

UNIVERSITY OF GENOVA
PHYSICS DEPARTMENT



DOCTORAL THESIS

NOTES FROM THE BULK

Author :
ERICA BERTOLINI

Advisor :
Prof. Nicola MAGGIORE

*A thesis submitted in fulfillment of the requirements
for the Doctoral degree*

in

Physics and Nanoscience - XXXVI cycle

March 2024

À ma grande-mère

ABSTRACT

In most cases Quantum Field Theories (QFTs) are considered without boundaries and have been successful in providing descriptions of fundamental interactions, including gravity and cosmology. This is because one is generally interested in bulk effects, where the boundary can be neglected. Nevertheless, boundaries do exist, and in some cases, their effects are self-evident and dominant. Important phenomena pertaining to condensed matter physics, like the Fractional Quantum Hall Effect and the behavior of Topological Insulators have been explained in terms of topological QFTs with boundaries. This is rather counterintuitive : topological QFTs, when considered without boundaries, have a vanishing Hamiltonian and no energy-momentum tensor. They might appear as the least physical theories one can imagine. Despite this, when a boundary is introduced, an extremely rich physics emerges, which can be observed experimentally. The scope of this Ph.D thesis is to study the effects of the presence of a boundary from a Quantum Field Theoretical perspective, searching for new physics and explanations of observed phenomena. In particular, thanks to the formal QFT setting, the issue of the existence of local, accelerated, edge modes in Hall systems is analyzed and understood in terms of the bulk-to-boundary approach as related to a curved background in topological QFTs with boundary. Within this formalism the induced metric on the boundary can be associated to the ad hoc potential introduced in the phenomenological models in order to obtain such non-constant edge velocities. This also leads to the prediction of local modes for Topological Insulators, and Quantum Spin Hall systems in general. The paradigm for which only topological QFTs have a physical content on the boundary is broken, and also non-Topological Quantum Field Theories such as fracton models and Linearized Gravity are shown to have non-trivial boundary dynamics. Indeed due to the breaking of their defining symmetry both models have a current algebra of the Kac-Moody type on the boundary. In the case of fractons this algebra is in a generalized form, which also appears in some kinds of higher order Topological Insulators, a sign of a possible relation between these materials and edge states of fracton quasiparticles. Concerning the theory of Linearized Gravity, instead, the algebra is a standard Kac-Moody one, whose presence was suspected, but never proved before. Physical results on the boundary range between condensed matter, elasticity and (massive) gravity models. A collateral result, which enrich this Thesis, is the building of a new covariant QFT for fractons with a peculiar gauge structure. This new model better highlight the properties of these quasiparticles.

PUBLICATIONS

The original content of this Thesis is mainly based on

- [1] Erica Bertolini and Nicola Maggiore. «Holographic Projection of Electromagnetic Maxwell Theory». *Symmetry* 12.7 (2020), p. 1134. DOI: [10.3390/sym12071134](https://doi.org/10.3390/sym12071134). arXiv: [2006.14902](https://arxiv.org/abs/2006.14902) [hep-th] (Chapter 6)

- [2] Erica Bertolini, Giulio Gambuti, and Nicola Maggiore. «Notes from the bulk: Metric dependence of the edge states of Chern-Simons theory». *Phys. Rev. D* 104.10 (2021), p. 105011. DOI: [10.1103/PhysRevD.104.105011](https://doi.org/10.1103/PhysRevD.104.105011). arXiv: [2110.13203](https://arxiv.org/abs/2110.13203) [hep-th] (Section 5.1)

- [3] Erica Bertolini, Filippo Fecit, and Nicola Maggiore. «Topological BF Description of 2D Accelerated Chiral Edge Modes». *Symmetry* 14.4 (2022), p. 675. DOI: [10.3390/sym14040675](https://doi.org/10.3390/sym14040675). arXiv: [2203.13520](https://arxiv.org/abs/2203.13520) [hep-th] (Section 5.2)

- [4] Erica Bertolini and Nicola Maggiore. «Maxwell theory of fractons». *Phys. Rev. D* 106.12 (2022), p. 125008. DOI: [10.1103/PhysRevD.106.125008](https://doi.org/10.1103/PhysRevD.106.125008). arXiv: [2209.01485](https://arxiv.org/abs/2209.01485) [hep-th] (Section 7.2)

- [5] Erica Bertolini, Alberto Blasi, Andrea Damonte, and Nicola Maggiore. «Gauging Fractons and Linearized Gravity». *Symmetry* 15.4 (2023), p. 945. DOI: [10.3390/sym15040945](https://doi.org/10.3390/sym15040945). arXiv: [2304.10789](https://arxiv.org/abs/2304.10789) [hep-th] (Section 7.3)

- [6] Erica Bertolini, Nicola Maggiore, and Giandomenico Palumbo. «Covariant fracton gauge theory with boundary». *Phys. Rev. D* 108.2 (2023), p. 025009. DOI: [10.1103/PhysRevD.108.025009](https://doi.org/10.1103/PhysRevD.108.025009). arXiv: [2306.13883](https://arxiv.org/abs/2306.13883) [hep-th] (Section 7.4)

- [7] Erica Bertolini and Nicola Maggiore. «Theory of a symmetric tensor field with boundary: Kac-Moody algebras in linearized gravity». *Phys. Rev. D* 108.10 (2023), p. 105012. DOI: [10.1103/PhysRevD.108.105012](https://doi.org/10.1103/PhysRevD.108.105012). arXiv: [2310.20303](https://arxiv.org/abs/2310.20303) [hep-th] (Section 8.2)

Простите, не поверю, — ответил Воланд, —
этого быть не может. Рукописи не горят.

Il Maestro e Margherita - M.A.Bulgakov

ACKNOWLEDGEMENTS

First of all I want to thank people I had the pleasure to work with : Alberto Blasi, Andrea Damonte, Filippo Fecit, Giulio Gambuti and Giandomenico Palumbo. I also want to thank Dario Ferraro, Paolo Meda, Maura Sassetti and Niccolò Traverso Ziani for enlightening discussions and information about some phenomenological and experimental scenarios. Special thanks to all those people I encountered during my last year of traveling, especially the international group I met in Vietnam, in particular Samuel Valach and Osama Khlaif, I hope to see you soon guys ! Last but not least (on the contrary) the most important thanks goes to Nicola Maggiore, there is too much I would say, but in the end, here, a "thank you" summarize it perfectly.

I also thank the Galileo Galilei Institute (GGI) in Firenze, Italy, the International Center for Interdisciplinary Science and Education (ICISE) in Quy Nhon, Vietnam, and the Nordita Institute in Stockholm, Sweden, for hospitality during part of these works.

These works have been partially supported by INFN Scientific Initiative GSS : “Gauge Theory, Strings and Supergravity” and by MIUR grant “Dipartimenti di Eccellenza” (100020-2018-SD-DIP-ECC_001).

CONTENTS

I	STATE OF THE ART - THE PHYSICS IS IN THE BOUNDARY	1
1	BOUNDARIES IN TIME	3
1.1	1948 : Casimir Effect	5
1.2	1981 : Symanzik's method	6
1.3	1997 : Holography and AdS/CMT	8
2	THE QFT APPROACH IN THIS THESIS	11
2.1	The symmetry, the action and the boundary	11
2.2	The breaking of the symmetry and current algebra	13
2.3	The induced lower-dimensional theory	15
II	TOPOLOGICAL FIELD THEORIES	19
3	AN OVERVIEW OF TQFTS (1988)	21
4	CHERN-SIMONS AND BF WITH BOUNDARY	25
4.1	Chern-Simons and the edge states of FQHE (1992)	25
4.2	The 3D BF model and Topological Insulators (2010)	28
5	METRIC DEPENDENCE OF THE EDGE STATES OF TQFTS	31
5.1	Edge states of Chern-Simons theory	32
5.2	BF description of 2D accelerated chiral edge modes	47
III	NON-TOPOLOGICAL FIELD THEORIES	67
6	AN OVERVIEW FOR THE NON-TOPOLOGICAL CASE	69
7	FRACTONS	75
7.1	What are fractons?	76
7.2	Maxwell Theory of Fractons	79
7.3	Gauge for Two	98
7.4	Fractons with Boundary	110
8	LINEARIZED GRAVITY	129
8.1	The theory of gravitational waves	129
8.2	Linearized gravity with boundary	132
IV	FINAL REMARKS	147
9	CONCLUDING REMARKS	149
	APPENDIX	155
A	SOME TECHNICAL REMARKS	157
A.1	Derivative of the Heaviside step distribution in curved spacetime	157
A.2	Basis for the Ω -tensors	158
B	DETAILED CALCULATIONS	161
B.1	Solutions of the boundary conditions for the BF model	161
B.2	Calculation of the fracton propagators	162
B.3	Fractons with boundary : detailed calculations	164
	BIBLIOGRAPHY	171

LIST OF TABLES

Table 1	Fracton phenomenology	78
Table 2	Properties of the fracton and Maxwell field strengths	83
Table 3	Gauge-fixing : comparison with the scalar case	109
Table 4	Holographic contacts in fracton and LG with boundary	122
Table 5	Comparison between EoM and 3D Maxwell	125
Table 6	Holographic contacts : constraints and discrete symmetries	127
Table 7	Holographic contacts in LG with boundary	145
Table 8	Fractons and LG with boundary <i>vs</i> LG with boundary	152

ACRONYMS

BC	Boundary Conditions
CFT	Conformal Field Theory
DoF	Degrees of Freedom
EoM	Equations of Motion
FQHE	Fractional Quantum Hall Effect
GNC	Gaussian Normal Coordinates
KM	Kac-Moody
LG	Linearized Gravity
QAH	Quantum Anomalous Hall
QFT	Quantum Field Theory
QSH	Quantum Spin Hall
TI	Topological Insulators
TQFT	Topological Quantum Field Theory
nD	n -dimensional

Part I

STATE OF THE ART - THE PHYSICS IS IN THE BOUNDARY

Boundaries exist in Nature. Their presence is usually swept under the rug, when teaching classes, except then saying, rather vaguely indeed, that “boundary effects” should be taken into account, which substantially affect the idealized bulk-only theories. Think, for instance, to the inexistent *infinitely long* solenoids, or to the ideal *infinitely extended* parallel plates of a capacitor. But boundaries have a long history of surprising successes.

BOUNDARIES IN TIME

Boundaries are everywhere in real life. What would it even mean for an everyday object to be without boundaries? We also set conceptual boundaries, limits to our lives, properties, even countries. But not in physics. We want it to be ideal, thus everything is infinite and perfect. But physics is the language of Nature, and Nature does not care for what we want. Indeed the presence of boundaries in a model may have strong consequences, even leading to unexpected phenomena, or a whole new physical interpretation, and the aim of this Thesis is to dwell on these aspects, searching for new physics through the powerful tools of Quantum Field Theory (QFT). In the long history of physics, the first boundary effect has been conceptualized in the '40s by Casimir [8], where he related the existence of vacuum pressure as to the presence two conducting parallel plates. This was later experimentally verified [9]. In field theory, the pioneering work which must be referred to is [10], where Symanzik, using the Casimir effect as a playground, defined the boundary in field theory as the surface, or the line, which separates propagators, *i.e.* two-points Green functions, in the sense that propagators computed between points lying on opposite sides of the boundary must vanish. This approach relies on very general principles of field theory, like locality and power counting, and not much space is left to arbitrariness. For instance, the conditions which must be fulfilled by the quantum fields on the boundary (of the Dirichlet, Neumann or Robin type), are not put by hand in the theory, but are those which naturally come out from the request of separability of propagators. This approach has been very fruitful also in the study of Topological Quantum Field Theories (TQFTs) with boundary [11–13]. TQFTs are characterized by the absence of physical local observables, the only observables being global properties of the manifold they are built on, like the genus, or the numbers of holes and handles [14]. In other words, TQFTs have vanishing Hamiltonian and energy-momentum tensor, thus no physical content of their own. Things change when a boundary is taken into account. One of the first, and still more important, example has been the classification of all known rational Conformal Field Theories (CFTs), obtained by the introduction of a boundary in three-dimensional (3D) topological Chern-Simons theory [15]. In that case, the boundary is a circle on a flat bulk 3D manifold. Later, the introduction of a boundary in TQFTs gave a field theoretical predictive description of the edge modes of the Fractional Quantum Hall Effect (FQHE) [16–20] and of another kind of topological state of matter, the Topological Insulators (TI), both in 3D and 4D [21–24]. In particular, the Chern-Simons coupling constant has been found to be tightly related to the filling factor of FQHE [25] and to the central charge of the Kac-Moody (KM) algebras [26, 27] formed by conserved currents on the edge of Chern-Simons theory [28–30]. Similarly, another class of TQFTs, the BF models [14, 31–33], which can be defined on any spacetime dimensions, have been found to describe the bulk theory of the TI [34, 35]. Hence, the introduction of a boundary in TQFTs is the only way to give local, physical and measurable observables to theories which otherwise show “only” formal interest [36–38], and major results have been achieved in condensed matter theory, mainly. Thus what is striking is that

when dealing with a boundary in a TQFT, local Degrees of Freedom (DoF) arise, and set the appropriate conditions for uncovering new physics. However much less investigated is the fact that boundary physics is not a peculiarity of TQFTs only. For instance, to a completely different framework belongs the so called AdS/CFT correspondence, also known as *gauge-gravity duality*, where a higher dimensional gravity theory in Anti-de-Sitter space can be mapped into an induced, lower dimensional, CFT through a well established dictionary [39–41]. Interestingly, it started in the '90s in the context of string theory, and it was later shown to have deep consequences in condensed matter theory (again!), even leading to promising phenomenological results concerning, for instance, the theory of superconductivity and of strange metals [42–44]. This duality is sometimes called AdS/CMT [45]. This Thesis lies on the solid ground of first principles of QFT and takes inspiration from the work of Symanzik to treat a kind of holography without gravity [46, 47]. The aim is to draw insights from phenomenology, in order to apply the formal tools of QFT to physical problems. For instance we will update well established facts involving TQFTs with boundary and quantum Hall phenomena to explain new effects which cannot be justified within the standard context of a QFT with boundary. In particular we will extend the flat bulk-to-boundary paradigm to a curved one and we shall see that this will allow us to give a theoretical description of accelerated edge modes recently observed in some Hall systems [48] without any heuristic introduction, which is the usual approach. The induced metric on the bulk will then be interpreted as a confining/interacting potential in the sample, in the same way as the coupling constant of the Chern-Simons theory, for instance, is related to the filling factor of the FQHE. This approach also leads to new predictions for non-constant edge velocities in Quantum Spin Hall (QSH) systems. Concerning standard QFTs, recently it has been shown [1, 49] that also non-TQFTs has nontrivial boundary dynamics. We will also consider boundary effects in new physical theories. That is the case for instance of fracton models, which describe quasiparticles with restricted mobility that are widely drawing the attention of the community, both theoretical, and experimental, from many areas of physics and mathematical physics [50–56]. These models have the peculiarity of being related to new phases of matter, but also, quite surprisingly to the theory of Linearized Gravity (LG), thus with gravitational waves : from microscopic to “cosmological” sizes. This makes such models quite interesting from the point of view of QFT in general (with and without boundary). We shall indeed see that both the theory defined by the covariant fracton symmetry and LG share a rich boundary dynamics, in the context of condensed matter and 3D theories of gravity.

The Thesis is organized as follows. Part I is dedicated to the introduction : Chapter 1 gives the general setting and motivations, presenting some historically relevant examples of the role played by boundaries in QFT. We will briefly review the Casimir effect (Section 1.1), Symanzik’s paper on the Casimir effect [10] (Section 1.2), and we finally give an example of AdS/CFT and the corresponding gauge/gravity (Section 1.3). In Chapter 2 we present the approach and techniques applied in this Thesis to study QFTs with boundary. Part II concerns TQFTs, which are briefly introduced in Chapter 3. In Chapter 4 the 3D Chern-Simons and BF abelian theories with boundary are presented, in flat spacetime, with their physical consequences and phenomenology. This will serve as an introduction to Chapter 5, where the 3D Chern-Simons and BF models are studied with boundary in curved spacetime, in order to find an explanation

of observations for which the standard flat bulk-to-boundary approach is not enough. The study of 4D Maxwell theory with boundary [1], showed that also non-TQFTs display nontrivial physics on the boundary, and this leads to Part **iii**, which focuses on boundary effects on non-TQFTs. After a brief overview of the 4D Maxwell theory with boundary in Chapter 6, Chapter 7 is dedicated to fracton models, where a new covariant QFT for fractons is built (Section 7.2), which agrees with the Literature (Section 7.1). The peculiar gauge structure of the fracton theory is analyzed (Section 7.3) and finally the theory with boundary is studied (Section 7.4). The strong relation between fractons and LG gives a starting point for considering LG with boundary in Chapter 8. Part **iv** concludes the Thesis with some final remarks.

The original contributions that are collected in this Thesis concern in particular Chapter 5, Chapter 7 (specifically Sections 7.2, 7.3 and 7.4), and Chapter 8 (Section 8.2).

In this introductory Part it can be of interest to present some well known cases where boundaries had a leading. The scope of what follows is to both present the general physical idea which motivated the papers, and to describe the technique applied in each situation, in order to highlight differences and similarities which inspired this Thesis.

1.1 1948 : CASIMIR EFFECT

The first time boundary effects were investigated was in 1948 in [8]. Inspired by a previous work on the interaction between atoms and a plate [57], in [8], in the context of classical electrodynamics, H.B.G. Casimir studied the interaction between two perfectly conducting plates, focusing on the change of electromagnetic vacuum energy. The idea was to look at if and how two parallel flat boundaries (plates) of conducting material could alter the vacuum energy just as a consequence of their existence. The system is composed of a perfectly conducting cubic cavity of volume L^3 with an additional conducting square plate of side L placed parallel to the xy face of the cube. This will play the role of the boundary, first placed at a small distance $a \ll L$ from the xy face and then at a very large distance $a \sim L/2$. The interaction between the plate and the xy face is given by the difference in energy between the two situations, which has a finite value given by

$$\delta E = \frac{1}{2} \sum \hbar\omega|_a - \frac{1}{2} \sum \hbar\omega|_{L/2}, \quad (1.1.1)$$

where the sums are over all the possible resonance frequencies. Given the wave numbers $k_x = \frac{\pi}{L}n_x$, $k_y = \frac{\pi}{L}n_y$, $k_z = \frac{\pi}{a}n_z$, where n_x , n_y and n_z are positive integers, for a big enough cavity, *i.e.* for large L , one can treat the x , y components of the wave number (those parallel to the conductive boundary) as continuous variables for which the sum can be replaced by an integral. In polar coordinates, with $x^2 \equiv k_x^2 + k_y^2$, the first term at the right hand side (r.h.s.) of (1.1.1) is

$$\frac{1}{2} \sum \hbar\omega|_a = \hbar c \frac{L^2}{2\pi} \sum_0^\infty \theta_n \int_0^\infty x dx \sqrt{x^2 + n_z^2 \frac{\pi^2}{a^2}} \quad (1.1.2)$$

where θ_n is meant to indicate that the term with $n_z = 0$ has to be multiplied by 1/2 since to every k_i correspond two standing waves unless one of the n_i is zero. This can

be ignored in the case of the continuous variables. For very large a , corresponding to the second term at the r.h.s. of (1.1.1), also this last summation can be replaced by an integral and it is therefore easily seen that the interaction energy (1.1.1) is given by

$$\delta E = \hbar c \frac{\pi^2 L^2}{4a^3} \left[\sum_0^\infty \theta_n \int_0^\infty du \sqrt{n_z^2 + u} - \int_0^\infty \int_0^\infty du dn_z \sqrt{n_z^2 + u} \right], \quad (1.1.3)$$

with $u \equiv \frac{a^2}{\pi^2} x^2$. At this point the integral can still diverge for big enough k . This physically means that high frequency waves are not affected by the cavity, nor by the position of the plates: the zero pt energy would diverge as if unaffected. The solution is to introduce a function which cuts away the divergent contribution by going sufficiently rapidly to zero in k . This function and the cutting frequency depend on the material of the cavity, but will not intervene in the approximation that will be taken into account. Indeed by following this approach, and using Euler-Maclaurin formula to estimate the difference between the integrals, one finds

$$\frac{\delta E}{L^2} = -\hbar c \frac{\pi^2}{720} \frac{1}{a^3}. \quad (1.1.4)$$

There is thus a force given by

$$F = -\frac{1}{L^2} \frac{\partial \delta E}{\partial a} = -\hbar c \frac{\pi^2}{240} \frac{1}{a^4}. \quad (1.1.5)$$

This means that two uncharged parallel conductive plates (boundaries), in vacuum, experience an attractive force between them. The existence of such an attractive force between the two plates is the consequence of vacuum energy pressure of electromagnetic waves. In evaluating this force one can see that it is observable only for very small values of a , *i.e.* the distance between the plates. Indeed it took many years before a first direct measure, in 1997, was made possible and verified the effect [9].

1.2 1981 : SYMANZIK'S METHOD

Many years later Symanzik, in [10], applied the "high flexibility and perfection of the QFT formalism" (*sic*) to recover the above result of the casimir force (1.1.5) as a QFT with boundary for the first time. The Casimir effect was indeed the perfect playground to investigate the meaning of a boundary in QFTs. Indeed how could Boundary Conditions (BC) be implemented without spoiling the renormalizability of the theory? What are the consequences on the observables? These are all legit and fundamental questions when dealing with a QFT with boundary.

Boundary conditions

The model considered by Symanzik in [10] is that of a free massive scalar field in euclidean four-dimensional spacetime¹, described by the lagrangian density

$$\mathcal{L}_0 = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \quad (1.2.1)$$

¹ the original paper [10] works in generic ν -dimensional spacetime, but for the introductory scope of this Thesis it is sufficient to work within the standard four-dimensional spacetime (*i.e.* $\nu = 4$), in order to better highlight crucial points and compare the results and procedure with those of the previous Section.

The model is considered to have a 3D boundary $\partial\mathcal{M}$ (the third dimension corresponding to euclidean time) described by the equation $f(x) = 0$, which separates the space into two regions: \mathcal{M} with $f(x) > 0$, and its complementary \mathcal{M}' with $f(x) < 0$. This is therefore a two-sided boundary, and the massive scalar theory lives on both sides of it. As we shall see this is different to what will be considered in the original part of this Thesis, which is a one-sided boundary. While this last would be more of a “confined” situation, where the theory is in an interior region delimited by the boundary, the present one is more similar to a defect or interface [58–61]. Due to the presence of the boundary, an augmented lagrangian density needs to be considered

$$\mathcal{L} = \mathcal{L}_0 - \delta(f(x))\phi(x - \epsilon)\partial_\mu\phi(x)\partial_\mu f(x), \quad (1.2.2)$$

where ϵ is a vector normal to the boundary $\partial\mathcal{M}$. Since on the boundary the field $\phi(x)$ and its derivative with respect to the normal of the boundary ($\partial_n\phi(x)$) are independent, this additional contribution to the lagrangian implements Dirichlet and Neuman BC in the two sides of the boundary respectively, as follows

$$\phi(x) \rightarrow 0 \quad x \rightarrow \partial\mathcal{M} \text{ from } \mathcal{M} \quad (1.2.3)$$

$$\partial_n\phi(x) \rightarrow 0 \quad x \rightarrow \partial\mathcal{M} \text{ from } \mathcal{M}' . \quad (1.2.4)$$

From (1.2.2) the generating functional of Green function can be computed to be

$$Z[J] = \int [d\phi] e^{\int \mathcal{L} + \int J\phi} \propto e^{\frac{1}{2} \int_{\mathcal{M}} J(x)G_D(x,y)J(y) + \frac{1}{2} \int_{\mathcal{M}'} J(x)G'_N(x,y)J(y)}, \quad (1.2.5)$$

where G_D is the Dirichlet Green function in \mathcal{M} given by

$$\begin{aligned} (m^2 - \partial_x^2)G_D(x, y) &= \delta(x - y) & \text{if } x, y \in \mathcal{M} \\ G_D(x, y) &= 0 & \text{if } x \in \partial\mathcal{M}, y \in \mathcal{M}, \end{aligned} \quad (1.2.6)$$

and G'_N the Neumann Green function in \mathcal{M}'

$$\begin{aligned} (m^2 - \partial_x^2)G'_N(x, y) &= \delta(x - y) & \text{if } x, y \in \mathcal{M}' \\ \partial_n G'_N(x, y) &= 0 & \text{if } x \in \partial\mathcal{M}, y \in \mathcal{M}'. \end{aligned} \quad (1.2.7)$$

Therefore the presence of the boundary affects the two-pt functions (propagators) of the theory as shown by (1.2.6) and (1.2.7), as a consequence of the surface term introduced in (1.2.2). Indeed the main consequence is that in (1.2.5) there are no correlations between points in \mathcal{M} and in \mathcal{M}' , have thus been decoupled from each other by the boundary, and propagators of points separated by the boundary have to vanish.

