappa \mathbb{I}_n \tag{2.97}$$

for all $i \in \{1, \ldots, n_x\}$; $j \in \{1, \ldots, p_x\}$. Then, with the prediction gain $\mathbb{K} = \mathfrak{P}^{-1}\mathcal{X}^\top$, we have

$$\gamma_{\mathbb{K}} \leq \sqrt{\lambda}. \tag{2.98}$$

Proof. The proof straightforwardly follows from Lemma 2.4.14, bearing in mind that the inequality (3.61) is maintained, with a non-strict condition, while adhering to the supremum bound, which leads to (2.98).

□

With the prediction equation (2.89), the condition (iv) in Theorem 2.4.11 becomes

$$6\mu\, \gamma_{\mathbb{K}}^2 \eta^N < 1 \tag{2.99}$$

which means that

$$N \geq \mathcal{N}(\gamma_{\mathbb{K}}^2) \triangleq 1 + \left\lfloor -\frac{\ln(6\mu\gamma_{\mathbb{K}}^2)}{\ln \eta} \right\rfloor.$$

It follows that if $\mathbb{K}$ is designed from Theorem 2.4.15, we have $\gamma_{\mathbb{K}}^2 \leq \lambda$. By definition, $\mathcal{N}(.)$ is a non-decreasing function, then we have $\mathcal{N}(\lambda) \geq \mathcal{N}(\gamma_{\mathbb{K}}^2)$, and $\mathcal{N}(1/(6\mu)) = 1$. Consequently, if the prediction gain $\mathbb{K}$ is designed from Theorem 2.4.15 with $\lambda = 1/(6\mu)$, then the MHE_N with the prediction equation (2.89) is RES for any $N \geq 1$.
On other hand, the LMI conditions (2.96)-(2.97) are only sufficient conditions and they can be infeasible for $\lambda = 1/(6\mu)$. To get the smallest possible value of λ satisfying (2.96)-(2.97), we have to solve the minimization problem

$$\min_{\mathfrak{P}, \mathcal{X}, \kappa \geq 2} (\lambda) \text{ subject to } (2.96) - (2.97). \tag{2.100}$$

Let $\lambda^\star$ be the smallest value returned by (2.100). Then, the MHE_N with (2.89) is RES for any $N \geq \mathcal{N}(\lambda^\star)$.

Remark 2.4.16. *While the first prediction design method may improve the prediction quality of $\bar{x}_{t-N}$ and its performance without controlling the bound $\gamma_{\mathbb{K}}$; however, the second prediction method offers the possibility to systematically tune this bound. This may allow the MHE_N to be RES for small values of the window size, N, which is important for real-time applications.*

2.5 Illustrative Examples

2.5.1 Example 1: chaotic system

Consider the second-order nonlinear discrete-time system with

$$(x,w) \mapsto f(x,w) := (1 - ax_1^2 + x_2 + w_1\,,\, bx_1 + w_2)$$
$$(x,v) \mapsto h(x,v) := x_1 + v$$

where $x \in \mathbb{R}^2$, $w \in \mathbb{R}^2$, $v \in \mathbb{R}$, $a = 1.4$, and $b = 0.3$. The system exhibits a chaotic behavior [55] and, through the generation of sufficiently small system noises, its state trajectories belong to the compact set $[-1.3, 1.3] \times [-0.39, 0.39]$, on which the nonlinearity is globally Lipschitz.
The system exhibits a chaotic behavior and its state belongs to the bounded compact set $[-1,1] \times [-1,1]$, which is a bounded invariant compact set on which the nonlinearity is globally Lipschitz. Both system and measurement noises are generated according to zero-mean Gaussian distributions with covariances equal to 0.01.
We need only to determine the matrices $\mathcal{F}_j^x$ since we have linear outputs and the system depends linearly on the disturbance $\boldsymbol{\omega}_t$. To compute $\mathcal{F}_j^x$, we have to decompose the Jacobian matrix into a convex form. We have

$$\frac{\partial f}{\partial x}(z_f, \boldsymbol{\omega}) = \begin{bmatrix} -2az_f(t) & 1 \\ b & 0 \end{bmatrix}$$

where $z_f(t)$ comes from the differential mean value theorem in Lemma 2.2.6. Since $z_f(t) \in [-1,\ 1]$, then we have $-2a \leq -2az_f(t) \leq 2a$. By using the convex decomposition technique, there exists $0 < \alpha(t) \leq 1$ such that

$$-2az_f(t) = -2a\alpha(t) + 2a(1 - \alpha(t))$$

which means that $\alpha(t) = \frac{z_f(t)+1}{2} < 1$ since $z_f(t) \in [-1,\ 1]$. Hence, we can write the Jacobian matrix under the form

$$\frac{\partial f}{\partial x}(z_f, \boldsymbol{\omega}) = \alpha(t) \overbrace{\begin{bmatrix} -2a & 1 \\ b & 0 \end{bmatrix}}^{\mathcal{F}_1^x} + (1 - \alpha(t)) \overbrace{\begin{bmatrix} 2a & 1 \\ b & 0 \end{bmatrix}}^{\mathcal{F}_2^x}.$$

It follows that $\mathcal{V}_{f,x}$ in (4.10) is given by:

$$\begin{aligned} \mathcal{V}_{f,x} &\triangleq \left\{ \sum_{j=1}^{2} \alpha_j \mathcal{F}_j^x,\ \text{such that } \alpha_j \geq 0, \sum_{j=1}^{2} \alpha_j = 1 \right\} \\ &= \left\{ \frac{(z+1)}{2} \begin{bmatrix} -2a & 1 \\ b & 0 \end{bmatrix} + \frac{(1-z)}{2} \begin{bmatrix} 2a & 1 \\ b & 0 \end{bmatrix}, \right. \\ &\qquad \left. \text{such that } z \in [-1,\ 1] \right\}. \end{aligned} \tag{2.101}$$

In this case, we have $n_x = 2$ instead of $n_x = 2^{n^2} = 16$, since we have only one nonlinear component; the other components in the system are linear. Then we have two LMI conditions to solve in (2.55) to obtain the i-EIOSS related coefficients.
Utilizing *MATLAB Yalmip* toolbox, $\varrho = 0.1$, $c_x = 30.3171$, $c_\nu = 193.4562$, $c_\omega = 1.3585\text{e+}05$. Thus, we set $\eta = \varrho$, $\mu = 2c_x$, $\nu = c_v$ to complywith the conditions of Theorems 3.2.1, 2.4.7, and 2.4.11 by obtaining minimum values of N equal to 3, 4, and 4 for the *freeze-all*, standard *one-step-ahead*, and *output-driven* one-step-ahead MHE settings, respectively. Specifically, for the *output-driven* one-step-ahead predictor we obtained

$$\mathfrak{P} = \begin{bmatrix} 1.3607 & 0 \\ 0 & 3.0027 \end{bmatrix} \quad \mathcal{X} = \begin{bmatrix} 0 & 0.9008 \end{bmatrix} \quad \kappa = 4$$

as solution of (2.100) and thus $\mathbb{K} = \begin{bmatrix} 0 & 0.3000 \end{bmatrix}^\top$.
We solved the minimization problems by using the general-purpose Matlab routine *fmincon*.
Fig. 3.1 illustrates the simulation results in a noise-free run with initial values of the actual and estimated states $\begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^\top$ and $\begin{bmatrix} -0.5 & -0.5 \end{bmatrix}^\top$, respectively. As compared to Fig. 2.1,
system and measurement noises generated according to zero-mean Gaussian distributions with dispersions equal to 0.02 and 0.1 were introduced in the simulation run of Fig. 2.2. Figs. 2.3 and 2.4 show the boxplots of the RMSEs and computational times over 1000 simulation noisy runs with null initial conditions, respectively. In this case study, the three approaches provide about the same performances in terms of precision (see Fig. 2.3), but, as to the computational load, the standard and output-driven one-step-ahead predictions turn out to be more efficient, as shown in Fig. 2.4.

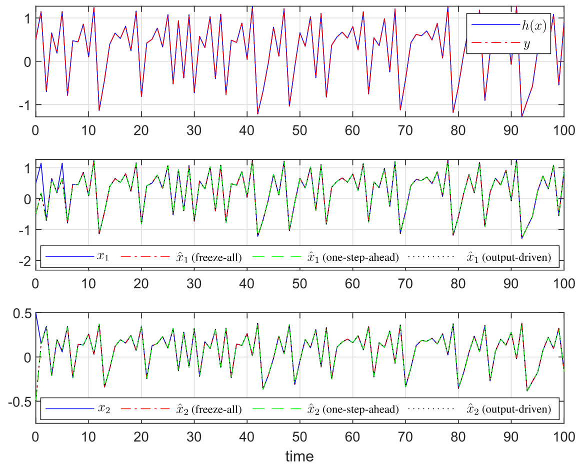

Figure 2.1: Simulation results in a noise-free run with $N = 4$.

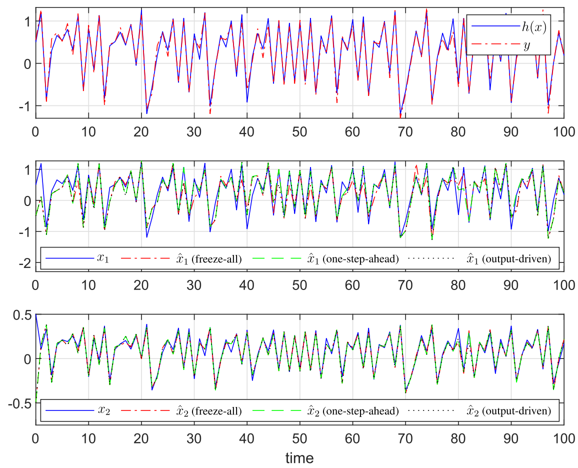

Figure 2.2: Simulation results in a noisy run with $N = 4$.

2.5.2 Example 2: Application to a Tumor Growth Model

2.5.2.1 Tumor Growth Mathematical Model

Mathematical modeling provides a low-cost, low-risk way of exploring the dynamics of biomedical processes.
In cancer research, mathematical models play a crucial role in analyzing various processes like incidence, pathogenesis, tumor growth, metastasis, immune reaction, and treatment. Although

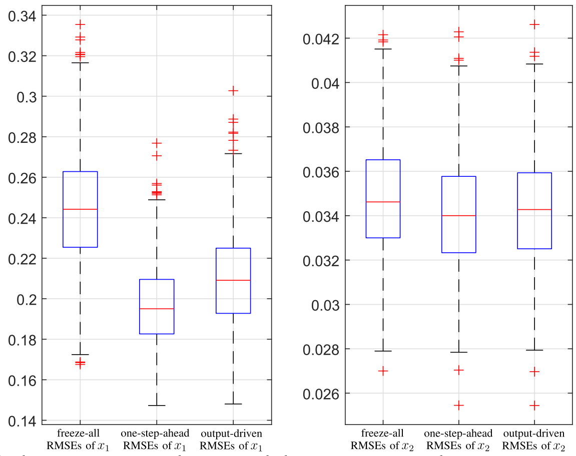

Figure 2.3: Boxplots of the RMSEs obtained by MHE with $N = 4$ (medians equal to 0.2442 and 0.0346; 0.1950 and 0.0340; 0.2091 and $.0343$).

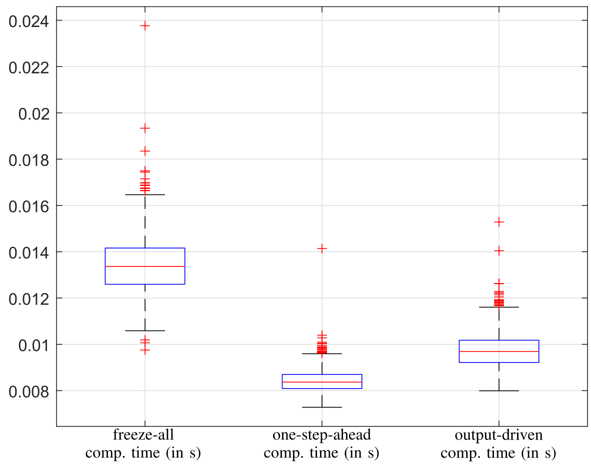

Figure 2.4: Boxplots of the computational times obained by MHE with $N = 4$ (medians equal to $0.0133, 0.0084, 0.0097$).

conventional cancer treatments such as surgery, radiotherapy, chemotherapy, immunotherapy, and stem cell transplants are employed, they are associated with harmful effects such as drug resistance and damage to healthy cells. This has led to a growing interest in the use of angiogenesis inhibitors, which have proven effective in curbing or slowing down the growth and spread of tumors. The process of inhibiting tumor angiogenesis is illustrated in Fig. 2.5.

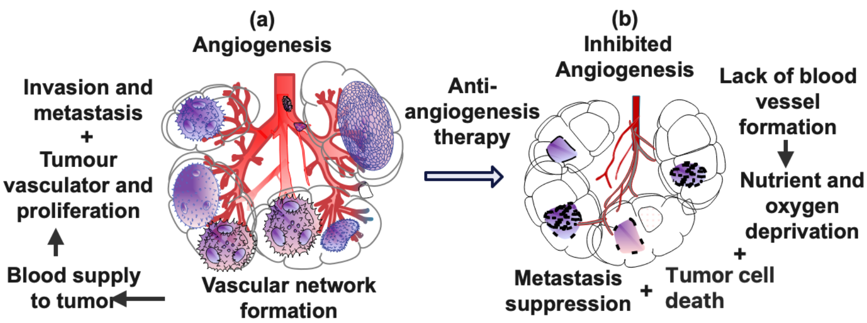

Figure 2.5: Tumor Angiogenesis Inhibition.

The concept of anti-angiogenic therapies was introduced in the early 1970s in [39] and has since been the subject of numerous theoretical and experimental studies [22,33,35,95]. By carrying out a number of experiments on mice with Lewis lung carcinoma, the authors in [39] and [49] developed and biologically validated a system of ODEs for the interactions between the primary tumor volume and the carrying capacity of the vasculature. The model they developed is a nonlinear one that takes into account angiogenic stimulation and inhibition and is referred to as the Hahnfeldt model. The ODEs system for this model is as follows:

$$\begin{cases} \dot{x}_1 &= -\lambda x_1 \ln\left(\dfrac{x_1}{x_2}\right), \\ \dot{x}_2 &= b x_1 - (\mu + d x_1^{\frac{2}{3}}) x_2 - c x_2 x_3, \end{cases} \tag{2.102}$$

where the model parameters are

- x_1 $[mm^3]$: the tumor volume ;
- x_2 $[mm^3]$: the carrying capacity of the vasculature.

The first equation describes the phenomenology of a tumor growth slowdown, as the tumor grows and resorts to its available support;

- λ $[day^{-1}]$: the tumor growth rate constant.

In the second equation the first term represents the stimulatory capacity of the tumor upon the inducible vasculature (bx_1), the second term accounts for a spontaneous loss (μx_2) and for tumor-dependent endogenous inhibition $(d x_1^{\frac{2}{3}} x_2)$ of previously generated vasculature; the third term refers to the vasculature inhibitory action performed by an exogenous drug administration $(c x_2 x_3)$, with

- u $[day^{-1}(mg/kg)]$: the actual control law ;
- b $[day^{-1}]$: the vascular birth rate ;
- d $[day^{-1} mm^{-2}]$: the endothelial cell death (death rate) ;
- c $[day^{-1}(mg/kg)^{-1}]$: the sensitivity to the drug;
- μ $[day^{-1}]$: the spontaneous vascular inactivation rate . According to the model literature [49], without loss of generality, parameter μ will be set equal to zero in the following.

Being the anti-angiogenic drug not directly administrated in the vein, a further compartment is considered to account for drug diffusion:

$$x_3(t) = \int_0^t e^{-\eta(t-t')} u(t') dt', \tag{2.103}$$

with

- $u(t')[day^{-1}(mg/kg)]$, is the rate of administration of inhibitor concentration at time t';
- $\eta[day^{-1}]$, being the diffusion rate into serum.

As a matter of fact, the whole system (2.102)-(2.103) (with $\mu = 0$) may be written in a compact ODE form:

$$\begin{cases} \dot{x}_1 &= -\lambda x_1 \ln\left(\dfrac{x_1}{x_2}\right), \\ \dot{x}_2 &= b x_1 - (\mu + d x_1^{\frac{2}{3}}) x_2 - c x_2 x_3, \\ \dot{x}_3 &= -\eta x_3 + u. \end{cases} \tag{2.104}$$

2.5.2.2 Simulation results

To perform the optimization, we utilized the Matlab routine called *fmincon*. The values of the cost function parameters are set to be equal to $\mu = 0.06, \nu = 1, \omega = 1$, and $\eta = 0.6$. The initial state and estimated state were set to be equal to $\begin{bmatrix} 200 & 620 & 0 \end{bmatrix}$ and $\begin{bmatrix} 20 & 20 & 20 \end{bmatrix}$, respectively. Fig. 2.6 illustrates the results obtained in simulation runs with system and measurement noises generated according to zero-mean Gaussian distributions with covariances equal to 0.01.

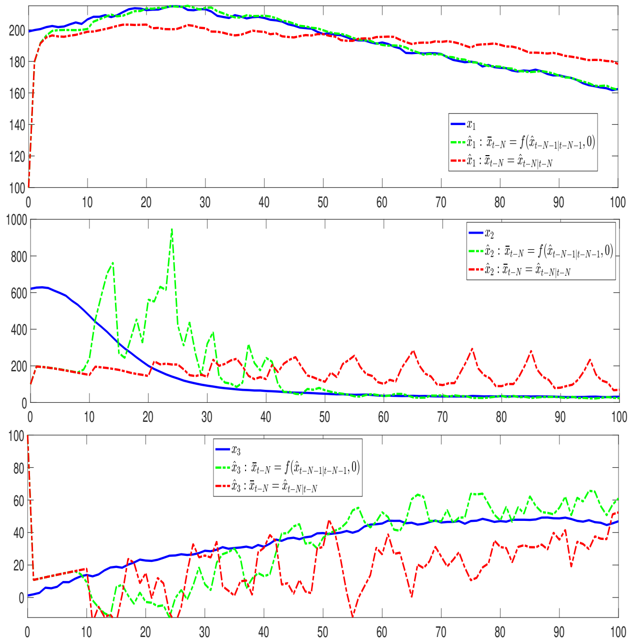

Figure 2.6: The system states and their estimates.

As said earlier, both freeze-all and enhanced one-step prediction-based techniques can be seen as alternative methods. Users must choose between them to determine which one suits their model best while ensuring that the model remains stable over time. Figure 2.6 shows that the freeze-all prediction-based method does not provide a good model estimation. In contrast, the improved one-step prediction method is more successful and offers better estimation for the tumor growth model.
The use of MHE for estimating the tumor growth model offers an advantage over observer-based methods [22, 33, 35]. Unlike high-gain observers, it does not require system transformation into a triangular form, which can be challenging and time-consuming for users.

2.6 Conclusion

In this chapter, we provided a simple but useful design method to check the i-EIOSS property of nonlinear systems. This method may be used easily for a nonlinear system without using its trajectories. Then, we explored the role played by prediction in MHE, while guaranteeing the robust stability of the estimation error for systems being i-EIOSS and relying on “ad hoc” developed mathematical tools. The three proposed prediction strategies are proven to ensure RES, while offering different degrees of conservatism and flexibility in their application. As part of future work, we plan to investigate further Lyapunov-based methods that can establish a rigorous mathematical connection between detectability and the i-EIOSS property for nonlinear discrete-time systems.

CHAPTER

3

Contributions to LMI-Based and High-Gain-Based Observer Design

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3.1 Introduction

Tremendous research activities have been paid to nonlinear observer design and various methods have been proposed in the literature. Among these methods, apart from the optimization/minimization of cost functions-based techniques, like the extended Kalman filter, and the moving horizon estimator, we can mention the famous high-gain observer design methodology [45], the sliding mode observer approach [36], and the LMI-based techniques [133].

Each of the above methods has advantages and drawbacks and offers reliable estimation for some families of systems. This explains why all these various methods have been proposed.

The high-gain observer design methodology offers the significant advantage of ensuring the feasibility of the Linear Matrix Inequality (LMI) and allows for state estimation. However, it necessitates that the system be triangular, which is a stringent requirement. If the system is not already in this form, it must be transformed into it using a diffeomorphism, a process that is often challenging to determine.

While the first two techniques guarantee the existence of the observer design under only some assumptions on the nonlinearity of the system, however, the last one provides only sufficient conditions expressed in terms of LMIs for which feasibility is not always ensured, which is the main drawback of LMI-based observer design approach. We will address this problem and analyze the feasibility of LMIs for some specific families of nonlinear systems, namely systems in companion form, and systems having feedforward structure in the first section of this chapter. The second part is dedicated to
This chapter is organized as follows: Section 3.2 defines the problem and discusses the necessary preliminary tools. The focus of this section is on proving an "Always LMI Feasibility Proof" for observer design in certain families of nonlinear systems including those in canonical form, systems with feedforward structures, and extensions to a broader class of systems. Next, Section 3.3 focuses on contributions to high-gain observer design for non-triangular systems. This section starts by outlining the motivation and formulation and presenting results specifically tailored to address non-triangular nonlinearities. Finally, it discusses additional refinements to the observer design, emphasizing the innovations and improvements made to overcome the challenges posed by non-triangular systems.

3.2 LMI Feasibility Analysis in Observer Design for Some Families of Systems

Several LMI-based techniques have been developed in the literature, where each technique attempts to reduce the conservatism of the LMI design conditions ensuring exponential convergence of the observer (3.2). Among these methods, there are the old techniques, which are conservative [48, 96, 97, 122], and the recent approaches [1], which provide feasible LMI conditions for a wider class of nonlinear systems. Feasibility of the LMI conditions depends on the Lipschitz constant and the structure of the nonlinearity of the system. To overcome these limitations, the recent LMI approaches use some mathematical tools in convenient ways to dominate the Lipschitz constant and to compensate for the structure of the nonlinearity due to additional decision variables. Despite the considerable efforts made to propose enhanced LMI conditions, this approach suffers from a major drawback, which is the absence of a guarantee of feasibility for any Lipschitz constant. This weakens the LMI techniques and sometimes makes them useless. Recently in [99], instead of guaranteeing the feasibility of LMI conditions, the authors proposed new results on guaranteeing *infeasibility* of the LMIs for systems where all the system components or all the output functions are non-monotonic. In spite of this result, the problem of guaranteeing feasibility is the most important and still remains open. It would therefore be interesting to work on the analysis and guarantee of the feasibility of LMIs for at least some particular families of nonlinear systems as is the case with some famous nonlinear observers, namely high-gain observers and sliding mode observers.

3.2.1 Problem Statement and Preliminary Results

Consider the class of systems described by the following equations:

$$\begin{cases} \dot{x} = Ax + f(x) \\ y = Cx + h(x) \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the system state and $y \in \mathbb{R}^p$ is the output measurement vector. Without loss of generality, and for the sake of brevity, we consider the system (3.1) without control input. We assume that the functions $f(.)$ and $h(.)$ are respectively γ_f-Lipschitz and γ_h-Lipschitz with respect to their arguments. Without loss of generality, the Lipschitz constraint is assumed to be global. Otherwise, we need to apply the *Hilbert* projection theorem [74, 90] or the *Kirszbraun–Valentine*

extension theorem [73, 125] to extend $f(.)$ and $h(.)$ to global Lipschitz functions. We only need the system (3.1) to admit a positively invariant compact set on which $f(.)$ and $h(.)$ are Lipschitz. The reader can also find details on this extension in [45].
As usual in the LMI context, which is the objective of the chapter, we consider the following Luenberger observer structure corresponding to (3.1):

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L\big(y - C\hat{x} - h(\hat{x})\big) \tag{3.2}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of x and the matrix $L \in \mathbb{R}^{n\times p}$ is the observer gain to be determined such that the estimation error $\epsilon \triangleq x - \hat{x}$ converges exponentially towards zero.
Of all the existing methods, the less conservative one is the LPV/LMI approach which is based on transforming the dynamics of the estimation error into a polytopic system, and then the application of the convexity principle leads to solving a finite number of LMI conditions without using strong upper bounds to dominate the nonlinearity of the system. For this reason, we will exploit this method and we will show that the LMIs are always feasible for some families of nonlinear systems. Hence, we will first recall the LPV/LMI technique.
By applying [131, Lemma 7], there exist functions $\psi_{ij}(.,.)$ and $\phi_{ij}(.,.)$ such that the dynamics of the estimation error is given as

$$\begin{aligned}\dot{\epsilon} &= \Big(A - LC\Big)\epsilon + \big[f(x) - f(\hat{x})\big] + \big[h(x) - h(\hat{x})\big]\\ &= \Big(\mathcal{A}(\psi) - L\mathcal{C}(\phi)\Big)\epsilon \end{aligned}\tag{3.3}$$

where

$$\mathcal{A}(\psi) \triangleq A + \sum_{i,j=1}^{n,n} \psi_{ij}\mathcal{H}_{ij}^{n,n} \tag{3.4}$$

$$\mathcal{C}(\phi) \triangleq C + \sum_{i,j=1}^{p,n} \phi_{ij}\mathcal{H}_{ij}^{p,n} \tag{3.5}$$

$$-\gamma_{f_i} \leq \underline{\gamma}_{\psi_{ij}} \leq \psi_{ij} \leq \bar{\gamma}_{\psi_{ij}} \leq \gamma_{f_i} \tag{3.6}$$

$$-\gamma_{h_i} \leq \underline{\gamma}_{\phi_{ij}} \leq \phi_{ij} \leq \bar{\gamma}_{\phi_{ij}} \leq \gamma_{h_i} \tag{3.7}$$

with

$$\psi_{ij} \triangleq \psi_{ij}\Big(x^{\hat{x}_{j-1}}, x^{\hat{x}_j}\Big), \;\; \phi_{ij} \triangleq \phi_{ij}\Big(x^{\hat{x}_{j-1}}, x^{\hat{x}_j}\Big).$$

It is clear from (3.6) and (3.7) that the parameters ψ and ϕ belong to the bounded convex sets

$$\mathcal{S}_f = \Big\{\varphi \in \mathbb{R}^{n\times n} : \underline{\gamma}_{\psi_{ij}} \leq \varphi_{ij} \leq \bar{\gamma}_{\psi_{ij}}\Big\}, \tag{3.8}$$

$$\mathcal{S}_h = \Big\{\varphi \in \mathbb{R}^{p\times n} : \underline{\gamma}_{\phi_{ij}} \leq \varphi_{ij} \leq \bar{\gamma}_{\phi_{ij}}\Big\} \tag{3.9}$$

for which the sets of vertices are respectively given by

$$\mathcal{V}_f = \Big\{\varphi \in \mathbb{R}^{n\times n} : \varphi_{ij} \in \{\underline{\gamma}_{\psi_{ij}}, \bar{\gamma}_{\psi_{ij}}\}\Big\} \tag{3.10}$$

and

$$\mathcal{V}_h = \Big\{\varphi \in \mathbb{R}^{p\times n} : \varphi_{ij} \in \{\underline{\gamma}_{\phi_{ij}}, \bar{\gamma}_{\phi_{ij}}\}\Big\}. \tag{3.11}$$

Hence, by using the quadratic Lyapunov function

$$\vartheta(\epsilon) \triangleq \epsilon^\top \mathbb{P}\epsilon$$

and developing its derivative along the trajectories of (3.3), we obtain following theorem.

Theorem 3.2.1 ([20]). *The estimation error ϵ satisfying (3.3) converges exponentially towards zero if there exists a positive definite matrix $\mathbb{P} = \mathbb{P}^\top$, a matrix $\mathcal{X} \in \mathbb{R}^{n \times p}$, and a scalar $\lambda > 0$ such that the following LMIs are feasible:*

$$\mathcal{A}(\psi)^\top \mathbb{P} + \mathbb{P}\mathcal{A}(\psi) - \mathcal{C}(\phi)^\top \mathcal{X}^\top - \mathcal{X}\mathcal{C}(\phi) + \lambda \mathbb{I}_n < 0 \tag{3.12}$$

$$\forall \psi \in \mathcal{V}_f, \ \forall \phi \in \mathcal{V}_h. \tag{3.13}$$

Moreover, the observer gain is computed by $L = \mathbb{P}^{-1}\mathcal{X}$.

Proof. The proof is straightforward from the LPV/LMI technique in [131]. The term $\lambda \mathbb{I}_n$ is added to get exponential convergence instead of asymptotic convergence.

□

Although (3.12) are the less restrictive LMI conditions that can exist in the literature, they are still strongly dependent on the Lipschitz constants of the nonlinearities, namely the set of vertices $\mathcal{V}_f$ and $\mathcal{V}_h$. They are not always feasible for all values of the bounds $\underline{\gamma}_{\phi_{ij}}, \bar{\gamma}_{\phi_{ij}}, \underline{\gamma}_{\psi_{ij}}$, and $\bar{\gamma}_{\psi_{ij}}$. To improve the feasibility, some guidelines have been given in [17]. Therefore, this note is a continuation of the work in [17]. We will not only improve the feasibility of LMI conditions as in [17], but we will show that LMIs (3.12) are still always feasible for some classes of nonlinear systems independently from the value of the Lipschitz constant of the nonlinearity of the system.

3.2.2 Feasible LMIs for Particular Families of Systems

For some classes of nonlinear systems, we can always guarantee feasibility of the LMIs for any bounds $\underline{\gamma}_{\phi_{ij}}, \bar{\gamma}_{\phi_{ij}}, \underline{\gamma}_{\psi_{ij}}$, and $\bar{\gamma}_{\psi_{ij}}$. This is the objective of this section.

3.2.2.1 Systems in canonical form

Here we will study the case where system (3.1) can be transformed into the following triangular form through a diffeomorphism $z = \Phi(x)$:

$$\begin{cases} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ f_z(z) \end{bmatrix} \\ y = z_1 \end{cases} \tag{3.14}$$

which can be written under the following compact form (3.15):

$$\begin{cases} \dot{z} = A_z z + B_z f_z(z) \\ y = C_z z \end{cases} \tag{3.15}$$

where A_z, C_z, and B_z have the companion structure as in [41]. Note that a more general class of systems with a nonlinearity $f_i(z_1, \ldots, z_i)$ in each component of the system can be considered, without loss of generality. However, for the sake of brevity, we investigate (3.14) with only a single nonlinear function in the last component of the system.
Now introduce the linear transformation

$$\zeta = \mathbb{T}_\tau z, \text{ where } \mathbb{T}_\tau \triangleq \operatorname{diag}\left(\frac{1}{\tau}, \ldots, \frac{1}{\tau^n}\right) \tag{3.16}$$

which transforms (3.14) into

$$\dot{\zeta} = \tau A_z \zeta + \frac{1}{\tau^n} f_z(\mathbb{T}_\tau^{-1}\zeta). \tag{3.17}$$

Let us consider the following state observer corresponding to (3.17):

$$\dot{\hat{\zeta}} = \tau A_z \hat{\zeta} + \frac{1}{\tau^n} f_z\big(\mathbb{T}_\tau^{-1}\hat{\zeta}\big) + L\left(y - C_z \mathbb{T}_\tau^{-1}\hat{\zeta}\right) \tag{3.18}$$

where L, independent from τ, is the constant observer gain to be determined. Then the dynamics of the estimation error $e_\zeta = \zeta - \hat{\zeta}$ is expressed as

$$\dot{e}_\zeta = \tau\Big(A_z - LC_z\Big)e_\zeta + B_z \Delta f_z \tag{3.19}$$

where

$$\Delta f_z \triangleq \frac{1}{\tau^n}\left[f_z(\mathbb{T}_\tau^{-1}\zeta) - f_z\big(\mathbb{T}_\tau^{-1}\hat{\zeta}\big)\right]. \tag{3.20}$$

Applying [131, Lemma 7], there exist functions

$$\psi_j \; : \; \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \; \mathbb{R}$$

and constants $\underline{\gamma}_j \leq 0$ and $\bar{\gamma}_j \geq 0$, such that

$$\Delta f_z = \Big[\sum_{j=1}^{j=n} \frac{\psi_j}{\tau^{n-j}} \mathbf{e}_n^\top(j)\Big] e_\zeta \tag{3.21}$$

and

$$\underline{\gamma}_j \leq \gamma_j \leq \bar{\gamma}_j, \tag{3.22}$$

where $\mathbf{e}_n^\top(j)$ is the j^{th} element of the canonical basis of $\mathbb{R}^n$.
Similarly to (3.4), we introduce the affine matrix $\mathcal{A}(\tau, \Psi)$ defined as

$$\mathcal{A}\Big(\tau, \Psi\Big) = A_z + \sum_{j=1}^{n}\Big[\frac{1}{\tau^{1+(n-j)}}\psi_j \mathbf{e}_n^\top(j)\Big] \tag{3.23}$$

where $\Psi = \big[\psi_1, \ldots, \psi_n\big]^\top$. Then, the parameter Ψ belongs to a bounded convex set for which the set of vertices is given by

$$\mathcal{V}_{f_z} \triangleq \Big\{v \in \mathbb{R}^n : v_j \in \{\underline{\gamma}_j, \bar{\gamma}_j\}\Big\}. \tag{3.24}$$

From (3.19), (3.21), and (3.23), it follows that the dynamics of the estimation error becomes

$$\dot{e}_\zeta = \tau\Big[\mathcal{A}\big(\tau, \Psi\big) - LC_z\Big]e_\zeta. \tag{3.25}$$

Consequently, we can state the following corollary as a particular case of Theorem 3.2.1.

Corollary 3.2.2 ([20]). *Let $\mathcal{P} = \mathcal{P}^\top > 0$ and $\mathcal{X}$ be matrices of appropriate dimensions, and $\tau > 0$ is a scalar, such that the following LMI conditions hold:*

$$\mathcal{A}\big(\tau, w\big)^\top \mathcal{P} + \mathcal{P}\mathcal{A}\big(\tau, w\big) - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z < 0, \tag{3.26}$$

$$\forall w \in \mathcal{V}_{f_z}. \tag{3.27}$$

Then the observer (3.18) corresponding to (3.17), with $L = \mathcal{P}^{-1}\mathcal{X}^\top$, converges exponentially towards zero. Moreover, the estimated state $\hat{x} = \Phi^{-1}\big(\mathbb{T}_\tau^{-1}\hat{\zeta}\big)$ converges exponentially to the state x of the original system (3.1).

Proof. The proof is omitted.

□

Corollary 3.2.2 is an intermediate gateway that leads straightforwardly to the next important result from the LMI point of view. Such a result is given in the following proposition.

Proposition 3.2.3 ([18]). *For any fixed values of the bounds $\underline{\gamma}_j$ and $\bar{\gamma}_j$, $j = 1, \dots, n$, there exists $\tau^* > 0$ such that the LMIs (3.26) are feasible for any $\tau \geq \tau^*$.*

Proof. Since (A_z, C_z) is observable, then there always exists a matrix $\mathcal{P} = \mathcal{P}^\top > 0$ and a matrix $\mathcal{X}$ such that

$$A_z^\top \mathcal{P} + \mathcal{P} A_z - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z < 0.$$

On the other hand, from the definition of $\mathcal{A}(\tau, \Psi)$ in (3.23), we have

$$\lim_{\tau \to +\infty} \big(\mathcal{A}(\tau, w)\big) = A_z, \ \forall w \in \mathcal{V}_{f_z}.$$

Then from continuity of $\mathcal{A}(\tau, w)$ with respect to τ, there exists $\tau^* > 0$ large enough such that the LMI (3.26) holds for any $\tau \geq \tau^*$.

□

Proposition 3.2.3 means that the LMIs (3.26) are always feasible for any global Lipschitz nonlinear function $f_z(.)$ independently from the value of its Lipschitz constant. This result is important in the LMI context since it always guarantees the design of an LMI-based exponential observer for any Lipschitz constant of the system. This is not the case in general for the arbitrary structure of the system where the feasibility of the LMIs depends strongly on the value of the Lipschitz constant of the system.

Remark 3.2.4 ([20]). *High-gain observer design is a particular case of the proposed methodology. Indeed, as can be seen in the proof of Proposition 3.2.3, a sufficiently large value of τ guarantees exponential stability of the estimation error. Such a sufficiently large value of τ leads to high values of the observer gain.*

Remark 3.2.5 ([20]). *The result of this section remains valid for the more general class of systems (3.28) with multi-nonlinearities, described by the following equations:*

$$\left\{ \begin{array}{l} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ 0 \end{bmatrix} + \begin{bmatrix} f_1(z_1) \\ f_2(z_1, z_2) \\ \vdots \\ f_{n-1}(z_1, z_2, \dots, z_{n-1}) \\ f_n(z) \end{bmatrix} \\ y = z_1. \end{array} \right. \tag{3.28}$$

The generalization is straightforward, therefore it is not necessary to provide the developments. On the other hand, we avoid repetition since in the next section, we will consider multi-nonlinearities in the system description.

3.2.2.2 Systems having feedforward structure

Consider the class of feedforward systems described by the equations below, which can be obtained by transforming (3.1) through the diffeomorphism $z = \Phi(x)$:

$$\begin{cases} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-2} \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2 + f_1(z_3, \dots, z_n) \\ z_3 + f_2(z_4, \dots, z_n) \\ \vdots \\ z_{n-1} + f_{n-2}(z_n) \\ z_n \\ u(t) \end{bmatrix} \\ \quad = A_z z + f^{\text{feed}}(z) + Bu(t) \\ y = z_1 \end{cases} \tag{3.29}$$

where $u(t)$ is any known signal. In general, when we consider system (3.1) with the presence of control input, namely $f(x, u)$ and $h(x, u)$ instead of $f(x)$ and $h(x)$, we get the structure (3.29) by using some backstepping transformation techniques [68]. Such a structure is encountered in several works in the literature, namely in [80, 89] and the references therein.
Similarly to the previous section, by using the transformation (4.37) and the observer (3.18), we obtain

$$\dot{e}_\zeta = \tau\Big[\mathcal{A}_{\text{feed}}(\tau, \Psi) - LC_z\Big]e_\zeta \tag{3.30}$$

with

$$\mathcal{A}_{\text{feed}}\Big(\tau, \Psi\Big) = A_z + \sum_{i=1}^{n}\sum_{j=i+2}^{n}\Big[\tau^{j-(i+1)}\psi_{ij}^{\text{feed}}\mathbf{e}_n(i)\mathbf{e}_n^\top(j)\Big] \tag{3.31}$$

where ψ_{ij}^{feed}, independent from τ, comes from [131, Lemma 7] with

$$-\gamma_{f_i^{\text{feed}}} \leq \underline{\gamma}_{\psi_{ij}^{\text{feed}}} \leq \psi_{ij}^{\text{feed}} \leq \bar{\gamma}_{\psi_{ij}^{\text{feed}}} \leq \gamma_{f_i^{\text{feed}}} \tag{3.32}$$

where $\underline{\gamma}_{\psi_{ij}^{\text{feed}}} \leq 0$ and $\bar{\gamma}_{\psi_{ij}^{\text{feed}}} \geq 0$. From the structure (3.29), in this case f^{feed}, we have $\psi_{ij}^{\text{feed}} \equiv 0$ for $i = n-1$, $i = n$, and $\forall j = 1, \dots, n$. It is obvious because from (3.29), the last two components of f^{feed} are zero. This means that the parameter Ψ belongs to an hyper-rectangle $\mathcal{S}_{f^{\text{feed}}}$ for which the set of vertices $\mathcal{V}_{f^{\text{feed}}}$ is defined as follows:

$$\mathcal{V}_{f^{\text{feed}}} = \Big\{\varphi \in \mathbb{R}^{n\times n} : \varphi_{ij} \in \{\underline{\gamma}_{\psi_{ij}^{\text{feed}}}, \bar{\gamma}_{\psi_{ij}^{\text{feed}}}\} \tag{3.33}$$

$$\varphi_{ij} = 0 \text{ for } i = n-1 \text{ and } i = n\Big\}. \tag{3.34}$$

It follows that

$$\lim_{\tau\to 0}\Big(\mathcal{A}_{\text{feed}}\Big(\tau, \Psi\Big)\Big) = A_z,\ \forall \Psi \in \mathcal{V}_{f^{\text{feed}}} \tag{3.35}$$

since ψ_{ij}^{feed} is bounded and independent from τ.
Now we are ready to state the following proposition.

Proposition 3.2.6 ([20]). *There exists $\tau^{feed} > 0$, such that the following LMI conditions hold $\forall \tau$: $0 < \tau \leq \tau^{feed}$:*

$$\mathcal{A}_{feed}\big(\tau, w\big)^\top\mathcal{P} + \mathcal{P}\mathcal{A}_{feed}\big(\tau, w\big) - C_z^\top\mathcal{X} - \mathcal{X}^\top C_z < 0, \forall w \in \mathcal{V}_{f^{feed}}, \tag{3.36}$$

where $\mathcal{P} = \mathcal{P}^\top > 0$ and $\mathcal{X}$ are matrices of appropriate dimensions, which are the decision variables of the LMIs (3.36). Then the observer (3.18) corresponding to (3.17), with $L = \mathcal{P}^{-1}\mathcal{X}^\top$, converges asymptotically. Moreover, the estimated state $\hat{x} = \Phi^{-1}\big(\mathbb{T}_\tau^{-1}\hat{\zeta}\big)$ converges asymptotically to the state x of the original system (3.1) for all τ satisfying $0 < \tau \leq \tau^{feed}$.

Proof. First, as for the Proposition 3.2.3, from observability of (A_z, C_z), we deduce there always exist a matrix $\mathcal{P} = \mathcal{P}^\top > 0$ and a matrix $\mathcal{X}$ such that

$$A_z^\top \mathcal{P} + \mathcal{P}A_z - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z < 0.$$

On the other hand, we have

$$\begin{aligned} \mathcal{A}_{\text{feed}}(\tau, w)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_{\text{feed}}(\tau, w) - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z \\ = \overbrace{A_z^\top \mathcal{P} + \mathcal{P}A_z - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z}^{<0} \\ +\mathbb{S}(\tau, w)\mathcal{P} + \mathcal{P}\mathbb{S}(\tau, w)^\top \end{aligned} \quad (3.37)$$

where

$$\mathbb{S}(\tau, w) \triangleq \sum_{i=1}^{n} \sum_{j=i+2}^{n} \left[\tau^{j-(i+1)} \psi_{ij}^{\text{feed}} \mathbf{e}_n(i) \mathbf{e}_n^\top(j)\right].$$

Then, from (3.35) and the continuity of $\mathcal{A}_{\text{feed}}(., w)$ with respect to τ (we can also use the *Archimedean* property), there exists $\tau^{\text{feed}} > 0$ such that

$$\mathcal{A}_{\text{feed}}(\tau^{\text{feed}}, w)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_{\text{feed}}(\tau^{\text{feed}}, w) - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z < 0 \forall w \in \mathcal{V}_{f^{\text{feed}}}. \quad (3.38)$$

On the other hand, we have

$$\mathcal{A}_{\text{feed}}(\tau, w) = \mathcal{A}_{\text{feed}}\left(\tau^{\text{feed}}, w^\tau\right)$$

with $w_{ij}^\tau = \left[\frac{\tau}{\tau^{\text{feed}}}\right]^{j-(i+1)} w_{ij}$. Since $\underline{\gamma}_{\psi_{ij}^{\text{feed}}} \leq 0$ and $\bar{\gamma}_{\psi_{ij}^{\text{feed}}} \geq 0$, and we have $w \in \mathcal{V}_{f^{\text{feed}}}$, then $w^\tau \in \mathcal{S}_{f^{\text{feed}}}$ for any $\tau \leq \tau^{\text{feed}}$, i.e.: $\frac{\tau}{\tau^{\text{feed}}} \leq 1$. Hence, from the convexity principle, the inequality (3.38) is preserved $\forall \tau \leq \tau^{\text{feed}}$, which ends the proof of Proposition 4.5.3.

□

3.2.3 Extension to a More General Class of Systems

This section is devoted to a more general class of systems, which does not contain necessarily linear parts like in (3.14). Consider the class of systems (3.1) which can be transformed to the following form through the diffeomorphism $z = \Phi(x)$:

$$\left\{ \begin{array}{l} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \phi_1(z_1, z_2) \\ \phi_2(z_1, z_2, z_3) \\ \vdots \\ \phi_{n-1}(z_1, z_2, \ldots, z_{n-1}, z_n) \\ \phi_n(z) \end{bmatrix} \\ = f^{\text{nl}}(z) \\ y = \varphi(z_1). \end{array} \right. \quad (3.39)$$

This extension can be useful in the sense that some real-world models are not in the form (3.14), and then they do not need to be transformed into (3.14) with complex structure of nonlinearities. For a motivating example, we can mention the tumor growth model investigated in [49] and the references therein. Such a tumor growth model is under the form (3.39). To avoid repetition

and cumbersome notations, we consider, in this section, only systems under the nonlinear canonical form. Extension to systems having nonlinear feedforward structures can be straightforwardly obtained.
We use again the same change of variable (4.37), $\zeta = \mathbb{T}_\tau z$, to design an observer for (3.39). We then, first, introduce the state observer corresponding to ζ:

$$\dot{\hat{\zeta}} = \mathbb{T}_\tau f^{\text{nl}}\left(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\right) + L\left[y - \varphi\left(\tau\hat{\zeta}_1\right)\right] \tag{3.40}$$

where we used $\mathbb{T}_\tau^{-1} = \mathbb{T}_{\frac{1}{\tau}}$; $\hat{\zeta}$ is the estimate of ζ. Then the estimation of z is expressed as $\hat{z} = \mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}$. Hence, the estimation of the original state x is given by $\hat{x} = \Phi^{-1}\left(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\right)$. Notice that in the case of the tumor growth model in [49], the system is under the form (3.39), then there is no need for nonlinear transformation. Therefore, we get directly from (3.40) an estimation of x as $\hat{x} = \mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}$.
Now let us return to the convergence analysis of the estimation error $e_\zeta = \zeta - \hat{\zeta}$. We have

$$\begin{aligned}\dot{e}_\zeta = \mathbb{T}_\tau & \overbrace{\left[f^{\text{nl}}\left(\mathbb{T}_{\frac{1}{\tau}}\zeta\right) - f^{\text{nl}}\left(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\right)\right]}^{\Delta f^{\text{nl}}} \\ & + L\left[\varphi\left(\tau\zeta_1\right) - \varphi\left(\tau\hat{\zeta}_1\right)\right].\end{aligned} \tag{3.41}$$

By applying [131, Lemma 7] and after isolating the terms corresponding to the $(i+1)^{\text{th}}$ component of the state and the i^{th} component of the nonlinearity, f_i^{nl}, we deduce that there exists functions

$$\psi_{ij}^{\text{nl}} \;:\; \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\varphi_1 \;:\; \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

and constants $\underline{\gamma}_{\psi_{ij}^{\text{nl}}}$, $\bar{\gamma}_{\psi_{ij}^{\text{nl}}}$, $\underline{\gamma}_{\varphi_1}$, and $\bar{\gamma}_{\varphi_1}$, such that

$$\Delta f^{\text{nl}} = \tau\left[\sum_{i=1}^{n-1}\psi_{i,i+1}^{\text{nl}}(t)\mathbf{e}_n(i)\mathbf{e}_n^\top(i+1)\right]e_\zeta + \left[\sum_{i=1}^{n}\sum_{j=1}^{i}\left[\frac{1}{\tau^{i-j}}\psi_{ij}^{\text{nl}}(t)\mathbf{e}_n(i)\mathbf{e}_n^\top(j)\right]\right]e_\zeta \tag{3.42}$$

$$\varphi\left(\tau\zeta_1\right) - \varphi\left(\tau\hat{\zeta}_1\right) = \tau\varphi_1(t)C_z e_\zeta \tag{3.43}$$

and

$$\underline{\gamma}_{\psi_{ij}^{\text{nl}}} \leq \psi_{ij}^{\text{nl}}(t) \leq \bar{\gamma}_{\psi_{ij}^{\text{nl}}} \tag{3.44}$$

$$\underline{\gamma}_{\varphi_1} \leq \varphi_1(t) \leq \bar{\gamma}_{\varphi_1} \tag{3.45}$$

where $\psi_{ij}^{\text{nl}}(t)$ and $\varphi_1(t)$ are independent from τ; $\psi_{ij}^{\text{nl}}(t) \triangleq \psi_{ij}^{\text{nl}}\left(\zeta^{\hat{\zeta}_{j-1}}, \zeta^{\hat{\zeta}_j}\right)$ and $\varphi_1(t) \triangleq \varphi_1(\zeta_1, \hat{\zeta}_1)$ are introduced for simplification. Furthermore, we assume, as in the previous section, that

$$\underline{\gamma}_{\psi_{ij}^{\text{nl}}} \leq 0 \text{ and } \bar{\gamma}_{\psi_{ij}^{\text{nl}}} \geq 0, \quad \text{for all } j \leq i, i = 1, \ldots, n. \tag{3.46}$$

More importantly, we need the following assumption for the existence of the observer we propose:

$$\underline{\gamma}_{\psi_{i,i+1}^{\text{nl}}} > 0 \text{ and } \underline{\gamma}_{\varphi_1} > 0. \tag{3.47}$$

Notice that conditions (3.47) are introduced first in [45] to guarantee the existence of a high-gain observer for the system (3.39). Authors in [45, Eq.(75), page 96] used a slightly different, but equivalent, condition, namely

$$0 < \alpha \leq \psi_{i,i+1}^{\text{nl}}(t) \leq \beta \text{ and } \alpha \leq \varphi_1(t) \leq \beta \tag{3.48}$$

which can be obtained from (3.44)-(3.45) and (3.47) with

$$\alpha = \min\left(\min_{i=1,\dots,n-1} \underline{\gamma}_{\psi^{\mathrm{nl}}_{i,i+1}}, \underline{\gamma}_{\varphi_1}\right), \text{ and}$$

$\beta = \max\left(\max_{i=1,\dots,n-1} \bar{\gamma}_{\psi^{\mathrm{nl}}_{i,i+1}}, \bar{\gamma}_{\varphi_1}\right).$
Now, we introduce the following notations:

$$\boldsymbol{\psi}_t \triangleq \begin{bmatrix} \psi^{\mathrm{nl}}_{1,2}(t) \\ \psi^{\mathrm{nl}}_{2,3}(t) \\ \vdots \\ \psi^{\mathrm{nl}}_{n-1,n}(t) \end{bmatrix} \in \mathbb{R}^{n-1}, \ \boldsymbol{\varphi}_t \triangleq \begin{bmatrix} \psi^{\mathrm{nl}}_{11}(t) \\ \psi^{\mathrm{nl}}_{21}(t) \\ \psi^{\mathrm{nl}}_{22}(t) \\ \psi^{\mathrm{nl}}_{31}(t) \\ \vdots \\ \psi^{\mathrm{nl}}_{n1}(t) \\ \vdots \\ \psi^{\mathrm{nl}}_{nn}(t) \end{bmatrix} \in \mathbb{R}^{\frac{n(n+1)}{2}} \tag{3.49}$$

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{\psi}_t) &\triangleq \sum_{i=1}^{n-1} \psi^{\mathrm{nl}}_{i,i+1}(t)\mathbf{e}_n(i)\mathbf{e}_n^\top(i+1) \\ &= \begin{bmatrix} 0 & \psi^{\mathrm{nl}}_{1,2}(t) & 0 & \dots & 0 & 0 \\ 0 & 0 & \psi^{\mathrm{nl}}_{2,3}(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \psi^{\mathrm{nl}}_{n-1,n}(t) \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \end{aligned} \tag{3.50}$$

$$\boldsymbol{\mathcal{A}}_{\mathrm{nl}}(\tau, \boldsymbol{\varphi}_t) \triangleq \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{1}{\tau^{1+i-j}} \psi^{\mathrm{nl}}_{ij}(t)\mathbf{e}_n(i)\mathbf{e}_n^\top(j) \tag{3.51}$$

$$\boldsymbol{C}(\varphi_1(t)) \triangleq \varphi_1(t)C_z.$$

The dynamic of the estimation error (3.41) can then be expressed under the following compact form:

$$\dot{e}_\zeta = \tau\Big[\boldsymbol{A}(\boldsymbol{\psi}_t) + \boldsymbol{\mathcal{A}}_{\mathrm{nl}}(\tau, \boldsymbol{\varphi}_t) - L\boldsymbol{C}(\varphi_1(t))\Big]e_\zeta. \tag{3.52}$$

By definition (3.49) and assumption (3.44), the time-varying parameters $\boldsymbol{\psi}_t$ and $\boldsymbol{\varphi}_t$ belong to bounded convex sets for which the sets of vertices are respectively, given as follows:

$$\mathcal{V}_{\boldsymbol{\psi}} = \Big\{\varphi \in \mathbb{R}^{n-1} : \varphi_i \in \{\underline{\gamma}_{\psi^{\mathrm{nl}}_{i,i+1}}, \bar{\gamma}_{\psi^{\mathrm{nl}}_{i,i+1}}\}, \text{where } \underline{\gamma}_{\psi^{\mathrm{nl}}_{i,i+1}} > 0, i = 1, \dots, n-1\Big\}, \tag{3.53}$$

$$\mathcal{V}_{\boldsymbol{\varphi}} = \left\{\boldsymbol{\rho} \in \mathbb{R}^{\frac{n(n+1)}{2}} : \boldsymbol{\rho} = \begin{bmatrix} \boldsymbol{\rho}_1 \\ \vdots \\ \boldsymbol{\rho}_n \end{bmatrix}, \boldsymbol{\rho}_i = \begin{bmatrix} \rho_{i1} \\ \vdots \\ \rho_{ii} \end{bmatrix}, \rho_{ij} \in \{\underline{\gamma}_{\psi^{\mathrm{nl}}_{ij}}, \bar{\gamma}_{\psi^{\mathrm{nl}}_{ij}}\}, \text{ s.t } (3.46)\right\}. \tag{3.54}$$

Now we are ready to state the main proposition of this section.