Casimir effect

We have seen in Section 1.1 that the presence of two parallel boundaries (plates) affects the vacuum energy through their presence, leading to a Casimir force (1.1.5) which attract one-another. In the context of a model described by the lagrangian (1.2.2), the vacuum energy for the scalar field is

$$E_{\mathcal{M}} = - \lim_{T \rightarrow \infty} \frac{1}{T} \ln \int [d\phi] e^{\int_0^T dt \int_{\mathcal{M}} dx \mathcal{L}(\phi, \partial\phi)}, \quad (1.2.8)$$

where $\bar{\mathcal{M}}$ denotes the spatial part of the boundary \mathcal{M} , *i.e.* $\mathcal{M} = \bar{\mathcal{M}} \times [0, T]$, $\partial\mathcal{M} = \partial\bar{\mathcal{M}} \times [0, T]$. The relevant quantity is the total energy, which is the ground-state energy in $\bar{\mathcal{M}}$, for instance with Dirichlet BC, plus the one in the complementary region $\bar{\mathcal{M}}'$ with Neumann BC. The boundary-independent contribution can be omitted, leaving surface terms only. Therefore in the context of the Casimir effect, considering Dirichlet conditions on the inner sides of two parallel plates, at distance a , the total energy is the same as if the two Neumann regions were absent. Thus for a QFT computation of the Casimir effect the surface graphs have merely the $\phi\partial_n\phi$ vertices on $\partial\mathcal{M}$ appearing in (1.2.2), for which the expansion is

$$E_{\bar{\mathcal{M}}}^{(D)} + E_{\bar{\mathcal{M}}}^{(N)} - const = - \lim_{T \rightarrow \infty} \frac{1}{2T} \left(2\text{Tr}\overline{\partial_n G} + \frac{1}{2}2^2\text{Tr}\overline{\partial_n G} \cdot \overline{\partial_n G} + \dots \right) \quad (1.2.9)$$

where Tr is the trace on the surface $\partial\mathcal{M}$, and $\overline{\partial_n G}$ is

$$\overline{\partial_n G} = -\frac{1}{2}\pi^{-2}\Gamma(2) [(X - X')^2 + a^2]^{-2} a, \quad (1.2.10)$$

with X and X' the $2 + 1$ -dimensional coordinates on the plates. While on the r.h.s. of (1.2.9) the odd terms all vanish, the even ones can be summed, giving

$$\frac{1}{area} \left(E_{\bar{\mathcal{M}}}^{(D)} + E_{\bar{\mathcal{M}}}^{(N)} - const \right) = -2^{-4}\pi^{-2}\Gamma(2)\zeta(4)a^{-3} = -\frac{1}{1440} \frac{\pi^2}{a^3}, \quad (1.2.11)$$

where $\zeta(4) = \sum_{n=1}^{\infty} n^{-4} = \pi^4/90$, and $\Gamma(2) = 1$ is the Gamma function. Eq.(1.2.9) is analogous to the Casimir potential obtained in Section 1.1, keeping in mind that here $\hbar = c = 1$.

1.3 1997 : HOLOGRAPHY AND ADS/CMT

The AdS/CFT correspondence, conjectured for the first time by Maldacena in 1997 [62], is another example of the successful approaches involving boundaries. It shows dualities between $d + 1$ -dimensional gravity bulk theories and their holographic counterparts on their d -dimensional boundaries, where the extra “energy” dimension run from zero to infinity [40, 41]. The AdS/CFT holographic correspondence, a.k.a. gauge/gravity duality, was originally conjectured in string theory and captured the attention of many physicists. For instance a typical paper on the topic had abstract like this [39]

Recently, it has been proposed by Maldacena that large N limits of certain conformal field theories in d dimensions can be described in terms of supergravity (and string theory) on the product of $d + 1$ -dimensional AdS space with a compact manifold. Here we elaborate on this idea and propose a precise correspondence between conformal field theory observables and those of supergravity: correlation functions in conformal field theory are given by the dependence of the supergravity action on the asymptotic behavior at infinity. In particular, dimensions of operators in conformal field theory are given by masses of particles in supergravity. As quantitative confirmation of this correspondence, we note that the Kaluza-Klein modes of Type IIB supergravity on $AdS_5 \times S^5$ match with the chiral operators of $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions. With some further assumptions, one can deduce a Hamiltonian version of the correspondence and show that the $\mathcal{N} = 4$ theory has a large N phase transition related to the thermodynamics of AdS black holes.

Later on the duality received much attention in condensed matter theory, enough to introduce for that case a new acronym (AdS/CMT). The bulk/boundary correspondence turned out to be a powerful new technique to study strongly coupled systems, reviewed for instance in [42–45, 63]. Despite the completely different context, the dictionary still involve a $d + 1$ -dimensional gravity theory in anti-de-Sitter (AdS) space-time, *i.e.* with negative curvature and metric of the form

$$ds^2 = \frac{L^2}{r^2} (dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu) , \quad (1.3.1)$$

whose d -dimensional boundary CFT is related to phenomena in condensed matter theory [44]. Starting from the “quantum gravity” $d + 1$ -dimensional bulk action S_{QG} and bulk field ϕ , the dictionary prescribes [39, 64] that the boundary d -dimensional QFT has source ϕ_0 related to the boundary value of the bulk field ϕ , and is obtainable as a boundary limit of the on-shell (*i.e.* saddle point) gravitational path integral as

$$Z_{QFT}[\phi_0] = \int [dA] e^{iS_{QFT} + i \int \phi_0 \mathcal{O}(A)} = Z_{QG}[\phi \rightarrow \phi_0] \sim e^{iS_{QG}|_{\phi \rightarrow \phi_0}} . \quad (1.3.2)$$

Thus the bulk field ϕ is related to the boundary operator \mathcal{O} through its boundary value ϕ_0 becoming a source for such operator

$$\phi|_{bd} = \phi_0 \leftrightarrow \mathcal{O} . \quad (1.3.3)$$

This is the core of the duality and the key to computing fundamental quantities (n-pt functions) in the boundary theory through

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle = \frac{\delta^{(n)} \log Z_{QFT}[\phi_0]}{\delta \phi_0(x_1) \dots \delta \phi_0(x_n)} \sim \frac{\delta^{(n)} Z_{QG}^{os}[\phi_0]}{\delta \phi_0(x_1) \dots \delta \phi_0(x_n)} . \quad (1.3.4)$$

An example : holographic conductivity

The easiest way to quickly show the peculiar features of this duality is to try and compute conductivity applying this the dictionary. This means that we want to compute Ohm’s law through the AdS/CFT correspondence, thus starting from a theory general relativity with boundary. In a 3+1-dimensional asymptotically AdS theory of gravity, electromagnetism is described by the following Einstein-Maxwell action

$$S_{bulk} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(R + \frac{6}{L^2} \right) - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \right] , \quad (1.3.5)$$

where L is the AdS radius, and $\mu\nu\dots$ are the bulk $d + 1$ indices ($m, n\dots$ will be those of the boundary). We have thus a bulk theory with a charged black hole, which we say to be at a chemical potential μ [65]. The Reissner-Nördstrom black hole solution, which comes from solving Einstein’s Equations of Motion (EoM), is

$$ds^2 = \frac{L^2}{r^2} \left[-f(r) dt^2 + \frac{dr^2}{f(r)} + \eta_{\mu\nu} dx^\mu dx^\nu \right] , \quad (1.3.6)$$

which is of the form (1.3.1) and with

$$f(r) = 1 - \left(1 + \frac{r_h^2 \mu^2}{\gamma^2} \right) \left(\frac{r}{r_h} \right)^3 + \frac{r_h^2 \mu^2}{\gamma^2} \left(\frac{r}{r_h} \right)^4 ; \quad \gamma \equiv \frac{2e^2 L^2}{\kappa^2} , \quad (1.3.7)$$

where r_h is the radius of the outer horizon of the black hole. It is also known that one can identify a temperature at the horizon of black holes, which is the Hawking temperature [66]. In the case of this Reissner-Nördstrom black hole it is

$$T = \frac{1}{4\pi r_h} \left(3 - \frac{r_h^2 \mu^2}{\gamma^2} \right) \Rightarrow r_h = r_h(\mu, T). \quad (1.3.8)$$

This means that black holes in the bulk correspond to placing the boundary field theory at some finite temperature, and the bulk spacetime corresponds to real time dynamics in the boundary thermal field theory. The duality bulk fields-boundary sources (1.3.3) coming from the prescription (1.3.2) here says

$$g_{mn} \leftrightarrow T_{mn} \text{ (stress)} \quad ; \quad A_m \leftrightarrow J_m \text{ (current)}. \quad (1.3.9)$$

Concerning the gauge field A_m , from solving its EoM with proper conditions at the horizon such that it vanishes, one finds that the temporal component takes the form

$$A_0 = \phi_0(x) + \phi_1(x)r = \mu \left(1 - \frac{r}{r_h} \right). \quad (1.3.10)$$

Then the dictionary tells us that the *leading* ϕ_0 and *subleading* ϕ_1 terms of this solution can be identified respectively with the source for the current J^0 and the expectation value $\langle J^0 \rangle$ [42–45]. Which means that the source for J^0 is the chemical potential μ , while the expectation value $\langle J^0 \rangle$ is the charge density. The presence of an electric field with frequency ω is related to the x -component of the gauge field through $\partial_t A_x = E e^{i\omega t}$, for which $A_x = \frac{E}{i\omega} e^{i\omega t}$ on the boundary. In the bulk, the leading order terms in A_x take the form

$$A_x = \frac{E}{i\omega} e^{i\omega t} + \langle J_x \rangle r + \dots, \quad (1.3.11)$$

from which the subleading term $\langle J_x \rangle$ can be determined through the EoM of the bulk. Considering the fact that this implies the presence of a g_{tx} component of the metric [42], from Maxwell and Einstein equations, together with ingoing BC at the horizon, we get

$$(f A'_x)' + \frac{\omega^2}{f} A_x = \frac{4\mu^2}{\gamma^2 r_h^2} A_x. \quad (1.3.12)$$

Although this cannot be solved analytically, it can be done numerically and from the Ohm's relation $J = \sigma E$, we can thus determine the response $\langle J_x \rangle$ in terms of the source, from which the ratio is the conductivity

$$\sigma(\omega) = \frac{1}{e^2} \frac{A'_x}{i\omega}. \quad (1.3.13)$$

What is surprising of this result, which highlight the power of this duality, is that starting from general relativity with a charged black hole and a boundary, one obtain the “conductivity” (1.3.13), which plotted with the numerical solution recovered from (1.3.12), reproduces a behaviour seen in graphene and typical of any CFT in 2 + 1 dimensions [42, 67]².

² Notice that a delta function appear in the analysis, which can however be eliminated when considering the theory on a lattice (so to break translational invariance). This discussion anyway lies outside the scope of this introductory Section and Thesis.

THE QFT APPROACH IN THIS THESIS

We have seen that it is possible to treat theories with boundary in the context of different formulations : through standard electrodynamics as in the Casimir effect, by means of the formalism of QFT, in the case of Symanzik's work, or looking at holography as a theory of gravity displaying microscopic condensed matter phenomena on its asymptotic conformal boundary. All of this with fruitful results. In this Section an additional case is considered, representing the technique and attitude in terms of which the original works contained in this Thesis is built. The principles around which the Thesis develops are those of QFT : symmetry, covariance, locality and power-counting, the only external input is given by the introduction of the boundary itself. No additional requests are made to recover the induced physics, it will be the theory itself that will give all the information needed.

2.1 THE SYMMETRY, THE ACTION AND THE BOUNDARY

The main ingredient from which we start building the theory is the symmetry, described by a transformation δ_α acting on the fields, which here are collectively represented as $\phi_A(x)$, as follows

$$\delta_\alpha \phi_A = \mathcal{W}_{\alpha A} , \quad (2.1.1)$$

and leaving the most general, power-counting-compatible, action invariant, *i.e.*

$$\delta_\alpha S_{inv}[\phi_A] = 0 . \quad (2.1.2)$$

Here the index α in δ_α is a generic one, meaning that the transformation might change the nature of the involved field (like the case of supersymmetry [68]). In the following we will only deal with transformation where no α -index appear. This is the case for instance of the standard gauge transformation

$$\delta_{gauge} A_\mu = \partial_\mu \Lambda \quad (2.1.3)$$

whose invariant 4D action describes Maxwell theory

$$S_{Max} = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} \quad (2.1.4)$$

with

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu . \quad (2.1.5)$$

In 3D the same symmetry give rise to the topological Chern-Simons action [69], or the Maxwell-Chern-Simons one [70] depending on the mass dimension of the gauge field. From just one symmetry and one field, three completely different models come out : we have the theory of electromagnetism, and by lowering the dimensions one can either have a theory of mostly mathematical interest, as is Chern-Simons, or one which

gives the 3D photon a topological mass [70]. Not to mention non-abelian extensions. Infinitesimal diffeomorphisms

$$\delta_{diff} h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (2.1.6)$$

for a rank-2 symmetric tensor field define the theory of LG, where the gauge parameter $\xi_\mu(x)$ is a vector. There are also more “exotic” cases that we will encounter, and the discussion will be postponed to the relevant Chapter. In Path Integral formalism the fundamental objects (observables) are related to the generating functional $Z[J]$ and in particular to the generating functional of the connected Green’s functions $Z_c[J]$, defined by

$$Z[J] = e^{iZ_c[J]} = \int [d\phi_A] e^{iS_{inv} + S_J}. \quad (2.1.7)$$

In (2.1.7) a source $J^A(x)$ is coupled to the field $\phi_A(x)$

$$S_J = \int d^d x J^A \phi_A, \quad (2.1.8)$$

which allows to compute the Green functions $G^{(n)}(x_1, \dots, x_n)$ as

$$\frac{\delta^n Z_c[J]}{\delta J^{A_1}(x_1) \dots \delta J^{A_n}(x_n)} \Big|_{J=0} = i^{n-1} \langle T(\phi_{A_1}(x_1) \dots \phi_{A_n}(x_n)) \rangle = G_{A_1 \dots A_n}^{(n)}(x_1, \dots, x_n), \quad (2.1.9)$$

where T represents the time-ordered product. Of particular interest are the 1- and 2-pt functions, *i.e.* mean value of the field and propagators

$$\frac{\delta Z_c[J]}{\delta J^A(x)} \Big|_{J=0} = \langle \phi_A \rangle \quad (2.1.10)$$

$$\frac{\delta^2 Z_c[J]}{\delta J^A(x) \delta J^B(y)} \Big|_{J=0} = i \langle T(\phi_A(x) \phi_B(y)) \rangle = G_{AB}^{(2)}(x - y). \quad (2.1.11)$$

If the transformation (2.1.1) is linear, which is the case for an abelian theory, the symmetry of (2.1.2) can be written in a functional way as

$$\left[\int d^d x \sum_{\phi_A} \mathcal{W}_{\alpha A} \frac{\delta}{\delta \phi_A} \right] S_{inv} = 0, \quad (2.1.12)$$

known as Ward identity. The symmetry implies a conserved current equation for the source $J^A(x)$ in (2.1.8). This can be immediately visualized if we consider for instance the gauge symmetry (2.1.3) and the Maxwell action (2.1.4) with a source term (2.1.8) where the current is $J^\mu(x)$. In that case the identity (2.1.12), for every value of the gauge parameter $\Lambda(x)$ (2.1.3), gives

$$\partial_\mu J^\mu = 0. \quad (2.1.13)$$

A gauge theory described by the action (2.1.2), in order to be well defined, needs a gauge-fixing so to render the path integral finite by identifying one representative for each gauge orbit [71], and allowing for the computation of observables (propagators for instance). Physical results should not depend on the gauge choice. In the case of theories with boundary that choice, usually, but not necessarily, is the axial one, where

a component normal to the boundary is set to be zero through a proper gauge fixing term S_{gf} . This term breaks covariance [72], as does the presence of the boundary itself. The boundary is introduced in the theory by means of a Heaviside theta distribution $\theta(f(x))$ in the action, where $f(x) = 0$ describes the boundary ∂M of the manifold \mathcal{M} . In this case the theory has a one-sided boundary. This differs from the approach of Symanzik [10], where the theory was defined on both sides of the boundary. Thus in our case we will not have to require any separability condition as (1.2.6) on the propagators. However still the question naturally arises of which BC should be imposed on the bulk fields or, rather, which are the most general BC which naturally emerge as a consequence of the presence of a boundary itself, without introducing them by hand. Indeed a statement of the procedure we want to build in this Thesis is that we want to get rid of the dependence on any particular choice. To do so we consider an additional action term S_{bd} analogous to Symanzik's term in his "augmented lagrangian" (1.2.2), compatible with power-counting and locality, which could have a form like the following

$$S_{bd} = \int d^d x \delta(f(x)) \phi_A T^{AB} \phi_B, \quad (2.1.14)$$

where T^{AB} is a generic operator that may contain spacetime derivatives. While Symanzik imposes BC and then sees which is the boundary term in the action which implements them, we generalize that approach not imposing any BC *a priori*, but writing the most general boundary term, compatible with locality and power counting, to find out which are the most general BC compatible with it. The question as whether to require covariance or not in the boundary term will depend on the model. Clearly one could for instance start from a non-covariant term and recover covariance as a particular case, however this could introduce an unnecessary high number of free parameters, which may render the analysis more complicated, hiding physical results in a proliferation of coefficients. On the other hand the request of covariance could prevent from reaching any result at all, as we shall see for instance in the case of Chern-Simons theory in Section 4.1. The presence of the boundary spoils the gauge invariance of the bulk action. This boundary term S_{bd} (2.1.14) allows to obtain the most general BC for the theory as a variational principle for the full action

$$S_{tot} = S_{inv} + S_{bd} + S_{gf} + S_J. \quad (2.1.15)$$

2.2 THE BREAKING OF THE SYMMETRY AND CURRENT ALGEBRA

We started this Chapter by saying that the symmetry is the main ingredient of a QFT, the first step from which a theory can be built. But it is even more important in the present case, since the symmetry, or better, its breaking by the introduction of the boundary, is also the first step to recover the induced lower-dimensional theory. As a consequence, the content of this Section depends on the specific symmetry considered, and the scope here is to give an idea of the procedure that we will follow in this Thesis. Indeed the presence of the boundary translates in a breaking of the symmetry which thus results in a breaking of the Ward identity (2.1.12), which acquires a nonzero value at its r.h.s. Such Ward identity is known [73, 74] to imply conservation laws, in the sense that, taken on-shell (*i.e.* $J = 0$), it can be written as an equation of the form

$$\partial \cdot (\dots)|_{\partial \mathcal{M}} = 0, \quad (2.2.1)$$

associated to the breaking, where “(...)” is a linear function of the bulk fields $\phi_A(x)$. This will be one of the ingredients thanks to which one should be able to build the boundary theory, as we shall see.

Commutator algebra and Kac-Moody

Through the relations (2.1.10) and (2.1.11), by taking functional derivatives with respect to the source of the broken Ward identity, it is possible to obtain an algebraic structure on the boundary $\partial\mathcal{M}$

$$\frac{\delta}{\delta J(x')}(\text{Broken Ward id.}) \rightarrow \text{boundary algebra} . \quad (2.2.2)$$

Since the on-shell breaking of the identity represents a conserved current on the boundary (2.2.1), this means that the commutation relations coming from (2.2.2) represents an algebra of currents. In fact typically this algebraic structure turns out to be of the KM type [26, 27, 75], which is a recurrent one in the context of theories with boundary [11, 29, 76, 77]. For instance one can think of the two-dimensional case, for which the current algebra takes the following form

$$[J^a(x), J^b(x')] = i\hbar f^{abc} J^c(x)\delta(x-x') + \frac{i\hbar^2}{2\pi} k\delta^{ab}\partial^{(x)}\delta(x-x') , \quad (2.2.3)$$

where f^{abc} are the structure constants of an algebra g , and k is known as the central charge. When the theory is abelian the structure constants disappear, and the r.h.s. is proportional to a derivative of the Dirac delta function with the central charge as coefficient. This is also called the Schwinger term [78] and is a second order quantum effect. To relate this current algebra to the proper KM one, which is associated to a compact and finite-dimensional Lie algebra g defined by the commutation relation

$$[T_m^a, T_n^b] = if^{abc}T_{m+n}^c + km\delta^{ab}\delta_{m,-n} , \quad (2.2.4)$$

where $m, n \in \mathbb{Z}$, and $a, b, c \in \{1, \dots, \dim g\}$, it is necessary to introduce a KM field

$$T^a(z) = \sum_n T_{-n}^a z^n , \quad (2.2.5)$$

related to the current in (2.2.3) through

$$J^a(x) = \frac{\hbar}{L} T^a e^{2\pi i \frac{x}{L}} , \quad (2.2.6)$$

where the definition is taken on a unitary circle of period L which can then be extended to infinity. Following these definitions the current algebra (2.2.3) can be written in terms of the KM one (2.2.4), where one can see that the central charge k is the element that commutes with all the other elements of the algebra T_m^a . Algebraic structures of this kind are frequent in physics, indeed, KM algebras are tightly related to QFTs with boundary. Starting from the pioneering paper [15], where all rational 2D conformal field theories were derived from the 3D Chern-Simons theory with a boundary, it is now almost paradigmatic that conserved currents exist on the boundary of topological field theories, forming KM-type algebras with a central charge proportional to the inverse of the coupling constant of the bulk theory. This will be particularly relevant in this Thesis to better interpret the boundary physics and identify physical observables.

2.3 THE INDUCED LOWER-DIMENSIONAL THEORY

Section 2.1 was the building of the theory starting from the symmetry, and the introduction of the boundary. That procedure is always valid : given a symmetry, this defines an invariant action and, with the addition of the boundary, proper gauge-fixing and boundary term are considered to deal with the model. On the other hand Section 2.2 deals with the consequences of the aforementioned introduction of boundary : the breaking of the symmetry/Ward identity, the emergence of a conserved current on the boundary, and of an algebraic structure. Thus the results are highly model/symmetry dependent. Therefore that part should only be seen as a guideline on how to proceed and what to possibly expect. Even though in all the cases presented in this Thesis everything works fine, this does not mean, a priori, that it always should. Indeed results may depend on how much “exotic” the symmetry is, or the general setting that is chosen. For instance the theory could be defined on a flat or curved background spacetime, and in the second case one should expect computations to be more tricky, if possible at all. One has to keep in mind that it is not necessarily always true that an induced boundary physics can be recovered from any theory just by adding a boundary. Indeed, until recently, it was believed that this only worked for a class of theories, *i.e.* TQFTs, and not in general. What we shall see in this Thesis is that apparently it is more universal than expected.

If after the building of the theory as Section 2.1 prescribes, results of the kind mentioned in Section 2.2 are recovered, the scope of the present Section is to show how to interpret the obtained results, and translate them into the induced theory, always keeping in mind that the spirit is to not impose anything by hand. All the results obtained up to this level should be enough to determine the boundary theory, indeed :

1. we can identify the DoF of the induced model by solving the on-shell conserved current equation (2.2.1). This will give a relation between the bulk fields $\phi(x)$ evaluated on the boundary, and new boundary fields $\varphi(X)$, obtained as the **general** solution of (2.2.1). This relation must be preserved, which implies that the boundary fields $\varphi(X)$ must transform in a way that does not affect it. That is what defines the transformation on the fields $\delta_{bd}\varphi$ under which the lower dimensional action $S_{d-1}[\varphi]$ must be invariant, *i.e.*

$$\delta_{bd}S_{d-1} = 0 . \quad (2.3.1)$$

2. The commutation relations obtained through (2.2.2), when written in terms of the new boundary fields $\varphi(X)$, could be interpreted as an equal time canonical commutation relation

$$[q(X) , p(X')] = i\delta^{(d-2)}(X - X') , \quad (2.3.2)$$

from which canonical variables can be identified which constrain the free parameters of the action S_{d-1} through

$$p = \frac{\delta S_{d-1}}{\delta \dot{q}} . \quad (2.3.3)$$

3. To consolidate the relation between the bulk, represented by the action $S_{tot}[\phi]$ (2.1.15), and the boundary, whose action is $S_{d-1}[\varphi]$ satisfying (2.3.1) and (2.3.3),

the remaining step is to match the BC of the bulk theory, written in terms of the boundary fields $\varphi(X)$, with the EoM of the $d - 1$ -dimensional theory, *i.e.*

$$\text{bulk BC} \leftrightarrow \text{boundary EoM} . \quad (2.3.4)$$

This, which we call “holographic contact” makes it possible to relate (some of) the remaining free parameters of S_{d-1} with those of the bulk, which may also involve the “coupling” constant.

Again it is important to remark that up to this point nothing has been imposed besides the presence of the boundary, and everything that is obtained simply comes from applying first principles of QFT. Once we have properly collected the information that the theory is giving us by following these steps, nothing is left if not to physically interpret the theory that has just been recovered. This can be done for instance by computing propagators, analyzing the EoM, or looking at the energy momentum tensor.

NOTATIONS

If not otherwise stated, the following notations concerning indices and coordinates will be used throughout the Thesis

$$\alpha, \beta, \dots, \mu, \nu, \dots = \{0, \dots, d - 1\} \quad (2.3.5)$$

$$a, b, c, \dots = \{0, \dots, d - 2\} \quad (2.3.6)$$

$$A, B, C, \dots = \{1, \dots, d - 2\} . \quad (2.3.7)$$

In the case $d = 3$ the last line (that corresponds to the index referring to the single spatial coordinate on the boundary) we will just use the corresponding number (or letter). Moreover bulk and boundary coordinates are respectively

$$x^\mu = (x^0, \dots, x^{d-1}) \quad (2.3.8)$$

$$X^m = (x^0, \dots, x^{d-2}) . \quad (2.3.9)$$

For instance in Chapter 5 we will have radial coordinates as follows

$$\begin{aligned} \mu, \nu, \rho, \dots &= \{0, 1, 2\} = \{t, r, \theta\} \\ i, j, k, \dots &= \{0, 2\} = \{t, \theta\} , \end{aligned} \quad (2.3.10)$$

$$x = (x^0, x^1, x^2) = (t, r, \theta) \quad \text{on the 3d bulk} \quad (2.3.11)$$

$$X = (x^0, x^2) = (t, \theta) \quad \text{on the 2d boundary } r = R . \quad (2.3.12)$$

Exceptions to this notation will be in Section 7.2, 7.3, and 8.1, where no boundary is present, and thus the distinction will be between space and time indices as

$$\alpha, \beta, \dots, \mu, \nu, \dots = \{0, \dots, d - 1\} \quad (2.3.13)$$

$$a, b, c, \dots = \{1, \dots, d - 1\} . \quad (2.3.14)$$

Mostly plus Minkowskian metric will be used $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$, and when dealing with curved spacetime we remark that the Levi-Civita tensor $\epsilon^{\mu\nu\rho}(x)$ is written in terms of the corresponding symbol $\tilde{\epsilon}^{\mu\nu\rho}$ as follows

$$\epsilon^{\mu\nu\rho}(x) = \frac{\tilde{\epsilon}^{\mu\nu\rho}}{\sqrt{-g(x)}} , \quad (2.3.15)$$

where $g(x)$ is the determinant of the bulk metric tensor $g_{\mu\nu}(x)$, with Lorentzian signature. Finally, concerning the constants, we will adopt the standard QFT convention where $\hbar = c = 1$ dimensionless, thus everything will be expressed in terms of mass dimensions, keeping in mind that $[\partial_x] = 1$ and $[x] = -1$.