Proposition 3.2.7 ([20]). *There exists $\tau^\star_{nl} > 0$, such that $\forall \tau \geq \tau^\star_{nl}$, there exist matrices $\mathcal{P} = \mathcal{P}^\top > 0$ and $\mathcal{X}$ of appropriate dimensions, and a scalar $\lambda > 0$, such that the following LMI holds:*

$$\big[\boldsymbol{A}(v) + \boldsymbol{\mathcal{A}}_{nl}(\tau, w)\big]^\top \mathcal{P} + \mathcal{P}\big[\boldsymbol{A}(v) + \boldsymbol{\mathcal{A}}_{nl}(\tau, w)\big] - \boldsymbol{C}(\kappa)^\top\mathcal{X} - \mathcal{X}^\top\boldsymbol{C}(\kappa) < -\lambda\mathbb{I}_n, \tag{3.55}$$

$$\forall v \in \mathcal{V}_{\boldsymbol{\psi}}, \forall w \in \mathcal{V}_{\boldsymbol{\varphi}}, \forall \kappa \in \{\underline{\gamma}_{\varphi_1}, \bar{\gamma}_{\varphi_1}\}. \tag{3.56}$$

Then the observer (3.40) with $L = \mathcal{P}^{-1}\mathcal{X}^\top$, converges exponentially. Moreover, the estimated state $\hat{x} = \Phi^{-1}\big(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\big)$ converges exponentially to the state x of the original system (3.1), $\forall \tau \geq \tau^\star_{nl} > 0$.

Proof. Since $v \in \mathcal{V}_{\psi}$, $\kappa \in \{\underline{\gamma}_{\varphi_1}, \bar{\gamma}_{\varphi_1}\}$, and taking in mind (3.47), it follows from [45, Lemma 2.1, page 96] that there exist matrices $\mathcal{P} = \mathcal{P}^\top > 0$ and $\mathcal{X}$ of appropriate dimensions, and a scalar $\lambda > 0$, such that

$$\boldsymbol{A}(v)^\top \mathcal{P} + \mathcal{P}\boldsymbol{A}(v) - \boldsymbol{C}(\kappa)^\top \mathcal{X} - \mathcal{X}^\top \boldsymbol{C}(\kappa) < -2\lambda \mathbb{I}_n, \tag{3.57}$$

$$\forall v \in \mathcal{V}_{\psi}, \forall w \in \mathcal{V}_{\varphi}, \forall \kappa \in \{\underline{\gamma}_{\varphi_1}, \bar{\gamma}_{\varphi_1}\}. \tag{3.58}$$

Also, since

$$\lim_{\tau \to 0} \left(\boldsymbol{\mathcal{A}}_{\text{nl}}(\tau, w)\right) = 0_{n\times n}, \ \forall w \in \mathcal{V}_{\varphi}$$

then it is obvious that $\exists \tau^\star_{\text{nl}} > 0$ large enough, such that

$$\boldsymbol{\mathcal{A}}_{\text{nl}}(\tau, w)^\top \mathcal{P} + \mathcal{P}\boldsymbol{\mathcal{A}}_{\text{nl}}(\tau, w) < \lambda \mathbb{I}_n, \forall w \in \mathcal{V}_{\varphi}, \ \forall \tau \geq \tau^\star_{\text{nl}}. \tag{3.59}$$

Hence, summing (3.57) and (3.59), the relation (3.55) is inferred.

□

Remark 3.2.8 ([20]). *Notice that the condition (3.47) may also be replaced by*

$$\bar{\gamma}_{\psi^{\text{nl}}_{i,i+1}} < 0 \text{ and } \bar{\gamma}_{\varphi_1} < 0 \tag{3.60}$$

and the result remains valid. Indeed, the condition for the existence of solution to (3.57) is the strict monotonicity of the nonlinearities ϕ_i and φ with respect to the variables z_{i+1} and z_1, respectively.

3.3 Contributions to High-Gain Observer for Non-Triangular Systems

This section focuses on the high-gain observer design methodology while exploiting the LMI approach for improvements. To complete the work established in various works in the literature, namely [67, 139] and the references therein, we propose a generalization to systems for which the nonlinearities are not in the triangular form. This extension, however, requires an extra constraint on the high-gain tuning parameter. To improve the result and reduce the conservatism of the design conditions, we exploit the LMI approach to transform the extra constraint into a set of LMI conditions. A design algorithm is then provided and a simple academic example is given to show the validity of the proposed methodologies.
The advantages of the proposed methodology, with respect to the standard high-gain observer and the LMI-based observer, are summarized in the following items:

- Compared to the standard high-gain methodology, there is no need for any transformation of the system to get a triangular form. The non-triangular terms are handled by the extra constraint or the set of LMIs;
- Compared to the LPV/LMI approach in [131], we get a significant reduction of the exponential number of LMIs since only a part of the nonlinearity is handled by the LMI approach.

Beyond the above advantages, the approach opens the door to important applications, such as systems with nonlinear outputs with arbitrary structure and systems with delayed or sampled measurements.
There are several other observer design techniques, such as LMI-based techniques [1, 131], high-gain observer in the sense of *Thau et al.* [38, 98, 102], and many others, where different Lipschitz formulations and structure of the linear part in the system are used to reduce the conservatism of the design conditions. It is worthy noticing that we focus on the high-gain observer methodology developed by *J-P. Gauthier et al.* in the sense of [41] for triangular systems. We propose to generalize such a method, while keeping the main design conditions of the same methodology. Only one extra-condition on the tuning parameter is involved in addition to the well-known high-gain synthesis conditions.

3.3.1 Motivation and problem formulation

It is quite clear and well-known from the previous section that the result of Theorem 4.3.1 was possible due to the structure (3.14). Without such a structure, the condition (2.93) is not sufficient to ensure the exponential convergence of the estimation error. Indeed, in such a situation, we can easily show that the upper bound k_f in (1.39) depends on τ; namely, we get $k_f(\tau)$, which leads to the following similar bound:

$$\tau > \max\left(1, \frac{2k_f(\tau)\lambda_{\max}(P)}{\lambda}\right). \tag{3.61}$$

However, the bound (3.61) might not have explicit solutions, or even be infeasible. This means that (3.61) is useless and the high-gain observer methodology cannot guarantee exponential convergence of the estimation error towards zero. The followinf is devoted to solving this problem by proposing a general observer design framework.

3.3.2 Results for Non-Triangular Nonlinearities

In this section, we propose results for a general class of systems with arbitrary structure of the nonlinearities. We will show that for some families of nonlinearities with Lipschitz constants satisfying some explicit conditions, the design of an observer of the form (3.18) remains possible despite the non-triangular form of the nonlinearity.

3.3.2.1 System description and assumptions

In this section, we introduce the. class of systems and the related main assumptions needed for the proposed design methodology. First, consider the following class of nonlinear systems:

$$\begin{cases} \dot{x} = \phi(x) \\ y = x_1 = Cx \end{cases} \tag{3.62}$$

where $x \in \mathbb{R}^n$ is the system state and $y \in \mathbb{R}^p$ is the output measurement. The function $\phi(.)$ is assumed to be $\gamma_\phi-$Lipschitz with respect to x. Without loss of generality, $\phi(.)$ is differentiable with respect to x. We also assume that there exist two positive scalars α and β such that the following double inequality holds $\forall x \in \mathbb{R}^n$:

$$0 < \alpha \leq \frac{\partial \phi_j}{\partial x_{j+1}}(x) \leq \beta,\ \forall\ j = 1, \dots, n-1. \tag{3.63}$$

We use the linear transformation (4.37) and propose the observer

$$\dot{\hat{\zeta}} = \mathbb{T}_\tau \phi\left(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\right) + L\left(y - C\hat{\zeta}_1\right) \tag{3.64}$$

corresponding to the transformed system

$$\dot{\zeta} = \mathbb{T}_\tau \phi\left(\mathbb{T}_{\frac{1}{\tau}}\zeta\right), \tag{3.65}$$

which leads to the error dynamics

$$\dot{e}_\zeta = \mathbb{T}_\tau \overbrace{\left[\phi\left(\mathbb{T}_{\frac{1}{\tau}}\zeta\right) - \phi\left(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\right)\right]}^{\Delta\phi} + \tau L C e_\zeta. \tag{3.66}$$

By applying [131, Lemma 7], we deduce that there exist functions $\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and constants $\underline{\gamma}_{\psi_{ij}}$, $\bar{\gamma}_{\psi_{ij}}$, such that

$$\mathbb{T}_\tau \Delta\phi(\zeta, \hat{\zeta}) = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{1}{\tau^{i-j}} \psi_{ij}(t) \mathbf{e}_n(i) \mathbf{e}_n^\top(j) \right] \right] e_\zeta. \tag{3.67}$$

and

$$\underline{\gamma}_{\psi_{ij}} \leq \psi_{ij}(t) \leq \bar{\gamma}_{\psi_{ij}} \tag{3.68}$$

where $\psi_{ij}(t)$ are independent from τ and the notation $\psi_{ij}(t) \triangleq \psi_{ij}\left(\zeta^{\hat{\zeta}_{j-1}}, \zeta^{\hat{\zeta}_j}\right)$ is introduced for simplification.

3.3.2.2 Convenient transformation of (3.66)

For establishing high-gain results for the general class of nonlinear systems (3.62), the transformation (3.67) is not convenient. For that, the idea consists in decomposing the right-hand side of (3.67) into three terms as follows:

$$\begin{aligned} \mathbb{T}_\tau \Delta\phi(\zeta, \hat{\zeta}) = {} & \tau \left[\sum_{i=1}^{n-1} \psi_{i,i+1}(t) \mathbf{e}_n(i) \mathbf{e}_n^\top(i+1) \right] e_\zeta \\ & + \underbrace{\left[\sum_{i=1}^{n} \sum_{j=1}^{i} \left[\frac{1}{\tau^{i-j}} \psi_{ij}(t) \mathbf{e}_n(i) \mathbf{e}_n^\top(j) \right] \right] e_\zeta}_{\mathcal{N}_1(e_\zeta)} \\ & + \underbrace{\left[\sum_{i=1}^{n-2} \sum_{j=i+2}^{n} \left[\tau^{j-i} \psi_{ij}(t) \mathbf{e}_n(i) \mathbf{e}_n^\top(j) \right] \right] e_\zeta}_{\mathcal{N}_2(e_\zeta)} \end{aligned} \tag{3.69}$$

where $\mathbf{e}_n(i) = \big[\underbrace{0, ..., 0, \overbrace{1}^{i^{\text{th}}}, 0, ..., 0}_{n \text{ components}} \big]^\top \in \mathbb{R}^n, n \geq 1$ is a vector of the canonical basis of $\mathbb{R}^n$.

By setting

$$\boldsymbol{A}(\boldsymbol{\psi}_t) \triangleq \sum_{i=1}^{n-1} \psi_{i,i+1}(t) \mathbf{e}_n(i) \mathbf{e}_n^\top(i+1) \tag{3.70}$$

the estimation error (3.66) is reduced to

$$\dot{e}_\zeta = \tau \Big(\boldsymbol{A}(\boldsymbol{\psi}_t) - LC \Big) e_\zeta + \mathcal{N}_1(e_\zeta) + \mathcal{N}_2(e_\zeta). \tag{3.71}$$

3.3.2.3 Design conditions for arbitrary nonlinear structure

From (3.63), the parameter $\boldsymbol{\psi}_t$ belongs to a bounded convex set for which the set of vertices is defined as:

$$\mathcal{V}_{\boldsymbol{\psi}} = \Big\{ \varphi \in \mathbb{R}^{n-1} : \varphi_i \in \{\alpha, \beta\}, i = 1, \ldots, n-1 \Big\}. \tag{3.72}$$

In addition, since the nonlinearities are Lipschitz with respect to their arguments and due to the structure of $\mathcal{N}_1(e_\zeta)$ and $\mathcal{N}_2(e_\zeta)$ with respect to τ, it is easy to show that there exist two positive constants κ_1 and κ_2, independent of τ, such that

$$\|\mathcal{N}_1(e_\zeta)\| \leq \kappa_1 \|e_\zeta\|, \tag{3.73}$$

$$\|\mathcal{N}_2(e_\zeta)\| \leq \tau^{n-1}\kappa_2 \|e_\zeta\|. \tag{3.74}$$

Hence, we are ready to state the following more general theorem.

Theorem 3.3.1 ([18]). *Let $P = P^\top > 0$ and $\mathcal{X}$ be matrices of appropriate dimensions, $\tau > 0$ and $0 \leq \epsilon \leq 1$ are scalars, such that the following conditions hold:*

$$\boldsymbol{A}(\varphi)^\top P + P\boldsymbol{A}(\varphi) - C^\top \mathcal{X} - \mathcal{X}^\top C + \lambda \mathbb{I}_n < 0, \quad \forall \varphi \in \mathcal{V}_{\boldsymbol{\psi}}, \tag{3.75}$$

$$\tau\epsilon > \max\left(1, \frac{2\kappa_1 \lambda_{\max}(P)}{\lambda}\right), \tag{3.76}$$

$$\tau \sqrt[n-2]{(1-\epsilon)} < \left(\frac{\lambda}{2\kappa_2 \lambda_{\max}(P)}\right)^{\frac{1}{n-2}}. \tag{3.77}$$

Then the observer (3.64) corresponding to (3.65), with $L = \mathcal{P}^{-1}\mathcal{X}^\top$, converges exponentially towards zero. Moreover, the estimated state $\hat{x}(t) = \mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}(t)$ exponentially converges to the state $x(t)$ of the original system (3.62).

Proof. By using the Lyapunov function $V(e_\zeta) \triangleq e_\zeta^\top P e_\zeta$ and computing its derivative along the trajectories of (3.71), we obtain

$$\dot{V}(e_\zeta) = e_\zeta^\top \left[\Big(\boldsymbol{A}(\boldsymbol{\psi}_t) - LC\Big)^\top P + P\Big(\boldsymbol{A}(\boldsymbol{\psi}_t) - LC\Big)\right] e_\zeta + 2e_\zeta^\top P\mathcal{N}_1(e_\zeta) + 2e_\zeta^\top P\mathcal{N}_2(e_\zeta). \tag{3.78}$$

Then, by using the change of variable $\mathcal{X} \triangleq L^\top P$ and from the convexity principle [31, 131], we deduce that (3.75) leads to

$$\Big(\boldsymbol{A}(\boldsymbol{\psi}_t) - LC\Big)^\top P + P\Big(\boldsymbol{A}(\boldsymbol{\psi}_t) - LC\Big) < -\lambda \mathbb{I}_n. \tag{3.79}$$

On other hand, from (3.73) and (3.74), we have

$$2e_\zeta^\top P\mathcal{N}_1(e_\zeta) \leq 2\lambda_{\max}(P)\kappa_1 \|e_\zeta\|^2 \tag{3.80}$$

and

$$2e_\zeta^\top P\mathcal{N}_2(e_\zeta) \leq 2\lambda_{\max}(P)\kappa_2 \tau^{n-1} \|e_\zeta\|^2. \tag{3.81}$$

Hence, by combining (3.78), (3.79), and (3.80), we get

$$\dot{V}(e_\zeta) \leq -\Big(\lambda\tau - 2\lambda_{\max}(P)[\kappa_1 + \tau^{n-1}\kappa_2]\Big)\|e_\zeta\|^2$$

Now, from the convex decomposition $\tau = \epsilon\tau + (1-\epsilon)\tau$, for any $0 \leq \epsilon \leq 1$, the inequality

$$\lambda\tau - 2\lambda_{\max}(P)[\kappa_1 + \tau^{n-1}\kappa_2] > 0$$

holds if

$$\epsilon\tau > \frac{2\kappa_1 \lambda_{\max}(P)}{\lambda}$$

and

$$(1-\epsilon)\tau > \frac{2\kappa_2\lambda_{\max}(P)}{\lambda}\tau^{n-1}$$

are satisfied, which allows us to conclude.

□

3.3.2.4 Comments and discussion

This section is dedicated to some constructive remarks on the previous results.

3.3.2.4.a On the feasibility of (3.75)

Notice that the linear matrix inequalities (3.75) are always feasible due to the required condition (3.63). For more details on this issue, we refer the reader to [45, Eq. (75), page 96]. Then, if there is any conservatism in the design procedure provided by Theorem 4.4.6, it could not come from (3.75). On the other hand, however, since it is useful to minimize the value of $\frac{2\kappa_1\lambda_{\max}(P)}{\lambda}$, we need to introduce additional constraints to solve jointly with (3.75). From homogeneity of (3.75), we can fixe $\lambda = 1$ and introduce a constraint like

$$P \leq \alpha\mathbb{I}_n \tag{3.82}$$

where $\alpha > 0$ is to be minimized. By using the Matlab/Yalmip software, we can efficiently solve this optimization problem.

3.3.2.4.b On the conservatism of the bounds (3.76)–(3.77)

It is quite clear that both (3.76)–(3.77) are satisfied jointly if there exist $P = P^\top > 0, \lambda > 0$ and $0 \leq \epsilon \leq 1$ such that

$$\max\left(1, \frac{2\kappa_1\lambda_{\max}(P)}{\epsilon\lambda}\right) < \left(\frac{(1-\epsilon)\lambda}{2\kappa_2\lambda_{\max}(P)}\right)^{\frac{1}{n-2}}. \tag{3.83}$$

The above inequality (3.83) is more likely to be satisfied if the Lipschitz constant κ_2 is small enough compared to κ_1. For instance, we can easily show that for $\kappa_2 \geq \kappa_1$, inequality (3.83) does not admit a solution. On the other hand, the proposed approach allows designing a high-gain observer for a non-triangular system without performing any nonlinear transformation. It is only needed that the nonlinearity of the system, namely κ_1 and κ_2, satisfies the above inequality:

$$\kappa_1 < \frac{\epsilon\lambda}{2\lambda_{\max}(P)}\left(\frac{(1-\epsilon)\lambda}{2\kappa_2\lambda_{\max}(P)}\right)^{\frac{1}{n-2}} \tag{3.84}$$

by taking in mind that for $\kappa_1 \geq 1$, we often have $\max\left(1, \frac{2\kappa_1\lambda_{\max}(P)}{\epsilon\lambda}\right) = \frac{2\kappa_1\lambda_{\max}(P)}{\epsilon\lambda}$.
To avoid the conservatism related to (3.83), we propose in Section 3.3.3 a relaxation by exploiting LMI-based techniques. As we can see in Section 3.3.3, the LMI technique makes it possible to increase the value of the bound (3.77), or even to make it vanish completely.

3.3.2.4.c Case of triangular systems

Although the Theorem 4.4.6 provides conservative design conditions for some systems, it proposes a more general design than the standard high-gain observer. Indeed, the standard high-gain observer for triangular systems may be viewed as a particular case of Theorem 4.4.6. Theorem 4.3.1 corresponds to the case $\kappa_2 = 0$, $\epsilon = 1$, and $\boldsymbol{A}(\varphi) \equiv A$ in Theorem 4.4.6. The advantage of the proposed general design method is that for some specific systems where the non-triangular part of the nonlinearity, namely κ_2, satisfies (3.77), it is not necessary to perform any change of coordinates to design a high-gain observer.

3.3.2.4.d On the sector condition (3.63)

Notice that (3.63) is a common condition to most of the existing work on high-gain observer in the sense of observer developed by *J-P. Gauthier et al.* Such a condition is required for the existence of solutions to the inequalities (3.75). Condition (3.63) is not exactly the sector condition only, but it represents also the strict monotonicity of the nonlinear component ϕ_j with respect to the state component x_{j+1}. The bound $\alpha > 0$ ensures the monotonicity of ϕ_j and then the observability of the pair $(\boldsymbol{A}(\varphi), C)$, $\forall \varphi \in \mathcal{V}_{\boldsymbol{\psi}}$, then the feasibility of (3.75). On the other hand, the upper bound β in (3.63) is required for the convexity principle leading to a finite number of LMI conditions (3.75). To sum up, without (3.63) we cannot follow the technique proposed in this section, as is the case with the standard high-gain observer methodology. Nevertheless, if we follow the proposed methodology and consider relaxing the upper bound in (3.63), i.e.: $\beta = +\infty$, therefore, according to (3.76), we need to prove that the inequalities (3.75) admit solutions with bounded matrices $P = P^\top > 0$ and $\mathcal{X}$.

3.3.3 Further Results: Improved Design Procedure

To overcome the conservatism of the previous generalized high-gain design method, we propose improved conditions by using LMI tools. Indeed, we will merge the constraint (3.77) with the inequality (3.75) to obtain only one of them expressed as a set of LMIs to solve. To this end, we have to gather the term $\mathcal{N}_2(e_\zeta)$ with the first sum on the right-hand side of (3.69).

3.3.3.1 Transformation of the error dynamics

We start by rewriting (3.69) as follows:

$$\mathbb{T}_\tau \Delta\phi(\zeta, \hat{\zeta}) = \tau\Big(\boldsymbol{A}(\boldsymbol{\psi}_t) + \boldsymbol{\mathcal{A}}(\boldsymbol{\varphi}^\tau(t))\Big)e_\zeta + \mathcal{N}_1(e_\zeta) \tag{3.85}$$

where

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\varphi}^\tau(t)) \triangleq \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} \boldsymbol{\varphi}^\tau_{(i,j)}(t)\mathbf{e}_n(i)\mathbf{e}_n^\top(j) \tag{3.86}$$

$$\boldsymbol{\varphi}^\tau(t) \triangleq \begin{bmatrix} \boldsymbol{\varphi}^\tau_{(1,3)}(t) \\ \vdots \\ \boldsymbol{\varphi}^\tau_{(1,n)}(t) \\ \vdots \\ \boldsymbol{\varphi}^\tau_{(n-2,n)}(t) \end{bmatrix} \in \mathbb{R}^{\frac{(n-1)(n-2)}{2}} \tag{3.87}$$

$$\boldsymbol{\varphi}^\tau_{(i,j)}(t) \triangleq \tau^{j-(i+1)}\psi_{ij}(t). \tag{3.88}$$

By introducing the notation

$$\mathcal{F}(\boldsymbol{\psi}_t, \boldsymbol{\varphi}^{\tau}(t)) \triangleq \boldsymbol{A}(\boldsymbol{\psi}_t) + \boldsymbol{\mathcal{A}}(\boldsymbol{\varphi}^{\tau}(t))$$

the dynamics equation (3.71) becomes

$$\dot{e}_{\zeta} = \tau\Big(\mathcal{F}(\boldsymbol{\psi}_t, \boldsymbol{\varphi}^{\tau}(t)) - LC\Big)e_{\zeta} + \mathcal{N}_1(e_{\zeta}). \tag{3.89}$$

Notice that the parameter $\boldsymbol{\psi}_t \in \mathcal{H}_{\psi}$ for which the set of vertices, $\mathcal{V}_{\psi}$, is defined in (3.72). From (3.68), the hyper parameter $\boldsymbol{\varphi}^{\tau}(t)$ belongs to a convex set $\mathcal{H}_{\tau}$ for which the set of vertices is defines as

$$\mathcal{V}_{\tau} = \Big\{\varphi \in \mathbb{R}^{\frac{(n-1)(n-2)}{2}} : \varphi_{ij} \in \{\underline{\gamma}_{\psi_{ij}}, \bar{\gamma}_{\psi_{ij}}\},\ j = i+2, \ldots, n; i = 1, \ldots, n-2\Big\}. \tag{3.90}$$

By considering $\underline{\gamma}_{\psi_{ij}} \leq 0$ and $\bar{\gamma}_{\psi_{ij}} \geq 0$, we have the following set inclusion property:

$$\forall \tau \geq 1, \tau_{\max} \geq 1 : \tau \leq \tau_{\max} \Rightarrow \mathcal{H}_{\tau} \subseteq \mathcal{H}_{\tau_{\max}}. \tag{3.91}$$

The above property (3.91) plays an important role in the design procedure. Indeed, it guarantees that if given LMI conditions are feasible in $\mathcal{H}_{\tau_{\max}}$ then they remain feasible in $\mathcal{H}_{\tau}$ for any $1 \leq \tau \leq \tau_{\max}$.

3.3.3.2 Improved design procedure

Now we are ready to state the following theorem providing an improved version of Theorem 4.4.6.

Theorem 3.3.2 ([18]). *Let $\mathcal{P} = \mathcal{P}^{\top} > 0$ and $\mathcal{X}$ be two matrices of appropriate dimensions, and $\tau_{\max} \geq 1$, such that the following LMI conditions are fulfilled:*

$$\mathcal{F}(\varphi, \phi)^{\top} \mathcal{P} + \mathcal{P}\mathcal{F}(\varphi, \phi) - C^{\top}\mathcal{X} - \mathcal{X}^{\top}C + \lambda \mathbb{I}_n < 0, \forall \varphi \in \mathcal{V}_{\boldsymbol{\psi}}, \forall \phi \in \mathcal{V}_{\tau_{\max}}. \tag{3.92}$$

Then the observer (3.64) corresponding to (3.65), with $L = \mathcal{P}^{-1}\mathcal{X}^{\top}$, converges exponentially towards zero for all $\tau \geq 1$ satisfying the double inequality:

$$\boxed{\max\left(1, \frac{2\kappa_1 \lambda_{\max}(\mathcal{P})}{\lambda}\right) < \tau \leq \tau_{\max}} \tag{3.93}$$

Moreover, the estimated state $\hat{x}(t) = \mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}(t)$ exponentially converges to the state $x(t)$ of the original system (3.62).