Part II

TOPOLOGICAL FIELD THEORIES

Topological Quantum Field Theories represent a paradigmatic example of how boundaries may be relevant in quantum field theory. In fact TQFTs are characterized by the defining property of having *global* observables only, and not *local*. This means that the word “observable” in this context is a kind of oxymoron, being of geometrical, rather than physical, nature. These results are quite important, and motivated the great interest in the community of theoretical physicists which arose on TQFTs after a couple of seminal papers at the end of the eighties of the past century [79, 80], where problems typical of mathematics and mathematical physics were for the first time successfully faced by quantum field theory methods. On the other hand, the fact that TQFTs have vanishing energy-momentum tensor, hence vanishing energy density and, above all, vanishing Hamiltonian, justified as well the colder attitude of the larger part of less formal physicists toward TQFTs. The introduction of boundaries in TQFTs drastically changed this scenario.

AN OVERVIEW OF TQFTS (1988)

Topological Quantum Field Theories are characterized by the defining property that their observables do not depend on the metric of the background spacetime. As a consequence, TQFTs do not display *physical* observables, which are local, but “only” *geometrical* ones, *i.e.* global properties of the manifolds, like handles, knots and so on [14, 79]. An easy way to highlight this peculiar property is to compute the energy-momentum tensor of a TQFT: the result is zero or, more precisely, the only contribution comes from the unphysical gauge-fixing sector. The situation drastically changes if a boundary is introduced in the background spacetime. The presence of a boundary breaks everything which can be broken, starting from translations and rotations, hence Lorentz invariance. In TQFTs the boundary also breaks gauge invariance, thus, if a boundary is present, integration by parts gives a nonvanishing contribution, and on the lower dimensional boundary unbroken residual symmetries survive. This will allow for local DoF to emerge on the boundary, and promote the theory to a physical one. However, before discussing the introduction of a boundary in a TQFT, it is interesting to briefly display the properties that defines a theory a topological one. In [14] it is stated that four are the characteristics that makes a theory a TQFT. Indeed A topological quantum field theory consists of

1. a collection of fields $\Phi(x)$ (which are Grassmann graded) defined on a Riemannian manifold (\mathcal{M}, g) ;
2. a nilpotent operator δ , which is odd with respect to the Grassmann grading ;
3. physical states defined to be δ -cohomology classes ;
4. an energy-momentum tensor which is δ -exact, *i.e.*

$$T_{\alpha\beta} = \delta V_{\alpha\beta}(\Phi, g) \quad (3.1)$$

for some functional $V_{\alpha\beta}$ of the fields and the metric.

The definition comes from BRS formulation [81, 82], where $\Phi(x)$ includes the gauge field, ghosts, and multipliers, and δ is the nilpotent, *i.e.* $\delta^2 = 0$, BRS operator. This is the content of 1. and 2., while point 3. says that the physical Hilbert space is defined by the condition

$$\delta|phys\rangle = 0, \quad (3.2)$$

and has thus states of the form

$$|phys\rangle' = |phys\rangle + \delta|x\rangle \sim |phys\rangle, \quad (3.3)$$

due to nilpotency, where “ \sim ” implies an equivalence between the states. This equivalence means that the physical Hilbert space is composed of states which are δ -closed

($\delta|x\rangle=0$) modulo δ -exact ($|x\rangle = \delta|y\rangle$) states, *i.e.* are δ -cohomology classes. The variation of any functional \mathcal{O} of the fields $\Phi(x)$ is a BRS transformation, therefore from BRS invariance of the vacuum

$$\langle \delta\mathcal{O} \rangle = 0, \quad (3.4)$$

where “ $\langle \dots \rangle$ ” represents the vacuum expectation value. The fourth prerequisite has a very important consequence on the generating functional

$$Z = \int [d\Phi] e^{-S_q}, \quad (3.5)$$

where S_q is the complete quantum action, which includes the classical action S_c together with the necessary gauge fixing and ghost terms and is BRS-invariant by construction *i.e.* $\delta S_q = 0$. Considering the theory on a manifold \mathcal{M} with metric $g_{\alpha\beta}(x)$, the energy momentum tensor given in pt.4. by (3.1) is such that the generating functional Z (3.5) is zero under an infinitesimal change in the metric³, in fact

$$\begin{aligned} \delta_g Z &= \int [d\Phi] e^{-S_q} \left(-\frac{1}{2} \int_{\mathcal{M}} d^n x \sqrt{-g} T_{\alpha\beta} \delta g^{\alpha\beta} \right) \\ &= \int [d\Phi] e^{-S_q} \delta\chi = \langle \delta\chi \rangle = 0, \end{aligned} \quad (3.6)$$

where we used the BRS invariance of the vacuum (3.4), the definition of energy-momentum tensor as the variation of the action with respect to the metric *i.e.*

$$\delta_g S_q = \frac{1}{2} \int_{\mathcal{M}} d^n x \sqrt{-g} T_{\alpha\beta} \delta g^{\alpha\beta}, \quad (3.7)$$

and where we defined

$$\chi \equiv -\frac{1}{2} \int_{\mathcal{M}} d^n x \sqrt{-g} V_{\alpha\beta} \delta g^{\alpha\beta}. \quad (3.8)$$

The term “topological” can thus be interpreted as this **metric independence**. The generating functional Z (3.5) does not depend on the local structure of the manifold, but only on global properties : Z is a topological invariant. However, it is not the only invariant. The vacuum expectation value of an observable

$$\langle \mathcal{O} \rangle = \int [d\Phi] e^{-S_q} \mathcal{O}(\Phi) \quad (3.9)$$

is another topological invariant, *i.e.* $\delta_g \langle \mathcal{O} \rangle = 0$ if [79]

$$\delta_g \mathcal{O} = \delta R \quad ; \quad \delta\mathcal{O} = 0, \quad (3.10)$$

which means that \mathcal{O} has to be in the δ -cohomology class (*i.e.* it is a BRS invariant operators which is not δ -exact), and which satisfy $\delta_g \mathcal{O} = \delta R$ for some R . Then we have that

$$\delta_g \langle \mathcal{O} \rangle = \int [d\Phi] e^{-S_q} (\delta_g \mathcal{O} - \delta_g S_q \cdot \mathcal{O}) = \langle \delta(R + \chi\mathcal{O}) \rangle = 0, \quad (3.11)$$

³ Metric independence of the functional measure is assumed. To show that this assumption is in fact realized, one needs to check for metric anomalies [14].

again due to (3.4). Another property of topological field theories is the absence of dynamical excitations, *i.e.* there are no propagating degrees of freedom. Indeed, from 1.-4., we have

$$\langle phys' | H | phys \rangle = \langle phys' | T_{00} | phys \rangle = \langle phys' | \delta V_{00} | phys \rangle = 0, \quad (3.12)$$

where H is the Hamiltonian, which means that the energy of any physical state is zero, and hence there are no physical excitations.

The classification

The above definition of a TQFT allows for a classification of such theories : one can have Witten type or Schwarz type TQFTs [69]. The discriminant comes from the shape of the quantum action, for which the two models acquire different features. For instance this difference also reflects on the names [83, 84] : Witten type theories are also called *cohomological* due to the structure of its observables, while the Schwarz type models are said to be *quantum* as a consequence of its non-triviality (classical action).

- Concerning **Witten type** theories [79, 85], the defining feature is that the complete quantum action S_q can be written as a BRS variation, *i.e.*

$$S_q = \delta V, \quad (3.13)$$

i.e. it is exact for some functional $V(\Phi, g)$ of the fields. This immediately implies

$$T_{\alpha\beta} = \delta \left(\frac{2}{\sqrt{-g}} \frac{\delta V}{\delta g^{\alpha\beta}} \right), \quad (3.14)$$

which through 4. ensures of the topological nature of the model. It is also possible to show [14] that the main property of Witten type theories (3.13), implies exactness of the functional Z at the semiclassical level (*i.e.* small \hbar). Examples of Witten theories are topological Yang-Mills and topological σ -model.

- Concerning **Schwarz type** theories [80, 86], the quantum action is of the form

$$S_q(\Phi, g) = S_c(\Phi) + \delta V(\Phi, g), \quad (3.15)$$

where $S_c(\Phi)$ is the metric-independent classical action, while the rest comes from gauge-fixing procedure. Metric independence of the classical action ensures that its energy momentum tensor vanishes, and the complete one thus depends only on gauge-fixing and ghost sector as

$$T_{\alpha\beta} = \delta \left(\frac{2}{\sqrt{-g}} \frac{\delta V}{\delta g^{\alpha\beta}} \right), \quad (3.16)$$

which, again, satisfies feature 4. of a TQFT.

To the second class, *i.e.* Schwarz theories, belong both Chern-Simons and BF theories⁴, which will be the subject of study in the next Chapters, for the case of the abelian 3D models, described by the following classical actions

$$S_{CS} = \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \quad (3.17)$$

$$S_{BF} = \int d^3x \epsilon^{\mu\nu\rho} B_\mu \partial_\nu A_\rho, \quad (3.18)$$

⁴ the higher dimensional $n > 3$ class of non-Abelian BF theories possess some “non-standard” properties [14].

where metric independence is ensured by the presence of the Levi-Civita symbol, which in curved spacetime is a density and comes with a metric contribution at the denominator of its tensorially-extended quantity (2.3.15) $\epsilon^{\mu\nu\rho}(x) = \epsilon^{\mu\nu\rho} / \sqrt{-g(x)}$, which compensate the integration measure $d^3x \sqrt{-g}$. As a curiosity, notice that the name of the BF theory is not an acronym, but directly comes from the aspect of the 3D theory. From the definition of the Maxwell field strength

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (3.19)$$

the abelian classical action (3.18) is indeed written as “BF”. From (3.17) and (3.18) one can see that both Schwarz-type Lagrangians transform under gauge the transformations

$$\delta A_\mu = \partial_\mu \lambda \quad ; \quad \delta' B_\mu = \partial_\mu \lambda', \quad (3.20)$$

as total derivatives

$$\delta S_{CS} = - \int d^3x \partial_\mu (\epsilon^{\mu\nu\rho} \lambda \partial_\nu A_\rho) \quad (3.21)$$

$$\delta' S_{BF} = - \int d^3x \partial_\mu (\epsilon^{\mu\nu\rho} \lambda' \partial_\nu A_\rho) \quad ; \quad \delta S_{BF} = 0, \quad (3.22)$$

which is a sign of the existence of a lower dimensional theory in the presence of a boundary, as we shall see in the following Chapter.

CHERN-SIMONS AND BF WITH BOUNDARY

The scope of this Chapter is to briefly review the physics that emerges when one introduces a boundary in a TQFT in flat spacetime, and in particular to the 3D abelian Schwarz theories discussed above : Chern-Simons and BF. The first Section 4.1 concerns the abelian Chern-Simons theory with boundary and its relation to the edge states of FQHE. This will be the first practical example of the results that one can obtain by applying the QFT procedure presented in Chapter 2 to a specific model, as was done in [87]. The theoretical observation of physical edge states on the boundary of Chern-Simons theory was done for the first time in 1992 with a different approach [16], and we shall see that both analysis are in agreement. The Section will thus also serve as a testing ground for the general procedure displayed in Chapter 2 by means of a practical example. After looking at the Chern-Simons case with boundary, in Section 4.2 we will apply the same steps to the abelian 3D BF model with boundary, following [88], and analyzing its relation with TI, discovered in 2010.

4.1 CHERN-SIMONS AND THE EDGE STATES OF FQHE (1992)

In 1990 the low energy effective field theory of the edge states of FQHE was derived, and shown to describe properties of chiral Luttinger liquids [89]. Under this regard, it is known [90, 91] that the abelian Chern-Simons theory, when coupled to matter sources as

$$S = S_{CS} + S_J = \kappa \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \int d^3x J^\mu A_\mu, \quad (4.1.1)$$

where κ is the “coupling” constant. This is relevant in the context of planar electrodynamics, indeed the spatial component of its EoM

$$\frac{\delta S}{\delta A_a} = -\kappa \epsilon^{0ab} F_{0b} + J^a = 0 \quad (4.1.2)$$

describes Hall conductivity

$$J^a = \sigma^{ab} E_b = \kappa \epsilon^{0ab} E_b \quad (4.1.3)$$

i.e. the presence of a current which is perpendicular to the usual electric field $E_b = F_{0b}$. Inspired by this relation, two years after the discovery of the edge states of FQHE, in 1992, it was shown [16] that these Hall edge states, represented by chiral Luttinger liquids, could be obtained as boundary modes of the abelian Chern-Simons model. Following [87] and the steps of Chapter 2, we will see how such a result can be recovered in the context of QFTs with boundary. The abelian Chern-Simons theory is described by the action (3.17), where a flat planar boundary on $x^2 = 0$ can be introduced by means of Heaviside step distribution. Thus the action is

$$S_{CS} = \frac{\kappa}{2} \int d^3x \theta(x^2) \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \quad (4.1.4)$$

Notice that the coupling constant κ could be reabsorbed by a redefinition of the gauge field $A_\mu(x)$, however when dealing with boundaries, it can be useful to maintain it explicit in order to keep track of the contribution of the boundary. Indeed we will see that this will be identified with a physical quantity, namely the filling fraction ν of the quantum Hall effect [90, 91]. The full action is given by (2.1.15), *i.e.*

$$S_{tot} = S_{CS} + S_{gf} + S_J + S_{bd} , \quad (4.1.5)$$

where in this case the gauge-fixing, source and boundary terms are of the following form

$$S_{gf} = \int d^3x \theta(x^2) b A_2 \quad (4.1.6)$$

$$S_J = \int d^3x \theta(x^2) J^i A_i \quad (4.1.7)$$

$$S_{bd} = \int d^3x \delta(x^2) \left(\frac{a_1}{2} A_0^2 + a_2 A_0 A_1 + \frac{a_3}{2} A_1^2 \right) . \quad (4.1.8)$$

The axial gauge fixing $A_2 = 0$ has been chosen and implemented through the EoM of the multiplier field $b(x)$, and where the matrix T^{AB} in (2.1.14) has the form

$$T^{ij} \equiv \frac{1}{2} \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} , \quad (4.1.9)$$

depending on the three constants a_1, a_2, a_3 . No derivatives in the boundary term appears due to power-counting, since the gauge field has mass dimensions $[A_\mu] = 1$. From the EoM of the full action, through the variational principle $\lim_{\epsilon \rightarrow 0} \int_0^\epsilon dx^2 \text{EoM}$, one recovers the BC of the theory, which, neglecting those that require for the gauge field to vanish, are of the form

$$A_0 - v A_1|_{x^2=0} = 0 , \quad (4.1.10)$$

where v is a constant coefficient that depends on the bulk coupling κ and on the boundary coefficient a_i , *i.e.* $v = v(\kappa, a_i)$. This BC will serve as a trigger to identify the physical theory once we have built the 2D induced action. It is interesting to notice that this same fundamental equation (4.1.10), that here comes naturally as a general BC for the theory through Symanzik's principle, in [16] was imposed as a gauge-fixing condition. This would however raise the ambiguity of whether the results would just be peculiar of that specific gauge choice. But physical results should not depend on the gauge choice: what would happen if, for instance, a covariant gauge-fixing, which is of a completely different nature, was to be considered? In [16] the gauge-fixing directly intervene on the physical results through its equation (4.1.10), while in terms of the QFT procedure outlined in Chapter 2 that equation naturally comes out as a BC for the theory, while the gauge choice only simplifies the computations, without affecting the final results. For instance, the KM algebra on the boundary of Chern-Simons theory has been shown to exist, for flat spacetime, both in covariant [28] and axial [29] gauge. The presence of the boundary at $x^2 = 0$ breaks the gauge invariance, which also implies a breaking of the Ward identity, as discussed in Section 2.2. By means of the EoM of the theory one can thus recover the following integrated broken Ward identity

$$\int_0^\infty dx^2 \partial_i J^i = -\kappa \epsilon^{0ij} \partial_i A_j|_{x^2=0} , \quad (4.1.11)$$

where ϵ^{0ij} is the 2D Levi-Civita symbol. By means of functional derivatives with respect to the current, this broken identity gives an equal-time algebraic structure of the KM type [26, 27]

$$[A_1(X), A_1(X')] = -\frac{i}{\kappa} \partial_1 \delta(x^1 - x'^1), \quad (4.1.12)$$

where we notice that the central charge is proportional to the inverse of the coupling. Finally, the identity (4.1.11), taken on-shell, gives a conserved current equation, which can be solved as

$$\epsilon^{0ij} \partial_i A_j|_{x^2=0} = 0 \quad \Rightarrow \quad A_i|_{x^2=0} = \partial_i \phi. \quad (4.1.13)$$

With this equation we now have all the information needed to build the 2D model at the boundary of Chern-Simons, and we can do this by following the steps of Section 2.3:

1. The boundary DoF are identified as the scalar field $\phi(X)$ in (4.1.13), and the symmetry preserving its definition is the shift symmetry

$$\delta_s \phi = \eta, \quad (4.1.14)$$

with η a constant.

2. Through the KM algebra, canonical variables are identified to be

$$q = \phi \quad ; \quad p = -\kappa \partial_1 \phi. \quad (4.1.15)$$

3. From 1. and 2., together with power-counting and locality, we get the following shift-invariant action

$$S_{2D} = \int d^2 X \left[-\kappa \partial_0 \phi \partial_1 \phi + c (\partial_1 \phi)^2 \right], \quad (4.1.16)$$

for which the matching between the BC (4.1.10) and its EoM as $\partial_1 \text{BC} = \text{EoM}$ gives a constraint on the free parameter c to be

$$c = \kappa v, \quad (4.1.17)$$

where v depends on the parameters a_i of the boundary action S_{bd} (4.1.8).

The 2D induced theory is thus described by the action

$$S_{2D} = \kappa \int d^2 X \left[-\partial_0 \phi \partial_1 \phi + v (\partial_1 \phi)^2 \right], \quad (4.1.18)$$

where the EoM is

$$\frac{\delta S_{2D}}{\delta \phi} = \partial_0 \partial_1 \phi - v \partial_1 \partial_1 \phi = 0. \quad (4.1.19)$$

The phenomenology

What's the physics described by this boundary theory? One can notice that the BC (4.1.10), written in terms of the boundary scalar field $\phi(x)$ through the solution (4.1.13) is

$$\partial_0 \phi + v \partial_1 \phi = 0, \quad (4.1.20)$$

which describe a scalar field (boson), that is chiral, *i.e.* satisfy

$$\phi(x^0, x^1) = \phi(x^1 - vx^0). \quad (4.1.21)$$

The BC (4.1.13), which is not imposed, as we said, plays a very important role in the identification of the nature of the edge DoF. Experimentally indeed, the edge states of the FQHE and of TI are fermionic, while the corresponding bulk theories are completely bosonic, being described in terms of gauge fields. But the BC have been recognized to be the conditions for the fermionization of bosonic DoF [12, 92, 93]. The scalar bosonic degree of freedom (4.1.21) found on the 2D boundary represents a chiral Weyl fermion. Indeed in Weyl fermion theory all local operators are of this form (4.1.21), and according to the fermion-boson correspondence [92, 94] one can associate a density operator $\rho(X)$ to the Weyl fermions $\psi(X)$ as

$$\rho =: \psi^\dagger \psi : , \quad (4.1.22)$$

which can be written as a chiral boson as

$$\rho = \frac{1}{2\pi} \partial_1 \phi . \quad (4.1.23)$$

In these terms the boundary EoM (4.1.19) appears as a wave equation

$$\partial_0 \rho - v \partial_1 \rho = 0 , \quad (4.1.24)$$

where the density $\rho(x)$ propagates along the boundary with constant velocity v . This corresponds to Tomonaga-Luttinger theory, whose action coincides to the 2D one we found (4.1.18) induced by the bulk Chern-Simons theory. The density $\rho(x)$ is known to satisfy [16] a commutation relation of the KM type

$$[\rho(x) , \rho(x')] = \frac{\nu}{4\pi} \partial_1 \delta(x - x') , \quad (4.1.25)$$

where ν is the filling factor of the FQHE. This algebra is exactly the one recovered in (4.1.12) from the broken Ward identity (4.1.11), since from (4.1.13) and (4.1.23) we have that $\rho \propto \partial_1 \phi \propto A_1|_{x^2=0}$. This finally allows to identify the coupling κ of Chern-Simons theory with a physical quantity, *i.e.* the filling factor ν , as

$$\kappa = \frac{1}{4\pi\nu} . \quad (4.1.26)$$

4.2 THE 3D BF MODEL AND TOPOLOGICAL INSULATORS (2010)

Many years after the first studies on the quantum Hall effect and its edge states, a new kind of topological phase of matter was observed : Topological Insulators [21, 22, 24, 95]. These new materials belong to the category of Hall systems, and are distinguished by the fact that their bulk is invariant under time-reversal (\mathcal{T}) transformation, and so are its edge modes, which are characterized by spin currents. This gave them the name of QSH systems. Indeed, while for standard Hall systems the presence of a time-reversal breaking magnetic field is necessary, in the case of the QSH the edge states are protected by that symmetry, and a manifestation of a spin-orbit interaction [22]. Following this discovery, a natural question raised as to whether edge states of QSH

systems could emerge from a TQFT with boundary, as it happened for the FQHE and Chern-Simons. In 2010 the answer arrived, and it was shown [34] that the edge states of three- and four-dimensional topological insulators could be related to the three- and four-dimensional BF models with boundary respectively. We remind that this is the other TQFT of Schwarz type, which in the abelian 3D case is represented by the classical action (3.18). Indeed one can see that the bulk BF theory, compared to the Chern-Simons one, has the additional property of being invariant under time-reversal. In fact defining the Time-Reversal transformation \mathcal{T} to act on coordinates as

$$\mathcal{T}x^0 = -x^0 \quad ; \quad \mathcal{T}x^1 = x^1 \quad ; \quad \mathcal{T}x^2 = x^2, \quad (4.2.1)$$

the 3D BF action (3.18) is invariant for fields that transform as follows

$$\mathcal{T}A_0 = A_0 \quad ; \quad \mathcal{T}A_{1,2} = -A_{1,2} \quad ; \quad \mathcal{T}B_0 = -B_0 \quad ; \quad \mathcal{T}B_{1,2} = B_{1,2}. \quad (4.2.2)$$

According to this the gauge field $A_\mu(x)$ transforms as an electromagnetic potential, while $B_\mu(x)$ as a spin current, remarking the physical relation with QSH. This invariance can also be transposed to the boundary states, thus mirroring the characteristic of topological insulators. In this Section we will review the 3D case that was studied as a QFT with boundary in [88]. This will serve as a starting point for the original work presented in the next Chapter.

The full action of the model with boundary is

$$S_{tot} = S_{BF} + S_{gf} + S_J + S_{bd}, \quad (4.2.3)$$

composed of the following bulk, gauge-fixing, source and boundary terms

$$S_{BF} = \kappa \int d^3x \theta(x^2) \epsilon^{\mu\nu\rho} B_\mu \partial_\nu A_\rho \quad (4.2.4)$$

$$S_{gf} = \int d^3x \theta(x^2) (bA_2 + d B_2) \quad (4.2.5)$$

$$S_J = \int d^3x \theta(x^2) (J^i A_i + K^i B_i) \quad (4.2.6)$$

$$S_{bd} = \int d^3x \delta(x^2) \left(\frac{a_1}{2} A_i A^i + \frac{a_2}{2} B_i B^i + a_3 A_i B^i \right), \quad (4.2.7)$$

where again a flat boundary at $x^2 = 0$ has been introduced through a theta term $\theta(x^2)$, an axial gauge fixing has been chosen for both $A_\mu(x)$ and $B_\mu(x)$ gauge fields, and a covariant boundary term has been defined in (4.2.7). Notice that here a Lorentz-invariant boundary term has been chosen, and a particular combination of the coefficients ($a_3 = 0$) allows for it to be invariant under the time-reversal \mathcal{T} transformation defined in (4.2.1) and (4.2.2), thus mimicking the property of the bulk. This reflects on the boundary and the BC, which, excluding Dirichlet solutions, give

$$a_1 A^i - \epsilon^{0ij} B_j = 0, \quad (4.2.8)$$

for $a_1 a_2 = \kappa - 1$, and $a_3 = 0$. The presence of the boundary implies a breaking of the Ward identity, one for each field, giving the following integrated quantities

$$\int_0^\infty dx^2 \partial_i J^i = -\kappa \epsilon^{0ij} \partial_i B_j |_{x^2=0} \quad (4.2.9)$$

$$\int_0^\infty dx^2 \partial_i K^i = -\kappa \epsilon^{0ij} \partial_i A_j |_{x^2=0}. \quad (4.2.10)$$

Going on-shell (*i.e.* at vanishing external sources $K = J = 0$) in (4.2.9) and (4.2.10) one can also recover the boundary DoF to be two scalar fields

$$\epsilon^{0ij} \partial_i B_j|_{x^2=0} = 0 \quad \Rightarrow \quad B_j|_{x^2=0} = \partial_j \zeta \quad (4.2.11)$$

$$\epsilon^{0ij} \partial_i A_j|_{x^2=0} = 0 \quad \Rightarrow \quad A_j|_{x^2=0} = \partial_j \Lambda . \quad (4.2.12)$$

A semidirect sum of KM algebras can be finally derived by applying functional derivatives with respect to the currents in the broken identities (4.2.9) and (4.2.10). For instance differentiating (4.2.9) with respect to $J^j(x')$ one gets, at equal time,

$$[B_1(X), A_1(X')] = \frac{i}{\kappa} \delta(x^1 - x'^1), \quad (4.2.13)$$

which can be interpreted as a canonical commutation relation for the boundary scalar fields, thus identifying the canonical variables to be

$$q = \kappa \zeta \quad ; \quad p = \partial_1 \Lambda . \quad (4.2.14)$$

The induced 2D shift-invariant action compatible with the canonical variables identified in (4.2.14) and whose EoM matches the Time-Reversal-invariant BC (4.2.8) of the bulk is the following

$$S_{2D} = \kappa \int d^2 X \left[\partial_0 \zeta \partial_1 \Lambda + \frac{1}{2a_1} (\partial_1 \zeta)^2 + \frac{a_1}{2} (\partial_1 \Lambda)^2 \right] . \quad (4.2.15)$$

Notice that it seems that starting with a covariant boundary action S_{bd} (4.2.7) leads to a 2D action that shows a strong-weak coupling duality *i.e.* $\zeta \leftrightarrow \Lambda$, $a_1 \leftrightarrow \frac{1}{a_1}$, and the EoM can be understood as the continuity equations for fluids,

$$\partial_0 \rho_i + v_i \partial_1 \rho_i = 0 , \quad (4.2.16)$$

with $i = 1, 2$, $v_1 \equiv \frac{1}{a_1}$, $v_2 \equiv a_1$, and densities $\rho_1 \equiv \partial_1 \zeta$, $\rho_2 \equiv \partial_1 \Lambda$. This does not represent a time-reversal invariant current, which is described by chiral bosons moving at equal and opposite velocities. This property is however recovered if $a_1 = 1$, for which

$$\partial_0 \rho_{\pm} \pm \partial_1 \rho_{\pm} = 0 , \quad (4.2.17)$$

where $\rho_{\pm} \equiv \partial_1 (\zeta \pm \Lambda)$, which describes bosons moving at the speed of light in opposite directions. We will see in the next Chapter, and in particular in Section 5.2, that by considering a non-covariant boundary action S_{bd} (4.2.7), we will be able to recover finite values for the velocities.

METRIC DEPENDENCE OF THE EDGE STATES OF TQFTS

We have just seen that, all measurable quantities related to edge states of Hall systems, like for instance filling factors ν and chiral velocities of the edge modes v , have been obtained for *flat* bulk manifolds, with planar, or radial (in the case of Chern-Simons on a disk [90] for instance) boundary. However experiments are pushing theoretical investigations, since recently *accelerated* chiral bosons have been observed on the edge of some particular Hall systems [48], which cannot be explained by the usual flat Chern-Simons with boundary paradigm (TQFT on flat spacetime with planar boundary), which unavoidably yields *constant* edge velocities, as we have shown in Chapter 4. The standard approach to deal with these cases is phenomenological, and basically consists in adding a suitable potential to the 2D Luttinger action, in order to reproduce time-dependent velocities [96–100]. The price of this way of solving the problem is that the whole “holographic” construction of finding the right lower dimensional action without any *ad hoc* extension fails. Similar experimental evidence is currently sought in the other relevant topological state of matter, *i.e.* the topological insulators. In all the cases we mentioned above, the lower dimensional edge dynamics depends on the bulk only through its parameters, and the details of the bulk manifold are somehow hidden by the particular flat choice. But is there any hidden dependence of the boundary physics on the bulk manifold? It appears interesting and reasonable to ask if, how and where the details of the metric reveal themselves in the edge observables, to see which are the quantities depending on the bulk metric and, maybe even more interestingly, which are the physical quantities really universal, or topologically protected. Thus the aim of this Chapter is to investigate an alternative approach to reproduce accelerated edge modes in Hall systems without the need of adding any empirical potential, while keeping the holographic construction intact. It consists in considering the bulk theory on a generic, rather than flat, background manifold. While the topological invariant action of course does not depend on the particular spacetime metric, Symanzik’s boundary term and the boundary itself certainly do, and it is an interesting issue to find out if, how and where this metric dependence reflects on the holographic 2D theory and, more interestingly, on physical observables. The holographic 2D theory induced on the boundary of the abelian Chern-Simons theory is the Floreanini-Jackiw action [101], describing edge modes moving with constant chiral velocities, which are indeed observed [102]. The velocity of the edge excitations is the main observable of both Hall systems and TI. We will see that a remnant of the bulk metric remains on these most relevant physical observables, *i.e.* the chiral velocity of the edge modes, which will become *local*. In other words, the edge modes of the Hall systems, when described by a TQFT built on a generic manifold, are accelerated and, moreover, the velocities also depend on the position of the quasiparticle on the boundary, not only on time. The dependence on the bulk metric manifests only through the determinant of the induced metric on the boundary, hence it is mild, as one might expect due to the topological character of the bulk theory. Still, the local effects are reproduced, without any empirical modification of the 2D holographic Luttinger theory.

The Chapter is composed of two main Sections, each concerning the effect of a curved spacetime in one of the two abelian Schwarz TQFT with boundary : Chern-Simons theory in Section 5.1, and the 3D BF model in Section 5.2.

5.1 EDGE STATES OF CHERN-SIMONS THEORY

The present Section, which concernns the Chern-Simons model in curved spacetime with boundary, is organized as follows

5.1.1	Chern-Simons with boundary at $r = R$	32
5.1.2	2D boundary theory	39
5.1.3	Holographic contact	41
5.1.4	Summary of results and discussion	43

In particular in Section 5.1.1 the Chern-Simons theory on a cylindrical spacetime $\mathbb{R} \times D$, in a not necessarily flat, Lorentzian background is introduced, and the EoM, together with the most general BC are derived. We find the Ward identity, broken by the boundary, and we study the existence of a KM algebra, with constant and positive central charge. In Section 5.1.2 we derive the 2D theory, holographically induced on the boundary of the 3D Chern-Simons theory. The holographic contact is imposed, which relates the parameters of the edge theory to the bulk ones. The dependence on the bulk through the induced metric on the boundary is discussed. Our concluding remarks are summarized in Section 5.1.4.

5.1.1 *Chern-Simons with boundary at $r = R$*

It is convenient, in this first approach, to work in Gaussian Normal Coordinates (GNC), where the line element takes the form

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = dr^2 + \gamma_{ij}(x)dX^i dX^j . \tag{5.1.1}$$

The adoption of the GNC is particularly useful for calculations in which one is given a hypersurface $\partial\mathcal{M}$, *i.e.* a $(d-1)$ -dimensional embedded submanifold of the d -dimensional manifold \mathcal{M} [103]. In the case treated here, being \mathcal{M} diffeomorphic to a cylinder $\mathcal{M} \simeq \mathbb{R} \times D$, this choice is the most natural one, since it immediately provides the induced metric on the boundary. The GNC are characterized by the presence of one coordinate (the one normal to the hypersurface, in our case r) such that $g_{rr} = 1$ and the off-diagonal terms vanish : $g_{rt} = g_{r\theta} = 0$ [104]. These are three extra conditions which might be seen as “gauge conditions” on the metric [105, 106], corresponding to the threefold coordinate freedom $x^\mu \rightarrow x'^\mu = x'^\mu(x)$. Simple examples of GNC are the Cartesian coordinates on Minkowski space, or polar coordinates in Euclidean 2 and 3-space, or the Robertson-Walker coordinates used in cosmology [105]. Using these coordinates, the determinant of the induced metric on the boundary

$$\gamma_{ij}(X) = \frac{\partial x^\mu}{\partial X^i} \frac{\partial x^\nu}{\partial X^j} g_{\mu\nu}(x) , \tag{5.1.2}$$

is [105]

$$\sqrt{-\gamma} = \sqrt{-g}, \quad (5.1.3)$$

which, in particular, holds on the boundary $r = R$. Lastly, in curved spacetime the Heaviside step distribution is a scalar, and its derivative, as in flat spacetime, is (see Appendix A.1)

$$\nabla_\mu \theta(R - r) = -\delta_\mu^r \delta(R - r). \quad (5.1.4)$$

The action

We consider the abelian 3D Chern-Simons theory on a cylindrical spacetime $\mathbb{R} \times D$, where the model is confined to the closed subspace $0 \leq r \leq R$. This is achieved by introducing a Heaviside step distribution $\theta(R - r)$ in the action. The Chern-Simons bulk term is then

$$S_{bulk} = \frac{\kappa}{2} \int d^3x \theta(R - r) \tilde{\epsilon}^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \quad (5.1.5)$$

We remark that although the coupling constant κ could be reabsorbed by a redefinition of the fields, it is useful to leave it explicit, in order to keep track of the contributions of the bulk theory to the physics of the boundary. Moreover, since the Chern-Simons theory, being topological, has vanishing energy momentum tensor, there is no constraint on the sign of κ from requiring a positive energy density. Still, κ should be a positive constant, as we shall see later. The gauge fixing term is chosen to be

$$S_{gf} = \int d^3x \sqrt{-g} \theta(R - r) b n^\mu A_\mu, \quad (5.1.6)$$

where $n^\mu = (0, 1, 0)$ is a unit vector. The field $b(x)$ is the Nakanishi-Lautrup Lagrange multiplier [107, 108] which implements the radial gauge choice

$$\frac{\delta S}{\delta b} = n^\mu A_\mu = A_r = 0. \quad (5.1.7)$$

The choice of the radial gauge-fixing $A_r = 0$ is analogous to the choice of the GNC (5.1.1). Both are the most convenient choices in presence of the boundary $r = R$: the latter on the metric $g_{\mu\nu}(x)$ and corresponds to the reparametrization invariance, the former on the gauge field $A_\mu(x)$ and comes from gauge invariance. Both transformations (ordinary gauge symmetry and diffeomorphism invariance) are broken by the boundary, and physical results should not depend on them [28, 29]. External sources are coupled to the $A_\mu(x)$ field, through the term

$$S_{ext} = \int d^3x \sqrt{-g} \theta(R - r) J^\mu A_\mu. \quad (5.1.8)$$

In addition, the presence of a boundary induces an extra term in the action, as the most general boundary term compatible with power counting:

$$S_{bd} = \int d^3x \sqrt{-g} \delta(r - R) \frac{1}{2} T^{ij} A_i A_j, \quad (5.1.9)$$

where $T^{ij} = T^{ji}$ is a symmetric matrix. Notice that in a curved spacetime all the coefficients appearing in the action may depend on the metric, which is dimensionless, and hence on the coordinates, but only through the metric. In particular, T^{ij} in S_{bd} might

depend implicitly on the coordinates $T^{ij} = T^{ij}(\gamma(X))$, where $\gamma(X)$ is the induced metric. An explicit dependence of T^{ij} on the coordinates “outside” the induced metric ($T^{ij} = T^{ij}(\gamma(X); X)$) is forbidden, since in the flat limit T^{ij} should reduce to a constant symmetric matrix : $T^{ij}(\gamma(X))|_{\gamma^{ij}=\eta^{ij}} = T^{ij}$. A similar boundary term appears also in [109] for Chern-Simons theory. In both cases, the main reason is to provide, by means of the modified equations of motion, the most general BC which need to be fixed. The introduction of such a local functional allows to introduce the BC in a systematic way by means of a variational principle, and not by hand, the boundary term must only satisfy the general requirements of power counting and locality. Gauge invariance and/or residual 2D covariance must not be required on it, otherwise, one would not recover the boundary dynamics which characterizes TQFT (see for instance [109], where the prescription of introducing a non covariant boundary term $\propto \int_{\partial Y} Tr A_1 A_2$ is adopted in order to get consistent BC). Finally, the total action of the theory, considering the bulk, gauge-fixing, external source and boundary terms, is

$$S = S_{bulk} + S_{gf} + S_{ext} + S_{bd} . \quad (5.1.10)$$

Equations of motion and boundary conditions

From the action S (5.1.10) we get the EoM

$$\frac{\delta S}{\delta A_\lambda} = \theta(R-r) \left(\kappa \epsilon^{\lambda\mu\nu} \partial_\mu A_\nu + b n^\lambda + J^\lambda \right) + \delta(R-r) \delta_j^\lambda \left(\frac{\kappa}{2} \epsilon^{ij} + T^{ij} \right) A_i = 0 . \quad (5.1.11)$$

Applying the operator $\lim_{\epsilon \rightarrow R} \int_\epsilon^R dr$ to the EoM (5.1.11), the following BC can be derived

$$\left(\frac{\kappa}{2} \frac{\tilde{\epsilon}^{1ij}}{\sqrt{-g}} + T^{ij} \right) A_j \Big|_{r=R} = 0 . \quad (5.1.12)$$

In a more compact way, the BC can be written in matricial form

$$M^{ij} A_j = 0 , \quad (5.1.13)$$

where

$$M = \begin{pmatrix} c_1 & c_2 - \tilde{\kappa} \\ c_2 + \tilde{\kappa} & c_3 \end{pmatrix} , \quad (5.1.14)$$

having defined

$$c_1 \equiv T^{00} ; c_2 \equiv T^{02} = T^{20} ; c_3 \equiv T^{22} \quad (5.1.15)$$

and

$$\tilde{\kappa} \equiv \frac{\kappa}{2} \frac{\tilde{\epsilon}^{012}}{\sqrt{-g}} . \quad (5.1.16)$$

Even though in (5.1.16) the value of the 012-component of the Levi-Civita symbol is $\tilde{\epsilon}^{012} = 1$, we choose to keep it explicit, to enhance the fact that $\tilde{\kappa}(X)$ is a scalar, and not a scalar density, as it would appear by hiding $\tilde{\epsilon}^{012}$. The most general solution of the BC (5.1.13) which does not involve vanishing components of the gauge field is

$$A_t + v A_\theta = 0 , \quad (5.1.17)$$

where the coefficient $v(X)$ in (5.1.17) must be different from zero and is not constant, since it depends on the parameters appearing in the action S (5.1.10) according to the following relations

- if $c_1 = 0$; $c_2 = \tilde{\kappa}$; $c_3 \neq 0$

$$2\tilde{\kappa}A_t + c_3A_\theta = 0 \quad \Rightarrow \quad v = \frac{c_3}{2\tilde{\kappa}}; \quad (5.1.18)$$

- if $c_1 \neq 0$; $c_2 = -\tilde{\kappa}$; $c_3 = 0$

$$c_1A_t - 2\tilde{\kappa}A_\theta = 0 \quad \Rightarrow \quad v = -\frac{2\tilde{\kappa}}{c_1}; \quad (5.1.19)$$

- if $c_1 \neq 0$; $c_2 \neq \pm\tilde{\kappa}$; $c_3 \neq 0$; $c_1c_3 - c_2^2 + \tilde{\kappa}^2 = 0$

$$\left\{ \begin{array}{l} c_1A_t + (c_2 - \tilde{\kappa})A_\theta = 0 \quad \Rightarrow \quad v = \frac{c_2 - \tilde{\kappa}}{c_1} \\ \text{or} \\ (c_2 + \tilde{\kappa})A_t + c_3A_\theta = 0 \quad \Rightarrow \quad v = \frac{c_3}{c_2 + \tilde{\kappa}}. \end{array} \right. \quad (5.1.20)$$

The parameter v appearing in the BC (5.1.17) depends on c_i (5.1.15), which are the components of the boundary parameter T^{ij} appearing in S_{bd} (5.1.9), and on $\tilde{\kappa}$ (5.1.16). Hence, v may depend on the coordinates through the induced metric γ^{ij} and on its determinant γ (which in GNC is equal to g). No explicit dependence on the coordinates $X = \{t, \theta\}$ is possible because, as we said, in the flat limit T^{ij} should be constant. The BC (5.1.17) is of the same type of that derived in [110] by an action principle, in analogy to our approach. The difference is that in [110] the analogue of the matrix T^{ij} is constant, which can be obtained after a rescaling by $\sqrt{-g}$. As a consequence, the parameter v appearing in the BC in [110] is constant. In other words, it is always possible to rescale v to a constant by means of a coordinate choice. The main goal of this Section, as we shall see, is that we will be able to relate the parameter v to a measurable quantity, which is the velocity of the chiral edge modes. For this reason we do not rescale v to an arbitrary constant value, but we let v to be determined by a phenomenological input. It is important not to rescale v to a constant value in order to be able to account for *accelerated* chiral bosons on the edge of certain Hall systems, which are not explained by a constant v . This is an important point, which concerns two different perspectives. Our approach is the same to the abelian Chern-Simons description of the Hall systems. From a pure field theoretical point of view, abelian Chern-Simons model is a free theory, with no coupling constant, for the simple reason that it can be reabsorbed by a rescaling of the gauge field. It is only the nonabelian extension of Chern-Simons theory which displays a true coupling constant. In condensed matter theory the perspective is different, somehow opposite. The coupling constants are not rescaled by field redefinitions, but are fixed by external phenomenological inputs. This reflects even in a different terminology. For instance, the abelian Chern-Simons “coupling constant”, which is oxymoric in field theory, is often called “Chern-Simons level” by a condensed matter-oriented reader, because it is fixed by its relation to the filling factor of Landau levels: $\kappa = \frac{1}{2\pi\nu}$. In curved spacetime the metric plays the role of a dynamical field, which we treat exactly in the same way: we do not rescale the metric by choosing a particular coordinates set, but we let the parameter v to be constrained by the observed chiral velocities of the edge modes. We have seen that v depends on the induced metric and its determinant, and hence we relate the induced metric (which depends on the bulk metric of Chern-Simons theory) to an observable quantity. And

this is the only way, so far, to take into account the observed accelerated chiral velocities, which are not explained by a flat background of Chern-Simons theory alone. If we were not interested in a phenomenological interpretation of our model, we would rescale κ to one and v to whatever value, including zero, which would correspond to a Dirichlet condition on one component of the gauge field.

Ward identity

The covariant derivative of the EoM (5.1.11) is

$$\begin{aligned} \nabla_\lambda \frac{\delta S}{\delta A_\lambda} &= \theta(R-r) \left[\kappa \epsilon^{\lambda\mu\nu} \nabla_\lambda \nabla_\mu A_\nu + \nabla_\lambda (b n^\lambda) + \nabla_\lambda J^\lambda \right] - \\ &\quad - \delta(R-r) \left(\kappa \epsilon^{1kj} \partial_k A_j + b + J^r \right) = 0, \end{aligned} \quad (5.1.21)$$

where the BC (5.1.12) have been used to cancel the $\delta(R-r)$ term on the r.h.s. of (5.1.11). Noting that

$$e^{\mu\nu\rho} \nabla_\mu \nabla_\nu A_\rho = 0, \quad (5.1.22)$$

we find

$$\theta(R-r) \nabla_\lambda (b n^\lambda + J^\lambda) - \delta(R-r) \left(\kappa \epsilon^{1kj} \partial_k A_j + b + J^r \right) = 0. \quad (5.1.23)$$

Keeping in mind that the covariant divergence is

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (V^\mu \sqrt{-g}) \quad ; \quad \nabla_k V^k = \frac{1}{\sqrt{-g}} \partial_k (V^k \sqrt{-g}), \quad (5.1.24)$$

where we used (5.1.3) in the second identity, multiplying (5.1.23) by $\sqrt{-g}$ and integrating along the coordinate normal to the boundary, we get⁵

$$\begin{aligned} 0 &= \int_0^\infty dr \left\{ \theta(R-r) \left[\partial_r (b \sqrt{-g}) + \partial_\lambda (J^\lambda \sqrt{-g}) \right] - \delta(R-r) \left(\kappa \tilde{\epsilon}^{1kj} \partial_k A_j \right) \right\} - \\ &\quad - \int_0^\infty dr \delta(R-r) [(b + J^r) \sqrt{-g}] \\ &= \int_0^\infty dr \left[\theta(R-r) \partial_k (J^k \sqrt{-g}) - \delta(R-r) \kappa \tilde{\epsilon}^{1kj} \partial_k A_j \right] - [(b + J^r) \sqrt{-g}]_{r=0} \\ &= \int_0^\infty dr \sqrt{-g} \left[\theta(R-r) \nabla_k J^k - \delta(R-r) \kappa \tilde{\epsilon}^{1kj} \partial_k A_j \right], \end{aligned} \quad (5.1.25)$$

where we integrated by parts and used the fact that, evaluating the EoM (5.1.11) for $\lambda = r$ and then going at $r = 0$, we have $[b(x) + J^r(x)]_{r=0} = 0$. In fact, (5.1.11) at $r = 0$ ($\lambda = r$) reads

$$\left[b + J^r - 2\tilde{\kappa} \left(\partial_\theta A_t - \partial_t A_\theta \right) \right]_{r=0} = 0, \quad (5.1.26)$$

and we notice that $A_\theta(x)$ necessarily vanishes at $r = 0$, together with its time derivatives, while $A_t(x)$, which in principle might not vanish at $r = 0$, at the origin must have vanishing angular derivatives, in order to be well defined, hence the result. Therefore, the contribution involving the Lagrange multiplier and J^r cancel out, leaving only

$$\int_0^R dr \sqrt{-g} \nabla_k J^k = \kappa \tilde{\epsilon}^{1kj} \partial_k A_j \Big|_{r=R}. \quad (5.1.27)$$

⁵ Notice that in general for an integration along r we should use the invariant measure $\sqrt{g_{rr}} dr$ in order to preserve the transformation properties under diffeomorphisms, but $g_{rr} = 1$ in GNG (5.1.1).

Eq.(5.1.27) is the Ward identity of the theory, broken at its r.h.s. by the presence of the boundary, which is crucial for the determination of the boundary algebra and of the 2D theory holographically induced on the boundary. Notice that it holds for any bulk metric, and it is simply the curved extension of its flat counterpart (4.1.11) [87]. From (5.1.27), at vanishing external sources $J^k(x) = 0$ (*i.e.* going on-shell), we find

$$\epsilon^{1kj} \partial_k A_j \Big|_{r=R; J=0} = 0, \quad (5.1.28)$$

which describes a conserved current on the closed boundary $r = R$, whose most general solution is [111, 112]

$$A_i(X) = \partial_i \Phi(X) + \delta_{i2} C, \quad (5.1.29)$$

where C is a constant (which we will show shortly being equal to zero), and $\Phi(X)$ is a scalar field that will play the role of DoF of the 2D boundary theory. The components of the gauge field on the boundary are then

$$A_t(t, \theta) = \partial_t \Phi(t, \theta) \quad ; \quad A_\theta(t, \theta) = \partial_\theta \Phi(t, \theta) + C. \quad (5.1.30)$$

Since we are considering a closed boundary, we also have to impose periodicity conditions on the fields :

$$A_i(t, \theta) = A_i(t, \theta + 2\pi) \quad \Rightarrow \quad \Phi(t, \theta) = \Phi(t, \theta + 2\pi). \quad (5.1.31)$$

The value of the constant C in (5.1.29) is found by applying the mean value theorem for holomorphic functions, which states that if f is analytic in a region D , and $a \in D$, then $f(a) = \frac{1}{2\pi} \oint_{\mathcal{C}(a)} f$, where $\mathcal{C}(a)$ is a circle centered in a . In our case (3D) taking for \mathcal{C} the circular boundary $r = R$ centered at $r = 0$ allows us to write

$$\oint_{ring R} A_\theta(x) = \Phi(t, \theta) \Big|_{\theta=0}^{\theta=2\pi} + 2\pi C, \quad (5.1.32)$$

where we used the periodicity condition of the boundary field $\Phi(X)$ (5.1.31). This is the value of the bulk field at the center of the ring, and using the fact that $A_\theta(t, r = 0, \theta) = 0$, we finally get

$$C = 0. \quad (5.1.33)$$

Algebra

The generating functional of the connected Green functions $Z_c[J]$ is defined, as usual, in the following way

$$e^{iZ_c[J]} = \int [dA][db] e^{iS[A, b; J]}, \quad (5.1.34)$$

where S is the total action (5.1.10). From (5.1.34) we get the 1- and 2-points Green functions

$$\left. \frac{\delta Z_c[J]}{\delta J^i(x)} \right|_{J=0} = \langle A_i(x) \rangle \quad (5.1.35)$$

$$\left. \frac{\delta^{(2)} Z_c[J]}{\delta J^i(x) \delta J^j(x')} \right|_{J=0} \equiv \Delta_{ij}(x, x') = i \langle T(A_i(x) A_j(x')) \rangle, \quad (5.1.36)$$

where the time-ordered product is defined as

$$\langle T(A_j(x) A_l(x')) \rangle \equiv \theta(t - t') \langle A_j(x) A_l(x') \rangle + \theta(t' - t) \langle A_l(x') A_j(x) \rangle. \quad (5.1.37)$$

The algebra is obtained by making the functional derivative with respect to $J^l(x')$ of the Ward identity (5.1.27), and then going on-shell, *i.e.* putting $J = 0$:

$$\frac{\delta}{\delta J^l(x')} \int_0^R dr \sqrt{-g} \nabla_k J^k(x) = \kappa \tilde{\epsilon}^{1kj} \partial_k \frac{\delta^{(2)} Z_c}{\delta J^l(x') \delta J^j(x)} \Big|_{r=R; J=0} . \quad (5.1.38)$$

Therefore the first step is to compute

$$\frac{\delta}{\delta J^l(x')} \nabla_k J^k(x) = \frac{\delta}{\delta J^l(x')} \left[\frac{1}{\sqrt{-g}} \partial_k \left(J^k \sqrt{-g} \right) \right] = \frac{1}{\sqrt{-g}} \partial_l \tilde{\delta}^{(3)}(x - x') , \quad (5.1.39)$$

where we used the relation (A.1.1) between the scalar and density delta function and the fact of working in GNC (5.1.3). Using (5.1.39) on the l.h.s. of (5.1.38), we obtain

$$\partial_l \tilde{\delta}^{(2)}(X - X') = i\kappa \tilde{\epsilon}^{1kj} \partial_k \langle T(A_j(x) A_l(x')) \rangle \Big|_{r=R} . \quad (5.1.40)$$

To write this we used the fact that for any $r' \leq R$ we have $\int_0^R dr \tilde{\delta}(r - r') = 1$, in fact, since by definition $\tilde{\delta}(r - r') \equiv -\partial_r \theta(r' - r)$:

$$\begin{aligned} \int_0^R dr \partial_r \theta(r' - r) f(r) &= \int_0^R \{ \partial_r [\theta(r' - r) f(r)] - \theta(r' - r) \partial_r f(r) \} \\ &= \theta(r' - r) f(r) \Big|_0^R - \int_0^R dr \theta(r' - r) \partial_r f(r) \\ &= \begin{cases} f(R) - f(0) - \int_0^R dr \partial_r f(r) = 0 & \text{if } r' > R \\ 0 - f(0) - \int_0^{r'} dr \partial_r f(r) = -f(r') & \text{if } r' \leq R \end{cases} . \end{aligned} \quad (5.1.41)$$

Choosing $l = \theta$ in (5.1.40) we have :

$$\begin{aligned} \partial_\theta \tilde{\delta}^{(2)}(X - X') &= i\kappa \tilde{\epsilon}^{1kj} \partial_k [\theta(t - t') \langle A_j(X) A_\theta(X') \rangle + \theta(t' - t) \langle A_\theta(X') A_j(X) \rangle] \\ &= i\kappa \tilde{\epsilon}^{102} (\partial_t \theta(t - t')) (\langle A_\theta(X) A_\theta(X') \rangle - \langle A_\theta(X') A_\theta(X) \rangle) \\ &\quad + \kappa \tilde{\epsilon}^{1kj} [i\theta(t - t') \langle \partial_k A_j(X) A_\theta(X') \rangle + i\theta(t' - t) \langle A_\theta(X') \partial_k A_j(X) \rangle] , \end{aligned} \quad (5.1.42)$$

where we used the on-shell condition (5.1.28). By defining

$$[A_j(X), A_\theta(X')] \equiv \langle A_j(X) A_\theta(X') \rangle - \langle A_\theta(X') A_j(X) \rangle , \quad (5.1.43)$$

we get

$$\partial_\theta \tilde{\delta}^{(2)}(X - X') = -i\kappa \tilde{\epsilon}^{012} \partial_t \theta(t - t') [A_\theta(X), A_\theta(X')] . \quad (5.1.44)$$

Finally, integrating over time we find the equal time commutator

$$\tilde{\epsilon}^{012} [A_\theta(X), A_\theta(X')] \Big|_{t=t'} = \frac{i}{\kappa} \partial_\theta \tilde{\delta}(\theta - \theta') , \quad (5.1.45)$$

By applying the same procedure to the case $l = t$ of (5.1.40), we find the equal time commutator

$$[A_\theta(X), A_t(X')] \Big|_{t=t'} = 0 . \quad (5.1.46)$$

Eq.(5.1.45) represents an abelian KM algebra identical to the one found in the case of Chern-Simons theory with planar boundary in flat space (4.1.12). We see that the

existence of a KM algebra and its central charge depends neither on the bulk metric nor on the details of the boundary. The central charge

$$c = \frac{1}{\kappa} \quad (5.1.47)$$

must be positive (for the unitarity of the associated CFT [73, 74]), and for this reason the coupling constant of the Chern-Simons theory has to be positive as well

$$\kappa > 0 . \quad (5.1.48)$$

Notice that this algebraic method is the only way to determine the sign of the Chern-Simons coupling constant, since this theory, being topological, has vanishing stress-energy tensor, and hence the usual argument based on the positivity of the energy density cannot be applied here.