Proof. From the convexity principle, if (3.92) holds for $\tau_{\max}$, then it holds for any $1 \leq \tau \leq \tau_{\max}$, according to the property (3.91). Hence, since $\mathcal{N}_1(.)$ satisfies (3.73), we can conclude as with the standard high-gain observer.

□

Remark 3.3.3 ([18]). *Compared to Theorem 4.4.6, Theorem 3.3.2 provides a less conservative upper bound on the tuning parameter τ. Due to the use of the LMI approach, the restrictive condition (3.77) vanished from the required design constraints. From numerical viewpoint, the LMI conditions (3.92) often provide high values of $\tau_{\max}$, compared to (3.77).*

3.3.3.3 Numerical design algorithm

This section provides a numerical algorithm to design the observer gain L and the tuning parameter τ. The algorithm consists of finding the maximum value of $\tau_{\max}$ for which the LMIs (3.92) are feasible. Hence, we deduce the parameters λ, P, $\mathcal{X}$, and $\tau_{\max}$ allowing the computation of $L = \mathcal{P}^{-1}\mathcal{X}^\top$ and the value of the tuning parameter τ according to (3.93). The numerical procedure is summarized in the following Algorithm 3.3.3.3. The algorithm is based on the use of the gridding method on $\tau_{\max}$. The gridding method is based on the introduction of a scaled variable $\sigma \triangleq \frac{\tau_{\max}}{1+\tau_{\max}}$, which means that $\tau_{\max} = \frac{\sigma}{1-\sigma}$. Consequently, the maximum value of $\sigma \in [\frac{1}{2}\ 1[$ for which the LMIs (3.92) are feasible will provide the maximum value of $\tau_{\max} \in [1\ +\infty[$ for which (3.92) are feasible.
ruled

Algorithm 3 Finding a maximum value of $\tau_{\max}$

Step 1: Choose a small $\delta > 0$ for the gridding and take $\sigma = \frac{1}{2}$. Then go to **Step 2**
Step 2: Solve LMIs (3.92) with the current value of σ
(3.92) is feasiblego to **Step 3**
if $\sigma > \frac{1}{2}$ **then** go to **Step 4**
if $\sigma = \frac{1}{2}$ **then** go to **Step 5**
Step 3:
while $\sigma + \delta < 1$ **do** $\sigma := \sigma + \delta$ and return to **Step 2**
Step 4: Take $\sigma_{\max} > \frac{1}{2}$ as the maximum value of σ for which LMIs (3.92) are feasible and check condition (3.93) with the corresponding $\tau_{\max} = \frac{\sigma_{\max}}{1-\sigma_{\max}}$ (3.93) is solvable go to **Step 5** go to **Step 6**
Step 5: Choose a value of τ satisfying (3.93) and compute $L = \mathcal{P}^{-1}\mathcal{X}^\top$
Step 6: STOP, return no solution.

3.3.3.4 Illustrative Example

To illustrate the proposed results, let us consider the following simple three-dimensional system:

$$\begin{cases} \dot{x}_1 = x_2 + \kappa_2 \sin(x_3) \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = h(x) \\ y = x_1 \end{cases} \tag{3.94}$$

where $h(.)$ is an arbitrary Lipschitz function with Lipschitz constant $k_h = 1$. Through this example, we provide a numerical comparison between Theorem 4.4.6 and Theorem 3.3.2 with respect to the value of the Lipschitz constant κ_2. Table 4.1 provides the $\kappa_2^{\max}$ tolerated by each method. It

Table 3.1: Numerical comparison

Method	Theorem 4.4.6	Theorem 3.3.2
$\kappa_2^{\max}$	$\approx 10^{-3}$	≈ 0.04

is quite clear from Table 4.1 that the value of $\kappa_2^{\max}$ tolerated by Theorem 3.3.2 is higher than that of Theorem 4.4.6. It is 40 times greater than that of Theorem 4.4.6. The estimation errors are depicted in Figure 3.1 for the case $\kappa_2 = 0.04$ and $h(x(t)) = \cos(x_2(t))$.

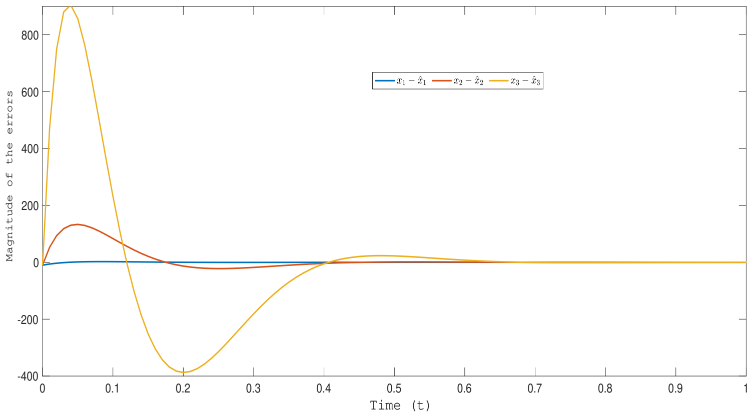

Figure 3.1: Behavior of the estimation error for $\kappa_2 = 0.04$.

3.4 Conclusion

In this paper, we proposed several observer design techniques for nonlinear systems in the presence of delayed and nonlinear outputs. Through a state augmentation technique and output transformation, the problem of the presence of delay and nonlinearities in the output measurement is easily solved by transferring the delay and the nonlinearities to the dynamic process. Such a transfer is achieved by creating a new output measurement and extending the dynamics of the system. For the specific classes of systems considered in this paper, namely systems in companion form and feedforward systems, novel specific synthesis conditions are proposed, which are less conservative than those existing in the literature.

Moreover, a more relaxed and alternative technique is introduced to avoid the conservative assumption regarding the boundedness of x_2. This approach is accompanied by several constructive comments and comparisons. For each class of systems, the new technique using the new observer structure with integral term is thoroughly compared to the method relaxing the boundedness of x_2 and to existing methods in the literature through rigorously expanded analytical arguments. Lastly, utilizing a Lyapunov–Krasovskii functional, we introduce an alternative method requiring only LMI conditions for ensuring the asymptotic convergence of the estimation error. This approach aims to refine existing techniques in the literature while eliminating additional constraints associated with the maximum allowable delay value. The efficiency of the proposed methods is demonstrated through a compelling illustrative example.

CHAPTER 4

Observer Design for Nonlinear Systems with Delayed Output Measurement

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4.1 Introduction

Delayed outputs are naturally encountered in remote estimation [129, 143], cyberattacks detection [58–60, 62], and multiagent systems in general [53, 126, 128]. Estimating the state of a nonlinear system from delayed measurements is a challenging yet critical task in control theory. While traditional observer design methods work well for systems with instantaneous measurements, The presence of delays not only complicates the estimation of the state of the system but also affects the stability and performance of the observer. Several methods have been proposed in the literature to handle this challenge, but many of them are limited in their ability to handle large delays [107].

The main objective of this work is to develop a novel approach for estimating the state of a nonlinear system from delayed measurements. Specifically, we aim to propose a method that can handle arbitrarily long delays in the measurements while ensuring robust and accurate state estimation.

Existing methods in the literature include using a chain of observers or predictors-based observers for handling long delays [3, 23, 34, 46, 52, 72, 88, 140, 142]. For feedforward systems, time-scaling techniques have been proposed, and for systems in triangular form, high-gain observer methodology has been explored. However, these methods have limitations in terms of the maximum allowable value of the delay.

To deal with arbitrarily long delays in the measurement, some methods are based on the use of a chain of observers [34,46] while other methods exploit predictors-based observers [4,72]. For some families of systems, namely feed-forward systems, the problem may be solved by using the time-scaling technique as in [88]. On the other hand, different methods based on the high-gain observer methodology have been proposed for systems in triangular form with some recent improvements. However, these methods are valid for only systems with small values of the upper bound of the delay, $\tau^\star$. The aim is to overcome this limitation and propose a novel approach that will be both simple and enhance the maximum allowable value of $\tau^\star$.

This chapter introduces a new approach designed to overcome the constraints present in existing methodologies. By leveraging a novel output-based dynamic extension technique, we substantially increase the maximum allowable value of the delay. This technique, preliminary presented in [16], allows the transfer of time-delay from the output to the dynamics of an augmented system. As a result, the stability analysis and observer design become more straightforward, consequently enabling to achieve a significantly larger maximum allowable delay value. Furthermore, the proposed approach incorporates the use of the Halanay inequality and the differential mean value theorem, along with other mathematical tools, in the derivation of Lyapunov functions. While existing observer designs for systems with delayed output measurements often focus on specific classes of systems, such as those in companion form or featuring feedforward structures, this work introduces a unified approach applicable to a broader range of systems with arbitrary structures. Additionally, specific results are derived for these particular classes of systems. Due to the dynamic extension technique and system transformations, we propose a novel state observer structure. Through rigorous analytical analysis, we demonstrate that this observer structure significantly enhance the upper bound of the maximum allowable value of the delay compared to conventional state observer structures. However, this enhancement requires fulfilling certain Lipschitz assumptions as a trade-off. For systems in companion form, it is shown that the maximum allowable delay value is increased and correlates explicitly with the tuning parameter of the high-gain observer proposed in this case. In contrast, the proposed design method for feedforward systems enables the accommodation of arbitrarily long delay in the output. Moreover, to avoid the additional constraints required for exponential convergence, which often introduce conservatism, we propose an alternative design approach using a specific Lyapunov–Krasovskii functional. While this method accommodates high values of the maximum allowable delay, it guarantees only asymptotic convergence of the error. An illustrative example showcases how these methods enhance the techniques presented in [119] and [92].

The rest of the chapter is organized as follows: Section 4.2 describes clearly the problem formulation of the observer design for systems with delayed outputs and presents the motivations of this work compared to the available methods in the literature. Section 4.3 provides the main idea of this work, starting by a preliminary result as a main tool. Section 4.4 proposes specific new results for particular families of systems, namely systems in companion form and feedforward systems. Section 4.5 gives a relaxation technique to avoid certain required assumptions on the system state. Section 4.6 introduces some constructive comments and analytical comparisons. Section 4.7 presents an alternative method relaxing the existing results in the literature. Section 4.8 introduces an illustrative example to show the validity and efficiency of the proposed methods. Finally, Section 4.9 concludes the work and discusses future endeavors.

4.2 Problem Formulation and Preliminary Tools

This section is devoted to formulating the observer design problem, namely addressing observer design in the presence of delayed output measurements. We start by defining the class of systems under investigation and revisiting conventional observer design methods available in existing literature. To provide a strong motivation for the results developed in this chapter, we give a succinct overview of the current state-of-the-art methodologies, thereby establishing the basis for the development of our proposed method. Then, before dealing with the main results, we introduce the primary mathematical tools used throughout the Lyapunov stability analysis.

4.2.1 Problem Formulation

The motivation of the work consists of developing a simple method to deal with nonlinear systems with delayed nonlinear output measurements. The aim is to establish novel design conditions allowing high values of the maximum allowable delay in the output measurement while ensuring the exponential convergence of the observer. We stand out from the literature by proposing a simple but useful method. The class of systems we consider is described by the following equations:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h\big(x(t-\tau(t))\big) \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^p$ represents the output measurements vector. The delay $\tau(t) \geq 0$ is assumed to be known and bounded, i.e.: there exists a positive constant $\tau^\star$ such that $\tau(t) \leq \tau^\star, \forall t \geq 0$.
The main objective consists of estimating the system state, $x(t)$, in real-time from the delayed measurements $y(t)$. While the problem in the case of delay-free outputs, is relatively easy to handle, however, the presence of the delay makes the problem challenging.
Usually in the literature, the following Luenberger state observer is proposed for the class of systems (4.1):

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + L\left[y(t) - h\big(\hat{x}(t-\tau(t))\big)\right], \tag{4.2}$$

where $\hat{x}(t)$ is the estimate of $x(t)$ and L is the observer gain to be determined. Several methods have been devoted in the literature for studying the convergence of this observer (4.2) for large values of the maximum allowable delay, $\tau^\star$. Except some results for feedforward systems [88], most of the contributions in this area concerns systems in companion form [3, 23, 34, 46, 72, 140, 142]. To the best of the authors' knowledge, no results exist for systems of arbitrary structure as defined by (4.1). Although recent improvements, as presented in [2], have leveraged the HG/LMI technique introduced in [140], the obtained results remain conservative. Moreover, the convergence analysis, based on Lyapunov theory and the use of the observer (4.2), requires extensive computations and cumbersome mathematical developments. Additionally, the observer gain L explicitly amplifies the delay in the output, requiring the application of the Jensen inequality in mathematical derivations to achieve appropriate synthesis conditions, as suggested by the conditions outlined in [2, Theorem 10], which are strongly interdependent, cumbersome, and conservative. All the aforementioned drawbacks associated with the utilization of observer (4.2) have motivated us to devise a novel approach to address both delay and nonlinearity in output measurements. Specifically, our objective is to develop a unified observer design method that:

- is applicable to systems of arbitrary structure without imposing specific forms;
- involves simple mathematical derivations and relies on less conservative tools, such as the Jensen inequality;
- avoids the need for observer (4.2), wherein the observer gain is directly multiplied by the delayed output, thereby amplifying the delay's effect;
- enhances the design conditions for specific nonlinear systems compared to existing literature.

To achieve the aforementioned advantages, a natural solution is to construct a new output measurement vector. Hence, the concept of employing a dynamic output-based extension technique arises, aimed at generating a novel system with linear and delay-free output measurements.

4.2.2 Preliminary tools

Before stating the main results, we introduce the following simple and well-known mathematical tools. First, we introduce the following Halanay inequality. The next Halanay lemma is important to conclude on the stability conditions. Although an elegant improved variant of Halanay inequality has been proposed in [87], the standard one is enough for the main results of this work.

Lemma 4.2.1 ([50, 87]). *Consider a continuous, piece-wise $\mathcal{C}^1$, and non-negative function ϑ defined in the interval $[-\tau^\star, +\infty)$ such that*

$$\dot{\vartheta}(t) \leq -c_1\vartheta(t) + c_2 \sup_{s\in[t-\tau^\star,t]} \vartheta(s). \tag{4.3}$$

Assume that $c_1 > c_2 > 0$. Then, there exist two scalars $\alpha > 0$ and $\beta > 0$ such that

$$\vartheta(t) \leq \alpha \mathrm{e}^{-\beta t} \sup_{s\in[-\tau^\star,0]} \vartheta(s), \forall t \geq 0. \tag{4.4}$$

Finally, we need the following Lemma 4.2.2.

Lemma 4.2.2. *Let $\phi : \mathbb{I} \to \mathbb{R}$ be a non-negative function, where $\mathbb{I}$ is an interval of $\mathbb{R}$. Then the following identity holds:*

$$\left[\sup_{s\in\mathbb{I}} \big(\phi(s)\big)\right]^2 = \sup_{s\in\mathbb{I}} \big(\phi^2(s)\big). \tag{4.5}$$

4.3 New Observer Design Results

This section is devoted to the main contributions of this chapter. We, first, present a preliminary result on which the main contributions are based. It will be used straightforwardly as a tool to conclude the main results.

4.3.1 Preliminary result as a main tool

Consider the class of systems described by the following equations:

$$\begin{cases} \dot{\zeta}(t) = f_\zeta\left(\zeta(t), u(t)\right) + B_\zeta \displaystyle\int_{t-\tau(t)}^{t} g(\zeta(s), u(s))\mathrm{d}s \\ y_\zeta(t) = C\zeta(t) \end{cases} \tag{4.6}$$

where $\zeta(t) \in \mathbb{R}^{n_\zeta}$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the control input, and $y_\zeta(t) \in \mathbb{R}^{p_\zeta}$ represents the output measurements vector. The delay $\tau(t) \geq 0$ is assumed to be known and bounded, i.e.: there exists a positive constant $\tau^\star$ such that $\tau(t) \leq \tau^\star, \forall t \geq 0$. Without loss of generality, we assume that the functions f_ζ and g are $\gamma_{f_\zeta}-$Lipschitz and γ_g-Lipschitz, respectively, with respect to ζ uniformly on $u(t)$. Assume also that $\zeta(t) = \zeta_0, \forall t \in [-\tau^\star, 0]$.
As a preliminary result, we develop a simple state observer design method for the system (4.6). To this end, we consider the following state observer :

$$\dot{\hat{\zeta}}(t) = f_\zeta\left(\hat{\zeta}(t), u(t)\right) + B_\zeta \int_{t-\tau(t)}^{t} g(\hat{\zeta}(s), u(s))\mathrm{d}s + L\left(y_\zeta(t) - C\hat{\zeta}(t)\right), \tag{4.7}$$

where $L \in \mathbb{R}^{n_\zeta \times p_\zeta}$ is the observer gain matrix to be determined such that the estimation error $\epsilon(t) \triangleq \zeta(t) - \hat{\zeta}(t)$ converges exponentially towards zero. Then, the estimation error dynamics is given as:

$$\dot{\epsilon}(t) = \Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right) - LC\epsilon(t) + B_\zeta \int_{t-\tau(t)}^{t} \Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\mathrm{d}s. \tag{4.8}$$

For the sake of obtaining convenient checkable stability conditions, we have to transform the nonlinear term $\Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right)$ by using Lemma 2.2.6. Then, there exists $\boldsymbol{z}_t \in \mathbf{Co}\left(\zeta(t), \hat{\zeta}(t)\right)$ as in (2.28) such that

$$\Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right) = \nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t)\epsilon(t)$$

where $\nabla_\zeta^{f_\zeta}$ is defined as in (2.30). Notice that here $\boldsymbol{z}_t$ depends on $u(t)$ but for the sake of brevity, we use $\boldsymbol{z}_t$ instead of $\boldsymbol{z}_t(u(t))$. It follows that the error system (4.8) is under the form:

$$\dot{\epsilon}(t) = \left[\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right]\epsilon(t) + B_\zeta \int_{t-\tau(t)}^{t} \Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\mathrm{d}s. \tag{4.9}$$

Since f_ζ is $\gamma_{f_\zeta}-$Lipschitz, then there exist constant matrices $\mathcal{A}_j^\zeta \in \mathbb{R}^{n_\zeta \times n_\zeta}$ and functions $\lambda_j(\boldsymbol{z}_t)$, $j = 1, \ldots \bar{n}_\zeta$ such that the generalized Jacobian $\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t)$ belongs to the convex polytopic set defined as:

$$\mathcal{H}_{f_\zeta} \triangleq \left\{ \sum_{j=1}^{\bar{n}_\zeta} \lambda_j(\boldsymbol{z}_t)\mathcal{A}_j^\zeta, \sum_{j=1}^{\bar{n}_\zeta} \lambda_j(\boldsymbol{z}_t) = 1, \lambda_j(\boldsymbol{z}_t) \geq 0 \right\} \tag{4.10}$$

Notice that the matrices $\mathcal{A}_j^\zeta \in \mathbb{R}^{n_\zeta \times n_\zeta}$, represent the vertices of the polytope $\mathcal{H}_{f_\zeta}$. Also, the jacobian $\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t)$ is affine on the variables $\lambda_j(\boldsymbol{z}_t)$, $j = 1, \ldots \bar{n}_\zeta$.
Before stating the preliminary proposition, notice that since the function g is γ_g-Lipschitz with respect to ζ, then we have

$$\left\|\Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\right\| \leq \gamma_g \|\epsilon(s)\|. \tag{4.11}$$

Now we are ready to state the main theorem based on the use of the standard quadratic Lyapunov function, i.e.: $\vartheta(\epsilon(t)) \triangleq \epsilon^\top(t)\mathcal{P}\epsilon(t)$, where $\mathcal{P} = \mathcal{P}^\top > 0$. We can use a more general Lyapunov function with a matrix $\mathcal{P}(\epsilon(t))$ depending on $\epsilon(t)$, however, we obtain non-constructive conditions difficult to deal with using numerical software algorithms. The second objective of using the standard quadratic Lyapunov function is to compare with available methods in the literature based on the same Lyapunov function.

Theorem 4.3.1 ([16]). *Assume that there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$, and a positive scalar μ such that the following conditions hold:*

$$\left(\mathcal{A}_j^\zeta\right)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_j^\zeta - C^\top \mathcal{R} - \mathcal{R}^\top C + \mu\mathcal{P} \leq 0, j = 1, \dots, \bar{n}_\zeta \tag{4.12a}$$

$$\tau^\star < \frac{\mu\lambda_{\min}(\mathcal{P})}{2\gamma_g\|\mathcal{P}B_\zeta\|} \tag{4.12b}$$

Then the observer (4.7), with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, converges exponentially.

Proof. By computing the derivative of the Lyapunov function $\vartheta(\epsilon(t)) \triangleq \epsilon^\top(t)\mathcal{P}\epsilon(t)$ along the trajectories of (4.9), we obtain

$$\begin{aligned}
\dot{\vartheta}(\epsilon(t)) = {}& \epsilon^\top(t)\left[\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)\right]\epsilon(t) \\
& + 2\epsilon^\top(t)\mathcal{P}B_\zeta \int_{t-\tau(t)}^{t} \Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right) \mathrm{d}s \\
\leq {}& \epsilon^\top(t)\left[\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)\right]\epsilon(t) \\
& + 2\gamma_g\|\mathcal{P}B_\zeta\|\|\epsilon(t)\| \int_{t-\tau^\star}^{t} \|\epsilon(s)\|\mathrm{d}s
\end{aligned} \tag{4.13}$$

Conditions (4.12a) and the convexity principe [32] lead to

$$\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right) \leq -\mu\mathcal{P}. \tag{4.14}$$

On the other hand, from Lemma 4.2.2, we get

$$\begin{aligned}
\|\epsilon(t)\| \int_{t-\tau^\star}^{t} \|\epsilon(s)\|\mathrm{d}s \leq {}& \tau^\star\|\epsilon(t)\| \sup_{s\in[t-\tau^\star,t]} \|\epsilon(s)\| \\
& \leq \tau^\star \left(\sup_{s\in[t-\tau^\star,t]} \|\epsilon(s)\|\right)^2 \\
& = \tau^\star \sup_{s\in[t-\tau^\star,t]} \|\epsilon(s)\|^2 \\
& \leq \frac{\tau^\star}{\lambda_{\min}(\mathcal{P})} \sup_{s\in[t-\tau^\star,t]} \vartheta(\epsilon(s)).
\end{aligned} \tag{4.15}$$

Hence, from (4.14) and (4.15), we deduce that

$$\dot{\vartheta}(\epsilon(t)) \leq -\mu\vartheta(\epsilon(t)) + \frac{2\gamma_g\|\mathcal{P}B_\zeta\|}{\lambda_{\min}(\mathcal{P})}\tau^\star \sup_{s\in[t-\tau^\star,t]} \vartheta(\epsilon(s)). \tag{4.16}$$

Consequently, from (4.12b) and Lemma 4.2.1, there exist two positive scalars α and β such that

$$\vartheta(t) \leq \alpha\mathrm{e}^{-\beta t} \sup_{s\in[-\tau^\star,0]} \vartheta(s), \forall t \geq 0, \tag{4.17}$$

which means that the estimation error $\epsilon(t)$ is exponentially stable. This completes the proof.

□

Remark 4.3.2 ([16]). *We are not interested in the values of α and β. We only need their existence to ensure exponential convergence. To have an idea on the construction of α and β, we refer the reader to [50, 87].*

Remark 4.3.3. *[16] The condition (4.12a) can be converted to LMIs by using the term $\mu\mathbb{I}_{n_\zeta}$ instead of $\mu\mathcal{P}$, i.e:*

$$\left(\mathcal{A}_j^\zeta\right)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_j^\zeta - C^\top\mathcal{R} - \mathcal{R}^\top C + \mu\mathbb{I}_{n_\zeta} \leq 0,\ j = 1, \ldots, \bar{n}_\zeta. \tag{4.18}$$

In this case, the inequality (4.12b) becomes

$$\tau^\star < \frac{\mu\lambda_{\min}(\mathcal{P})}{2\gamma_g\lambda_{\max}(\mathcal{P})\|\mathcal{P}B_\zeta\|}. \tag{4.19}$$

From a numerical point of view, to maximize the value of $\tau^\star$ tolerated by the proposed design conditions (4.18)-(4.19), we can introduce the additional constraint

$$\eta_1\mathbb{I}_{n_\zeta} < \mathcal{P} < \eta_2\mathbb{I}_{n_\zeta} \tag{4.20}$$

to solve jointly with (4.18) while minimizing $\eta_2 > 0$ and maximizing $\mu > 0$ and η_1. This leads, for instance, to solve the following optimization problem:

$$\min_{\mathcal{P}>0;\eta_1>0;\eta_2>0;\mu>0} \left(c_1\eta_2 - c_2\mu - c_3\eta_1\right),\ \textit{subject to}\ (4.18), (4.20) \tag{4.21}$$

where $c_i > 0, i = 1, 2, 3$ are known constant scalars. The user can use advanced multi-objective optimization algorithms to improve the value of $\tau^\star$.