5.1.2 2D boundary theory

The 2D action

In Section 5.1.1, we identified the solution of the conserved current equation (5.1.28) as the boundary DoF. This will allow us to find the 2D dynamics on the boundary, in fact we can express the equal time commutation relation (5.1.45) in terms of the boundary field $\Phi(X)$, by using (5.1.30)

$$[\partial_\theta \Phi(X), \tilde{\epsilon}^{012} \partial_{\theta'} \Phi(X')] = i \frac{1}{\kappa} \partial_\theta \tilde{\delta}(\theta - \theta') , \quad (5.1.49)$$

which, rewritten as

$$[\Phi(X), \kappa \tilde{\epsilon}^{012} \partial_{\theta'} \Phi(X')] = i \tilde{\delta}(\theta - \theta') , \quad (5.1.50)$$

can be interpreted as a canonical commutation relation in curved spacetime

$$[q(t, x), p(t, x')] = i \tilde{\delta}(x - x') , \quad (5.1.51)$$

with $p(t, x)$ a density [105], provided that we match the canonical variables of the 2D theory as follows

$$q(X) \equiv \Phi(X) \quad ; \quad p(X) \equiv \tilde{\epsilon}^{012} \kappa \partial_\theta \Phi(X) . \quad (5.1.52)$$

Here again, the presence of the $\tilde{\epsilon}^{012}$ factor in (5.1.52) is of great importance, since it makes $p(X)$ a scalar density, according to the standard definition [113]. Once identified the canonical variables, we can begin the search for the induced boundary theory, by writing the most general Lagrangian compatible with power counting. Noting that the scalar field has vanishing mass dimension, we find

$$\mathcal{L}_{2D} = \sqrt{-\gamma} (a^{ij} \partial_i \Phi \partial_j \Phi + b^i \partial_i \Phi + c) , \quad (5.1.53)$$

where the coefficients a^{ij} , b^i , c

1. must be tensor quantities, *i.e.* symmetric tensor of rank (2,0), contravariant vector and scalar respectively, in order that the Lagrangian (5.1.53) is a scalar density ;
2. must have mass dimension 0, 1 and 2, respectively ;

3. may depend on the scalar field $\Phi(X)$, the metric $\gamma_{ij}(X)$ and/or its determinant $\gamma(X)$ (but not on their derivatives), since in 2D the scalar field $\Phi(X)$, like the metric, is dimensionless, and thus the power counting is preserved. In other terms, the 2D action (5.1.53) is written as a derivative expansion.

We also notice that the scalar field $\Phi(X)$ is defined by means of (5.1.30) up to a shift transformation

$$\delta_{shift}\Phi = \alpha, \quad (5.1.54)$$

with α constant, which implies that the action

$$S_{2D} = \int d^2X \mathcal{L}_{2D} \quad (5.1.55)$$

describing the boundary theory should possess the same symmetry as well, *i.e.*

$$\delta_{shift}S_{2D} = 0. \quad (5.1.56)$$

An immediate consequence of the shift symmetry (5.1.56) is that a^{ij} and c must be constant with respect to Φ . For what concerns b^i , it is easily seen that it may admit a linear dependence on the scalar field. In fact, if $b^i = b_1^i + b_2^i\Phi$, where b_1^i and b_2^i do not depend on Φ , but may depend on the induced metric γ_{ij} and on its determinant γ , we have that

$$\delta_{shift} \int d^2X \sqrt{-\gamma} b^i \partial_i \Phi = \int d^2X \sqrt{-\gamma} \frac{\partial b^i}{\partial \Phi} \alpha \partial_i \Phi = \alpha \int d^2X \sqrt{-\gamma} \nabla_i \left(\frac{\partial b^i}{\partial \Phi} \Phi \right), \quad (5.1.57)$$

which is a vanishing boundary term. Therefore, the corresponding term in the 2D Lagrangian (5.1.53) does not contribute, being a boundary term as well.

$$\int d^2X \sqrt{-\gamma} b^i \partial_i \Phi = \int d^2X \sqrt{-\gamma} (b_1^i \partial_i \Phi + b_2^i \Phi \partial_i \Phi) = \int d^2X \sqrt{-\gamma} \nabla_i (b_1^i \Phi + \frac{1}{2} b_2^i \Phi^2). \quad (5.1.58)$$

The most general 2D action satisfying the shift symmetry is

$$\begin{aligned} S_{2D} &= \int d^2X \sqrt{-\gamma} (a^{ij} \partial_i \Phi \partial_j \Phi + c) \\ &= \int d^2X \sqrt{-\gamma} (a_0 \partial_t \Phi \partial_t \Phi + 2a_1 \partial_t \Phi \partial_\theta \Phi + a_2 \partial_\theta \Phi \partial_\theta \Phi + c), \end{aligned} \quad (5.1.59)$$

where $a_0 \equiv a^{00}$, $a_1 \equiv a^{02}$ and $a_2 \equiv a^{22}$. The last term does not contribute to the equations of motion of the scalar field Φ and will be omitted. The most general dependence of the dimensionless a^{ij} on the induced metric γ^{ij} is

$$a^{ij} = \alpha^{ij}(\gamma) + \beta(\gamma) \gamma^{ij}, \quad (5.1.60)$$

where $\alpha^{ij}(\gamma)$ and $\beta(\gamma)$ may depend at most on the metric determinant. What is left now is to verify under which conditions the Lagrangian \mathcal{L}_{2D} is compatible with the canonical variables identified from the KM algebra through (5.1.52). This is reached by requiring

$$\frac{\partial \mathcal{L}_{2D}}{\partial \dot{q}} = p = \tilde{\epsilon}^{012} \kappa \partial_\theta \Phi. \quad (5.1.61)$$

The l.h.s. of (5.1.61) is

$$\frac{\partial \mathcal{L}_{2D}}{\partial \dot{q}} = \frac{\partial \mathcal{L}_{2D}}{\partial (\partial_t \Phi)} = \sqrt{-\gamma} (2a_0 \partial_t \Phi + 2a_1 \partial_\theta \Phi), \quad (5.1.62)$$

therefore we must ask

$$a_0 = 0 \quad ; \quad a_1 = \tilde{\kappa} \quad (a_2 \text{ free}), \quad (5.1.63)$$

where $\tilde{\kappa}$ is the scalar function related to the Chern-Simons coupling constant through (5.1.16). Eq.(5.1.63) represents a constraint on the metric dependence of a^{ij} in (5.1.60), in fact it must be

$$0 = a_0 = a^{00} = \alpha^{00} + \beta \gamma^{00} \quad (5.1.64)$$

$$\tilde{\kappa} = a_1 = a^{02} = \alpha^{02} + \beta \gamma^{02}. \quad (5.1.65)$$

Since we do not want to impose any unnecessary condition on the components of the induced metric γ^{ij} , that leads to the request

$$\beta = 0, \quad (5.1.66)$$

therefore

$$a_2 = a^{22} = \alpha^{22}(\gamma), \quad (5.1.67)$$

which means that a_2 is a free parameter depending at most on the metric determinant : $a_2 = a_2(\gamma)$. Hence, up to terms which do not contribute to the equation of motion of the scalar field $\Phi(X)$, the action S_{2D} (5.1.55) is

$$S_{2D} = \int d^2 X \sqrt{-\gamma} (\tilde{\kappa} \partial_t \Phi + a_2 \partial_\theta \Phi) \partial_\theta \Phi. \quad (5.1.68)$$

5.1.3 Holographic contact

From the action S_{2D} (5.1.68) we get the following EoM

$$\partial_\theta \left(\partial_t \Phi + \frac{a_2}{\tilde{\kappa}} \partial_\theta \Phi \right) = 0, \quad (5.1.69)$$

where we divided by the Chern-Simons coupling constant κ . Our task is now to find under which conditions the BC (5.1.17), written in terms of the boundary field $\Phi(X)$ by means of (5.1.30) (with $C = 0$ (5.1.33))

$$\partial_t \Phi + v \partial_\theta \Phi = 0, \quad (5.1.70)$$

can be related to the EoM (5.1.69). The BC (5.1.70) is the equation of a chiral boson whose velocity v , defined by (5.1.17), may depend on the induced metric γ^{ij} through the coefficients T^{ij} appearing in S_{bd} (5.1.9). We recover here the physical interpretation of the parameter v appearing in the BC (5.1.17) as the chiral velocity of the edge modes living on the boundary of Chern-Simons theory, which are known, in the framework of FQHE, to be measurable quantities. In most Hall systems, the observed chiral velocity is constant, and this is achieved by a Chern-Simons theory built on a flat 3D space-time with planar boundary [87]. The novelty we are finding here, is that v is now a *local* quantity, in particular depending on time, which is a consequence of considering

the Chern-Simons theory on a curved, instead of flat, spacetime. This corresponds to *accelerated* edge chiral bosons, which have been indeed recently observed, and which cannot be explained by putting a boundary on a flat background. The fact that v is a phenomenological parameter requires that it should be determined by experimental inputs, hence, as stressed at the beginning of the Section, a choice of coordinates, which would set v to an arbitrary, possibly constant, or even vanishing, value, should not be done *a priori*. We might rather say that the *right* choice of coordinates is that of the laboratory frame where the chiral local velocity is measured. Compatibility between the BC (5.1.70) and the EoM (5.1.69) is reached if we require that $\text{EoM} \equiv \partial_\theta \text{BC}$, or introducing, as in Section 4.1, the field $\rho(t, \theta) \equiv \frac{1}{2\pi} \partial_\theta \Phi(t, \theta)$. In any case, the following condition must hold

$$v = \frac{a_2}{\tilde{\kappa}} . \quad (5.1.71)$$

Eq. (5.1.71) relates the parameters of the two theories, thus establishing a (holographic) link between the bulk ($v, \tilde{\kappa}$) and the induced boundary (a_2). From (5.1.71) we see that, because of (5.1.67), the holographic contact makes v depend only on the determinant γ of the induced metric γ_{ij} , and not on its components. This constitutes a restriction on all possible v appearing in the BC (5.1.17), which, instead, might depend on the determinant of the induced metric *and* on its components as well. This constraint, however, does not spoil the possible local character of v . Eq. (5.1.71) is a condition on the parameters which appears in S_{bd} for the holographic bulk-boundary contact to be possible. Therefore the edge velocity depends on the BC for the Chern-Simons gauge field. The issue of the determination of the chiral velocities has been discussed in details by Wen (in [96] and [97] for instance), who remarked that it cannot be determined from the bulk action (nor be rescaled to an arbitrary value), and that it is thus appropriate to take the velocity v as a phenomenological parameter. This is the phenomenological counterpart of the considerations made in [114] : Chern-Simons theory provides merely an effective large-distance description which captures the general symmetries of the manybody theory and has, as such, intrinsic limitations. For instance, it should not be expected to capture certain details such as the velocity of the edge chiral bosons. In [114] it is correctly argued that the determination of the chiral velocity should come from gauge-breaking terms present in the full Lagrangian, which, however cannot be identified with the gauge fixing term, since physical results should not depend on the gauge choice, hopefully. In our approach, it is clear that gauge-breaking term present in the full Lagrangian thanks to which the TQFT acquires local degrees of freedom is S_{bd} (2.5). It is the breaking of the gauge symmetry which starts the game: the breaking of the gauge symmetry due to the boundary manifests itself in the breaking of the Ward identity (5.1.27), which yields the irrotationality condition (5.1.28), which identifies the local bosonic boundary degrees of freedom. It is from the broken Ward identity that the algebra (5.1.45) is derived, and then all the rest follows, as described. None of these results depend on the particular gauge choice, as we already remarked. The 2D action (5.1.68) can now be entirely written in terms of the parameters appearing in the 3D bulk theory (5.1.10) as follows

$$S_{2D} = \frac{\kappa}{2} \int d^2 X \tilde{\epsilon}^{012} (\partial_t \Phi + v \partial_\theta \Phi) \partial_\theta \Phi , \quad (5.1.72)$$

where κ is the Chern-Simons coupling constant and $v(t, \theta)$, appearing in the BC (5.1.17), depends on the parameters of the boundary action S_{bd} (5.1.9) and, through $\tilde{\kappa}$, on the

determinant of the induced metric $\gamma(t, \theta)$. We observe that the only dependence of the action S_{2D} on the bulk metric is concealed in $v(t, \theta)$. We might say that, in this sense, the metric form of (5.1.72) is protected. Physically, the 2D action (5.1.72) can be immediately identified with the Luttinger theory [115], relevant example of the bosonization phenomenon, for which the density operator $n(t, \theta)$, written in terms of chiral fermions, is

$$n(t, \theta) =: \psi^\dagger(t, \theta) \psi(t, \theta) : . \quad (5.1.73)$$

We have seen for the flat case discussed in Section 4.1 that bosonization, *i.e.* fermion/-boson correspondence, is achieved through the identification

$$n(t, \theta) = \frac{1}{2\pi} \partial_\theta \Phi(t, \theta) , \quad (5.1.74)$$

where $\Phi(X)$ is just the chiral boson (5.1.70) found as the edge state of the Chern-Simons theory. The density operator (5.1.73) satisfies the following commutation relation

$$[n(t, \theta), n(t, \theta')] = i \frac{\nu}{2\pi} \partial_\theta \delta(\theta - \theta') , \quad (5.1.75)$$

where ν is the filling factor of the FQHE. Eq.(5.1.75), with the bosonisation relation (5.1.74), is exactly the KM algebra we found in (5.1.49), by means of which we can physically interpret the field $A_\theta(t, \theta)$ as the density operator $n(t, \theta)$ (5.1.73). Here we also notice how the boundary theory makes sense for *any* dependence of v on X . Indeed, from the perspective of (5.1.70), the EoM derived from the action (5.1.68) reads

$$\partial_t n + \partial_\theta (vn) = 0 , \quad (5.1.76)$$

which represents the continuity equation of a 1D fluid of velocity v and density n , as already remarked in the case of flat bulk metric in Section 4.1 and in [11]. This, again, confirms that v , from the perspective of the Chern-Simons theory, is simply a free phenomenological parameter. As a consequence, we can associate the parameters of the action S_{2D} (5.1.72) to physical quantities. In particular, by matching through (5.1.74) the algebra (5.1.49), written in terms of the boundary field $\Phi(X)$, with the algebra (5.1.75) written in terms of the density operator $n(t, \theta)$, we obtain the well known relation between the filling factor ν and the coupling constant of the Chern-Simons theory κ [11, 87]

$$\kappa = \frac{1}{2\pi\nu} . \quad (5.1.77)$$

Hence, from (5.1.70), we can identify $v(t, \theta)$ as the spacetime-dependent velocity of the chiral boson $\Phi(X)$. Notice that, because of (5.1.71), the chiral boson turn left or right depending on the sign of a_2 in (5.1.72), being $\tilde{\kappa}$ positive due to the positivity of the central charge. This is in agreement with [89], where the edge excitations were studied on a disc and on a cylinder, *i.e.* on curved boundaries in flat spacetime, which belong to the cases studied in this Section. It is interesting to remark that we recover those results, in particular the algebra and the Luttinger theory, in a quite different way.

5.1.4 Summary of results and discussion

The aim of the original work presented in this Section was to understand if the geometry of the bulk spacetime affected in some way the boundary physics of Chern-Simons

theory, which, in the flat case, we have seen to reproduce, on the boundary, the theory of edge states of the Fractional Quantum Hall Effect. Naively, starting from a TQFT one would expect a mild dependence on the bulk metric. Instead, our analysis shows that the velocity of the edge chiral boson indirectly depends on the bulk metric, through the determinant of the induced metric. Our work might therefore be the first step towards a theoretical framework for the recently observed accelerated chiral boson on the edge of particular Hall systems, which cannot be explained by the usual description in terms of flat Chern-Simons theory with boundary. On the other hand, on the boundary of Chern-Simons theory we find the usual KM algebraic structure of the flat background case, with the same central charge proportional to the inverse of the Chern-Simons coupling. Therefore, the KM algebra appears to be insensitive to both the bulk metric and the type of boundary (at least, it is the same for planar and radial boundary). What is certainly true is that the Chern-Simons bulk theory in flat spacetime, when considered with a boundary, generates local DoF on the 2D induced theory, corresponding to the edge states of FQHE, with *constant* chiral velocities. The possibility of local velocities is not captured by the flat approach alone.

A few remarks are in order.

- **Technical remarks** : the determinant of the metric depends on the coordinates that are chosen, and locally one might always find coordinates such that the determinant is a constant. Moreover, in 2+1 dimensions the boundary of Chern-Simons theory is well known to be a CFT, hence depends only upon the conformal class of the boundary metric, and in every conformal class there is a metric of constant determinant. This would correspond to choosing a reference frame where the velocity of the chiral boson is normalized at a given constant value. This represents a good, physical reason for not choosing a metric with constant determinant. The boundary conformal structure is ensured by the existence of an algebraic structure of the KM type. Had we finished here, we would not have a good reason for not choosing, amongst the metrics in GNC, those with constant determinant. This would have been a without loss of generality choice, and we would have achieved our result as far as only the conformal structure described by the boundary algebra is concerned. But this is only half of our results : on the *planar* boundary of Chern-Simons theory with *flat* bulk metric it is known to exist the Luttinger theory of the chiral boson. We derived the corresponding action following the procedure described in Section 5.1.2. The action (5.1.72) is not invariant under change of coordinates, which hence is not allowed at this stage. Nonetheless, the Luttinger theory which we found on the boundary of the Chern-Simons theory with non-flat metric is not independent from its bulk counterpart. We would not have found this result if, in the previous step, we had frozen the bulk metric by choosing coordinates with constant determinant (or a reference frame where the chiral velocity is constant). Instead, a memory of the bulk metric survives, in the velocity of the chiral boson, which depends only on the determinant of the bulk metric. Not having made the choice of constant determinant leaves therefore the possibility of highlighting the residual dependence on the bulk metric, and also of selecting the bulk metric by means of a physical input, which is the velocity of the edge chiral bosons, which is a physical observable. This is not so different from what is usually done with the Chern-Simons coupling constant,

which is fixed by means of the physical requirements of the incompressible Hall fluid, by normalizing the external current coupled to the electromagnetic gauge field to the known relationship between Chern-Simons coupling constant and filling factor $\kappa = \frac{1}{2\pi\nu}$ as in (4.1.3). Yet, speaking of a coupling constant for the *abelian* Chern-Simons theory is rather inappropriate, since it is always possible to reabsorb the coupling constant by means of a redefinition of the gauge fields. Only in the non-abelian case a true coupling constant for Chern-Simons theory exists. Nevertheless, normalizing to one the Chern-Simons coupling constant would be equivalent to losing the entire structure of Landau levels of (F)QHE, and for that reason we keep it alive. We see an analogy between the two situations: as we do not normalize to one the Chern-Simons coupling constant in the abelian case in order to describe the filling factors of the FQHE, for the same reason we do not choose coordinates such that the determinant is constant, in order to keep the possibility of fixing the bulk metric by measuring the (possibly time dependent) velocity of the edge chiral boson.

- **The phenomenological approach** : it is useful to make a comparison with the approach usually adopted in the Literature to describe non-constant velocities of the edge excitations in the Hall systems. It is known that the fermions in a quantum Hall droplet are constrained by a confining potential whose gradient determines the velocity of the chiral edge modes [98]. For instance, the confinement may involve potentials that depend on space and/or time due to deformations or interactions in the sample [99] and typically, in the phenomenological framework, one can invoke these considerations in order to take into account non-constant chiral velocities, which therefore, in the microscopic model, depend on the different possible potentials in the sample. In general, the paradigmatic way to describe the edge excitations has been illustrated by Wen [96, 97, 100], and well summarized in [98]. One starts with a 3D Chern-Simons action in *flat* spacetime, from which the bulk DoF are eliminated by integrating out the time-component of the gauge field $A_0(x)$. Then, the irrotational constraint on the spatial components of the gauge field is *imposed* : $\vec{\nabla} \times \vec{A} = 0$. The boundary DoF is then identified by the scalar field characterizing the solution $\vec{A} = \vec{\nabla}\Phi$. Finally, the “appropriate effective action” at the edge is written as the sum of two terms: the Luttinger action, describing a chiral boson with *constant* velocity, to which is possibly added an interaction term, which depends on the form of the edge confining potential and on the details of the electron-electron interactions. If this latter interaction is assumed to be local, this obviously implies non-constant velocities. Usually, to simplify the description, the interactions are taken as piecewise functions, in order to take into account, for instance, the screening effects which are present in the leads. More details can be found in [116] and in [117]. This latter paper is related to an interesting experiment [118]. Other experiments are described in [119, 120]. To make it short, the total interaction is split into two parts : a constant velocity piece v contained in the Luttinger action and an additional local velocity term in the interaction. This is the framework for the description of edge excitations with non-constant velocities. Alternatively, according to our approach, there is no need of *imposing* the irrotational condition which gives rise to the scalar boundary degree of freedom. Rather, it emerges as the breaking term of

the Ward identity (5.1.27), evaluated on shell. Similarly, the 2D action (5.1.72) is not *chosen* as an effective action, but is recovered by means of a general procedure which we described as an “holographic contact” between the BC on the bulk gauge field and the EoM of the 2D scalar field. Finally, we showed that a bulk Chern-Simons model on curved spacetime yields a possible *non-constant* chiral velocity, where the locality property is inherited solely from the induced metric determinant from a bulk-boundary correspondence. An intriguing way to interpret this fact, in relation to the microscopic models, is to consider it as an equivalent description : a change of potential can be effectively encoded in the Chern-Simons theory by a bulk metric. As the Chern-Simons coupling constant κ encodes all the possible values of the filling factor ν , similarly the Chern-Simons bulk metric encodes the spacetime properties of the (confining) potential, in terms of the induced metric determinant.

- On current experiments :** one of the main results of this Thesis is that the measurable chiral velocity v of the edge modes is related not only to the Chern-Simons level κ , as it is known already for a flat background, but also to the determinant of the metric, through which v can acquire a time dependence, which can be measured. It is indeed possible to detect time dependent edge chiral velocity, as well as to investigate non-standard geometries (for what concerns the quantum Hall effect). In [48] the boundary of a Hall system with $\nu = 2$ is considered. The two channels (left and right) interact, and the problem diagonalizes into one fast and one slow mode, and the velocity is not constant and is measured. For what concerns non conventional bulk geometries, recently, in [121], the FQHE has been observed in graphene. Here, the electrons live on a two-dimensional “suspended” membrane made by carbon atoms. There are some differences with respect to the standard case (Dirac-type linear dispersion relation rather than quadratic, presence of an additional degree of freedom like a pseudo-spin, for instance), which makes the phenomenology similar to that of FQHE, but with different values of the plateaux. The nice thing is that graphene technically allows to make samples with various shapes like Corbino discs, and observations seem sensitive to the bulk geometry. Moreover, always using graphene, there are attempts to realize quantum Hall systems in curved space, like described in [122] and in [123]. Now, we do not know if the results presented in this Section can immediately be applied to these particular experimental sets, but the remarkable fact is that a recent experimental research activity exists, which concerns the main topic of this Chapter : the possibility of having boundary chiral modes whose time dependent velocities are sensitive to the bulk geometry.

5.2 BF DESCRIPTION OF 2D ACCELERATED CHIRAL EDGE MODES

This Section, which concerns the study of the 3D BF model in curved space as a natural continuation of the results obtained in the previous Section for Chern-Simons, is organized as follows

5.2.1	The model : bulk and boundary	47
5.2.2	2D boundary theory	53
5.2.3	Holographic contact	56
5.2.4	The role of Time-Reversal symmetry	60
5.2.5	Summary of results and discussion	64

In particular in Section 5.2.1 we prepare the tools to face the problem : we introduce a radial boundary in the gauge-fixed 3D BF action, and derive à la Symanzik the most general BC on the two gauge fields of the theory. As we said, the boundary breaks gauge invariance, and this reflects in the breakings of the two Ward identities describing the broken gauge symmetry. The breaking are particularly fruitful, as they lead us to identify the 2D scalar degrees of freedom and the KM algebra formed by the edge conserved currents. Requiring the positivity of the KM central charge constrains the BF coupling to be positive. In Section 5.2.2 the 2D theory is derived as the holographic projection of the 3D bulk theory. The contact is realized in Section 5.2.3 by matching the BC on the bulk side and the EoM on the boundary side. The resulting action involves two scalar fields and is more complicated than the simple Luttinger theory found on the edge of Chern-Simons model. We give a physical interpretation of the 2D theory, and we find three possibilities for the motion of the edge quasiparticles: same directions, opposite directions and a single-moving mode. But, requiring that the Hamiltonian of the 2D theory is bounded by below, the case of edge modes moving in the same direction is ruled out. We are therefore left with physical situations characterized by edge excitations moving with opposite velocities, (examples are FQHE with $\nu = 1 - 1/n$, with n positive integer [124], and Helical Luttinger Liquids phenomena [125]) or a single-moving mode (Quantum Anomalous Hall [22, 126, 127]). In Section 5.2.4 a strong restriction is obtained by requiring Time Reversal symmetry, which uniquely identifies modes with equal and opposite velocities, and we know that this is the case of Topological Insulators. The novelty, with respect to the flat bulk background, is that the modes have local velocities, which corresponds to Topological Insulators with *accelerated* edge modes. In Section 5.2.5 we summarize and discuss our results.