Remark 4.3.4 ([16]). *The detailed feasibility analysis of the conditions (4.12a) in Theorem 4.3.1 is not introduced. Indeed, it is well-known in the literature that this LMI-based technique, called LPV/LMI technique introduced in [131], is less conservative than the other LMI-based methods in the observer design context. Conditions (4.12a) offer the less conservative way to handle Lipschitz nonlinearities, as demonstrated analytically and numerically in [17, 20, 131]. Moreover, it will be shown in the next sections that for some families of nonlinear systems, the feasibility of (4.12a) can always be guaranteed.*

4.3.2 Main result: New observer design technique

In this section, we propose a simple observer design method for the class of systems (4.1) with nonlinear delayed-output measurement, which is the main motivation of this work. As stated in Section 4.2.1, to handle the delay in the output measurements, several techniques have been proposed in the literature. We propose a novel and different observer design technique. To this end, we, first, introduce the new state variable, $z(t) \in \mathbb{R}^{n_z \times n_z}$, defined by:

$$\begin{cases} \dot{z}(t) = f_z(z(t), u(t)) + Y_z y(t) \\ z(0) = z_0, \end{cases} \tag{4.22}$$

where f_z is a known globally Lipschitz function, the matrix $Y_z \in \mathbb{R}^{n_z \times p}$ is known and constant, and $z_0 \in \mathbb{R}^{n_z}$ is a known constant vector. The idea consists in using a state augmentation approach to get a new system for which the created $z(t)$ is the output measurement. Indeed, since $f_z, u(t), Y_z$, and z_0 are all known, then the state $z(t)$ is known in real-time from the measured output $y(t)$

of the original system (4.1). Before introducing the main transformation, notice that from the *Newton-Leibniz* formula, $y(t)$ can be written under the form:

$$y(t) = h(x(t)) - \int_{t-\tau(t)}^{t} \frac{\partial h}{\partial x}\big(x(s)\big) f\big(x(s), u(s)\big)\mathrm{d}s. \tag{4.23}$$

By exploiting (4.22) and (4.23), the system (4.1) can be transformed into the form (4.6) with

$$\zeta(t) \triangleq \begin{bmatrix} z(t) \\ x(t) \end{bmatrix}, y_\zeta \triangleq z(t),\ C \triangleq \begin{bmatrix} \mathbb{I}_{n_z} & 0 \end{bmatrix}, \tag{4.24}$$

$$f_\zeta\big(\zeta(t), u(t)\big) \triangleq \begin{bmatrix} f_z(z(t), u(t)) + Y_z h(x(t)) \\ \\ f(x(t), u(t)) \end{bmatrix}, \tag{4.25}$$

$$g\big(\zeta(t), u(t)\big) \triangleq \frac{\partial h}{\partial x}\big(x(t)\big) f\big(x(t), u(t)\big),\ B_\zeta \triangleq \begin{bmatrix} -Y_z \\ 0 \end{bmatrix} \tag{4.26}$$

Then, we propose the following generalized state observer:

$$\dot{\hat{\zeta}}(t) = f_\zeta\left(\hat{\zeta}(t), u(t)\right) + B_\zeta \int_{t-\tau(t)}^{t} g(\hat{\zeta}(s), u(s))\mathrm{d}s + L\left(y_\zeta(t) - C\hat{\zeta}(t)\right) \tag{4.27a}$$

$$\hat{x}(t) = \begin{bmatrix} 0 & \mathbb{I}_n \end{bmatrix} \hat{\zeta}(t). \tag{4.27b}$$

Before summarizing the result in a corollary, we need the following assumption.

Assumption 4.3.5. *The function g defined in (4.26) is γ_g–Lipschitz with respect to $x(t)$, uniformly on $u(t)$.*

Now all the conditions to apply Theorem 4.3.1 are satisfied, we can summarize the result in the following corollary. [[16]] Consider the system (4.6) with the parameters and functions given in (4.24)–(4.26). Assume that Assumption 4.3.5 is satisfied and there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$, and a positive scalar μ such that the conditions (4.12a)–(4.12b) hold. Let $L = \mathcal{P}^{-1}\mathcal{R}^\top$ be the gain matrix of (4.27a). Then, the estimated state $\hat{x}(t)$ given by (4.27b) converges exponentially to the state $x(t)$ of the original system (4.1).

Remark 4.3.6 ([16]). *Without loss of generality, we consider in (4.22) a linear function f_z depending on $z(t)$ only, i.e: $f_z(z(t), u(t)) = A_z z(t)$. Even, for simplification, we can take $f_z(z(t), u(t)) \equiv 0$. In addition, these considerations allow reducing the dimension of the corresponding polytopic set $\mathcal{H}_{f_\zeta}$, which reduces then the number of LMIs (4.12a) to solve.*

4.4 Results for Particular Families of Systems

This section considers the high-gain observer and its robustness with respect to the delay in the output measurement. Although several techniques have been proposed in the literature for this class of systems, we show that our method is simple and applies straightforwardly to this class of systems under the companion form.

4.4.1 System description and assumptions

Without loss of generality, we consider the family of systems described by (4.1) with

$$\begin{aligned} f(x(t),u(t)) &= Ax(t) + Bf_x\big(x(t)\big) \\ h\big(x(t-\tau(t))\big) &= h_x\big(C_x x(t-\tau(t))\big) \\ (A)_{ij} &= \begin{cases} 1 & \text{if} \quad j = i+1 \\ 0 & \text{if} \quad j \neq i+1 \end{cases} \\ C_x &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^\top . \end{aligned} \tag{4.28}$$

where f_x and h_x are $\gamma_{f_x}-$Lipschitz and $\gamma_{h_x}-$Lipschitz, respectively, with respect to their arguments. For the sake of observability, the following assumption is necessary.

Assumption 4.4.1. *There exists $\delta_h > 0, \delta_h \leq \gamma_{h_x}$ such that*

$$\delta_h \leq \frac{\partial h_x}{\partial v}(v) \leq \gamma_{h_x}, \ \forall v \in \mathbb{R}. \tag{4.29}$$

Without loss of generality, for the sake of simplification, we assume that

$$\gamma_{h_x} \triangleq \sup_{v\in\mathbb{R}} \left(\frac{\partial h_x}{\partial v}(v) \right) = 1. \tag{4.30}$$

Otherwise, we use for the observer the output

$$y_{\text{new}}(t) \triangleq \frac{y(t)}{\sup\limits_{v\in\mathbb{R}} \left(\frac{\partial h_x}{\partial v}(v) \right)}. \tag{4.31}$$

As in the previous section, we create the following new variable, $z(t) \in \mathbb{R}$, as in (4.22):

$$\begin{cases} \dot{z}(t) = \gamma y(t) \\ z(t) = z(0) = z_0, \forall t \in [-t^\star, 0]. \end{cases} \tag{4.32}$$

where $\gamma > 0$ is a constant scalar, which is considered a tuning parameter. Also, in this case, from the *Newton-Leibniz* formula, (4.23) is reduced to

$$y(t) = h(x(t)) - \int_{t-\tau(t)}^{t} \frac{\partial h_x}{\partial x_1}\big(x_1(s)\big) x_2(s) \mathrm{d}s. \tag{4.33}$$

Remark 4.4.2 ([16])**.** *Notice that in the case of linear output $y(t) = Cx(t - \tau(t))$, which is often encountered in real application models, the identity (4.33) becomes*

$$y(t) = x_1(t) - \int_{t-\tau(t)}^{t} x_2(s) \mathrm{d}s. \tag{4.34}$$

In this case, Assumption 4.3.5 is clearly satisfied globally since $g(x(t)) = x_2(t)$ is linear.

Now, let us go back to (4.33). It is quite clear that the function

$$\phi(x_1, x_2) \triangleq \frac{\partial h_x}{\partial x_1}\big(x_1\big) x_2 \tag{4.35}$$

is globally Lipschitz with respect to x_2 because h_x is $\gamma_{h_x}-$Lipschitz, and then $\frac{\partial h_x}{\partial x_1}(x_1)$ is bounded. However, it is not globally Lipschitz with respect to x_1. Then, we need an additional assumption on x_2 that we will consider in the next theorem.

As in the previous section, we consider the transformed system (4.6) with the parameters (4.24)–(4.26) with $Y_z = \gamma$, and according to (4.28) as follows:

$$\begin{cases} \dot{\zeta}(t) = f_\zeta\left(\zeta(t)\right) + B_\zeta \displaystyle\int_{t-\tau(t)}^{t} g(\zeta(s))\mathrm{d}s \\ y_\zeta(t) = C\zeta(t) \end{cases} \tag{4.36}$$

where $g(\zeta) = \phi(\zeta_2, \zeta_3)$.

4.4.2 System transformation: High-gain observer

By construction of the corresponding augmented system (4.36), without the integral term, the triangular companion form is preserved. Then, we can apply the high-gain observer methodology. To this end, we perform a second transformation, which is usual in this context, although it is often applied to the error system. Let us introduce the following linear transformation:

$$\xi = \mathbb{T}_\theta \zeta, \text{ where } \mathbb{T}_\theta \triangleq \operatorname{diag}\left(\frac{1}{\theta}, \dots, \frac{1}{\theta^{n+1}}\right) \tag{4.37}$$

which transforms (4.36) into

$$\begin{cases} \dot{\xi}(t) = \mathbb{T}_\theta f_\zeta\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) + B_\zeta \displaystyle\int_{t-\tau(t)}^{t} g\left(\mathbb{T}_{\frac{1}{\theta}}\xi(s)\right)\mathrm{d}s \\ y_\zeta(t) = C\mathbb{T}_{\frac{1}{\theta}}\xi(t) \end{cases} \tag{4.38}$$

To make the developments easy to follow and for any convenience, we express the system (4.38) in the following detailed form:

$$\begin{cases} \dot{\xi}(t) = \begin{bmatrix} \frac{\gamma}{\theta} h_x(\theta^2\xi_2(t)) \\ \\ \theta[0 \;\; A]\xi(t) \end{bmatrix} + \dfrac{1}{\theta^{n+1}}\begin{bmatrix} 0 \\ B \end{bmatrix} f_x\left([0 \;\; \mathbb{I}_n]\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) \\ \qquad\quad + \dfrac{1}{\theta} B_\zeta \displaystyle\int_{t-\tau(t)}^{t} \phi\big(\theta^2\xi_2(s), \theta^3\xi_3(s)\big)\mathrm{d}s \\ \\ y_\zeta(t) = \theta C\xi(t) = \theta\xi_1(t) \end{cases} \tag{4.39}$$

Before summarizing the result, let us define the function $\breve{g}$ as the Lipschitz extension of ϕ introduced in (4.35):

$$\breve{g}(z_1, z_2) = \phi\big(z_1, \pi_{\mathcal{I}}(z_2)\big) \tag{4.40}$$

where $\pi_{\mathcal{I}}(z_2)$ stands for the *Hilbert* projection of z_2 on $\mathcal{I}$ for any closed interval $\mathcal{I} \subset \mathbb{R}$. Define also the matrix $\mathcal{A}_\gamma(.)$ as

$$\mathcal{A}_\gamma(\boldsymbol{v}) \triangleq \begin{bmatrix} 0 & [\gamma\boldsymbol{v} \quad 0_{1\times n-1}] \\ 0_{n\times 1} & A \end{bmatrix}, \; \forall \boldsymbol{v} \in \mathbb{R}. \tag{4.41}$$

Theorem 4.4.3 ([16]). *Assume that the component $x_2(t)$ of the system (4.1), with (4.28), belongs to a compact interval $\mathcal{I} \subset \mathbb{R}$ and that the function ϕ defined in (4.35) is Lipschitz in $\mathbb{R} \times \mathcal{I}$. Assume there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p\times n+1}$, and positive scalars μ and γ such that the following conditions hold:*

$$\left(\mathcal{A}_\gamma(\ell)\right)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_\gamma(\ell) - C^\top\mathcal{R} - \mathcal{R}^\top C + \mu\mathbb{I}_{n+1} \leq 0, \tag{4.42a}$$

$$\ell \in \{\delta_h, 1\} \tag{4.42b}$$

$$\theta > \max\left(1, \frac{2\kappa_f \lambda_{\max}(\mathcal{P})}{\mu}\right) \tag{4.42c}$$

$$\tau^\star < \frac{\lambda_{\min}(\mathcal{P})\left[\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\right]}{2\theta^2\gamma\kappa_g\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|} \tag{4.42d}$$

Then the output $\hat{x}(t)$ of the following observer

$$\begin{aligned}\dot{\hat{\xi}}(t) &= \mathbb{T}_\theta f_\zeta\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right) + B_\zeta \int_{t-\tau(t)}^{t} \breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right)\mathrm{d}s \\ &\quad + L\left(y_\zeta(t) - C\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right) \end{aligned} \tag{4.43a}$$

$$\hat{x}(t) = \begin{bmatrix}0_{n\times 1} & \mathbb{I}_n\end{bmatrix}\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t) \tag{4.43b}$$

with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, converges exponentially to the state $x(t)$ of the original system (4.1) with the particular parameters in (4.28).

Proof. It is sufficient to show that $\tilde{\xi}(t) \triangleq \xi(t) - \hat{\xi}(t)$ is exponentially stable. Then, the dynamics of the estimation error is given by:

$$\begin{aligned}\dot{\tilde{\xi}}(t) &= \theta\left[\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right]\tilde{\xi}(t) \\ &\quad + \left(\psi(\xi(t)) - \psi(\hat{\xi}(t))\right) \\ &\quad + B_\zeta\int_{t-\tau(t)}^{t}\left[g\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) - \breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right)\right]\mathrm{d}s \end{aligned} \tag{4.44}$$

where

$$\psi(\xi(t)) \triangleq \frac{1}{\theta^{n+1}}\begin{bmatrix}0\\B\end{bmatrix} f_x\left([0\ \ \mathbb{I}_n]\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) \tag{4.45}$$

and $h_x(\theta^2\xi_2(t)) - h_x(\theta^2\hat{\xi}_2(t)) = \theta^2\dfrac{\partial h_x}{\partial v}(w(t))\tilde{\xi}_2(t)$, $w(t) \in$ **Co** $\left(\theta^2\xi_2(t), \theta^2\hat{\xi}_2(t)\right)$, from the differential mean value theorem in Lemma 2.2.6 applied to the scalar function h_x.
In addition, since f_x is $\gamma_{f_x}-$Lipschitz and from the structure of ψ in (4.45), there exists a constant $\kappa_f \geq \gamma_{f_x}$, independent from θ, such that

$$\left\|\psi(\xi(t)) - \psi(\hat{\xi}(t))\right\| \leq \kappa_f\|\tilde{\xi}(t)\|. \tag{4.46}$$

Since $\theta^3\xi_3(t) = x_2(t) \in \mathcal{I}$ and the *Hilbert* projection preserves the Lipchitz constant in $\mathbb{R}^2$ and from the structure of g in (4.39), there exists $\kappa_g \geq \gamma_g$ such that

$$\left\|g\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) - \breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right)\right\| \leq \kappa_g\theta^2\|\tilde{\xi}(t)\|. \tag{4.47}$$

Now, after computing the derivative of $\vartheta(\tilde{\xi}(t)) \triangleq \tilde{\xi}^\top(t)\mathcal{P}\tilde{\xi}(t)$ along the trajectories of (4.44), and by considering the bounds (4.46)-(4.47) and writing $B_\zeta = -\gamma C^\top$, we get

$$\begin{aligned}\dot{\vartheta}(\tilde{\xi}(t)) \leq \tilde{\xi}^\top(t)&\left[\left(\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right)^\top\mathcal{P}\right. \\ &\left.+\mathcal{P}\left(\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right)\right]\tilde{\xi}(t) \\ &+ 2\theta^2\gamma\kappa_g\|\mathcal{P}C^\top\|\|\tilde{\xi}(t)\|\int_{t-\tau^\star}^{t}\|\tilde{\xi}(s)\|\mathrm{d}s \\ &+ 2\kappa_f\lambda_{\max}(\mathcal{P})\|\tilde{\xi}(t)\|^2. \end{aligned} \tag{4.48}$$

It follows from (4.42a) and the convexity principle that

$$\begin{aligned}\dot{\vartheta}(\tilde{\xi}(t)) \leq & -\frac{\Big(\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\Big)}{\lambda_{\max}(\mathcal{P})}\vartheta(\tilde{\xi}(t)) \\ & + \tau^\star\frac{2\theta^2\gamma\kappa_g\|\mathcal{P}C^\top\|}{\lambda_{\min}(\mathcal{P})}\sup_{s\in[t-\tau^\star,t]}\vartheta(\tilde{\xi}(s))\end{aligned} \tag{4.49}$$

Then, according to Lemma 4.2.1 and $\theta \geq 1$, the exponential convergence of $\tilde{\xi}(t)$ is inferred if

$$\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P}) > 0$$

and

$$\frac{\Big(\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\Big)}{\lambda_{\max}(\mathcal{P})} > \tau^\star\frac{2\theta^2\gamma\kappa_g\|\mathcal{P}C^\top\|}{\lambda_{\min}(\mathcal{P})}$$

which are equivalent to (4.42c) and (4.42d), respectively.

□

Remark 4.4.4 ([16]). *Notice that for systems with linear output, i.e: $h_x(x_1) = Cx$, the boundedness of $x_2(t)$ is not needed because the function g does not depend on x_1, then it is globally Lipschitz on x_1.*

Remark 4.4.5 ([16]). *Depending on the user's needs and the model at hand, the condition (4.42c) may be relaxed by applying the HG/LMI-based observer proposed in [140]. This leads to a smaller value of θ, which allows improving the bound of the tolerated value of $\tau^\star$.*

4.4.3 Results for feedforward systems: Low-gain observer

For this particular class of systems, we show as in [88] that the proposed method works for arbitrary long delays in the output measurement. To avoid repetition and cumbersome notations, in this section, we present the results without detailing the mathematical developments. The class of feedforward systems is described as (4.28) by replacing the matrix B and the function f_x by the following:

$$B \triangleq \begin{bmatrix}\mathbb{I}_{n-2}\\ 0\end{bmatrix},\ f_x(x) \triangleq \begin{bmatrix} f_{x_1}(x_3,\ldots,x_n)\\ \vdots \\ f_{x_{n-2}}(x_n)\end{bmatrix}. \tag{4.50}$$

We perform exactly the same transformation as in Section 4.4.2 but with $\theta \leq 1$ instead of $\theta \geq 1$. The only difference we get with the new structure of f_x in (4.50) (or ψ in (4.45)) is the inequality (4.46) which should be replaced by

$$\Big\|\psi(\xi(t)) - \psi(\hat{\xi}(t))\Big\| \leq \theta^2\kappa_f\|\tilde{\xi}(t)\|. \tag{4.51}$$

This difference leads to novel conditions which are always feasible for arbitrarily high values of $\tau^\star$. We summarize the new conditions in the next theorem.

Theorem 4.4.6 ([16]). *Assume that the component $x_2(t)$ of the system (4.1), with (4.28) and (4.50), belongs to a compact interval $\mathcal{I} \subset \mathbb{R}$ and that the function ϕ defined in (4.35) is Lipschitz in $\mathbb{R}\times\mathcal{I}$. Assume there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n+1\times n+1}$, a matrix $\mathcal{R} \in \mathbb{R}^{p\times n+1}$, and positive scalars μ and γ such that the following conditions hold:*

$$\left(\mathcal{A}_\gamma(\ell)\right)^\top\mathcal{P} + \mathcal{P}\mathcal{A}_\gamma(\ell) - C^\top\mathcal{R} - \mathcal{R}^\top C + \mu\mathbb{I}_{n+1} \leq 0, \ell \in \{\delta_h, 1\}. \tag{4.52}$$

Then, $\forall \tau^\star \geq 0$, there exists $\theta_{\tau^\star} > 0$ defined by

$$\theta_{\tau^\star} \triangleq \min\left(1, \frac{\mu}{2\kappa_f \lambda_{\max}(\mathcal{P})}, \frac{\lambda_{\min}(\mathcal{P})\mu}{2\tau^\star \lambda_{\max}(\mathcal{P})\left[\gamma\kappa_g \|\mathcal{P}C^\top\| + \kappa_f\right]}\right) \tag{4.53}$$

such that the output $\hat{x}(t)$ of the following observer

$$\begin{aligned} \dot{\hat{\xi}}(t) = {} & \mathbb{T}_\theta f_\zeta\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right) + B_\zeta \int_{t-\tau(t)}^{t} \breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right) \mathrm{d}s \\ & + L\left(y_\zeta(t) - C\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right) \end{aligned} \tag{4.54a}$$

$$\hat{x}(t) = \begin{bmatrix} 0_{n\times 1} & \mathbb{I}_n \end{bmatrix} \mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t) \tag{4.54b}$$

with $L = \mathcal{P}^{-1}\mathcal{R}^\top$ and $\theta < \theta_{\tau^\star}$, converges exponentially to the state $x(t)$ of the original system (4.1) with the parameters and functions in (4.28) and (4.50).

Proof. The proof follows the same steps as the Theorem 4.4.3, except the inequality (4.49) which becomes

$$\begin{aligned} \dot{\vartheta}(\tilde{\xi}(t)) \leq {} & -\frac{\left(\mu\theta - 2\theta^2\kappa_f\lambda_{\max}(\mathcal{P})\right)}{\lambda_{\max}(\mathcal{P})}\vartheta(\tilde{\xi}(t)) \\ & + \tau^\star \frac{2\theta^2\gamma\kappa_g\|\mathcal{P}C^\top\|}{\lambda_{\min}(\mathcal{P})} \sup_{s\in[t-\tau^\star,t]} \vartheta(\tilde{\xi}(s)) \end{aligned} \tag{4.55}$$

due to (4.51).
Then, according to Lemma 4.2.1 and $\theta \leq 1$, the exponential convergence of $\tilde{\xi}(t)$ is inferred if

$$\mu\theta - 2\theta^2\kappa_f\lambda_{\max}(\mathcal{P}) > 0 \tag{4.56}$$

and

$$\frac{\left(\mu\theta - 2\theta^2\kappa_f\lambda_{\max}(\mathcal{P})\right)}{\lambda_{\max}(\mathcal{P})} > \tau^\star \frac{2\theta^2\gamma\kappa_g\|\mathcal{P}C^\top\|}{\lambda_{\min}(\mathcal{P})} \tag{4.57}$$

which means that we have respectively

$$\theta < \frac{\mu}{2\kappa_f\lambda_{\max}(\mathcal{P})}$$

and

$$\theta < \frac{\lambda_{\min}(\mathcal{P})\mu}{2\tau^\star\lambda_{\max}(\mathcal{P})\left[\gamma\kappa_g\|\mathcal{P}C^\top\| + \kappa_f\right]}.$$

Consequently, for $L = \mathcal{P}^{-1}\mathcal{R}^\top$ and $\theta < \theta_{\tau^\star}$, where $\theta_{\tau^\star}$ is defined by (4.53), the exponential convergence is inferred. This completes the proof.

□

4.5 Relaxation of the Boundedness of x_2

The main drawback of the previous results is the Lipschitz assumption of the function g defined in (4.26), respectively, according to the corresponding case. In the case of Section 4.4.2 and Section 4.4.3, this requires the boundedness of the state $x_2(t)$ of the original system and the Lipchitz assumption on the function ϕ defined in (4.35). Although these conditions vanish in the case of linear outputs, however, such assumptions are conservative in general. The result of Section 4.3.2 is a straightforward consequence of the preliminary results of Section 4.3.1. Nevertheless, to avoid the previous strong assumptions, we can proceed differently. The idea consists of applying the *Newton-Leibniz* formula on the error system instead of on the output as in (4.23) and (4.33). To this end, we have to transform the original system (4.1) into the following one:

$$\begin{cases} \dot{\zeta}(t) = f_\zeta\left(\zeta(t), \zeta(t-\tau(t)), u(t)\right) \\ y_\zeta(t) = C\zeta(t) \end{cases} \tag{4.58}$$

and apply the following observer:

$$\dot{\hat{\zeta}}(t) = f_\zeta\left(\hat{\zeta}(t), \hat{\zeta}(t-\tau(t)), u(t)\right) + L\left(y_\zeta(t) - C\hat{\zeta}(t)\right). \tag{4.59}$$

Before stating the results, to avoid repetitions and cumbersome notations, it is worth noting that the developments of this section are introduced without detailing the computations. For the sake of organization and comparison with the results from the previous section, we present the findings of this section on a case-by-case basis, starting with the general case, followed by specific cases, namely companion systems and feedforward systems. This allows for clear and analytical comparisons with the previous results, thereby avoiding the need for comparisons through numerical examples.

4.5.1 General case

The transformation (4.58) can be obtained from the output-based dynamic extension technique, as in Section 4.3.2, by considering the augmented system:

$$\zeta(t) \triangleq \begin{bmatrix} z(t) \\ x(t) \end{bmatrix}, C \triangleq \begin{bmatrix} \mathbb{I}_{n_z} & 0 \end{bmatrix}, \tag{4.60}$$

$$\begin{aligned} f_\zeta\big(\zeta(t), \zeta(t-\tau(t)), u(t)\big) &\triangleq \begin{bmatrix} Y_z h(x(t-\tau(t))) \\ \\ f(x(t), u(t)) \end{bmatrix}, \\ &= A_\zeta f(x(t), u(t)) + B_\zeta h(x(t-\tau(t))) \end{aligned} \tag{4.61}$$

where for simplicity, the vector $z(t)$ is defined by

$$\begin{cases} \dot{z}(t) = Y_z y(t) \\ z(0) = z_0. \end{cases} \tag{4.62}$$

As a result, system (4.58) constitutes a time-delay system with output measurements free from delay. Consequently, the Luenberger observer (4.59) is employed without an integral term. Notably, contrarily to the standard observer structure discussed in Section 4.2.1, the observer gain, L, does not directly affect the original output measurements by multiplication, as the delay is shifted to the function f_ζ. This technique, unlike the results in [2], reduces the observer's sensitivity to time-delay and simplifies the complexity of both mathematical derivations and design conditions. The analytical comparisons presented in the next section validate this assertion. These comparisons show also that despite the simplicity of the observer (4.59), the observer (4.27) with integral term leads to a less conservative bound on the maximum allowable value of the delay.
From Lemma 2.2.6, there exist $\boldsymbol{z}_t$ and $\bar{\boldsymbol{z}}_t$ in $\mathbb{R}^n$ such that

$$f_\zeta\big(\zeta(t), \zeta(t-\tau(t)), u(t)\big) = A_\zeta \nabla_x^f(\boldsymbol{z}_t)\epsilon_x(t) + B_\zeta \nabla_x^h(\bar{\boldsymbol{z}}_t)\epsilon_x(t-\tau(t)) \tag{4.63}$$

where $\epsilon_x(s) \triangleq x(s) - \hat{x}(s)$. By taking

$$\nabla_x^{f,h}(\boldsymbol{z}_t, \bar{\boldsymbol{z}}_t) \triangleq \Big(A_\zeta \nabla_x^f(\boldsymbol{z}_t) + B_\zeta \nabla_x^h(\bar{\boldsymbol{z}}_t)\Big) \begin{bmatrix} 0 & \mathbb{I}_n \end{bmatrix} \tag{4.64}$$

the estimation error dynamics is expressed as follows:

$$\dot{\epsilon}(t) = \Big[\nabla_x^{f,h}(\boldsymbol{z}_t, \bar{\boldsymbol{z}}_t) - LC\Big]\epsilon(t) + B_\zeta \nabla_x^h(\bar{\boldsymbol{z}}_t) \begin{bmatrix} 0 & \mathbb{I}_n \end{bmatrix} \Big(\epsilon(t-\tau(t)) - \epsilon(t)\Big) \tag{4.65}$$

where $\epsilon(s) \triangleq \zeta(s) - \hat{\zeta}(s)$. Therefore, we can summarize the result in the following well-structured Proposition 4.5.1. Before stating the proposition, notice that by construction of the dynamic extension technique, f_ξ, in (4.61), similar to, the jacobian $\nabla_x^{f,h}(\boldsymbol{z}_t, \bar{\boldsymbol{z}}_t)$, introduced in (4.64), belongs to the convex bounded set $\mathcal{H}_{f_\zeta}$ defined in (4.10).

Proposition 4.5.1 ([16]). *Assume that there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$, and a positive scalar μ such that the conditions (4.12a) hold. Then the observer (4.59), with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, corresponding to (4.58) and (4.61)-(4.62) , converges exponentially if the maximum value of the delay, $\tau^\star$, satisfies the following bound:*

$$\tau^\star < \frac{\mu \lambda_{\min}(\mathcal{P})}{2\gamma_h \|\mathcal{P}B_\zeta\| \sqrt{\|Y_z\|^2\gamma_h^2 + \gamma_f^2 + \|LC\|}}. \tag{4.66}$$

Proof. From the convexity principle, inequalities (4.12a) imply the following inequality:

$$\Big(\nabla_x^{f,h}(\boldsymbol{z}_t, \bar{\boldsymbol{z}}_t) - LC\Big)^\top \mathcal{P} + \mathcal{P}\Big(\nabla_x^{f,h}(\boldsymbol{z}_t, \bar{\boldsymbol{z}}_t) - LC\Big) \leq -\mu\mathcal{P}. \tag{4.67}$$

It follows that the derivative of the Lyapunov function

$$\vartheta(\epsilon(t)) \triangleq \epsilon^\top(t)\mathcal{P}\epsilon(t)$$

satisfies the inequality

$$\dot{\vartheta}(\epsilon(t)) \leq -\mu\vartheta(\epsilon(t)) + \Theta(\mathcal{P}, L, Y_z)\,\tau^\star \sup_{s\in[t-2\tau^\star, t]} \vartheta(\epsilon(s)) \tag{4.68}$$

where

$$\Theta(\mathcal{P}, L, Y_z) \triangleq \frac{2\gamma_h \|\mathcal{P}B_\zeta\| \sqrt{\|Y_z\|^2\gamma_h^2 + \gamma_f^2 + \|LC\|}}{\lambda_{\min}(\mathcal{P})}. \tag{4.69}$$

The bound (4.68)–(4.69) is obtained after developing the computations, applying Lemma 4.2.2, and then by using the *Newton-Leibniz* formula on the term $\epsilon(t-\tau(t)) - \epsilon(t)$. The details to prove (4.68)–(4.69) are omitted to avoid cumbersome repetitions. Consequently, from Lemma 4.2.1, the estimation error converges exponentially if

$$\Theta(\mathcal{P}, L, Y_z)\,\tau^\star < \mu \tag{4.70}$$

which is equivalent to (4.66).

□

4.5.2 Companion system (4.28)

For particular structures of systems, the bound (4.66) can be simplified. This is mainly the case of systems under the companion form (4.28). Without expanding the developments, by following similar steps as in Section 4.2 and Section 4.5, we summarize the results. First, the transformation (4.37) converts the system (4.58) into the following one:

$$\begin{cases} \dot{\xi}(t) = \mathbb{T}_\theta f_\varsigma \left(\mathbb{T}_{\frac{1}{\theta}}\xi(t), \mathbb{T}_{\frac{1}{\theta}}\xi(t-\tau(t))\right) \\ y_\varsigma(t) = \theta C\xi(t) \end{cases} \tag{4.71}$$

with the corresponding observer

$$\dot{\hat{\xi}}(t) = \mathbb{T}_\theta f_\varsigma \left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t), \mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t-\tau(t))\right) + L\left(y_\varsigma(t) - \theta C\hat{\xi}(t)\right) \tag{4.72a}$$

$$\hat{x}(t) = \begin{bmatrix} 0_{n\times 1} & \mathbb{I}_n \end{bmatrix} \mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t) \tag{4.72b}$$

where $z(t)$ is defined in this case by (4.32).
By considering the same notations as in Section 1.3, with the new observer (4.72a), and taking into account (4.30), we get the following proposition.

Proposition 4.5.2 ([16]). *Let h_x satisfy Assumption 4.4.1 and (4.30), and f_x is γ_{f_x}–Lipschitz. Assume there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\varsigma \times n_\varsigma}$, a matrix $\mathcal{R} \in \mathbb{R}^{p\times n+1}$, and positive scalars μ and γ such that the conditions (4.42a)–(4.42c) hold. If the following condition is satisfied*

$$\tau^\star < \frac{\lambda_{\min}(\mathcal{P})\left[\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\right]}{\theta^2\sqrt{8}\gamma\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\| \max(1,|l_2|)} \tag{4.73}$$

where l_2 is the second component of the gain L, then the state $\hat{x}(t)$ given by (4.72b), with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, converges exponentially to the state $x(t)$ of the original system (4.1) with the parameters and functions in (4.28).

Proof. To avoid cumbersome repetitions, the proof is simplified. It uses the results and definitions of the previous sections without repetitions. The estimation error dynamic satisfies the following equation:

$$\begin{aligned} \dot{\tilde{\xi}}(t) = {} & \theta\left[\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right]\tilde{\xi}(t) \\ & + \left(\psi(\xi(t)) - \psi(\hat{\xi}(t))\right) \\ & + \gamma\theta^2\frac{\partial h_x}{\partial x_1}(w(t))C^\top \int_{t-\tau(t)}^{t} \overbrace{\left[\tilde{\xi}_3(s) - l_2\tilde{\xi}_1(s)\right]}^{\dot{\tilde{\xi}}_2(s)} \mathrm{d}s \end{aligned} \tag{4.74}$$

where ψ is defined in (4.45) and satisfies the Lipschitz inequality (4.46), and $\tilde{\xi}(t) = \xi(t) - \hat{\xi}(t)$. Then, by expanding the derivative of the Lyapunov function $\vartheta(\tilde{\xi}(t)) = \tilde{\xi}(t)^\top\mathcal{P}\tilde{\xi}(t)$, and from (4.42a)–(4.42c) and (4.46), we obtain:

$$\begin{aligned} \dot{\vartheta}(\tilde{\xi}(t)) \leq {} & -\frac{\left(\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\right)}{\lambda_{\max}(\mathcal{P})}\vartheta(\tilde{\xi}(t)) \\ & + \tau^\star\frac{2\sqrt{2}\theta^2\gamma\|\mathcal{P}C^\top\|\max(1,|l_2|)}{\lambda_{\min}(\mathcal{P})} \sup_{s\in[t-\tau^\star,t]} \vartheta(\tilde{\xi}(s)) \end{aligned} \tag{4.75}$$

which implies that the estimation error is asymptotically stable if $\tau^\star$ satisfies (4.73). This ends the proof.

□

4.5.3 Feedforward system (4.28)/(4.50)

By analogy to the previous sections, for this class of systems, we get similar results but with different bounds on $\tau^\star$. We consider the class of systems (4.1) described by (4.28) and (4.50), which is transformed through (4.37) into (4.71) with the corresponding state observer (4.72a). The only difference compared to the previous section lies in the structure of f_ζ which leads to different stability conditions. Then, we get the following proposition.

Proposition 4.5.3 ([16]). *Assume there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p \times n+1}$, and positive scalars μ and γ such that the conditions (4.52) hold. Then, $\forall \tau^\star \geq 0$, there exists $\theta_{\tau^\star} > 0$ defined by*

$$\theta_{\tau^\star} \triangleq \min\left(1, \frac{\mu}{2\kappa_f \lambda_{\max}(\mathcal{P})}, \frac{\lambda_{\min}(\mathcal{P})\mu}{2\tau^\star \lambda_{\max}(\mathcal{P})\left[\kappa_f + \sqrt{3}\gamma \|\mathcal{P}C^\top\| \max(1, |l_2|, \kappa_f)\right]}\right) \tag{4.76}$$

such that the observer (4.72a) corresponding to (4.71) under the structure (4.28) and (4.50), with $L = \mathcal{P}^{-1}\mathcal{R}^\top$ and $\theta < \theta_{\tau^\star}$, converges exponentially.

Proof. The proof follows the same steps as in the previous section for feedforward systems. Indeed, in this case, the estimation error dynamic is given as follows:

$$\begin{aligned}\dot{\tilde{\xi}}(t) = \theta \left[\mathcal{A}_\gamma \left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right] \tilde{\xi}(t) + \left(\psi(\xi(t)) - \psi(\hat{\xi}(t))\right) \\ + \gamma\theta \frac{\partial h_x}{\partial x_1}(w(t)) C^\top \int_{t-\tau(t)}^{t} \dot{\tilde{\xi}}_2(s)\mathrm{d}s\end{aligned} \tag{4.77}$$

where

$$\dot{\tilde{\xi}}_2(s) = \theta\left(\tilde{\xi}_3(s) - l_2\tilde{\xi}_1(s)\right) + \frac{1}{\theta^2}\left(\psi_2(\xi(t)) - \psi_2(\hat{\xi}(t))\right)$$

and ψ satisfies the Lipschitz inequality (4.51), with now $\theta < 1$. We can show easily that $\dot{\tilde{\xi}}_2(s)$ satisfies the inequality

$$\|\dot{\tilde{\xi}}_2(s)\| \leq \theta\sqrt{3}\max(1, |l_2|, \kappa_f)\|\tilde{\xi}(s)\|. \tag{4.78}$$

Therefore, by following similar steps as in the previous sections, we can show that if the LMIs (4.52) hold, then the derivative of the Lyapunov function $\vartheta(\tilde{\xi}(t)) = \tilde{\xi}(t)^\top \mathcal{P}\tilde{\xi}(t)$ satisfies the following inequality:

$$\begin{aligned}\dot{\vartheta}(\tilde{\xi}(t)) \leq -\frac{\left(\mu\theta - 2\theta^2\kappa_f\lambda_{\max}(\mathcal{P})\right)}{\lambda_{\max}(\mathcal{P})}\vartheta(\tilde{\xi}(t)) \\ + \tau^\star \frac{2\sqrt{3}\theta^2\gamma\|\mathcal{P}C^\top\|\max(1, |l_2|, \kappa_f)}{\lambda_{\min}(\mathcal{P})} \sup_{s\in[t-\tau^\star, t]} \vartheta(\tilde{\xi}(s)).\end{aligned} \tag{4.79}$$

Consequently, from Lemma 4.2.1, $\vartheta(\tilde{\xi}(t))$ converges exponentially towards zero for any $\theta < \theta_{\tau^\star}$, if $\theta_{\tau^\star}$ satisfies the bound (4.76). This ends the proof.

□

4.6 Comments and Discussion

This section is devoted to foster constructive discussions and insightful comments regarding the results presented in the previous sections, particularly in contrast to established methodologies within the field. The main goal is to clarify the contributions made and demonstrate the advantages of both the output-based dynamic extension technique and the new state observer structure, incorporating an integral term. To enhance readability, we split this section into distinct subsections, presenting each case separately: the general case, the companion forms, and finally, the feedforward systems.

4.6.1 On the general case

It is quite clear that although the estimation strategy proposed in Section 4.3 needs the Lipschitz assumption on the function g defined in (4.26), it leads, however, to a simpler and higher bound on $\tau^\star$. For an explicit comparison, let us define $\tau_1^\star$ and $\tau_2^\star$ the maximum bounds obtained by (4.12b) and (4.70), respectively. Then, we have $\tau_1^\star < \tau_2^\star$ if

$$\frac{\gamma_g}{\gamma_h} > \sqrt{\|Y_z\|^2\gamma_h^2 + \gamma_f^2 + \|LC\|} \tag{4.80}$$

which is not easy to evaluate in general. Indeed, this depends on the values of $\gamma_h, \gamma_f, \|Y_z\|$, and $\|LC\|$. For instance, for systems with linear outputs, the inequality (4.80) cannot be satisfied because $\gamma_g \le \gamma_h\gamma_f$. This means that for this class of systems with linear outputs, which is often encountered in real-world applications, the bound obtained by the strategy of Section 4.3.2 is less conservative than (4.70). In addition, the upper bound (4.70) depends on the observer gain L, which is not convenient for the user. Indeed, the higher is the norm $\|LC\|$ the lower is the bound of $\tau^\star$ in (4.70). Further results and analytical comparisons are given in the next sections.
Additionally, it is worth noting that an alternative bound to (4.70) can be derived without resorting to the state augmentation technique, as commonly seen in the literature [4, 23, 140]. However, employing the state augmentation strategy offers notable advantages: it streamlines the developments and simplifies the problem to systems with linear outputs. Conversely, the bound (4.70) is derived without the utilization of the well-known *Jensen inequality*, resulting, in general, in more conservative upper bounds. Moreover, without the dynamic extension approach, when utilizing the observer (4.2), the gain L directly multiplies the original output, $y(t)$, thereby increasing the impact of delay on estimation accuracy.

4.6.2 Case of the companion form

This category of systems holds particular significance, as it has attracted considerable attention in the literature, with numerous studies dedicated to developing methodologies for addressing high values of $\tau^\star$ in output measurements [3, 23, 34, 46, 52, 72, 142]. Recently, a novel approach using the standard high-gain observer and exploiting the HG/LMI technique has emerged, as showcased in [2]. The authors demonstrated notable enhancements in the maximum allowable value of $\tau^\star$. Therefore, the outcomes presented in this section for this class of systems will be compared to the work in [2].

4.6.2.1 On the maximum bound in (4.42d) and (4.73)

Here, we provide the value of θ for which the bounds in (4.42d) and (4.73) reach their maximum. We begin with the bound in (4.73). To do so, let us introduce the following function:

$$\rho_{76}(\theta) \triangleq \frac{\lambda_{\min}(\mathcal{P})\big[\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\big]}{\theta^2\sqrt{8}\gamma\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|\max(1,|l_2|)}.$$

It is sufficient to study the function $\rho_{76}(\theta)$ on the interval $\left[\max\left(1, \frac{2\kappa_f \lambda_{\max}(\mathcal{P})}{\mu}\right), +\infty\right[$ to find its maximum value. By simple calculations, we deduce that the derivative $\frac{\mathrm{d}\rho_{76}}{\mathrm{d}\theta}(.) = \rho'_{76}(.)$ is increasing on the interval $[\max\left(1, \frac{2\kappa_f \lambda_{\max}(\mathcal{P})}{\mu}\right), \frac{4\kappa_f \lambda_{\max}(\mathcal{P})}{\mu}]$ and decreasing on the interval $[\frac{4\kappa_f \lambda_{\max}(\mathcal{P})}{\mu}, +\infty[$, with $\rho'_{76}\left(\frac{4\kappa_f \lambda_{\max}(\mathcal{P})}{\mu}\right) = 0$. This means that the maximum value of $\tau^\star$ in (4.73) is obtained for $\theta^\star = \frac{4\kappa_f \lambda_{\max}(\mathcal{P})}{\mu}$. That is, we have

$$\begin{aligned}\tau^\star \leq \tau^\star_{76} &= \rho_{76}\left(\frac{4\kappa_f \lambda_{\max}(\mathcal{P})}{\mu}\right)\\ &= \frac{\mu^2 \lambda_{\min}(\mathcal{P})}{16\sqrt{2}\gamma\kappa_f \lambda^2_{\max}(\mathcal{P})\|\mathcal{P}C^\top\| \max(1, |l_2|)}. \end{aligned}\tag{4.81}$$

Of course, this is the case when $\kappa_f \lambda_{\max}(\mathcal{P}) > \frac{\mu}{4}$. Otherwise, the maximum value $\tau^\star_{76}$ will be

$$\tau^\star_{76} = \rho_{76}(1) = \frac{\lambda_{\min}(\mathcal{P})\left[\mu - 2\kappa_f \lambda_{\max}(\mathcal{P})\right]}{\sqrt{8}\gamma\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\| \max(1, |l_2|)}. \tag{4.82}$$

We do the same to obtain the value of θ for which the bound in (4.42d) reaches its maximum. We consider the function

$$\rho_{45}(\theta) \triangleq \frac{\lambda_{\min}(\mathcal{P})\left[\mu\theta - 2\kappa_f \lambda_{\max}(\mathcal{P})\right]}{2\theta^2\gamma\kappa_g \lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|}$$

and for the same reasons as for ρ_{76}, we get the following maximum bound in this case:

$$\begin{aligned}\tau^\star \leq \tau^\star_{45} &= \rho_{45}\left(\frac{4\kappa_f \lambda_{\max}(\mathcal{P})}{\mu}\right)\\ &= \frac{\mu^2 \lambda_{\min}(\mathcal{P})}{16\gamma\kappa_g\kappa_f \lambda^2_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|}. \end{aligned}\tag{4.83}$$

However, if $\kappa_f \lambda_{\max}(\mathcal{P}) > \frac{\mu}{4}$, the bound $\tau^\star_{45}$ should be the following one:

$$\tau^\star_{45} = \rho_{45}(1) = \frac{\lambda_{\min}(\mathcal{P})\left[\mu - 2\kappa_f \lambda_{\max}(\mathcal{P})\right]}{2\gamma\kappa_g \lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|}. \tag{4.84}$$

4.6.2.2 Comparison between (4.42d) and (4.73)

This subsection is devoted to analytical comparisons between the bound (4.42d) derived from employing the new observer structure with an integral term, and the bound (4.73) obtained by using the observer (4.72). These comparisons enable us to avoid the necessity for numerical evaluations. To this end, we compute the ratio, $\ell_{76/45}$, between the bounds (4.73) and (4.42d) as

$$\ell_{76/45} \triangleq \frac{\rho_{76}(\theta)}{\rho_{45}(\theta)} = \frac{\kappa_g}{\sqrt{2}\max(1, |l_2|)}. \tag{4.85}$$

We remind that κ_g represents the Lipschitz constant of the function $g(\zeta) = \phi(\zeta_2, \zeta_3)$, where ϕ is defined in (4.35). Clearly, the bound (4.42d) surpasses (thus, is higher than) that of (4.73), if $\ell_{76/45} < 1$. However, this condition relies on the values of κ_g and l_2. According to (4.30), we have

$$\begin{aligned}\kappa_g &= \sqrt{\sup_{\zeta_2\in\mathbb{R}, \zeta_3\in\mathcal{I}}\left(\left|\frac{\partial^2 h_x}{\partial\zeta_2^2}(\zeta_2)\zeta_3\right|^2 + \left|\frac{\partial h_x}{\partial\zeta_2}(\zeta_2)\right|^2\right)}\\ &= \sqrt{1 + \sup_{\zeta_2\in\mathbb{R}, \zeta_3\in\mathcal{I}}\left(\left|\frac{\partial^2 h_x}{\partial\zeta_2^2}(\zeta_2)\zeta_3\right|^2\right)}. \end{aligned}\tag{4.86}$$

Therefore, $\ell_{76/45} < 1$ if the following inequality holds:

$$\sup_{\zeta_2 \in \mathbb{R}, \zeta_3 \in \mathcal{I}} \left| \frac{\partial^2 h_x}{\partial \zeta_2^2}(\zeta_2)\zeta_3 \right| < 2\left[\max(1, |l_2|)\right]^2 - 1$$
$$= \max\left(1, 2|l_2|^2 - 1\right). \quad (4.87)$$

Otherwise, if $\ell_{76/45} > 1$, the bound (4.73) surpasses (4.42d). In general, direct comparison between the two bounds is challenging. However, the advantage of (4.42d) over (4.73) stems from the fact that (4.73) explicitely depends on the observer gain component $|l_2|$, which has the potential to be high. More notably, in systems with linear output–common in the techniques developed for this class of systems–the bound (4.42d) consistently exceeds that of (4.73). This is obvious because, in the case of linear output, $\kappa_g = 1$ due to $\frac{\partial^2 h_x}{\partial \zeta_2^2} \equiv 0$. Consequently, (4.87) holds for any l_2. This underscores the significance of employing the new observer structure with an integral term.

4.6.2.3 Comparison with the bound in [2]

To provide a comprehensive comparison between (4.42d) and the bounds delineated in [2], it is imperative to establish an analytical connection between the solutions of the LMI condition (18) outlined in [2, Theorem 10] and those of our LMI conditions (4.42a). Although the observer presented in [2, Theorem 10] operates within a dimension of n, contrarily to our observer, which operates in a dimension of $n+1$, our design methods offer distinct advantages, notably:

- The LMI conditions presented in [2, Eq.(18), Theorem 10] depends on the maximum allowable delay value, $\tau^\star$, presenting a notable limitation that necessitates the utilization of specific numerical algorithms to solve the resulting inequalities. Furthermore, compared to our LMI conditions (4.42a), those in [2, Eq.(18), Theorem 10] exhibit a certain conservatism from a feasibility point of view, while notably, our LMI conditions remain consistently feasible for the considered class of triangular systems.

- In contrast to our approach, the LMIs proposed in [2, Eq. (18), Theorem 10] incorporate an additional decision variable beyond the Lyapunov matrix and the observer gain, which contributes to the determination of the bounds on $\tau^\star$. This introduces an additional conservatism since this decision variable may decrease significantly the maximum allowable delay value, $\tau^\star$.

- In contrast to our method, the approach in [2] gives two different bounds on $\tau^\star$, namely τ_1 and τ_2, where $\tau^\star < \min(\tau_1, \tau_2)$. However, the bound τ_1 depends on $\lambda\theta - 2\kappa_f \lambda_{\max}(\mathcal{P})$ while τ_2 depends on $\frac{1}{\lambda\theta - 2\kappa_f \lambda_{\max}(\mathcal{P})}$. Hence, regardless of the value of θ, whether it is close to $\lambda\theta - 2\kappa_f \lambda_{\max}(\mathcal{P})$ or not, it affects the value of $\tau^\star$ since either τ_1 or τ_2 is very small.

Furthermore, the design methods proposed in herein are characterized by their simplicity and straightforwardness, contrary to the method presented in [2], which involves several intermediate results and cumbersome developments. For a deeper understanding of this comparative analysis, we refer the readers to explore [2] for a more detailed exposition.

4.6.3 Case of the feedforward form

Firstly, it should be noted that for this class of systems, regardless of the observer used, whether it is the standard observer (4.72a) or the new structure (4.54), it converges for any arbitrarily high value of $\tau^\star$, as shown in Theorem 4.4.6 and Proposition 4.5.3 through the bounds (4.53) and (4.76), respectively. Unlike other families of systems where the challenge lies in obtaining the maximum allowable value of $\tau^\star$, for feedforward systems, the objective is to achieve the maximum allowable value of $\theta_{\tau^\star}$. Indeed, a higher value of $\theta_{\tau^\star}$ corresponds to a faster convergence rate. For the comparison, let θ_{56} and θ_{79} be the bound $\theta_{\tau^\star}$ in Theorem 4.4.6 and Proposition 4.5.3, respectively. Then, for a given maximum value of the delay, $\tau^\star$, Theorem 4.4.6 provides faster convergence rate than Proposition 4.5.3 if $\pi_{79/45} \triangleq \frac{\theta_{79}}{\theta_{56}} < 1$. From (4.53) and (4.76), we have

$$\pi_{79/45} < \frac{\gamma\kappa_g\|\mathcal{P}C^\top\| + \kappa_f}{\kappa_f + \sqrt{3}\gamma\|\mathcal{P}C^\top\|\max(1,|l_2|,\kappa_f)}. \tag{4.88}$$

Hence, according to the definition of κ_g in (4.86), inequality (4.88) holds true if the following condition is satisfied:

$$\begin{aligned}\sup_{\zeta_2\in\mathbb{R},\zeta_3\in\mathcal{I}}\left|\frac{\partial^2 h_x}{\partial\zeta_2^2}(\zeta_2)\zeta_3\right| &< 3\left[\max(1,|l_2|,\kappa_f)\right]^2 - 1\\ &= \max\left(2, 3|l_2|^2 - 1, 3\kappa_f^2 - 1\right).\end{aligned} \tag{4.89}$$

This condition is always fulfilled in the context of systems with linear outputs due to $\frac{\partial^2 h_x}{\partial\zeta_2^2} \equiv 0$. Consequently, for linear outputs, Theorem 4.4.6 consistently yields a faster convergence rate than Proposition 4.5.3.

It is worth noting that this class of systems has been studied in [88]; however, the condition stated it is implicit. Only the existence of a constant $\tau^\star$ is mentioned, and determining its value necessitates numerical computation through methods such as the groping method, which entails simulating a given system. While the result in [88] guarantees exponential convergence despite arbitrarily long delays in the output, the proposed techniques offer explicit bounds on the observer tuning parameter θ, providing a more straightforward and practical solution.

4.7 Further Result

This section introduces a new approach for handling high values of delays in output measurements. The method is based on solving a set of LMI conditions, sidestepping the necessity for additional constraints on the maximum allowable delay, denoted as $\tau^\star$, such as that required in (4.12b), for instance. For more generality, we tackle the general case, leading to general outcome applicable for any particular family of systems. As we can see from the previous sections, by exclusively employing the standard quadratic Lyapunov function, we have provided notable enhancements over the existing findings in the literature, despite the utilization of more general Lyapunov functions as studied in [2] and related references. In this section, by using a Lyapunov–Krasovskii functional containing a double integral term, we avoid constraints on $\tau^\star$ similar to those described in (4.12b). The aim of this section is to enhance the existing LMI-based design approaches in the literature for a similar class of systems, as in [119] and [92]. What distinguishes this section apart from the preceding results is the utilization of a Lyapunov-Krasovskii functional to ensure asymptotic convergence of the observer, instead of exponential convergence. This enables us to derive LMI conditions without the need for additional constraints similar to (4.12b).