5.2.1 *The model : bulk and boundary**The action*

We consider the abelian 3D BF model on a manifold diffeomorphic to a cylinder of radius R . The boundary is introduced by means of a Heaviside step function in the bulk action, constraining the radial coordinate to $r \leq R$. The BF bulk action is

$$S_{BF} = \kappa \int d^3x \theta(R - r) \tilde{\epsilon}^{\mu\nu\rho} \partial_\mu A_\nu B_\rho, \quad (5.2.1)$$

where $A_\mu(x)$ and $B_\mu(x)$ are two gauge fields with mass dimensions $[A] = [B] = 1$ and κ is a constant which will be determined by physical inputs, as we shall see later. Differently from the previous case, in this Section we decide not to fix GNC (we will see that indeed this does not affect the final results). According to this the properties of distributions in general curved spacetimes (such as the derivative of the Heaviside step function) are given in Appendix A.1. We choose the radial gauge, implemented by the following gauge-fixing term

$$S_{gf} = \int d^3x \sqrt{-g} \theta(R-r) [(bA_\mu + d B_\mu) n^\mu] , \quad (5.2.2)$$

where $n^\mu = (0, 1, 0)$ is a vector and $b(x)$, $d(x)$ are scalar Nakanishi-Lautrup Lagrange multipliers [107, 108]:

$$\frac{\delta S}{\delta b} = n^\mu A_\mu = A_r = 0 \quad ; \quad \frac{\delta S}{\delta d} = n^\mu B_\mu = B_r = 0 . \quad (5.2.3)$$

The external source term is

$$S_{ext} = \int d^3x \sqrt{-g} \theta(R-r) \left(J^\mu A_\mu + \hat{J}^\mu B_\mu \right) , \quad (5.2.4)$$

where $J^\mu(x)$ and $\hat{J}^\mu(x)$ are vectors. The presence of a boundary induces, as an additional contribution, the most general boundary term compatible with power-counting and locality [10]

$$S_{bd} = \int d^3x \sqrt{-\gamma} \delta(r-R) \left(\frac{\alpha^{ij}}{2} A_i A_j + \frac{\beta^{ij}}{2} B_i B_j + \zeta^{ij} A_i B_j \right) , \quad (5.2.5)$$

where $\alpha^{ij} = \alpha^{ji}$, $\beta^{ij} = \beta^{ji}$ and ζ^{ij} are dimensionless tensors which depend, at most, on the induced metric (components γ^{ij} and/or determinant γ) in the following way

$$\alpha^{ij} = \hat{\alpha}^{ij}(\gamma) + \hat{\alpha}(\gamma) \gamma^{ij} = \hat{\alpha}^{ij} + \hat{\alpha} \gamma^{ij} \quad (5.2.6)$$

$$\beta^{ij} = \hat{\beta}^{ij}(\gamma) + \hat{\beta}(\gamma) \gamma^{ij} = \hat{\beta}^{ij} + \hat{\beta} \gamma^{ij} \quad (5.2.7)$$

$$\zeta^{ij} = \hat{\zeta}^{ij}(\gamma) + \hat{\zeta}(\gamma) \gamma^{ij} = \hat{\zeta}^{ij} + \hat{\zeta} \gamma^{ij} . \quad (5.2.8)$$

Thus, as in the previously discussed Chern-Simons case, a non-covariant boundary term has been chosen. Notice that covariance could be recovered by taking $\hat{\alpha}^{ij} = \hat{\beta}^{ij} = \hat{\zeta}^{ij} = 0$. In the flat limit the parameters α^{ij} , β^{ij} and ζ^{ij} in S_{bd} are constant. Finally, the total action, containing BF bulk, gauge fixing, external sources and boundary terms, is

$$S = S_{BF} + S_{gf} + S_{ext} + S_{bd} . \quad (5.2.9)$$

Equations of motion and boundary conditions

From the total action S (5.2.9), the EoM of the theory follow

$$\frac{\delta S}{\delta A_\lambda} = \theta(R-r) \left[-\kappa \epsilon^{\mu\lambda\rho} \partial_\mu B_\rho + b n^\lambda + J^\lambda \right] + \delta_i^\lambda \frac{\sqrt{-\gamma}}{\sqrt{-g}} \delta(r-R) \left[-\kappa \frac{\tilde{\epsilon}^{i1j}}{\sqrt{-\gamma}} B_j + \alpha^{ij} A_j + \zeta^{ij} B_j \right] = 0 \quad (5.2.10)$$

$$\frac{\delta S}{\delta B_\lambda} = \theta(R-r) \left[\kappa \epsilon^{\mu\nu\lambda} \partial_\mu A_\nu + d n^\lambda + \hat{J}^\lambda \right] + \delta_i^\lambda \frac{\sqrt{-\gamma}}{\sqrt{-g}} \delta(r-R) \left[\beta^{ij} B_j + \zeta^{ij} A_j \right] = 0 . \quad (5.2.11)$$

From the EoM (5.2.10), (5.2.11) we get the BC of the theory by applying the operator $\lim_{\epsilon \rightarrow R} \int_{\epsilon}^R$, *i.e.*

$$\lim_{\epsilon \rightarrow R} \int_{\epsilon}^R dr (5.2.10) : \quad \alpha^{ij} A_j + \left(-\kappa \frac{\tilde{\epsilon}^{1j}}{\sqrt{-\gamma}} + \zeta^{ij} \right) B_j \Big|_{r=R} = 0 \quad (5.2.12)$$

$$\lim_{\epsilon \rightarrow R} \int_{\epsilon}^R dr (5.2.11) : \quad \zeta^{ji} A_j + \beta^{ij} B_j \Big|_{r=R} = 0. \quad (5.2.13)$$

Eq.(5.2.12) and (5.2.13) can be written in a compact form as follows

$$\Lambda^{IJ} X_J \Big|_{r=R} = 0, \quad (5.2.14)$$

where $I, J = \{i; j\} = \{0, 2; 0, 2\}$,

$$\Lambda^{IJ} \equiv \begin{pmatrix} \alpha^{ij} & \zeta^{ij} - \kappa \tilde{\epsilon}^{1j} \\ \zeta^{ji} & \beta^{ij} \end{pmatrix} = \begin{pmatrix} \alpha^{00} & \alpha^{02} & \zeta^{00} & \zeta^{02} - \hat{\kappa} \\ \alpha^{20} & \alpha^{22} & \zeta^{20} + \hat{\kappa} & \zeta^{22} \\ \zeta^{00} & \zeta^{20} & \beta^{00} & \beta^{02} \\ \zeta^{02} & \zeta^{22} & \beta^{02} & \beta^{22} \end{pmatrix}, \quad (5.2.15)$$

and

$$X_J \equiv \begin{pmatrix} A_j \\ B_j \end{pmatrix}, \quad (5.2.16)$$

where we defined

$$\hat{\kappa} \equiv \frac{\kappa \tilde{\epsilon}^{012}}{\sqrt{-\gamma}}. \quad (5.2.17)$$

Here again we leave $\tilde{\epsilon}^{012}$ explicit (instead of simply putting it equal to 1) to keep explicit the tensor nature of all quantities. For instance, in this way, it is immediate to see that $\hat{\kappa}$ is a scalar function. The BC (5.2.14) defines a linear, homogeneous system of four equations and four variables $A_i|_{r=R}$ and $B_i|_{r=R}$, for which, requiring $\det \Lambda = 0$, it is possible to write three of the fields in terms of one:

$$B_\theta(X) = -l_1 B_t(X) \quad (5.2.18)$$

$$A_\theta(X) = -l_2 B_t(X) \quad (5.2.19)$$

$$A_t(X) = -l_3 B_t(X), \quad (5.2.20)$$

where $l_{1,2,3}$ depend on the coefficients of the total action (5.2.9) and therefore, in general, are local functions of the induced metric on the boundary $\gamma_{ij}(X)$. Their explicit form can be found in Appendix B.1 ((B.1.8), (B.1.9) and (B.1.10)). Notice that to exclude Dirichlet-like solutions (*i.e.* $A_i|_{r=R} = B_i|_{r=R} = 0$), which would trivialize the boundary 2D physics, we must require $l_i \neq 0$ and $l_i^{-1} \neq 0$.

Ward identities

The covariant divergence of the EoM (5.2.10) is

$$\begin{aligned} \nabla_\lambda \frac{\delta S}{\delta A_\lambda} &= \nabla_\lambda \left[\theta(R-r) \left(-\kappa \epsilon^{\mu\lambda\rho} \partial_\mu B_\rho + n^\lambda b + J^\lambda \right) \right] \\ &= \frac{1}{\sqrt{-g}} \delta(r-R) \kappa \tilde{\epsilon}^{1j} \partial_i B_j + \frac{1}{\sqrt{-g}} \partial_\lambda \left[\theta(R-r) n^\lambda b \sqrt{-g} \right] + \nabla_\lambda \left[\theta(R-r) J^\lambda \right] = 0, \end{aligned} \quad (5.2.21)$$

where we used the BC (5.2.12), the fact that $\epsilon^{\mu\nu\rho}\nabla_\mu\partial_\nu B_\rho = 0$, the definition of covariant derivative of the step function (A.1.14) and the formula for the covariant divergence (5.1.24)

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu (V^\mu\sqrt{-g}) . \quad (5.2.22)$$

By multiplying (5.2.21) by $\sqrt{-g}$ and integrating over the coordinate normal to the boundary $r = R$, we get

$$\begin{aligned} 0 &= \int_0^{+\infty} dr \left\{ \delta(r-R) \kappa \tilde{\epsilon}^{ij} \partial_i B_j + \partial_\lambda \left[\theta(R-r) n^\lambda b \sqrt{-g} \right] + \sqrt{-g} \nabla_\lambda \left[\theta(R-r) J^\lambda \right] \right\} \\ &= \kappa \tilde{\epsilon}^{ij} \partial_i B_j \Big|_{r=R} - b \sqrt{-g} \Big|_{r=0} + \int_0^{+\infty} dr \sqrt{-g} \nabla_\lambda \left[\theta(R-r) J^\lambda \right] , \end{aligned} \quad (5.2.23)$$

where we adopted the convention according to which the θ components of the fields, their θ -derivatives and the Lagrange multipliers vanish at $r = 0$:

$$A_\theta = B_\theta = \partial_\theta A_t = \partial_\theta B_t = b = d|_{r=0} = 0 . \quad (5.2.24)$$

A comment here may be useful : the invariant measure for integrating along r can be identified as induced from the bulk, as follows

$$\int d^3x \sqrt{-g} \delta^{(2)}(X - X') = \int d^3x \sqrt{-g} \frac{\tilde{\delta}^{(2)}(X - X')}{\sqrt{-\gamma}} = \int dr \frac{\sqrt{-g}}{\sqrt{-\gamma}} = \int dr \sqrt{g_{rr}} , \quad (5.2.25)$$

where we used (A.1.12), and $X = (t, \theta)$ are the boundary coordinates. However, in the specific case of (5.2.23), the $\sqrt{g_{rr}}$ factor in the integration can be omitted, simply dividing (5.2.21) by $\sqrt{g_{rr}}$. Using (A.1.14) in (5.2.23), we finally find

$$\int_0^R dr \sqrt{-g} \nabla_\lambda J^\lambda = \left(-\kappa \tilde{\epsilon}^{ij} \partial_i B_j + \sqrt{-g} J^r \right)_{r=R} , \quad (5.2.26)$$

which is the Ward identity corresponding to the gauge transformation of the gauge field $A_\mu(x)$, broken at its r.h.s. by the presence of the boundary. By applying the same procedure to the EoM (5.2.11), with $d|_{r=0} = 0$, we obtain a second broken Ward identity :

$$\int_0^R dr \sqrt{-g} \nabla_\lambda \hat{J}^\lambda = \left(-\kappa \tilde{\epsilon}^{ij} \partial_i A_j + \sqrt{-g} \hat{J}^r \right)_{r=R} . \quad (5.2.27)$$

From (5.2.26) and (5.2.27), going on-shell (*i.e.* $J = \hat{J} = 0$), we find

$$\tilde{\epsilon}^{ij} \partial_i B_j \Big|_{r=R} = 0 \quad (5.2.28)$$

$$\tilde{\epsilon}^{ij} \partial_i A_j \Big|_{r=R} = 0 , \quad (5.2.29)$$

which describe two conserved currents on the boundary $r = R$. The most general solutions to these equations are [111, 112]

$$B_i(X) = \partial_i \psi(X) + \delta_{i2} \hat{C} \quad (5.2.30)$$

$$A_i(X) = \partial_i \varphi(X) + \delta_{i2} C , \quad (5.2.31)$$

where C and \hat{C} are two constants and $\varphi(X)$ and $\psi(X)$ are scalar boundary fields with zero mass dimensions, *i.e.* $[\varphi] = [\psi] = 0$, which should be identified with the boundary DoF. Being on a closed, periodic boundary, we need to specify periodicity conditions

$$A_i(t, \theta) = A_i(t, \theta + 2\pi) \Rightarrow \varphi(t, \theta) = \varphi(t, \theta + 2\pi) \quad (5.2.32)$$

$$B_i(t, \theta) = B_i(t, \theta + 2\pi) \Rightarrow \psi(t, \theta) = \psi(t, \theta + 2\pi) . \quad (5.2.33)$$

The values of the constants C and \hat{C} in (5.2.30), (5.2.31) are found by applying the mean value theorem for holomorphic functions, as in the previous Section 5.1, which states that if f is analytic in a region D , and $a \in D$, then $f(a) = \frac{1}{2\pi} \oint_{\mathcal{C}(a)} f$, where $\mathcal{C}(a)$ is a circle centered in a . In our case taking for \mathcal{C} the circular boundary $r = R$ centered at $r = 0$ allows us to write

$$A_\theta(t, r = 0, \theta) = \oint_{ring R} A_\theta(x) = \varphi(t, \theta) \Big|_{\theta=0}^{\theta=2\pi} + 2\pi C . \quad (5.2.34)$$

From the requirement $A_\theta(t, r = 0, \theta) = 0$, it follows that

$$C = 0 . \quad (5.2.35)$$

Analogously, we also get

$$\hat{C} = 0 . \quad (5.2.36)$$

Algebra

The generating functional of the connected Green functions $Z_c[J, \hat{J}]$ is defined

$$e^{iZ_c[J, \hat{J}]} = \int [dA][dB][db][dd] e^{iS[A, B, b, d; J, \hat{J}]} , \quad (5.2.37)$$

from which the 1- and 2-points Green functions are derived

$$\left. \frac{\delta Z_c[J]}{\delta J^i(x)} \right|_{J=0} = \langle A_i(x) \rangle \quad (5.2.38)$$

$$\left. \frac{\delta^{(2)} Z_c[J]}{\delta J^i(x) \delta J^j(x')} \right|_{J=0} \equiv \Delta_{ij}(x, x') = i \langle T(A_i(x) A_j(x')) \rangle , \quad (5.2.39)$$

where T is the time-ordered product

$$\langle T(A_i(x) A_j(x')) \rangle \equiv \theta(t - t') \langle A_i(x) A_j(x') \rangle + \theta(t' - t) \langle A_j(x') A_i(x) \rangle . \quad (5.2.40)$$

In order to compute the propagator (5.2.39), we need the following result

$$\frac{\delta}{\delta J^i(x')} \nabla_\lambda J^\lambda(x) = \frac{\delta}{\delta J^i(x')} \left[\frac{1}{\sqrt{-g}} \partial_\lambda \left(J^\lambda \sqrt{-g} \right) \right] = \frac{1}{\sqrt{-g}} \partial_i \tilde{\delta}^{(3)}(x - x') , \quad (5.2.41)$$

where we used (5.2.22) and the relation between scalar and density Dirac delta distributions (A.1.1). Taking now the functional derivative with respect to the external sources J, \hat{J} of the broken Ward identities (5.2.26) and (5.2.27), we get

$\frac{\delta}{\delta J^k(x')}$ (5.2.26) :

$$\begin{aligned} \int_0^R dr \partial_k \tilde{\delta}^{(3)}(x-x') &= -\kappa \tilde{\epsilon}^{i1j} \partial_i \frac{\delta^{(2)} Z_c}{\delta J^k(X') \delta \hat{J}^j(X)} \Big|_{J=\hat{J}=0} \\ \partial_k \tilde{\delta}^{(2)}(X-X') &= -i\kappa \tilde{\epsilon}^{i1j} \partial_i \langle T(A_k(X') B_j(X)) \rangle \\ &= -i\kappa \tilde{\epsilon}^{012} [B_\theta(X), A_k(X')] \partial_t \theta(t-t'), \end{aligned} \quad (5.2.42)$$

where we used (5.2.39), (5.2.40) and (5.2.41). Going on-shell, *i.e.* using (5.2.28), we have

$$\tilde{\epsilon}^{012} [B_\theta(X), A_k(X')] \partial_t \theta(t-t') = \frac{i}{\kappa} \partial_k \tilde{\delta}^{(2)}(X-X'). \quad (5.2.43)$$

Setting $k=t$ in (5.2.43) and integrating over time, we get the equal time commutator

$$[B_\theta(X), A_t(X')] = 0. \quad (5.2.44)$$

Analogously, choosing $k=\theta$ we get

$$\tilde{\epsilon}^{012} [B_\theta(X), A_\theta(X')] \Big|_{t=t'} = \frac{i}{\kappa} \partial_\theta \tilde{\delta}(\theta-\theta'). \quad (5.2.45)$$

We can repeat this to find the whole current algebra

$\frac{\delta}{\delta \hat{J}^k(x')}$ (5.2.26) :

$$\begin{aligned} 0 &= \tilde{\epsilon}^{i1j} \partial_i \frac{\delta^{(2)} Z_c}{\delta \hat{J}^k(X') \delta \hat{J}^j(X)} \Big|_{J=0} = \tilde{\epsilon}^{i1j} \partial_i \langle T(B_k(X') B_j(X)) \rangle \\ &= \tilde{\epsilon}^{012} [B_\theta(X), B_k(X')] \partial_t \theta(t-t'), \end{aligned} \quad (5.2.46)$$

which leads to

$$[B_\theta(X), B_k(X')] \Big|_{t=t'} = 0. \quad (5.2.47)$$

In the same way, from $\frac{\delta}{\delta J^k(x')}$ (5.2.27) we get

$$[A_\theta(X), A_k(X')] \Big|_{t=t'} = 0. \quad (5.2.48)$$

Finally, from $\frac{\delta}{\delta \hat{J}^k(x')}$ (5.2.27) we have

$$[A_\theta(X), B_t(X')] \Big|_{t=t'} = 0 \quad (5.2.49)$$

$$\tilde{\epsilon}^{012} [A_\theta(X), B_\theta(X')] \Big|_{t=t'} = \frac{i}{\kappa} \partial_\theta \tilde{\delta}(\theta-\theta'). \quad (5.2.50)$$

Summarizing, the equal time commutators are

$$[B_\theta(X), A_t(X')] = 0 \quad (5.2.51)$$

$$\tilde{\epsilon}^{012} [B_\theta(X), A_\theta(X')] = \frac{i}{\kappa} \partial_\theta \tilde{\delta}(\theta-\theta') \quad (5.2.52)$$

$$[B_\theta(X), B_k(X')] = 0 \quad (5.2.53)$$

$$[A_\theta(X), A_k(X')] = 0 \quad (5.2.54)$$

$$[A_\theta(X), B_t(X')] = 0 \quad (5.2.55)$$

$$\tilde{\epsilon}^{012} [A_\theta(X), B_\theta(X')] = \frac{i}{\kappa} \partial_\theta \tilde{\delta}(\theta-\theta'). \quad (5.2.56)$$

One can observe that by using the property of the delta function $\delta'(x) = -\delta'(-x)$, the commutators (5.2.52) and (5.2.56) represent the same relation. Eq.(5.2.51)-(5.2.56) describe a semidirect sum of KM algebras [26, 27] with central charge

$$c = \frac{1}{\kappa}, \quad (5.2.57)$$

which, as one can expect, inherit the topological nature of the bulk theory, being independent from the metric. Again we remark that from the positivity of the central charge, being necessary for the unitarity of the CFT [73, 74], we get a constraint on the coupling constant of the BF bulk model

$$\kappa > 0. \quad (5.2.58)$$

Typically, the constraint on the coupling constants of QFTs is derived by asking the positivity of the energy density, which is the 00-component of the energy momentum tensor T^{00} . This cannot be achieved in TQFTs, which have vanishing Hamiltonian, as it is well known. The constraint in that case is obtained by asking the positivity of the central charge of the edge current algebra, as in the present case, and in the previous (Section 5.1).

5.2.2 2D boundary theory

We now focus on the construction of the boundary theory, whose DoF are the boundary scalar fields $\varphi(X)$ and $\psi(X)$ defined by the solutions (5.2.30) and (5.2.31) of the conserved currents equations (5.2.28) and (5.2.29). To build the 2D induced theory we follow three steps :

1. identification of the 2D canonical variables in terms of boundary fields ;
2. derivation of the most general 2D action ;
3. bulk-boundary correspondence (holographic contact).

2D canonical variables

The first step is to write the commutator (5.2.56) in terms of the boundary fields $\varphi(X)$, $\psi(X)$, using (5.2.30) and (5.2.31) (with $C = \hat{C} = 0$) :

$$\tilde{\epsilon}^{012} [\partial_\theta \varphi(X), \partial_{\theta'} \psi(X')] = \frac{i}{\kappa} \partial_\theta \tilde{\delta}(\theta - \theta') \quad \Rightarrow \quad \tilde{\epsilon}^{012} [\varphi(X), \partial_{\theta'} \psi(X')] = \frac{i}{\kappa} \tilde{\delta}(\theta - \theta'). \quad (5.2.59)$$

Analogously, from (5.2.52) we get

$$\tilde{\epsilon}^{012} [\psi(X), \partial_{\theta'} \varphi(X')] = \frac{i}{\kappa} \tilde{\delta}(\theta - \theta'). \quad (5.2.60)$$

Both the relations (5.2.59) and (5.2.60) can be interpreted as canonical commutation relations of the type

$$[q(X), p(X')] = i \tilde{\delta}(\theta - \theta'), \quad (5.2.61)$$

once the following identifications are done :

$$q_1 \equiv \varphi \quad ; \quad p_1 \equiv \kappa \tilde{\epsilon}^{012} \partial_\theta \psi \quad (5.2.62)$$

$$q_2 \equiv \psi \quad ; \quad p_2 \equiv \kappa \tilde{\epsilon}^{012} \partial_\theta \varphi . \quad (5.2.63)$$

Therefore the DoF of the 2D theory are equivalently described by either of the sets of canonical variables (5.2.62) or (5.2.63).