4.7.1 Lyapunov-Krasovskii analysis

Consider the system (4.6) with the parameters and functions given in (4.24)–(4.26) and the observer (4.27), with the function g satisfying Assumption 4.3.5. This leads to the estimation error dynamic (4.9). By using Lemma 2.2.6, there exists $\bar{z}_t \in \mathbf{Co}\left(\zeta(t),\hat{\zeta}(t)\right)$ as in (2.28) such that

$$\Delta g\left(\zeta(s),\hat{\zeta}(s),u(s)\right) = \nabla_\zeta^g(\bar{z}_s)\epsilon(s)$$

where ∇_{ζ}^{g} is defined as in (2.30). Then, the error dynamic (4.9) is rewritten under the following form:

$$\dot{\epsilon}(t) = \left[\nabla_{\zeta}^{f_\zeta}(z_t) - LC\right]\epsilon(t) + B_\zeta \int_{t-\tau(t)}^{t} \nabla_{\zeta}^{g}(\bar{z}_s)\epsilon(s)\mathrm{d}s. \tag{4.90}$$

Since the function g satisfies Assumption 4.3.5, then there exist constant matrices $\mathcal{B}_j^g$ of appropriate dimensions, and functions $\bar{\lambda}_j(\bar{z}_t)$, $j = 1, \dots \bar{n}_g$ such that the generalized Jacobian $\nabla_{\zeta}^{g}(\bar{z}_s)$ belongs to the convex polytopic set defined as:

$$\mathcal{H}_g \triangleq \left\{ \sum_{j=1}^{\bar{n}_g} \bar{\lambda}_j(\bar{z}_s)\mathcal{B}_j^g, \sum_{j=1}^{\bar{n}_g} \bar{\lambda}_j(\bar{z}_s) = 1, \bar{\lambda}_j(\bar{z}_s) \geq 0 \right\}. \tag{4.91}$$

Now let us consider the following Lyapunov–Krasovskii functional:

$$\vartheta(\epsilon(t)) = \epsilon^\top(t)\mathcal{P}\epsilon(t) + \int_{-\tau^\star}^{0}\int_{v}^{0} \epsilon(t+s)^\top \Pi(t+s)\epsilon(t+s)\mathrm{d}s\mathrm{d}v \tag{4.92}$$

with

$$\Pi(s) \triangleq \left(\nabla_{\zeta}^{g}(\bar{z}_s)\right)^\top \mathbb{S}\, \nabla_{\zeta}^{g}(\bar{z}_s)$$

where $\mathcal{P}$ and $\mathbb{S}$ are two symmetric and positive definite matrices of appropriate dimensions. By expanding the derivative of the ϑ along the trajectories of (4.90), we get

$$\begin{aligned}\dot{\vartheta}(\epsilon(t)) = {} & \epsilon^\top(t)\left[\left(\nabla_{\zeta}^{f_\zeta}(z_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_{\zeta}^{f_\zeta}(z_t) - LC\right)\right]\epsilon(t) \\ & + 2\int_{t-\tau(t)}^{t} \epsilon^\top(t)\mathcal{P}B_\zeta\nabla_{\zeta}^{g}(\bar{z}_s)\epsilon(s)\mathrm{d}s + \tau^\star\epsilon(t)^\top\Pi(t)\epsilon(t) \\ & - \int_{t-\tau^\star}^{t} \epsilon^\top(s)\Pi(s)\epsilon(s)\mathrm{d}s\end{aligned} \tag{4.93}$$

where we used

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\tau^\star}^{0}\int_{v}^{0} \varphi(t+s)\mathrm{d}s\mathrm{d}v = \tau^\star\varphi(t) - \int_{t-\tau^\star}^{t} \varphi(\nu)\mathrm{d}v$$

for any continuous function $\varphi(.)$ defined on $[-\tau^\star, +\infty[\to \mathbb{R}$. No, by using the Young inequality on the first integral term in (4.93) and the fact that $\tau(t) \leq \tau^\star$, we obtain

$$\begin{aligned}2\int_{t-\tau(t)}^{t} \epsilon^\top(t)\mathcal{P}B_\zeta\nabla_{\zeta}^{g}(\bar{z}_s)\epsilon(s)\mathrm{d}s \leq {} & \tau^\star\epsilon^\top(t)\left(\mathcal{P}B_\zeta\right)^\top \mathbb{S}^{-1}\mathcal{P}B_\zeta\epsilon(t) \\ & + \int_{t-\tau^\star}^{t} \epsilon^\top(s)\Pi(s)\epsilon(s)\mathrm{d}s.\end{aligned}$$

It follows that

$$\dot{\vartheta}(\epsilon(t)) \leq \epsilon^\top(t)\mathbb{M}\left(\mathcal{P}, \mathbb{S}, t\right)\epsilon(t) \tag{4.94}$$

where

$$\begin{aligned}\mathbb{M}\left(\mathcal{P}, \mathbb{S}, t\right) \triangleq {} & \left(\nabla_{\zeta}^{f_\zeta}(z_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_{\zeta}^{f_\zeta}(z_t) - LC\right) \\ & + \tau^\star\mathcal{P}B_\zeta\mathbb{S}^{-1}\left(\mathcal{P}B_\zeta\right)^\top + \tau^\star\Pi(t).\end{aligned}$$

As a consequence, we deduce that $\dot{\vartheta}(\epsilon(t)) < 0, \forall\epsilon(t) \neq 0$ if

$$\mathbb{M}\left(\mathcal{P}, \mathbb{S}, t\right) < 0, \forall t \geq 0. \tag{4.95}$$

In the next section, we present sufficient LMI conditions guaranteeing the inequality (4.95).

$$\begin{bmatrix} \left(\mathcal{A}_j^{\zeta}\right)^{\top}\mathcal{P}+\mathcal{P}\mathcal{A}_j^{\zeta}-C^{\top}\mathcal{R}-\mathcal{R}^{\top}C & (\mathcal{P}B_{\zeta})^{\top} & \mathcal{B}_k^{g\top}\mathbb{S} \\ \mathcal{P}B_{\zeta} & -\frac{1}{\tau^{\star}}\mathbb{S} & 0 \\ \mathbb{S}\mathcal{B}_k^{g} & 0 & -\frac{1}{\tau^{\star}}\mathbb{S} \end{bmatrix} < 0, j=1,\ldots,\bar{n}_{\zeta}; k=1,\ldots,\bar{n}_g. \tag{4.96}$$

4.7.2 Delay-dependent and extra constraints-free LMIs

This section is devoted to a new LMI-based method, where the provided LMIs are delay-dependent and free of any extra constraint on the maximum allowable delay $\tau^{\star}$. The result is summarized in the following theorem.

Theorem 4.7.1. *Assume that there exist two symmetric and positive definite matrices $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$ and $\mathbb{S} \in \mathbb{R}^{n_s \times n_s}$, and a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$ such that the LMI conditions (4.96) hold. Then the observer (4.27), with $L = \mathcal{P}^{-1}\mathcal{R}^{\top}$, corresponding to the system (4.6) with the parameters and functions given in (4.24)–(4.26) converges asymptotically.*

Proof. Once the Lyapunov-Krasovskii functional is expanded to derive inequality (4.95), the subsequent steps of the proof unfold straightforwardly. Specifically, invoking the Schur Lemma confirms the satisfaction of inequality (4.95) provided the following conditions are met:

$$\begin{bmatrix} \left(\nabla_{\zeta}^{f_\zeta}(\boldsymbol{z}_t)-LC\right)^{\top}\mathcal{P}+\mathcal{P}\left(\nabla_{\zeta}^{f_\zeta}(\boldsymbol{z}_t)-LC\right) & (\mathcal{P}B_{\zeta})^{\top} & \left(\nabla_{\zeta}^{g}(\bar{\boldsymbol{z}}_t)\right)^{\top}\mathbb{S} \\ \mathcal{P}B_{\zeta} & -\frac{1}{\tau^{\star}}\mathbb{S} & 0 \\ \mathbb{S}\,\nabla_{\zeta}^{g}(\bar{\boldsymbol{z}}_t) & 0 & -\frac{1}{\tau^{\star}}\mathbb{S} \end{bmatrix} < 0 \tag{4.97}$$

for all $t \geq 0$. Therefore, using the convexity principle and introducing the change of variable $\mathcal{R} = L^{\top}\mathcal{P}^{-1}$, the inequality (4.97) is fulfilled if (4.96) holds true. Consequently, we infer that $\dot{\vartheta}(\epsilon(t)) < 0, \forall\epsilon(t) \neq 0$, implying asymptotic stability of the estimation error around the origin. There is no guarantee of exponential convergence.

□

Remark 4.7.2. *The design approach outlined in Theorem 4.7.1 is notably straightforward, requiring only the fulfillment of a set of LMI conditions without imposing any additional constraints on the delay. Nonetheless, it solely guarantees the asymptotic convergence of the observer. To attain exponential convergence, even when using the Lyapunov-Krasovskii functional (4.92), extra constraints on the delay become necessary. Furthermore, the utilization of the Halanay inequality remains unavoidable in such cases. Further exploration in this research direction needs a deeper investigation along with meticulous development and the utilization of advanced mathematical tools.*

4.8 Illustrative Example

This section focuses on an illustrative example aimed at validating the proposed observer design method. As discussed in earlier sections, thorough analytical comparisons have been made between the newly proposed observer structure and existing techniques in the literature. Given the fact that the analytical comparisons concern the companion systems and feedforward systems, numerical comparisons are unnecessary. Hence, we present an illustrative example that covers the general case, for which, to the best of the authors' knowledge, there are only few existing results in the literature. As an application example, we consider the two-dimensional bioreactor model described by the following equations:

$$\begin{aligned}\dot{x} &= \begin{bmatrix}-l_1 & c_1\\ 0 & 0\end{bmatrix}x + \begin{bmatrix}0\\ c_2\sin(x_2)+c_3\cos(x_2)+c_3u\end{bmatrix}\\ y(t) &= x_1(t-\tau(t))\end{aligned} \tag{4.98}$$

where the input $u(t)$ is given by $u(t) = \sin(0.35t)$ and the constants l_1 and $c_i, i = 1, \dots, 4$, are given by $l_1 = 1$, $c_1 = 1$, $c_2 = c_3 = 0.02$, and $c_4 = 8$. In this case, the function g in (4.26) is reduced to $g(\zeta, u) = -l_1\zeta_2 + c_1\zeta_3$, which satisfies Assumption 4.3.5 with Lipschitz constant $\gamma_g = \sqrt{l_1^2 + c_1^2} = \sqrt{2}$. We use the output augmentation technique by using the new variable $z(t)$ introduced in (4.32). Then, we have

$$B_\zeta = \begin{bmatrix}-\gamma & 0 & 0\end{bmatrix}^\top, \; f_\zeta(\zeta, u) = \begin{bmatrix}\gamma\zeta_2 & f(\zeta,u)\end{bmatrix}^\top$$

with

$$f(\zeta, u) \triangleq \begin{bmatrix}-l_1\zeta_2 + c_1\zeta_3\\ c_2\sin(\zeta_3)+c_3\cos(\zeta_3)+c_3u\end{bmatrix}.$$

Therefore, the Jacobian of the function f_ζ is given by

$$\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) = \begin{bmatrix}0 & \gamma & 0\\ 0 & -l_1 & c_1\\ 0 & 0 & c_2\cos(\boldsymbol{z}_t) - c_3\sin(\boldsymbol{z}_t)\end{bmatrix}.$$

Also, the jacobian of the scalar function g is given by $\nabla_\zeta^g(\bar{\boldsymbol{z}}_t) = \mathcal{B}_1^g = \begin{bmatrix}0 & -l_1 & c_1\end{bmatrix}$. Utilizing the LMI conditions (4.12a) subject to (4.12b) outlined in Theorem 4.3.1, and the LMIs (4.96) of Theorem 4.7.1, we obtain the results presented in Table 4.1, contrasting them with the methodologies introduced in prior works such as [2], [23], [119], and [92]. These references explore the same example to show and compare the efficiency and validity of the approaches proposed therein.

Table 4.1: Comparison between different methods.

Method	$\tau^\star$
[23, Theorem 1]	< 0.01
[2, Theorem 17]	< 0.02
Theorem 4.3.1–LMI conditions (4.12)	≈ 0.03
[119, Theorem 2]	≈ 1
[92, Theorem 3]	≈ 23.60
Theorem 4.7.1–LMI conditions (4.96)	≈ 34.50

It is quite clear from Table 4.1 that whether based fully on LMIs or requiring extra constraints on $\tau^\star$, our methods significantly enhance existing techniques in the literature.
For $\tau^\star = 5s$ and $\gamma = 1$, we obtain the following observer gain from the LMIs (4.96):

$$L = \begin{bmatrix}5.1004 & 1.0124 & 0.7387\end{bmatrix}^\top.$$

Figure 4.1 shows that the estimation errors converge towards zero despite the delay $\tau^\star = 5s$.
To demonstrate the efficiency of the proposed output-based dynamic extension technique, we introduce Gaussian noise with a mean of zero and a standard deviation of σ to disturb the output measurement. Two distinct scenarios are examined in our simulations. In Figure 4.2, we illustrate

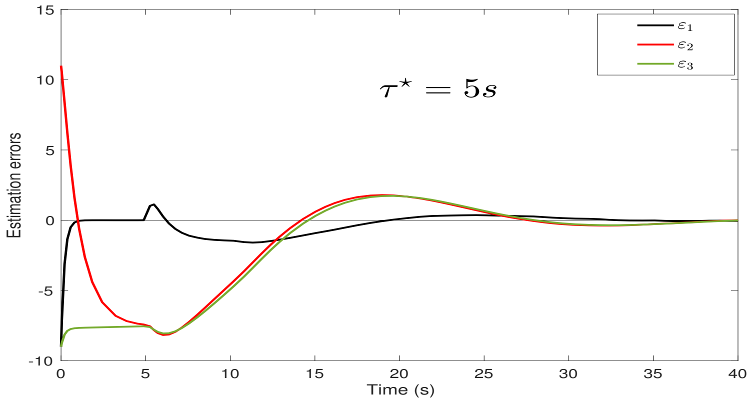

Figure 4.1: Behavior of the estimation errors for $\tau^\star = 5s$.

the behavior of estimation errors for $\sigma = 1$, while Figure 4.3 presents those for $\sigma = 10$. Notably, even with significant values of σ, our new observer structure, augmented with an integral term, exhibits commendable performance, showcasing robustness to measurement noise. This robustness can be attributed to the inherent filtering capabilities of the new observer, which simultaneously addresses substantial delays and noise in the output signal.

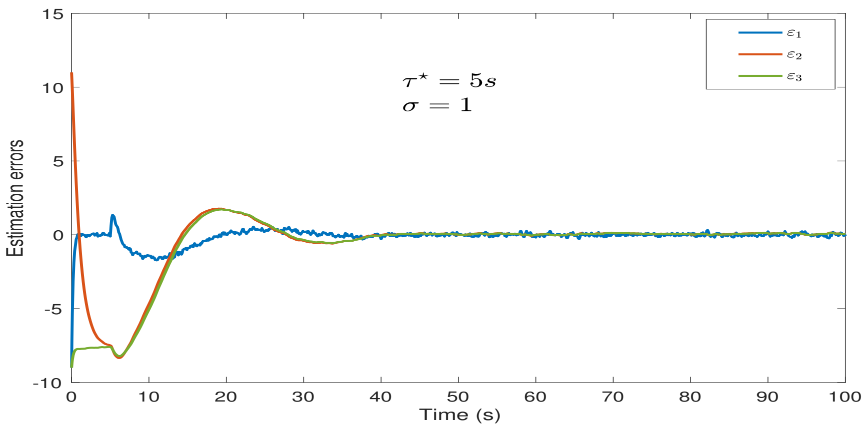

Figure 4.2: Behavior of the estimation errors for $\tau^\star = 5s$.

4.9 Conclusion

We proposed several observer design techniques for nonlinear systems in the presence of delayed and nonlinear outputs. Through a state augmentation technique and output transformation, the problem of the presence of delay and nonlinearities in the output measurement is easily solved by transferring the delay and the nonlinearities to the dynamic process. Such a transfer is achieved by creating a new output measurement and extending the dynamics of the system. For the specific classes of systems considered in this chapter, namely systems in companion form and feedforward systems, novel specific synthesis conditions are proposed, which are less conservative than those existing in the literature. Moreover, a more relaxed and alternative technique is introduced to avoid the conservative assumption regarding the boundedness of x_2. This approach is accompanied by several constructive comments and comparisons. For each class of systems, the new technique

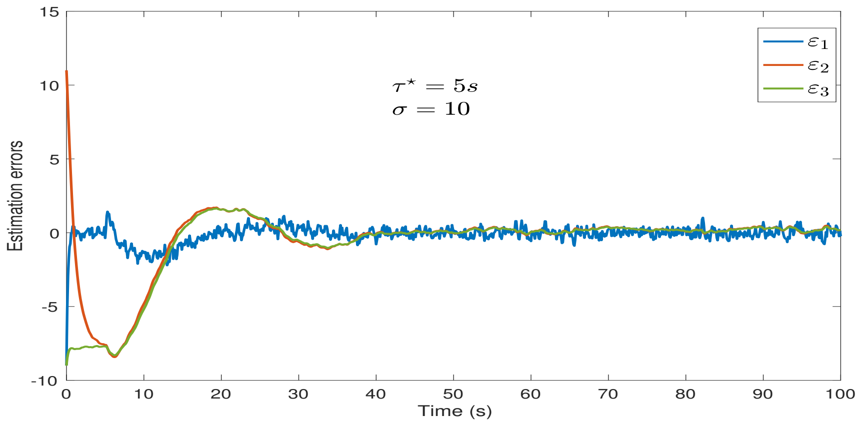

Figure 4.3: Behavior of the estimation errors for $\tau^\star = 5s$.

using the new observer structure with the integral term is thoroughly compared to the method relaxing the boundedness of x_2 and to existing methods in the literature through rigorously expanded analytical arguments. Lastly, utilizing a Lyapunov–Krasovskii functional, we introduce an alternative method requiring only LMI conditions for ensuring the asymptotic convergence of the estimation error. This approach aims to refine existing techniques in the literature while eliminating additional constraints associated with the maximum allowable delay value. The efficiency of the proposed methods is demonstrated through a compelling illustrative example.

Conclusion

In this Thesis, we first provided a brief overview of observer and estimation methods. Following this, we provided an LMI-based design method to check the i-EIOSS property of nonlinear systems. This method may be used easily for a nonlinear system without needing to use its trajectories. To develop such an LMI method, we proposed a mathematical tool, which allowed us to develop a general Lyapunov function-based result. The established outcome is significant within the field of robust estimation techniques, as it simplifies the process of tuning the parameters of the estimator. This is primarily because these tuning parameters directly rely on the coefficients associated with the i-EIOSS property. based on the use of the i-EIOSS property.
Next, we demonstrated that LMI-based approaches can also guarantee the design of nonlinear observers for a large class of nonlinear systems. We proposed LMI conditions for the synthesis of nonlinear observers and showed that the feasibility of such LMIs is guaranteed for some families of nonlinear systems. While the feasibility of the LMIs is not ensured for the arbitrary structure of the nonlinearities, it is shown that such feasibility guarantee is applicable to important families of systems, namely triangular systems and systems having feedforward structures.
Lastly, we proposed novel results on high-gain observer design for some families of nonlinear non-triangular systems. The contributions are both simple but very useful for certain nonlinear models. Indeed, in case the proposed new conditions are satisfied, there is no need to perform any nonlinear transformation of the model to design a high-gain observer. By combining the LMI approach with the standard high-gain methodology, an efficient design procedure is proposed in Theorem 3.3.2. Many questions remain open in this area, offering several directions for future research based on the results presented in this manuscript.

Future Work

1. The proposed results in Section 3.2 are not only motivated by the non-triangularity of nonlinearities of the model. They also build a bridge to important applications even for triangular dynamic models. Among these applications, the results may be generalized to systems with any nonlinear measurements and systems with delayed outputs. For these applications, we could access the tuning parameter κ_2 and choose it large enough to satisfy the constraint (3.77) in Theorem 4.4.6 or (3.93) in Theorem 3.3.2. Although the LMI-based

technique allows providing a less conservative bound bound (3.77), however, such a bound may still be conservative for nonlinearities with a large value of κ_2. Overcoming this limitation is one of the future work we aim to address. One other relevant challenge we aim to solve in the future consists of overcoming the sector condition (3.63). To address this challenging limitation, one of the possible potential avenues is to propose an observer with a time-varying tuning parameter, $\tau(t)$, which may lead to a general Riccati equation, instead of (3.75), for which we need to prove the existence of bounded solutions.

2. In future work, we aim to extend the idea in Chapter 4 to several estimation problems, namely unknown input observer design for systems with nonlinear outputs, adaptive observers for systems with unknown parameters in the output measurements, and systems with delay in the inputs. Combining the dynamic extension technique with output error saturation could improve the performance of the estimation problem. We also aim to apply the proposed results to real-world models, namely applications for connected and autonomous vehicle tracking. To highlight the benefits of the proposed dynamic extension technique together with the new state observer structure, we aim to establish several further theoretical results. First, an extension to systems with disturbances in the measurements will be tackled by exploiting a generalized version of the Halanay inequality. In this case, an Input-to-State Stable (ISS) bound on the estimation error will be given. Finally, an extension to systems with unknown time delay in the output will be investigated.

Bibliography

Bibliography

[1] B. Acikmese and M. Corless. Observers for systems with nonlinearities satisfying incremental quadratic constraints. *Automatica*, 47(7):1339–1348, 2011.

[2] A. Adil, A. Hamaz, I. N'doye, A. Zemouche, T. M. Laleg-Kirati, and F. Bedouhene. On high-gain observer design for nonlinear systems with delayed output measurements. *Automatica*, 141:110281, 2022.

[3] T. Ahmed-Ali, I. Karafyllis, and F. Lamnabhi-Lagarrigue. Global exponential sampled-data observers for nonlinear systems with delayed measurements. *Systems & Control Letters*, 62(7):539–549, 2013.

[4] T. Ahmed-Ali, V. Van Assche, J. F. Massieu, and F. Dorleans. Continuous-discrete observer for state affine systems with sampled and delayed measurements. *IEEE Transactions on Automatic Control*, 58(4):1085–1091, 2013.

[5] A. Alessandri, M. Baglietto, and G. Battistelli. Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes. *Automatica*, 44(7):1753–1765, 2008.

[6] A. Alessandri, M. Baglietto, G. Battistelli, and M. Gaggero. Moving-horizon state estimation for nonlinear systems using neural networks. *IEEE Transactions on Neural Networks*, 22(5):768–780, 2011.

[7] A. Alessandri, M. Baglietto, G. Battistelli, and V. Zavala. Advances in moving horizon estimation for nonlinear systems. In *49th IEEE Conference on Decision and Control (CDC)*, pages 5681–5688. IEEE, 2010.

[8] A. Alessandri and P. Coletta. Design of Luenberger observers for a class of hybrid linear systems. In *International Workshop on Hybrid Systems: Computation and Control*, pages 7–18. Springer, 2001.

[9] A. Alessandri and A. Rossi. Time-varying increasing-gain observers for nonlinear systems. *Automatica*, 49(9):2845–2852, 2013.

[10] A. Alessandri, M. Zasadzinski, and A. Zemouche. Optimistic vs pessimistic moving-horizon estimation for quasi–lpv discrete-time systems. *IFAC-PapersOnLine*, 53(2):5004–5009, 2020.

[11] D. A. Allan. *A Lyapunov-like function for analysis of model predictive control and moving horizon estimation*. The University of Wisconsin-Madison, 2020.

[12] D. A. Allan, J. Rawlings, and A. R. Teel. Nonlinear detectability and incremental input/output-to-state stability. *SIAM Journal on Control and Optimization*, 59(4):3017–3039, 2021.

[13] V. Andrieu, L. Praly, and A. Astolfi. High gain observers with updated gain and homogeneous correction terms. *Automatica*, 45(2):422–428, 2009.

[14] H. Arezki, A. Alessandri, and A. Zemouche. Robust moving-horizon estimation for quasi-LPV discrete-time systems. In *22nd IFAC World Congress*, 2023.

[15] H. Arezki, A. Alessandri, and A. Zemouche. Robust stability analysis of moving-horizon estimator for LPV discrete-time systems. In *22nd IFAC World Congress*, 2023.

[16] H. Arezki, B. Bagnerini, and A. Zemouche. Observer design for nonlinear systems with delayed nonlinear output measurement. In *Accepted in 2024 European Control Conference*. IEEE, 2024.

[17] H. Arezki, D. Bouhadjra, K. Chaib-Draa, A. Zemouche, I N'Doye, and T. M. Laleg-Kirati. LMI feasibility improvement to design observers for a class of Lipschitz nonlinear systems. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 6151–6155, Texas, USA, 2021. IEEE.

[18] H. Arezki and A. Zemouche. Simple but useful contributions to high-gain observer for non-triangular systems. *IEEE Control Systems Letters*, 7:3343–3348, 2023.

[19] H. Arezki, A. Zemouche, A. Alessandri, and P. Bagnerini. LMI design procedure for incremental input/output-to-state stability in nonlinear systems. *IEEE Control Systems Letters*, 7:3403–3408, 2023.

[20] H. Arezki, A. Zemouche, P Bagnerini, and S. Djennoune. LMI feasibility analysis in observer design for some families of nonlinear systems. In *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 2645–2650. IEEE, 2023.

[21] H. Arezki, A. Zemouche, F. Bedouhene, and A. Alessandri. State observer design method for a class of non-linear systems. *IET Control Theory & Applications*, 14(12):1648–1655, 2020.

[22] H. Arezki, F. Zhang, and A. Zemouche. A HG/LMI-Based Observer for a Tumor Growth Model. In *2022 European Control Conference (ECC)*, pages 259–263. IEEE, 2022.

[23] V. V. Assche, T. Ahmed-Ali, C.A.B. Hann, and F. Lamnabhi-Lagarrigue. High gain observer design for nonlinear systems with time-varying delayed measurements. *IFAC Proceedings Volumes*, 44(1):692–696, 2011. 18th IFAC World Congress.

[24] D. Astolfi and L. Marconi. A high-gain nonlinear observer with limited gain power. *IEEE Transactions on Automatic Control*, 60(11):3059–3064, 2015.

[25] D. Astolfi and L. Marconi. A high-gain nonlinear observer with limited gain power. *IEEE Transactions on Automatic Control*, 60(11):3059–3064, 2015.

[26] J. D Avilés and J. A. Moreno. Dissipative interval observer design for discrete-time nonlinear systems. *Asian Journal of Control*, 22(4):1422–1436, 2020.

[27] J. D Avilés, J. A. Moreno, and F. J. Bejarano. Dissipative state observer design for nonlinear time-delay systems. *Journal of the Franklin Institute*, 360(2):887–909, 2023.