The 2D action

To find the most general 2D action, we make a derivative expansion in the boundary fields $\varphi(X)$ and $\psi(X)$, compatible with power-counting (we remind that $[\varphi] = [\psi] = 0$)

$$\begin{aligned} S_{2D}[\varphi, \psi] &= \int d^2 X \mathcal{L}_{2D} \quad (5.2.64) \\ &= \int d^2 X \sqrt{-\gamma} (a^{ij} \partial_i \varphi \partial_j \varphi + b^{ij} \partial_i \psi \partial_j \psi + c^{ij} \partial_i \varphi \partial_j \psi + d^i \partial_i \varphi + f^i \partial_i \psi + h) , \end{aligned}$$

where $a^{ij} = a^{ji}$, $b^{ij} = b^{ji}$ and c^{ij} are tensors, d^i and f^i are vectors and h is a scalar, with mass dimensions

$$[a] = [b] = [c] = 0 \quad ; \quad [d] = [f] = 1 \quad ; \quad [h] = 2 . \quad (5.2.65)$$

The coefficients appearing in (5.2.64) may depend on the induced metric $\gamma_{ij}(X)$ and/or on the boundary fields, but not on their derivatives. The definitions of the scalar fields $\varphi(X)$ and $\psi(X)$ (5.2.30) and (5.2.31) are invariant under the shift transformations δ_s, δ'_s defined as

$$\delta_s \varphi = \eta \quad (5.2.66)$$

$$\delta'_s \psi = \eta' . \quad (5.2.67)$$

Consequently, the 2D lagrangian \mathcal{L}_{2D} in (5.2.64) must be shift-invariant as well

$$\delta_s \mathcal{L}_{2D} = \delta'_s \mathcal{L}_{2D} = 0 , \quad (5.2.68)$$

which implies

$$S_{2D}[\varphi, \psi] = \int d^2 X \sqrt{-\gamma} (a^{ij} \partial_i \varphi \partial_j \varphi + b^{ij} \partial_i \psi \partial_j \psi + c^{ij} \partial_i \varphi \partial_j \psi + d^i \partial_i \varphi + f^i \partial_i \psi) , \quad (5.2.69)$$

with

$$\begin{aligned} \frac{\partial a^{ij}}{\partial \varphi} &= \frac{\partial b^{ij}}{\partial \varphi} = \frac{\partial c^{ij}}{\partial \varphi} = 0 \\ \frac{\partial a^{ij}}{\partial \psi} &= \frac{\partial b^{ij}}{\partial \psi} = \frac{\partial c^{ij}}{\partial \psi} = 0 , \end{aligned} \quad (5.2.70)$$

and

$$\begin{aligned} \frac{\partial d^i}{\partial \varphi} \partial_i \varphi + \frac{\partial f^i}{\partial \varphi} \partial_i \psi &= 0 \\ \frac{\partial d^i}{\partial \psi} \partial_i \varphi + \frac{\partial f^i}{\partial \psi} \partial_i \psi &= 0 . \end{aligned} \quad (5.2.71)$$

Hence, the coefficients d^i and f^i may still depend on the boundary fields $\varphi(X)$ and $\psi(X)$, provided that the constraints (5.2.71) hold. In $S_{2D}[\varphi, \psi]$ (5.2.69) we omitted the scalar term h because, being metric-dependent only, it does not contribute to the EoM of the boundary theory. As we did for S_{bd} (5.2.5), we parametrize the metric dependence of the rank-2 tensors as follows :

$$a^{ij} = \hat{a}^{ij} + \hat{a} \gamma^{ij} \quad (5.2.72)$$

$$b^{ij} = \hat{b}^{ij} + \hat{b} \gamma^{ij} \quad (5.2.73)$$

$$c^{ij} = \hat{c}^{ij} + \hat{c} \gamma^{ij} , \quad (5.2.74)$$

where the hat means dependence on the metric determinant at most. As observed for the coefficients of the boundary action S_{bd} (5.2.5), in the flat limit the tensors appearing in the action (5.2.69) must reduce to constant matrices, and, in particular, c^{ij} to a constant symmetric matrix. The compatibility of the 2D Lagrangian in (5.2.69) with the canonical boundary structure is ensured if the relation

$$\frac{\partial \mathcal{L}_{2D}}{\partial \dot{q}} = p , \quad (5.2.75)$$

holds for both q_1, p_1 in (5.2.62) and q_2, p_2 in (5.2.63), being equivalent descriptions of the boundary DoF. From (5.2.69) we have

$$\frac{\partial \mathcal{L}_{2D}}{\partial \dot{\varphi}} = \sqrt{-\gamma} (2a^{0i} \partial_i \varphi + c^{0i} \partial_i \psi + d^0) \quad (5.2.76)$$

$$\frac{\partial \mathcal{L}_{2D}}{\partial \dot{\psi}} = \sqrt{-\gamma} (2b^{0i} \partial_i \psi + c^{i0} \partial_i \varphi + f^0) , \quad (5.2.77)$$

and the request (5.2.75) is fulfilled if

$$a^{0i} = b^{0i} = c^{00} = d^0 = f^0 = 0 \quad ; \quad c^{02} = c^{20} = \hat{\kappa} \quad ; \quad (a^{22}, b^{22}, c^{22} \text{ free}) , \quad (5.2.78)$$

where $\hat{\kappa}(\gamma)$ has been defined in (5.2.17) and d^2, f^2 are constrained by (5.2.71). Eq.(5.2.78) represents a constraint on the metric dependence of the coefficients (5.2.72)-(5.2.74), for which we have

$$\begin{aligned} a^{0i} = \hat{a}^{0i} + \hat{a} \gamma^{0i} = 0 \quad ; \quad c^{00} = \hat{c}^{00} + \hat{c} \gamma^{00} = 0 \\ b^{0i} = \hat{b}^{0i} + \hat{b} \gamma^{0i} = 0 \quad ; \quad c^{02} = \hat{c}^{02} + \hat{c} \gamma^{02} = \hat{\kappa} . \end{aligned} \quad (5.2.79)$$

Since we do not want to impose unnecessary conditions on the induced metric γ^{ij} (we are interested in determining if and how the 2D theory keeps memory of the bulk through the induced metric on the boundary), we must ask

$$\hat{a} = \hat{b} = \hat{c} = 0 \quad \Rightarrow \quad a^{ij} = \hat{a}^{ij}(\gamma) \quad ; \quad b^{ij} = \hat{b}^{ij}(\gamma) \quad ; \quad c^{ij} = \hat{c}^{ij}(\gamma) . \quad (5.2.80)$$

From (5.2.78) and (5.2.80), we also get

$$\hat{a}^{0i} = \hat{b}^{0i} = \hat{c}^{00} = 0 \quad ; \quad \hat{c}^{02} = \hat{c}^{20} = \hat{\kappa} . \quad (5.2.81)$$

Applying (5.2.78) and (5.2.80) to the action S_{2D} (5.2.69), we obtain

$$S_{2D}[\varphi, \psi] = \int d^2 X \sqrt{-\gamma} \left[\hat{a}^{22} (\partial_\theta \varphi)^2 + \hat{b}^{22} (\partial_\theta \psi)^2 + 2\hat{\kappa} \partial_t \varphi \partial_\theta \psi + \hat{c}^{22} \partial_\theta \varphi \partial_\theta \psi + d^2 \partial_\theta \varphi + f^2 \partial_\theta \psi \right] , \quad (5.2.82)$$

where all the coefficients (but d^2 and f^2) may depend on the determinant $\gamma(X)$ of the induced metric $\gamma_{ij}(X)$, but not on its components. The EoM of the action $S_{2D}[\varphi, \psi]$ are

$$\frac{\delta S_{2D}[\varphi, \psi]}{\delta \varphi} = -\frac{1}{\sqrt{-\gamma}} \partial_\theta \left[\sqrt{-\gamma} (2\hat{a}^{22} \partial_\theta \varphi + 2\hat{\kappa} \partial_t \psi + \hat{c}^{22} \partial_\theta \psi + d^2) \right] = 0 \quad (5.2.83)$$

$$\frac{\delta S_{2D}[\varphi, \psi]}{\delta \psi} = -\frac{1}{\sqrt{-\gamma}} \partial_\theta \left[\sqrt{-\gamma} (2\hat{b}^{22} \partial_\theta \psi + 2\hat{\kappa} \partial_t \varphi + \hat{c}^{22} \partial_\theta \varphi + f^2) \right] = 0, \quad (5.2.84)$$

where we used (5.2.71) and the fact that $\sqrt{-\gamma} \hat{\kappa}$ is constant, in order to write these equations as $\frac{1}{\sqrt{-\gamma}} \partial_\theta [\dots]$, *i.e.* as a θ -derivative.

5.2.3 Holographic contact

We now consider the generic solutions of the BC (5.2.18)-(5.2.20), where the bulk gauge fields $A_i(X)$ and $B_i(X)$ are now replaced by their boundary values $\partial_i \varphi(X)$ and $\partial_i \psi(X)$, defined in (5.2.30)-(5.2.31) and with $C = \hat{C} = 0$ (5.2.35)-(5.2.36) :

$$\partial_\theta \psi = -l_1 \partial_t \psi \quad (5.2.85)$$

$$\partial_\theta \varphi = -l_2 \partial_t \psi \quad (5.2.86)$$

$$\partial_t \varphi = -l_3 \partial_t \psi. \quad (5.2.87)$$

Eq.(5.2.85) describes a chiral boson $\psi(X)$ moving at the 2D edge of the bulk with velocity $v_\psi = \frac{1}{l_1}$. In the same way, by using (5.2.86) in (5.2.87), we find that $\varphi(X)$ is a chiral boson as well, satisfying

$$\partial_t \varphi - \frac{l_3}{l_2} \partial_\theta \varphi = 0, \quad (5.2.88)$$

moving with velocity $v_\varphi = -\frac{l_3}{l_2}$. What is important to remark now is that, differently from what happens in flat spacetime (Chapter 4 and [11, 87, 88]) and in analogy to the case of Chern-Simons theory in curved spacetime with radial boundary (Section 5.1 and [2]), on the edge of a generic bulk manifold we find two chiral bosons moving with *local*, rather than constant, velocities. In fact, both velocities, at this stage, can depend on the determinant and/or on the components of γ_{ij} which, in general, are local quantities. To establish the holographic contact, we consider the EoM (5.2.83), (5.2.84) with $d^2 = f^2 = 0$ (since the BC are homogeneously linear in the derivatives)

$$\partial_\theta \left(2 \frac{\hat{a}^{22}}{\hat{\kappa}} \partial_\theta \varphi + 2 \partial_t \psi + \frac{\hat{c}^{22}}{\hat{\kappa}} \partial_\theta \psi \right) = 0 \quad (5.2.89)$$

$$\partial_\theta \left(2 \frac{\hat{b}^{22}}{\hat{\kappa}} \partial_\theta \psi + 2 \partial_t \varphi + \frac{\hat{c}^{22}}{\hat{\kappa}} \partial_\theta \varphi \right) = 0. \quad (5.2.90)$$

The holographic contact is realized by inserting into the EoM (5.2.89), (5.2.90) the BC solutions (5.2.85)-(5.2.87), which gives

$$-2 \frac{l_2}{\hat{\kappa}} \hat{a}^{22} + 2 - \frac{l_1}{\hat{\kappa}} \hat{c}^{22} = 0 \quad (5.2.91)$$

$$-2 \frac{l_1}{\hat{\kappa}} \hat{b}^{22} - 2l_3 - \frac{l_2}{\hat{\kappa}} \hat{c}^{22} = 0. \quad (5.2.92)$$

We can write two of the three boundary parameters (e.g. $\hat{a}^{22}, \hat{b}^{22}$) in terms of the remaining one (\hat{c}^{22}) and of the bulk coefficients ($\hat{\kappa}, l_i$):

$$\hat{a}^{22} = +\hat{\kappa} \frac{1}{l_2} \left(1 - \frac{l_1}{2\hat{\kappa}} \hat{c}^{22} \right) \quad (5.2.93)$$

$$\hat{b}^{22} = -\hat{\kappa} \frac{l_3}{l_1} \left(1 + \frac{l_2}{2\hat{\kappa}l_3} \hat{c}^{22} \right). \quad (5.2.94)$$

Remember that \hat{a}^{22} and \hat{b}^{22} , defined in (5.2.72) and (5.2.73), must depend on the determinant of the induced metric only, and not on its components. On the other hand, the coefficients l_i (B.1.8)-(B.1.10) may depend on both the determinant and the components of γ_{ij} . Therefore, we have to tune the parameters of the boundary action (5.2.5) in order that \hat{a}^{22} and \hat{b}^{22} have the right dependence. For instance, one easy way to achieve this is to set $\hat{\alpha} = \hat{\beta} = \hat{\zeta} = 0$ in (5.2.6)-(5.2.8). These two equations are consequence of the bulk (BC)-boundary (EoM) correspondence, from which we can find out the physics of the 2D induced theory. We do this by inserting them back into the 2D action (5.2.82) (with $d^2 = f^2 = 0$):

$$S_{2D}[\varphi, \psi] = \kappa \int d^2 X \tilde{\epsilon}^{012} \left\{ \left[\frac{1}{l_2} (\partial_\theta \varphi)^2 - \frac{l_3}{l_1} (\partial_\theta \psi)^2 \right] - \frac{\hat{c}^{22}}{2\hat{\kappa}} \frac{l_1}{l_2} \left(\partial_\theta \varphi - \frac{l_2}{l_1} \partial_\theta \psi \right)^2 + 2\partial_t \varphi \partial_\theta \psi \right\}. \quad (5.2.95)$$

Physical interpretation

The holographic contact has been imposed by inserting into the EoM the BC which represent two chiral bosons. Therefore we expect that the 2D theory should describe two chiral bosons as well. This fact appears evident by considering the following linear combination

$$\Phi^\pm \equiv \varphi \pm \psi. \quad (5.2.96)$$

In terms of these new fields the action $S_{2D}[\varphi, \psi]$ (5.2.95) writes

$$S_{2D}[\Phi^+, \Phi^-] = S_{2D}[\Phi^+] + S_{2D}[\Phi^-] + \kappa \int d^2 X \tilde{\epsilon}^{012} \partial_\theta \Phi^+ \partial_\theta \Phi^- \left(\frac{2\hat{\kappa}(l_1 + l_2 l_3) - \hat{c}^{22}(l_1^2 - l_2^2)}{4\hat{\kappa}l_1 l_2} \right), \quad (5.2.97)$$

where

$$S_{2D}[\Phi^\pm] = \frac{\kappa}{2} \int d^2 X \tilde{\epsilon}^{012} \partial_\theta \Phi^\pm (\pm \partial_t \Phi^\pm + v_\pm \partial_\theta \Phi^\pm), \quad (5.2.98)$$

and

$$v_\pm = \frac{2\hat{\kappa}(l_1 - l_2 l_3) - \hat{c}^{22}(l_1 \mp l_2)^2}{4\hat{\kappa}l_1 l_2}. \quad (5.2.99)$$

The action $S_{2D}[\Phi^+, \Phi^-]$ (5.2.97) decouples into the sum of the Luttinger actions (5.2.98)

$$S_{2D}[\Phi^+, \Phi^-] = S_{2D}[\Phi^+] + S_{2D}[\Phi^-], \quad (5.2.100)$$

provided that the following condition on the parameters of the theory holds

$$2\hat{\kappa}(l_1 + l_2 l_3) - \hat{c}^{22}(l_1^2 - l_2^2) = 0. \quad (5.2.101)$$

Once decoupled, we may identify the fields $\Phi^+(X)$ and $\Phi^-(X)$ as Right (R) and Left (L) modes moving at the radial edge of the 3D bulk theory with velocities $\pm v_\pm$ respectively [94], where

$$v_\pm = \frac{1 \mp l_3}{l_2 \pm l_1} \quad \text{if } l_1^2 - l_2^2 \neq 0, \quad (5.2.102)$$

and

$$\begin{aligned} v_+ &= \frac{1}{l_1} & ; & \quad v_- = \frac{1}{l_1} - \frac{\hat{c}^{22}}{\hat{\kappa}} & \quad \text{if } l_1 = l_2 \quad ; \quad l_3 = -1 \\ v_+ &= -\frac{1}{l_1} + \frac{\hat{c}^{22}}{\hat{\kappa}} & ; & \quad v_- = -\frac{1}{l_1} & \quad \text{if } l_1 = -l_2 \quad ; \quad l_3 = 1 . \end{aligned} \quad (5.2.103)$$

As a consequence of the holographic contact, the metric dependence of the boundary parameters (5.2.80) is transferred to the l_i coefficients through (5.2.93) and (5.2.94). This makes v_{\pm} depend on the determinant of the induced metric $v_{\pm} = v_{\pm}(\gamma)$. We therefore remark the crucial point that the fact of dealing with a curved bulk spacetime has the primary consequence that the velocities of the edge modes depend on both time and space $v_{\pm} = v_{\pm}(t, \theta)$, differently to what happens for flat backgrounds. Hence, from v_{\pm} we see that the edge action S_{2D} (5.2.100) may describe three classes of physical situations, tuned by the *local* bulk parameters l_i and by \hat{c}^{22} :

1. $\mathbf{v_+v_- > 0}$: LR movers with opposite velocities.

It is realized if

$$\frac{1 - l_3^2}{l_2^2 - l_1^2} > 0 \quad \text{if } l_2^2 - l_1^2 \neq 0 , \quad (5.2.104)$$

or

$$\hat{\kappa} - \hat{c}^{22}l_1 > 0 \quad \text{if } l_1 = \pm l_2 , \quad l_3 = \mp 1 . \quad (5.2.105)$$

This situation describes generic chiral Luttinger liquids [128], but also helical ones [125]. In fact ordinary TI [21–24, 34], characterized by edge modes moving in opposite directions with equal velocities

$$v_+ = v_- \quad (\text{Topological Insulators}) , \quad (5.2.106)$$

fall into this category. It is easy to see that the condition (5.2.106) is satisfied provided that

$$l_1 + l_2l_3 = 0 \quad (5.2.107)$$

$$\hat{c}^{22} = 0 . \quad (5.2.108)$$

The equal and opposite edge velocities therefore are

$$v_+ = v_- = \frac{1}{l_2} , \quad (5.2.109)$$

which still, for a generic bulk manifold, may have a spacetime dependence.

2. $\mathbf{v_+v_- < 0}$: LR movers in the same direction.

It is realized if

$$\frac{1 - l_3^2}{l_2^2 - l_1^2} < 0 \quad \text{if } l_2^2 - l_1^2 \neq 0 \quad (5.2.110)$$

or

$$\hat{\kappa} - \hat{c}^{22}l_1 < 0 \quad \text{if } l_1 = \pm l_2 , \quad l_3 = \mp 1 . \quad (5.2.111)$$

Also in this case we can recover the particular case of a pair of Hall systems [124], with edge excitations moving in the same direction with the same velocity

$$v_+ = -v_- \quad (\text{pair of Hall systems}) , \quad (5.2.112)$$

realized if

$$l_2 + l_1 l_3 = 0 \quad (5.2.113)$$

$$\hat{c}^{22} = \frac{2\hat{\kappa}}{l_1}. \quad (5.2.114)$$

The velocities of the edge modes in this case are

$$v_+ = -v_- = \frac{1}{l_1}. \quad (5.2.115)$$

3. $v_+ v_- = 0$: L or R mover not moving, which characterizes the Quantum Anomalous Hall (QAH) Insulators [22]. This happens when

$$l_3 = \pm 1 \quad \text{if } l_1 \mp l_2 \neq 0, \quad (5.2.116)$$

which means

$$v_{\pm} = 0 \quad ; \quad v_{\mp} = \frac{2}{l_2 \mp l_1} \quad ; \quad \hat{c}^{22} = \frac{2\hat{\kappa}}{l_1 \mp l_2}. \quad (5.2.117)$$

Some comments are in order. First of all we notice that, since the BF coupling constant κ must be positive, $\hat{c}^{22} = 0$ uniquely identifies L and R modes moving on the edge of the 3D bulk with opposite velocities ($v_+ = -v_-$). Hence, as for chiral velocities, \hat{c}^{22} should be determined either by a phenomenological input or by a symmetry principle. Now, TI belong to this class of edge excitations, and are topological phases of electrons which respect Time Reversal (\mathcal{T}) symmetry [22, 34]. This suggests that $\hat{c}^{22} = 0$ might be related to the conservation of \mathcal{T} -symmetry and, conversely, $\hat{c}^{22} \neq 0$ to its violation, like it happens, for instance, in the case of the QAH Insulators [22], described by case 3. We shall come back to this point in the next Subsection. A second comment comes from the fact that it is necessary that the Hamiltonian corresponding to the action (5.2.100) is positive definite. This request yields constraints on the bulk parameters of the model, *i.e.* the ‘‘coupling’’ constant $\hat{\kappa}$ (5.2.17), the parameters α^{ij} (5.2.6), β^{ij} (5.2.7) and ζ^{ij} (5.2.8) appearing in S_{bd} (5.2.5), together with the parameter \hat{c}^{22} . We recall that the parameters l_i depend on the bulk through (B.1.8), (B.1.9) and (B.1.10) and appear in the action $S_{2D}[\Phi^+, \Phi^-]$ through v_{\pm} . The canonical variables defined in (5.2.62) and (5.2.63) in terms of Φ^{\pm} (5.2.96) write

$$\begin{aligned} q_1 &= \frac{\Phi^+ + \Phi^-}{2} \quad ; \quad p_1 = \frac{\kappa \hat{c}^{012}}{2} \partial_{\theta} (\Phi^+ - \Phi^-) \\ q_2 &= \frac{\Phi^+ - \Phi^-}{2} \quad ; \quad p_2 = \frac{\kappa \hat{c}^{012}}{2} \partial_{\theta} (\Phi^+ + \Phi^-). \end{aligned} \quad (5.2.118)$$

The Hamiltonian density of the model therefore is

$$\begin{aligned} \mathcal{H}_{2D} &= p_1 \dot{q}_1 + p_2 \dot{q}_2 - \mathcal{L}_{2D} \\ &= -\frac{1}{2\kappa \hat{c}^{012}} [v_+(p_1 + p_2)^2 + v_-(p_1 - p_2)^2] \\ &= -\frac{\kappa \hat{c}^{012}}{2} [v_+(\partial_{\theta} \Phi^+)^2 + v_-(\partial_{\theta} \Phi^-)^2]. \end{aligned} \quad (5.2.119)$$

Positive energy density means $\mathcal{H}_{2D} > 0$, and since $\kappa > 0$ (5.2.58), requiring the coefficients of the squared terms to be positive gives the following constraint

$$\mathcal{H}_{2D} > 0 \quad \Leftrightarrow \quad v_+ \leq 0, \quad v_- \leq 0. \quad (5.2.120)$$

Therefore, we observe that the physical situation of edge modes moving in the same direction ($\mathbf{v}_+ \mathbf{v}_- < \mathbf{0}$) is not compatible with the positivity conditions (5.2.120). The fact that the Hamiltonian is not bounded by below would lead us to discard this case, leaving us only with the cases 1 and 3.

5.2.4 *The role of Time-Reversal symmetry*

In this Section we present two alternative ways, a) and b), of introducing the Time Reversal \mathcal{T} -symmetry in the theory with boundary. It will turn out that these two approaches, although seemingly quite different, are indeed equivalent. \mathcal{T} -transformation is defined in the usual way as $\mathcal{T}x^0 = -x^0$. Due to the invariance of the line element ds^2 , we have

$$\mathcal{T}\gamma_{t\theta} = -\gamma_{t\theta}^6. \quad (5.2.121)$$

The only components of the gauge fields which change sign under \mathcal{T} are:

$$\begin{aligned} \mathcal{T}A_r(t, r, \theta) &= -A_r(-t, r, \theta) \\ \mathcal{T}A_\theta(t, r, \theta) &= -A_\theta(-t, r, \theta) \\ \mathcal{T}B_t(t, r, \theta) &= -B_t(-t, r, \theta). \end{aligned} \quad (5.2.122)$$

According to this definition $A_\mu(x)$ may be associated to an electric potential and $B_\mu(x)$ to a spin current, like in the flat case [88].

a) *When $\mathcal{T}S_{bd} = S_{bd}$*

It is immediate to see that the bulk action S_{BF} (5.2.1) is \mathcal{T} -invariant⁷, and it is interesting to study which are the consequences of imposing Time-Reversal also on the boundary term S_{bd} (5.2.5), *i.e.*

$$\mathcal{T}S_{bd} = S_{bd}. \quad (5.2.123)$$

Due to (5.2.121), to the fact that $\mathcal{T}\gamma = \gamma$ and to the form of the coefficients (5.2.6), (5.2.7) and (5.2.8), the parameters appearing in S_{bd} (5.2.5) which transform non-trivially under \mathcal{T} are

$$\mathcal{T}\alpha^{02} = \hat{\alpha}^{02} - \hat{\alpha}\gamma^{02} \quad (5.2.124)$$

$$\mathcal{T}\beta^{02} = \hat{\beta}^{02} - \hat{\beta}\gamma^{02} \quad (5.2.125)$$

$$\mathcal{T}\zeta^{02} = \hat{\zeta}^{02} - \hat{\zeta}\gamma^{02} \quad (5.2.126)$$

$$\mathcal{T}\zeta^{20} = \hat{\zeta}^{20} - \hat{\zeta}\gamma^{20}. \quad (5.2.127)$$

Requiring the invariance (5.2.123), from (5.2.122) and (5.2.124)-(5.2.127), we get the following constraints

$$\hat{\alpha}^{02} = \hat{\alpha}^{20} = 0 \quad ; \quad \hat{\beta}^{02} = \hat{\beta}^{20} = 0 \quad ; \quad \hat{\zeta}^{00} = 0 \quad ; \quad \hat{\zeta}^{22} = 0 \quad ; \quad \hat{\zeta} = 0. \quad (5.2.128)$$

⁶ Notice that if the metric is stationary, *i.e.* $\partial_t \gamma_{ij} = 0$, then the \mathcal{T} -invariance (5.2.121) requires that $\gamma_{t\theta} = 0$.

⁷ Other choices of \mathcal{T} are possible which leave S_{BF} invariant, like for instance $\mathcal{T}A_t = -A_t$, $\mathcal{T}B_r = -B_r$, $\mathcal{T}B_\theta = -B_\theta$, which correspond to $A_\mu \leftrightarrow B_\mu$. One can show that these choices are equivalent [88].

The resulting \mathcal{T} -invariant boundary term S_{bd} (5.2.5) is

$$S_{bd} = \int d^3x \sqrt{-\gamma} \delta(r-R) \left[\frac{\gamma^{ij}}{2} (\hat{\alpha} A_i A_j + \hat{\beta} B_i B_j) + \frac{\hat{\alpha}^{00}}{2} A_t A_t + \frac{\hat{\alpha}^{22}}{2} A_\theta A_\theta + \frac{\hat{\beta}^{00}}{2} B_t B_t + \frac{\hat{\beta}^{22}}{2} B_\theta B_\theta + \hat{\zeta}^{02} A_t B_\theta + \hat{\zeta}^{20} A_\theta B_t \right], \quad (5.2.129)$$

and we recall that all the coefficients appearing in (5.2.129) might still depend on the determinant γ of the induced metric γ_{ij} , being therefore local quantities and not simply constants. It will be interesting to investigate which are the consequences, if any, of imposing \mathcal{T} on S_{bd} on the holographically induced 2D theory.