[28] E. W. Bai and S. Sastry. Observability of nonlinear systems: A geometric approach. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 1249–1254. IEEE, 2005.

[29] G. Besançon. Remarks on nonlinear adaptive observer design. *Systems & control letters*, 41(4):271–280, 2000.

[30] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[31] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, June 1994.

[32] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. Linear matrix inequalities in system and control theory. *Studies in Applied Mathematics. Philadelphia, PA: SIAM*, 15, 1994.

[33] F. Cacace, V. Cusimano, A. Germani, P. Palumbo, and F. Papa. Closed-loop control of tumor growth by means of anti-angiogenic administration. *Mathematical Biosciences & Engineering*, 15(4):827, 2018.

[34] F. Cacace, A. Germani, and C. Manes. A chain observer for nonlinear systems with multiple time-varying measurement delays. *SIAM Journal on Control and Optimization*, 52(3):1862–1885, 2014.

[35] V. Cusimano, P. Palumbo, and F. Papa. Closed-loop control of tumor growth by means of anti-angiogenic administration. In *2015 54th IEEE Conference on Decision and Control (CDC)*, pages 7789–7794. IEEE, 2015.

[36] C. Edwards and S. Spurgeon. On the development of discontinuous observers. *Int. J. of Control*, 59(5):1211–1229, 1994.

[37] Christopher Edwards and Sarah Spurgeon. *Sliding mode control: theory and applications*. Crc Press, 1998.

[38] M. Ekramian, S. Hosseinnia, and F. Sheikholeslam. Observer design for non-linear systems based on a generalised lipschitz condition. *IET control theory & applications*, 5(16):1813–1818, 2011.

[39] J. Folkman. Anti-angiogenesis: new concept for therapy of solid tumors. *Annals of surgery*, 175(3):409, 1972.

[40] R. A. Freeman and P. V. Kokotović. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Birkhäuser, Boston, MA, 1996.

[41] J. P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems, applications to bioreactors. *IEEE Transactions on Automatic Control*, 37(6):875–880, 1992.

[42] J. P. Gauthier and I.A.K. Kupka. Observability and observers for nonlinear systems. *SIAM J. on Control and Optimization*, 32(4):975–994, 1994.

[43] Jean-Paul Gauthier, Hassan Hammouri, and Sami Othman. A simple observer for nonlinear systems applications to bioreactors. *IEEE Transactions on automatic control*, 37(6):875–880, 1992.

[44] Jean-Paul Gauthier and Ivan AK Kupka. Observability and observers for nonlinear systems. *SIAM journal on control and optimization*, 32(4):975–994, 1994.

[45] J.P. Gauthier and I. Kupka. *Deterministic Observation Theory and Applications*. Cambridge University Press, Cambridge, UK, 2001.

[46] A. Germani, C. Manes, and P. Pepe. A new approach to state observation of nonlinear systems with delayed output. *IEEE Transactions on Automatic Control*, 47(1):96–101, 2002.

[47] M. S. Grewal and A. P. Andrews. *Kalman Filtering: Theory and Practice Using MATLAB*. Wiley, 2001.

[48] Q. P. Ha and H. Trinh. State and input simultaneous estimation for a class of nonlinear systems. *Automatica*, 40:1779–1785, 2004.

[49] P. Hahnfeldt, D. Panigrahy, J. Folkman, and L. Hlatky. Tumor development under angiogenic signaling: a dynamical theory of tumor growth, treatment response, and post vascular dormancy. *Cancer research*, 59(19):4770–4775, 1999.

[50] A. Halanay. *Differential Equations: Stability, Oscillations, Time Lags*, volume 23. Academic Press, New York, 1966.

[51] E. L. Haseltine and J. B. Rawlings. Critical evaluation of extended kalman filtering and moving-horizon estimation. *Industrial & Engineering Chemistry Research*, 44(8), 2005.

[52] L. Hassan, A. Zemouche, and M. Boutayeb. $\mathcal{H}_\infty$ observers design for a class of nonlinear time-delay systems in descriptor form. *International Journal of Control*, 84(10):1653–1663, 2011.

[53] L. Hassan, A. Zemouche, and M. Boutayeb. Robust unknown input observers for nonlinear time-delay systems. *SIAM Journal on Control and Optimization*, 51(4):2735–2752, 2013.

[54] Simon Haykin. *Communication Systems*. Wiley, Hoboken, NJ, 5th edition, 2008.

[55] Michel Hénon. Chaotic systems: The hénon map and its dynamical properties. *Journal of Chaos Theory*, 12(4):415–425, 2004.

[56] R. Hermann and J. P. Gauthier. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22(5):728–740, 1977.

[57] John H. Hubbard and Beverly H. West. *Differential Equations: A Dynamical Systems Approach*. Springer-Verlag, New York, 1995.

[58] D. C. Huong, V. T. Huynh, and H. Trinh. Integral outputs-based robust state observers design for time-delay systems. *SIAM Journal on Control and Optimization*, 57(3):2214–2239, 2019.

[59] V. T. Huynh, A. Arogbonlo, H. Trinh, and A. M. T. Oo. Design of observers for positive systems with delayed input and output information. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 67(1):107–111, 2019.

[60] V.T. Huynh, C. M. Nguyen, and H. Trinh. Static output feedback control of positive linear systems with output time delays. *International Journal of Systems Science*, 50(15):2815–2823, 2019.

[61] A. Isidori. *Nonlinear Control Systems*. Springer, London, 3rd edition, 1995.

[62] W. Jeon, Z. Xie, A. Zemouche, and R. Rajamani. Simultaneous cyber-attack detection and radar sensor health monitoring in connected ACC vehicles. *IEEE Sensors Journal*, 21(14):15741–15752, 2020.

[63] Z. P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for iss systems and applications. *Mathematics of Control, Signals, and Systems*, 7(2):95–120, 1994.

[64] Thomas Kailath, A. H. Sayed, and B. Hassibi. *Linear Estimation*. Prentice Hall, New Jersey, 2000.

[65] R. E. Kalman. A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, 82:35–45, 1960.

[66] R.E. Kalman et al. A new approach to linear filtering and prediction problems. *Journal of basic Engineering*, 82(1):35–45, 1960.

[67] H. Khalil. Cascade high-gain observers in output feedback control. *Automatica*, 80:110–118, 2017.

[68] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River NJ, 1996.

[69] Hassan K. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ, 3rd edition, 2002.

[70] Hassan K. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ, 3rd edition, 2002.

[71] H. Kheloufi, F Bedouhene, A Zemouche, and A. Alessandri. Observer-based stabilisation of linear systems with parameter uncertainties by using enhanced lmi conditions. *International Journal of Control*, 88(6):1189–1200, 2015.

[72] A. Khosravian, J. Trumpf, and R. Mahony. State estimation for nonlinear systems with delayed output measurements. In *54th IEEE Conference on Decision and Control (CDC)*, pages 6330–6335, 2015.

[73] M. Kirszbraun. Über die zusammenziehende und Lipschitzsche transformationen. *Fundamenta Mathematicae*, 22(1):77–108, 1934.

[74] P. Klein. Hilbert spaces and the projection theorem.

[75] S. Knüfer and M.A. Müller. Nonlinear full information and moving horizon estimation: Robust global asymptotic stability. *Automatica*, 150:110603, 2023.

[76] M. Krichman, E. D. Sontag, and Y. Wang. Input-output-to-state stability. *SIAM Journal on Control and Optimization*, 39(6):1874–1928, 2001.

[77] D. Kurtulus and T. H. Lee. *State Observers: Theory and Applications*. Springer, Berlin, 2004.

[78] H. Le Van and T. Hieu. Exponential stability of time-delay systems via new weighted integral inequalities. *Applied Mathematics and Computation*, 275:335–344, 2016.

[79] H. S. Lee and V. Utkin. Discrete-time sliding mode observers: Application to fault detection and isolation. *International Journal of Robust and Nonlinear Control*, 17(3-4):313–341, 2007.

[80] X. Liu, W. Lin, C. Zhao, and Y. Hu. Digital control of a family of time-delay feedforward systems with sparse sampling. *Systems & Control Letters*, 163:105200, 2022.

[81] L. Ljung. Asymptotic behaviour of the extended Kalman filter as a parameter estimator for linear systems. *IEEE Trans. on Automatic Control*, AC-24(1):36–50, February 1979.

[82] David G. Luenberger. *An Introduction to Observers*, volume 16. IEEE, 1971.

[83] A. M. Lyapunov. The general problem of the stability of motion. *International journal of control*, 55(3):531–534, 1992.

[84] M. Malisoff, F. Mazenc, and S. Ahmed. Output feedback stabilization by reduced order finite time observers using a trajectory based approach. In *2019 American Control Conference (ACC)*, pages 1134–1138. IEEE, 2019.

[85] A. M. Matthias. Nonlinear moving horizon estimation in the presence of bounded disturbances. *Automatica*, 79:306–314, 2017.

[86] P. S. Maybeck. *Stochastic Models, Estimation, and Control*. Academic Press, 1979.

[87] F. Mazenc, M. Malisoff, and M. Krstic. Stability and observer designs using new variants of Halanay's inequality. *Automatica*, 123:109299, 2021.

[88] F. Mazenc, A. Zemouche, and S. I. Niculescu. Observer with small gains in the presence of a long delay in the measurements. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 4327–4332, Melbourne, Australia, Dec 2017.

[89] F. Mazenc, A. Zemouche, and S.I. Niculescu. Observer with small gains in the presence of a long delay in the measurements. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 4327–4332. IEEE, 2017.

[90] T. Mazumdar. Generalized projection theorem and weak noncoercive evolution problems in Hilbert space. *Journal of Mathematical Analysis and Applications*, 46(1):143–168, 1974.

[91] Hannah Michalska and David Q Mayne. Moving horizon observers and observer-based control. *IEEE Transactions on Automatic Control*, 40(6):995–1006, 1995.

[92] H. Nechaf, A. Zemouche, M. Mostefai, T.M. Laleg-Kirati, N. Djeghali, and F. Bedouhene. Linear matrix inequality–based observer design methods for a class of nonlinear systems with delayed output measurements. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, 237(3):433–446, 2023.

[93] H. Nijmeijer and A. J. Van der Schaft. *Nonlinear Dynamical Control Systems*. Springer, New York, 1990.

[94] K. Ogata. *Modern Control Engineering*. Prentice Hall, New Jersey, 5th edition edition, 2010.

[95] C. Pouchol and E. Trélat. Combattre les tumeurs résistantes à la chimiothérapie grâce au contrôle optimal. 2019.

[96] S. Raghavan and J. K. Hedrick. Observer design for a class of nonlinear systems. *Int. J. of Control*, 59(2):515–528, 1994.

[97] R. Rajamani. Observers for Lipschitz nonlinear systems. *IEEE Trans. on Automatic Control*, 43(3):397–401, March 1998.

[98] R. Rajamani. Observers for Lipschitz nonlinear systems. *IEEE transactions on Automatic Control*, 43(3):397–401, 1998.

[99] R. Rajamani, W. Jeon, H. Movahedi, and A. Zemouche. On the need for switched-gain observers for non-monotonic nonlinear systems. *Automatica*, 114:108814, 2020.

[100] Christopher V Rao, James B Rawlings, and David Q Mayne. Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations. *IEEE transactions on automatic control*, 48(2):246–258, 2003.

[101] James B Rawlings and Luo Ji. Optimization-based state estimation: Current status and some new results. *Journal of Process Control*, 22(8):1439–1444, 2012.

[102] K. Röbenack. Improving high gain observer design for nonlinear systems using the structure of the linear part. *Control Autom. Syst.*, 1:57–73, 2016.

[103] R.T. Rockafellar. *Convex analysis*, volume 11. Princeton university press, 1997.

[104] J. Schiller, S. Muntwiler, J. Köhler, M. Zeilinger, and M.A. Müller. A Lyapunov function for robust stability of moving horizon estimation. *IEEE Trans. on Automatic Control*, 68(12):7466–7481, 2023.

[105] Julian D Schiller, Sven Knüfer, and Matthias A Müller. Robust stability of suboptimal moving horizon estimation using an observer-based candidate solution. *IFAC-PapersOnLine*, 54(6):226–231, 2021.

[106] O. Sename, P. Gaspar, and volume=437 year=2013 publisher=Springer Bokor, J. *Robust control and linear parameter varying approaches: application to vehicle dynamics*.

[107] A. Seuret, T. Floquet, J. P. Richard, and S. Spurgeon. Observer design for systems with non small and unknown time-varying delay. *Lecture notes in control and information sciences*, 388:233–242, 2009.

[108] Yuri Shtessel, Christopher Edwards, Leonid Fridman, Arie Levant, et al. *Sliding mode control and observation*, volume 10. Springer, 2014.

[109] J. J. E. Slotine and W. Li. *Applied Nonlinear Control*. Prentice Hall, Englewood Cliffs, NJ, 1991.

[110] Y. Song and J. W. Grizzle. The extended Kalman filter as a local asymptotic obeserver for nonlinear discrete-time systems. In *Proc. 1992 ACC*, pages 3365–3369, Chicago, USA, 1992.

[111] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE transactions on automatic control*, 34(4):435–443, 1989.

[112] E. D. Sontag. Input to state stability: Basic concepts and results. In *Nonlinear and optimal control theory*, pages 163–220. Springer, 2008.

[113] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995.

[114] E. D. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. *Systems & Control Letters*, 29(5):279–290, 1997.

[115] E. D. Sontag and Y. Wang. Notions of input to output stability. *Systems & Control Letters*, 38(4-5):235–248, 1999.

[116] E.D. Sontag and Yuan. W. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, 1996.

[117] H. W. Sorenson. *Kalman Filtering Theory and Application*. IEEE Press, 1990.

[118] S. K. Spurgeon. Sliding mode observers: a survey. *International Journal of Systems Science*, 39(8):751–764, 2008.

[119] B. Targui, O. Hernández-González, C. M. Astorga-Zaragoza, G.V. Guerrero-Ramírez, and M. E. Guerrero-Sánchez. A new observer design for systems in presence of time-varying delayed output measurements. *International Journal of Control, Automation and Systems*, 17(1):117–125, 2019.

[120] A. R. Teel. Connections between razumikhin-type theorems and the iss nonlinear small gain theorem. *IEEE Transactions on Automatic Control*, 41(4):552–556, 1996.

[121] A. R. Teel, TT Georgiou, L Praly, and E Sontag. Input-output stability. *The control handbook*, 4250, 1996.

[122] F. E. Thau. Observing the state of nonlinear dynamic systems. *International Journal of Control*, 17(3):471–479, 1973.

[123] H. Trinh, C. H. Dinh, and N. Saeid. Observer design for positive fractional-order interconnected time-delay systems. *Transactions of the Institute of Measurement and Control*, 41(2):378–391, 2019.

[124] V. I. Utkin. *Sliding Modes in Control and Optimization*. Springer, 1992.

[125] F. A. Valentine. A Lipschitz condition preserving extension for a vector function. *American Journal of Mathematics*, 67(1):83–93, 1945.

[126] B. Wang, W. Chen, B. Zhang, P. Shi, and H. Zhang. A nonlinear observer-based approach to robust cooperative tracking for heterogeneous spacecraft attitude control and formation applications. *IEEE Transactions on Automatic Control*, 68(1):400–407, 2022.

[127] J. C. Willems. Dissipative dynamical systems part i: General theory. *Archive for Rational Mechanics and Analysis*, 45(5):321–351, 1972.

[128] M. Wooldridge. *An introduction to multiagent systems*. John wiley & sons, 2009.

[129] W. Xing, X. Zhao, T. Başar, and W Xia. Optimal transmission scheduling for remote state estimation in CPSs with energy harvesting two-hop relay networks. *Automatica*, 152:110963, 2023.

[130] A. Zemouche. *Sur l'observation de l'état des systèmes dynamiques non linéaires*. PhD thesis, 2007.

[131] A. Zemouche and M. Boutayeb. On LMI conditions to design observers for Lipschitz nonlinear systems. *Automatica*, 49(2):585–591, 2013.

[132] A. Zemouche, M. Boutayeb, and G.I. Bara. Observers for a class of Lipschitz systems with extension to $\mathcal{H}_\infty$ performance analysis. *SCL*, 57(18):18–27, 2008.

[133] A. Zemouche and R. Rajamani. LMI-based observer design for non-globally Lipschitz systems using Kirszbraun–Valentine extension theorem. *IEEE Control Systems Letters*, 6:2617–2622, 2022.

[134] A. Zemouche and R. Rajamani. Observer design for non-globally Lipschitz nonlinear systems using Hilbert projection theorem. *IEEE Control Systems Letters*, 6:2581–2586, 2022.

[135] A. Zemouche, R. Rajamani, G. Phanomchoeng, B. Boulkroune, H. Rafaralahy, and M. Zasadzinski. Circle criterion-based $\mathcal{H}_\infty$ observer design for Lipschitz and monotonic nonlinear systems: Enhanced LMI conditions and constructive discussions. *Automatica*, 85:412 – 425, 2017.

[136] A. Zemouche, R. Rajamani, H. Trinh, and M. Zasadzinski. A new lmi based hâàû observer design method for lipschitz nonlinear systems. In *2016 European Control Conference (ECC)*, pages 2011–2016, 2016.

[137] A. Zemouche, R. Rajamani, H. Trinh, and M. Zasadzinski. A new LMI based $\mathcal{H}_\infty$ observer design method for Lipschitz nonlinear systems. In *IEEE American Control Conference*, Aalborg, Denmark., July 2016.

[138] A. Zemouche, F. Zhang, F. Mazenc, and R. Rajamani. High-gain nonlinear observer with lower tuning parameter. *IEEE Transactions on Automatic Control*, 64(8):3194–3209, 2019.

[139] A. Zemouche, F. Zhang, F. Mazenc, and R. Rajamani. High-gain nonlinear observer with lower tuning parameter. *IEEE Transactions on Automatic Control*, 64(8):3194–3209, 2019.

[140] A. Zemouche, F. Zhang, F. Mazenc, and R. Rajamani. High-gain nonlinear observer with lower tuning parameter. *IEEE Transaction on Automatic Control*, 64(8):3194–3209, 2019.

[141] Ali Zemouche, Fan Zhang, Frédéric Mazenc, and Rajesh Rajamani. High-gain nonlinear observer with lower tuning parameter. *IEEE Transactions on Automatic Control*, 64(8):3194–3209, 2018.

[142] D. Zhang and Y. Shen. Continuous sampled-data observer design for nonlinear systems with time delay larger or smaller than the sampling period. *IEEE Transactions on Automatic Control*, 62(11):5822–5829, 2017.

[143] X. Zhao, L. Liu, M.V. Basin, and Z. Fei. Event-triggered reverse attacks on remote state estimation. *IEEE Transactions on Automatic Control*, 2023.

Abstract

A dynamical system models how a system evolves over time, governed by specific laws or equations. Ensuring the stability of such systems, under a given control action, is vital for predicting their long-term behavior. Stability analysis methods, such as LyapunovâÄôs theory and Input-to-State Stability (ISS), provide essential tools to assess whether a system will remain stable. When direct measurements of all system states are not possible, estimation techniques, such as observers and estimators, play a crucial role in reconstructing internal states from available measurements, enabling effective feedback control. Then, developing advanced estimation schemes and observer design methods is the main motivation of this thesis. Towards this end, three major contributions are proposed in this thesis, as detailed below:

1. Analysis of Robust Moving Horizon Estimation (MHE) Schemes: Since the incremental exponential input/output-to-state stability (i-EIOSS) property is required to synthesize the parameters of the cost function in the MHE context, then, first, two numerical design procedures are proposed to ensure that a nonlinear system achieves i-EIOSS property and to compute the associated i-EIOSS parameters. Then, the robust stability of moving horizon estimation (MHE) is proven. Novel design conditions are established and advanced prediction techniques are introduced.

2. Contributions to LMI-based and high-gain-based Observers: This chapter is split into two parts: the first part deals with observer design for nonlinear systems via Linear Matrix Inequalities (LMIs). The main goal consists of showing that for some families of nonlinear systems, the LMI-based observer design techniques always provide exponential convergent observer. The second part deals with high-gain observer methodology. A novel design method is proposed for systems with arbitrary nonlinear structures contrary to the standard results on high-gain observer methodology developed for triangular nonlinearities.

3. Contributions to observer design for nonlinear systems with delayed measurements: The main idea behind consists of using a dynamic extension technique to transform a system with delayed nonlinear outputs into a system with linear outputs and a delay-dependent integral term in the dynamic process. Based on this transformation, novel and less conservative synthesis conditions are established, ensuring the exponential convergence of the observer despite the large values of the delay in the output measurements.

Key words: Moving Horizon Estimation; Observer design; LMIs; ISS; i-EIOSS; Delayed-outputs.

Résumé

Un système dynamique modélise l'évolution d'un système au fil du temps, régie par des lois ou des équations spécifiques. Assurer la stabilité de ces systèmes, sous une action de contrôle donnée, est essentiel pour prévoir leur comportement à long terme. Les méthodes d'analyse de stabilité, telles que la théorie de Lyapunov et la stabilité entrée-état (ISS), fournissent des outils essentiels pour évaluer si un système restera stable. Lorsque la mesure directe de tous les états du système n'est pas possible, des techniques d'estimation, telles que les observateurs et les estimateurs, jouent un

rôle crucial en reconstruisant les états internes à partir des mesures disponibles, permettant ainsi un contrôle en retour de sortie efficace. Le développement de schémas d'estimation avancés et de méthodes de conception d'observateurs constitue la principale motivation de cette thèse. Dans ce cadre, trois contributions majeures sont proposées dans cette thèse, détaillées ci-dessous :

1. Analyse des schémas d'estimation à horizon glissant robuste (MHE) : Comme la propriété de stabilité exponentielle incrémentale entrée/sortie-état (i-EIOSS) est nécessaire pour synthétiser les paramètres de la fonction de coût dans le contexte du MHE, deux procédures de conception numérique sont d'abord proposées pour garantir qu'un système non linéaire réalise la propriété i-EIOSS et pour calculer les paramètres i-EIOSS associés. Ensuite, la stabilité robuste de l'estimation en horizon glissant (MHE) est prouvée. De nouvelles conditions de conception sont établies et des techniques de prédiction avancées sont introduites.

2. Contributions aux observateurs basés sur les LMIs et les observateurs à grand-gain : Ce chapitre est divisé en deux parties : la première partie traite de la conception d'observateurs pour des systèmes non linéaires via des inégalités matricielles linéaires (LMIs). L'objectif principal est de montrer que, pour certaines familles de systèmes non linéaires, les techniques de conception d'observateurs basées sur les LMIs fournissent toujours des observateurs exponentiellement convergents. La deuxième partie concerne la méthodologie des observateurs à grand-gain. Une nouvelle méthode de conception est proposée pour des systèmes avec des structures non linéaires arbitraires, contrairement aux résultats standards sur la méthodologie des observateurs à grand-gain développés pour les non-linéarités triangulaires.

3. Contributions à la conception d'observateurs pour des systèmes non linéaires avec des mesures retardées : L'idée principale consiste à utiliser une technique d'extension dynamique pour transformer un système avec des sorties non linéaires retardées en un système avec des sorties linéaires et un terme intégral dépendant du retard dans le processus dynamique. Sur la base de cette transformation, de nouvelles conditions de synthèse moins contraignantes sont établies, garantissant la convergence exponentielle de l'observateur malgré les valeurs élevées du retard dans les mesures de sortie.

Mots-clés : Estimation à horizon glissant ; Observer design ; LMIs ; i-EIOSS; mesures retardées.

Riassunto

Un sistema dinamico è un sistema evolve nel tempo, governato da leggi o equazioni specifiche. Garantire la stabilità di tali sistemi, sotto un'azione di controllo specifica, é fondamentale per prevedere il loro comportamento a lungo termine. I metodi di analisi della stabilità, come la teoria di Lyapunov e la stabilità ingresso-stato (ISS), forniscono strumenti essenziali per valutare se un sistema rimarrà stabile. Quando non é possibile misurare direttamente tutti gli stati del sistema, le tecniche di stima, come gli osservatori e gli stimatori, svolgono un ruolo cruciale nel ricostruire gli stati interni a partire dalle misurazioni disponibili, permettendo un controllo a retroazione efficace. Lo sviluppo di schemi avanzati di stima e metodi di progettazione di osservatori éla principale motivazione di questa tesi. A tal fine, in questa tesi vengono proposte tre principali contribuzioni, come dettagliato di seguito:

1. Analisi degli schemi di stima a orizzonte mobile robusti (MHE): Poich la proprietà di stabilità esponenziale incrementale ingresso/uscita-stato (i-EIOSS) énecessaria per sintetizzare i parametri della funzione di costo nel contesto MHE, vengono proposte due procedure di

progettazione numerica per garantire che un sistema non lineare raggiunga la proprietà i-EIOSS e per calcolare i parametri i-EIOSS associati. Successivamente, viene dimostrata la stabilità robusta della stima a orizzonte mobile (MHE). Vengono stabilite nuove condizioni di progettazione e introdotte tecniche di previsione avanzate.

2. Contributi agli osservatori basati su LMIs e osservatori a guadagno elevato: Questo capitolo édiviso in due parti: la prima parte tratta della progettazione di osservatori per sistemi non lineari tramite disuguaglianze matriciali lineari (LMIs). L'obiettivo principale édimostrare che, per alcune famiglie di sistemi non lineari, le tecniche di progettazione di osservatori basate su LMIs forniscono sempre un osservatore esponenzialmente convergente. La seconda parte riguarda la metodologia degli osservatori a guadagno elevato. Viene proposta una nuova metodologia di progettazione per sistemi con strutture non lineari arbitrarie, in contrasto con i risultati standard sulla metodologia degli osservatori a guadagno elevato sviluppati per le non-linearità triangolari.

3. Contributi alla progettazione di osservatori per sistemi non lineari con misurazioni ritardate: L'idea principale consiste nell'usare una tecnica di estensione dinamica per trasformare un sistema con uscite non lineari ritardate in un sistema con uscite lineari e un termine integrale dipendente dal ritardo nel processo dinamico. Sulla base di questa trasformazione, vengono stabilite nuove condizioni di sintesi meno conservative, garantendo la convergenza esponenziale dell'osservatore nonostante i grandi valori del ritardo nelle misurazioni in uscita.

Parole chiave: Stima a orizzonte mobile; Progettazione di osservatori; LMIs; ISS; i-EIOSS; Uscite ritardate.