Generic non-diagonal metric $\gamma_{t\theta} \neq 0$

As a consequence of (5.2.128), in the hypothesis of $\gamma_{t\theta} \neq 0$ and if $\hat{\alpha}$ and/or $\hat{\beta}$ are/is non-vanishing, the BC (5.2.12) and (5.2.13) become

$$\alpha^{00} A_t + \hat{\alpha} \gamma^{t\theta} A_\theta + (\hat{\zeta}^{02} - \hat{\kappa}) B_\theta \Big|_{r=R} = 0 \quad (5.2.130)$$

$$\hat{\alpha} \gamma^{t\theta} A_t + \alpha^{22} A_\theta + (\hat{\zeta}^{20} + \hat{\kappa}) B_t \Big|_{r=R} = 0 \quad (5.2.131)$$

$$\hat{\zeta}^{20} A_\theta + \beta^{00} B_t + \hat{\beta} \gamma^{t\theta} B_\theta \Big|_{r=R} = 0 \quad (5.2.132)$$

$$\hat{\zeta}^{02} A_t + \hat{\beta} \gamma^{t\theta} B_t + \beta^{22} B_\theta \Big|_{r=R} = 0, \quad (5.2.133)$$

or, using the notation already adopted in (5.2.14),

$$\Lambda_{\mathcal{T}}^{IJ} X_J \Big|_{r=R} = 0, \quad (5.2.134)$$

where $\Lambda_{\mathcal{T}}$ is the matrix (5.2.15) evaluated at (5.2.128). Notice the explicit dependence on the off-diagonal component $\gamma_{t\theta}$ of the induced metric and on $\hat{\alpha}$ or $\hat{\beta}$. The linear system (5.2.134) has nontrivial solutions if $\det \Lambda_{\mathcal{T}} = 0$, *i.e.*

$$\begin{aligned} 0 &= \det \Lambda_{\mathcal{T}} \\ &= \hat{\alpha} \left(\gamma^{t\theta} \right)^2 \left\{ -\hat{\alpha} \det \beta + \hat{\beta} \left[\hat{\zeta}^{20} \hat{\kappa} - \hat{\zeta}^{02} \left(2\hat{\zeta}^{20} + \hat{\kappa} \right) \right] \right\} + \\ &+ \alpha^{00} \left[\alpha^{22} \det \beta - \beta^{22} \hat{\zeta}^{20} \left(\hat{\zeta}^{20} + \hat{\kappa} \right) \right] + \hat{\zeta}^{02} \left(\hat{\zeta}^{02} - \hat{\kappa} \right) \left[\hat{\zeta}^{20} \left(\hat{\zeta}^{20} + \hat{\kappa} \right) - \alpha^{22} \beta^{00} \right]. \end{aligned} \quad (5.2.135)$$

It is easily seen that the solutions (5.2.18)-(5.2.20) are recovered, with

$$l_1 \rightarrow l_1|_{(5.2.128)} \quad ; \quad l_2 \rightarrow l_2|_{(5.2.128)} \quad ; \quad l_3 \rightarrow l_3|_{(5.2.128)}, \quad (5.2.136)$$

and $l_{1,3}|_{(5.2.128)} \propto \gamma^{t\theta}$, as shown in Appendix B.1. Notice that for a diagonal (but not necessarily static) metric ($\gamma_{t\theta} = 0$) or when $\hat{\alpha} = \hat{\beta} = 0$, these solutions must be discarded, since they imply Dirichlet BC on both fields ($l_{1,3}|_{(5.2.128)} = 0 \Rightarrow B_\theta(X) = A_t(X) = 0$), which, as we already remarked, would trivialize the 2D physics. The case of diagonal metric or $\hat{\alpha} = \hat{\beta} = 0$ will be analyzed in the next Subsection. The procedure we followed to recover the 2D theory does not change: the bulk-boundary correspondences (5.2.93) and (5.2.94) still hold, with the replacements (5.2.136). From (B.1.11) and (B.1.13) we see that, due to the \mathcal{T} -invariance request (5.2.128), the coefficients l_1 and l_3

appearing in the BC (5.2.18) and (5.2.20) explicitly depend on $\gamma^{t\theta}$. As a consequence, \hat{a}^{22} (5.2.93) and \hat{b}^{22} (5.2.94) would depend on $\gamma^{t\theta}$ as well, but we know that, due to the holographic contact, the coefficients appearing in the action S_{2D} (5.2.82) should depend on the determinant of the induced metric only, and not on its components. The only way to realize this is to set $\hat{c}^{22} = 0$. In fact, in this case we have

$$\hat{c}^{22} = 0 \quad \Rightarrow \quad \frac{l_1|_{(5.2.128)}}{l_3|_{(5.2.128)}} = -l_2|_{(5.2.128)}, \quad (5.2.137)$$

which does not depend on $\gamma^{t\theta}$, and, from (5.2.93) and (5.2.94),

$$\hat{a}^{22} = \frac{1}{l_2|_{(5.2.128)}} \hat{\kappa} \quad ; \quad \hat{b}^{22} = -\frac{l_3|_{(5.2.128)}}{l_1|_{(5.2.128)}} \hat{\kappa}. \quad (5.2.138)$$

Eq. (5.2.137) coincides with Eqs.(5.2.108) and (5.2.107), which belong to the case 1 considered in the previous Section, where we have seen that the physical situation described by the decoupled action $S_{2D}[\Phi^+, \Phi^-]$ (5.2.100) together with the conditions (5.2.108) and (5.2.107), is that of a Luttinger model for two chiral currents with non-constant and opposite velocities. We thus established a link between the parameter \hat{c}^{22} and \mathcal{T} -invariance on the boundary, which enforces the physical interpretation as edge states of TI, as anticipated in the previous Section.

Diagonal metric $\gamma_{t\theta} = 0$

When considering a diagonal metric or boundary coefficients (5.2.6)-(5.2.8) which depend at most on the determinant of the metric (*i.e.* when $\gamma_{t\theta} = 0$ or $\hat{\alpha} = \hat{\beta} = 0$), the BC (5.2.130)-(5.2.133) become

$$\alpha^{00} A_t + (\hat{\zeta}^{02} - \hat{\kappa}) B_\theta \Big|_{r=R} = 0 \quad (5.2.139)$$

$$\alpha^{22} A_\theta + (\hat{\zeta}^{20} + \hat{\kappa}) B_t \Big|_{r=R} = 0 \quad (5.2.140)$$

$$\hat{\zeta}^{20} A_\theta + \beta^{00} B_t \Big|_{r=R} = 0 \quad (5.2.141)$$

$$\hat{\zeta}^{02} A_t + \beta^{22} B_\theta \Big|_{r=R} = 0, \quad (5.2.142)$$

which represent two systems of homogeneous linear equations. Non-Dirichlet solutions are

$$A_t(X) = -l_a B_\theta(X) \quad (5.2.143)$$

$$A_\theta(X) = -l_b B_t(X), \quad (5.2.144)$$

where

$$l_a = \frac{\beta^{22}}{\hat{\zeta}^{02}} \quad \text{and} \quad l_b = \frac{\beta^{00}}{\hat{\zeta}^{20}}, \quad (5.2.145)$$

provided that

$$(\hat{\zeta}^{02} - \hat{\kappa}) \hat{\zeta}^{02} - \alpha^{00} \beta^{22} = 0 \quad (5.2.146)$$

$$(\hat{\kappa} + \hat{\zeta}^{20}) \hat{\zeta}^{20} - \alpha^{22} \beta^{00} = 0. \quad (5.2.147)$$

Following the same steps described in Section 5.2.3, we still land on the action S_{2D} (5.2.82). The holographic contact is realized crossing the EoM (5.2.83) and (5.2.84)

with the BC (5.2.143) and (5.2.144), which in terms of the boundary fields $\varphi(X)$, $\psi(X)$ read :

$$\partial_t \varphi = -l_a \partial_\theta \psi \quad (5.2.148)$$

$$\partial_\theta \varphi = -l_b \partial_t \psi . \quad (5.2.149)$$

The correspondence is achieved if

$$\hat{a}^{22} = \frac{1}{l_b} \hat{\kappa} \quad ; \quad \hat{b}^{22} = l_a \hat{\kappa} \quad ; \quad \hat{c}^{22} = 0 . \quad (5.2.150)$$

From the properties of the tensors \hat{a}^{ij} , \hat{b}^{ij} , \hat{c}^{ij} in (5.2.80), it is immediate to check that $l_{a,b}$ depend only on the determinant of the induced metric γ , *i.e.* $l_{a,b} = l_{a,b}(\gamma)$. The 2D action (5.2.82) decouples into a pair of Luttinger models, provided that the following condition holds

$$l_a = \frac{1}{l_b} , \quad (5.2.151)$$

and describes two chiral modes travelling on the edge of the 3D bulk with equal and opposite local velocities $v_\pm = l_a(\gamma)$

$$\partial_t \Phi^+ + l_a \partial_\theta \Phi^+ = 0 \quad (5.2.152)$$

$$\partial_t \Phi^- - l_a \partial_\theta \Phi^- = 0 , \quad (5.2.153)$$

characterizing Π [21, 22, 34].

b) *Inherited \mathcal{T} -transformation*

It is possible to impose the \mathcal{T} -symmetry on the theory with boundary in an alternative way with respect to what we did in the previous Subsection, reaching the same physical conclusions (*i.e.* chiral edge modes moving with opposite velocities as the unique physical outcome of putting a boundary onto 3D BF theory). The bulk gauge fields $A_\mu(x)$ and $B_\mu(x)$ transform under \mathcal{T} according to (5.2.122). Consequently, the boundary scalar fields $\psi(X)$ and $\varphi(X)$, being defined by (5.2.30) and (5.2.31), should inherit the following \mathcal{T} -transformations

$$\mathcal{T}\varphi(t, \theta) = -\varphi(-t, \theta) \quad ; \quad \mathcal{T}\psi(t, \theta) = \psi(-t, \theta) , \quad (5.2.154)$$

hence, due to (5.2.96), we have

$$\mathcal{T}\Phi^+(t, \theta) = -\Phi^-(-t, \theta) \quad ; \quad \mathcal{T}\Phi^-(t, \theta) = -\Phi^+(-t, \theta) . \quad (5.2.155)$$

The action $S_{2D}[\Phi^+, \Phi^-]$ (5.2.97) is \mathcal{T} -invariant if

$$v_+ = v_- , \quad (5.2.156)$$

and if the decoupling condition (5.2.101) holds. This is precisely the situation considered in case 1 treated in the physical interpretation of Section 5.2.3, uniquely identified by the vanishing of the parameter \hat{c}^{22} (5.2.108).

We therefore checked our guess on the peculiar role played by the parameter \hat{c}^{22} appearing in the action $S_{2D}[\varphi, \psi]$ (5.2.82), and also that the two alternative ways described in Subection 5.2.4 of imposing the \mathcal{T} -symmetry (on the boundary term S_{bd}

(5.2.5) or directly on $S_{2D}[\varphi, \psi]$ through the defining relations (5.2.30) and (5.2.31)) are indeed physically equivalent. Lastly, we remark that the physical situation represented by the TI, *i.e.* the existence of chiral edge modes moving with equal and opposite velocities, is singled out by imposing \mathcal{T} -symmetry (in either way), while asking that the Hamiltonian \mathcal{H}_{2D} is bounded by below, as discussed in Section 5.2.3, admits a larger class of physical situations, including \mathcal{T} -breaking effects, but only for opposite-moving modes. The possibility of edge modes moving in the same direction is ruled out by both argumentations, and \mathcal{T} -symmetry is more restrictive than the positive energy condition. According to this analysis, case 3 is not compatible with \mathcal{T} -symmetry, however lower-bounded Hamiltonian does admit the possibility of a single-moving edge state. Hence it can be interpreted as a \mathcal{T} -breaking effect associated to QAH insulators [22]. In the same way the situations with $\hat{c}^{22} \neq 0$ belonging to case 1 can be seen as other examples of symmetry breaking effects in Quantum Spin Hall systems [21, 22, 125].

It is known [22, 34] that \mathcal{T} -symmetry is peculiar to the edge states of TI in 2D, which are described by a helical Luttinger model [125]. This is exactly the situation we observed for $\hat{c}^{22} = 0$ (5.2.108). Therefore we may finally claim that BF model with boundary together with \mathcal{T} -invariance is an effective description of the edge states of TI with possibly non-constant chiral velocities. The existence of accelerated edge modes is a direct consequence of the bulk/boundary correspondence in curved spacetime : flat spacetime analysis provides only for constant velocities. The $\hat{c}^{22} \neq 0$ situation now can be related to \mathcal{T} -breaking phenomena in helical Luttinger liquids [125]. For instance (as we already remarked) we can associate the case 3 of Section 5.2.3 to the Quantum Anomalous Hall Effect [22, 126, 127].

5.2.5 *Summary of results and discussion*

In this Section we considered the abelian 3D BF model on a manifold described by a generic metric, with a radial boundary which spoils the topological character of the theory. Following a method introduced by Symanzik, we added to the action a boundary term constrained only by locality and power counting, in order to find out the most general BC on the two gauge fields involved in the theory. The boundary breaks gauge invariance, and this reflects in two broken Ward identities, from which we identified the boundary degrees of freedom, represented by two scalar fields, deriving also a semidirect sum of two Kac-Moody algebras formed by two on-shell conserved currents. As in the Chern-Simons case of Section 5.1, the central charge of the Kac-Moody algebra is proportional to the inverse of the BF coupling, and this constrains the BF coupling to be positive. Once written in terms of the scalar boundary degrees of freedom, the Kac-Moody algebra can be interpreted as the commutation relation of canonical variables, which led us to derive the corresponding 2D action. The bulk/boundary holographic correspondence is achieved by matching the BC on the 3D gauge fields and the equations of motion of the 2D action. The resulting 2D action is a complicated functional of the pair of scalar fields but, by means of a simple linear redefinition, the action can be written in terms of two Luttinger actions for two chiral fields, plus a mixed term, which we force to vanish by imposing a decoupling condition. At this point, the parameters

which survived to the holographic contact and to the decoupling condition allow for interesting physical interpretations of the theory, which may indeed describe :

1. two edge excitations moving in opposite directions. This is realized in Hall systems, like Fractional Quantum Hall Effect with $\nu = 1 - 1/n$ [124], and edge modes of Quantum Spin Hall systems, like Topological Insulators (when $v_+ = v_-$), possibly interacting [129], or nanowires [130] with additional magnetic fields acting on the velocities up to switching one off [131, 132]. In higher dimensions an effect of renormalization of chiral velocities (*i.e.* $v_+v_- > 0$) can be achieved by adding magnetic fields [133, 134], or by structural deformations [135] ;
2. two chiral bosons moving in the same direction. This physically corresponds to Hall systems like, for instance, Quantum Hall with $\nu = 2$, or Fractional Quantum Hall Effect with $\nu = 2/5$ [48, 124, 136] possibly with non-constant interactions or confining potentials [96–100] ;
3. one static and one moving mode. This is the situation of Quantum Anomalous Hall, where magnetic impurities break the \mathcal{T} -symmetry of Topological Insulators [22, 126, 127]. This could also be explained as an extremal effect of a magnetic field acting on the modes of the nanowires mentioned above.

In all cases described above, the chiral velocities depend on the induced metric on the boundary, hence on time and space. Therefore, their local nature is a direct consequence of the non-flatness of the bulk metric, which thus appear as a general property of (abelian) TQFTs in curved spacetimes. Moreover, we calculated the Hamiltonian corresponding to the 2D action and we found that the request of the existence of a lower bound rules out the case 2 : Hall systems with parallel velocities, like Fractional Quantum Hall with $\nu = 2/5$ [124] or Integer Quantum Hall with $\nu = 2$ [136], cannot be described by a BF theory with boundary. We then considered the role of Time Reversal symmetry, which distinguishes the two main Schwarz-type TQFTs : \mathcal{T} -violating Chern-Simons and \mathcal{T} -respecting BF. \mathcal{T} -symmetry can be introduced in two different ways : from the bulk side asking that the “Symanzik’s” boundary term of the 3D action is \mathcal{T} -invariant, which has consequences on the subsequent steps till the holographically induced 2D action. \mathcal{T} -symmetry may be imposed also on the boundary side directly on the 2D action, starting from the definition of the scalar degrees of freedom. These two seemingly inequivalent ways of requiring \mathcal{T} -invariance lead to the same outcome : the only physical case which respects \mathcal{T} -symmetry is a subclass of case 1 above, namely the one involving two edge excitations moving in opposite directions with the same velocity, which, in the general case of non-Minkowskian bulk metric, might be local, *i.e.* time and space dependent. This is the case of Topological Insulators. Hence, imposing \mathcal{T} -symmetry is much more restrictive than asking for a lower bounded Hamiltonian. Topological phases of matter displaying accelerated edge modes have been observed for instance in the Integer Quantum Hall with $\nu = 2$ [48]. But, to our knowledge, generalized Topological Insulators with accelerated chiral edge modes have not been discovered yet. Here we thus predict that they should, and we presented a theoretical framework for their existence, as a direct consequence of a non-Minkowskian bulk background. In our opinion, this represents a cleaner alternative to adding an *ad hoc* local potential to the pair of the decoupled Luttinger actions, which would spoil the whole holographic construction.

Part III

NON-TOPOLOGICAL FIELD THEORIES

In most cases, Quantum Field Theories are considered without boundaries and have been successful in providing descriptions of fundamental interactions, including gravity and cosmology. This is because one is generally interested in bulk effects, where the boundary can be neglected. Nevertheless, boundaries do exist, and in some cases, their effects are self-evident and dominant. As we have seen in the previous Chapters, recently, important phenomena pertaining to condensed matter physics, like the Fractional Quantum Hall Effect and the behaviour of Topological Insulators, have been explained in terms of topological QFTs with boundaries. This is rather counterintuitive : topological QFTs, when considered without boundaries, have a vanishing Hamiltonian and no energy-momentum tensor. They might appear as the least physical theories one can imagine. Despite this, when a boundary is introduced, an extremely rich physics emerges, which can be observed experimentally. This is an example of the power of the boundary, and for a long time it has been believed that such power was peculiar only of these kind of theories. Until now...

AN OVERVIEW FOR THE NON-TOPOLOGICAL CASE

Since the introductory part of Chapter 1, we have encountered many cases of physical results induced by the presence of boundaries. The Casimir effect [8] perhaps is the first highly nontrivial example of boundary effect which has been thoroughly studied in a systematic way. The role of boundaries has been largely discussed in 2D Conformal Field Theory [15, 137]. In particular, in [15] the zoo of Conformal Field Theories has been tamed by means of a boundary put on the 3D topological Chern-Simons theory. In [137], instead, the role of the boundary, and in particular of the BC, has been exploited for the study of the Virasoro algebras and their extensions (Kac-Moody, superconformal, W -algebras). We have seen that TQFTs have vanishing Hamiltonian and energy-momentum tensor, and it is rather surprising that for such non-physical theories it has been possible to establish [29, 76, 138] that on their lower dimensional edge, conserved currents exist, which form KM algebras [26, 27], whose central charge is inversely proportional to the coupling constant of the bulk theory, and directly related to the velocity of the boundary propagating DoF. This property seems to be a common feature of different physical situations, like the 3D FQHE [11, 139] and TI in 3D [34, 77, 140] and 4D [95, 141, 142]. We have also seen that in the Symanzik's approach, the boundary separates two half-spaces : left and right hand side with respect to a plane. Single-sided boundaries can also be considered, as shown throughout this whole work, which correspond to quite different physical situations : in these cases, the Symanzik's separation requirement on propagators is not the most natural one. It is easier, and more intuitive, as we discussed, to implement the confinement of the theory in a half-space by means of a theta Heaviside step function directly introduced in the bulk action. The theories with single-sided boundaries are therefore treated with a different approach, which was explained in Chapter 2, and the physical results on the boundary do not necessarily coincide with those of the separating boundaries. This is the case, for instance, of 3D Maxwell-Chern-Simons theory, where the Maxwell term is completely transparent in case of double-sided boundary [143] and algebraically active in the single-sided case [37, 144]. It is precisely the different role of the non-topological Maxwell term that motivated the study of non-TQFTs, or theories with non-topological terms, defined in half-spaces [145, 146]. In fact, all the results mentioned until now were obtained for boundaries in TQFTs : 3D Chern-Simons and BF theories [14, 31, 32]. What is missing is the study of physical, realistic, hence entirely non-topological, theories, defined on a half-space, and it is intriguing to investigate the role of the boundary in these cases : which are the edge DoF? are there conserved currents, like in the topological cases? do they form an algebra? of which type? is there a lower-dimensional holographic counterpart of the non-topological bulk theory? is this unique? what kind of physics does it describe? The first example which came to mind was of course that of the 4D Maxwell theory of electromagnetism [1], whose nontrivial results, which we will review shortly in this Chapter, motivated the present Part of the Thesis. We will thus dwell into more exotic cases, like fracton theories [54, 55], or models which nowa-

days are of high interest, like Linearized Gravity, with the aim of answering the above questions and possibly acquire some phenomenological insights.

The case of the 4D Maxwell theory with boundary : a new beginning

The Minkowskian 4D Maxwell theory can be confined in the half-spacetime $x^3 \geq 0$ by means of the introduction in the action of the Heaviside step function $\theta(x^3)$

$$S_M = -\frac{\kappa}{4} \int d^4x \theta(x^3) F_{\mu\nu} F^{\mu\nu} , \quad (6.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, and $A_\mu(x)$ is the gauge field, with canonical mass dimension $[A] = 1$. In (6.1) $\kappa > 0$ is a constant which must be positive in order to have a positive-definite energy density. Maxwell theory, being a free field theory, does not display a coupling constant, which can always be reabsorbed by redefining the gauge field $A_\mu(x)$. Nonetheless, here again, we do not normalize κ to one, in order to be able to identify at any time the role played by the bulk action in the physics on the boundary. The gauge fixing term

$$S_{gf} = \int d^4x \theta(x^3) b A_3 \quad (6.2)$$

implements, through the Lagrange multiplier field $b(x)$ [107, 108], the axial gauge condition

$$A_3(x) = 0 . \quad (6.3)$$

On the boundary $x^3 = 0$, the fields and their ∂_3 -derivatives must be treated as independent fields [49, 147]. To highlight this fact, we adopt the following notation :

$$\tilde{A}_a(X) \equiv \partial_3 A_a|_{x^3=0} , \quad (6.4)$$

whose mass dimension is $[\tilde{A}] = 2$. Therefore we must introduce another term in the action, coupling these two independent fields, $A_\mu(x)$ and, on the boundary, $\tilde{A}_a(X)$, to the external sources $J^\mu(x)$ and $\tilde{J}^a(X)$ respectively:

$$S_J = \int d^4x \left[\theta(x^3) J^a A_a + \delta(x^3) \tilde{J}^a \tilde{A}_a \right] . \quad (6.5)$$

The existence of the boundary requires an additional contribution to the action, subject to the general constraints of locality, power counting and 3D covariance. Taking into account the presence of an additional independent gauge field $\tilde{A}_3(X)$ (6.4), the most general boundary term has to be the following

$$S_{bd} = \int d^4x \delta(x^3) \left(a^{ab} A_a A_b + b^{abc} \partial_a A_b A_c + c^{ab} \tilde{A}_a A_b \right) , \quad (6.6)$$

where

$$a^{ab} = a^{ba}, \quad b^{abc} = -b^{acb}, \quad c^{ab} \quad (6.7)$$

are constant matrices, with mass dimensions $[a^{ab}] = 1$, $[b^{ab}] = [c^{ab}] = 0$. Notice that here again we do not require covariance, however, we shall see that in this case the

theory will require it in order to recover an induced physics. The total action, consisting in bulk term, gauge fixing, external sources and boundary contribution, finally is

$$S_{tot} = S_M + S_{gf} + S_J + S_{bd} . \quad (6.8)$$

Broken Ward identities can be recovered from both the gauge field and its ∂_3 -derivative as follows

$$\int_0^{+\infty} dx^3 \partial^a J_a = \kappa \partial^a \tilde{A}_a \Big|_{x^3=0} \quad (6.9)$$

$$\partial^a \tilde{J}_a \Big|_{x^3=0} = -\kappa \partial^a A_a \Big|_{x^3=0} , \quad (6.10)$$

noticing that the Ward identity (6.10), differently from (6.9), is local and not integrated. Both identities are broken, because of the presence of the boundary, by a linear term at their r.h.s., and, at vanishing external sources $\tilde{J} = J = 0$, *i.e.* going on shell, we find

$$\partial^a \tilde{A}_a \Big|_{x^3=0} = 0 \quad (6.11)$$

$$\partial^a A_a \Big|_{x^3=0} = 0 , \quad (6.12)$$

which show the existence of a couple of conserved currents on the 3D edge of 4D Maxwell theory. From the broken Ward identities (6.9) and (6.10), through the procedure of Section 2.2, *i.e.* by taking functional derivatives with respect to the currents $\tilde{J}^a(x)$, $J^a(x)$, we also get the following equal time commutators for the conserved currents $A_a(X)$ and $\tilde{A}_a(X)$, which are of KM type

$$[\tilde{A}_0(X), A_M(X')] = -\frac{i}{\kappa} \partial_M \delta^{(2)}(X - X') \quad (6.13)$$

$$[A_0(X), \tilde{A}_M(X')] = \frac{i}{\kappa} \partial_M \delta^{(2)}(X - X') \quad (6.14)$$

$$[\tilde{A}_0(X), \tilde{A}_a(X')] = [A_0(X), A_a(X')] = [\tilde{A}_0(X), A_0(X')] = 0 . \quad (6.15)$$

In order to identify the correct DoF on the 3D boundary, it is convenient to introduce a field $B_a(X)$ defined by the linear transformations

$$\begin{aligned} B_0 &\equiv \mu A_0 + \nu \tilde{A}_0 \\ B_M &\equiv \rho A_M + \sigma \tilde{A}_M , \end{aligned} \quad (6.16)$$

where μ , ν , ρ and σ are constant parameters, which can be set later at our convenience, with mass dimensions constrained by the request of the dimensional homogeneity of (6.16)

$$[\mu] = [\nu] + 1 , \quad [\rho] = [\sigma] + 1 , \quad [\mu] = [\rho] \quad \text{and} \quad [\nu] = [\sigma] . \quad (6.17)$$

In terms of $B_a(X)$, the algebra (6.13)-(6.15) reduces to the only nonvanishing commutator

$$[B_0(X), B_M(X')] = i \frac{\mu\sigma - \nu\rho}{\kappa} \partial_M \delta^{(2)}(X - X') , \quad (6.18)$$

which describes an abelian KM algebra exactly like (2.2.3) described in Section 2.2, and whose central charge is proportional to the inverse of the Maxwell coupling κ :

$$\frac{1}{\tilde{\kappa}} \equiv \frac{\mu\sigma - \nu\rho}{\kappa} . \quad (6.19)$$

Therefore this shows that also for non-TQFT an algebraic structure of KM type might exist on the boundary. The 3D current conservation relations (6.11) and (6.12) can be solved by

$$\tilde{A}_a(X) = \epsilon_{abc} \partial^b \tilde{\xi}^c(X) \quad (6.20)$$

$$A_a(X) = \epsilon_{abc} \partial^b \xi^c(X) , \quad (6.21)$$

where the fields $\tilde{\xi}_a(X)$ and $\xi_a(X)$ have canonical dimensions

$$[\tilde{\xi}] = 1 \quad \text{and} \quad [\xi] = 0 . \quad (6.22)$$

Consequently, we have

$$B_0 = \epsilon_{0MN} \partial^M (\mu \xi^N + \nu \tilde{\xi}^N) = \epsilon_{0MN} \partial^M \lambda^N \quad (6.23)$$

$$B_M = \epsilon_{Mab} \partial^a (\rho \xi^b + \sigma \tilde{\xi}^b) = \epsilon_{Mab} \partial^a \tilde{\lambda}^b , \quad (6.24)$$

where we defined

$$\lambda_a \equiv \mu \xi_a + \nu \tilde{\xi}_a \quad (6.25)$$

$$\tilde{\lambda}_a \equiv \rho \xi_a + \sigma \tilde{\xi}_a . \quad (6.26)$$

The equations (6.25) and (6.26) define the 3D vector fields $\lambda_a(X)$ and $\tilde{\lambda}_a(X)$ which, as we shall show in what follows, are the dynamical variables in terms of which the 3D theory induced on the boundary of 4D Maxwell theory will be constructed, according to the QFT procedure of Section 2.3. Notice that the defining relations (6.23) and (6.24) are left invariant under the transformations

$$\lambda_a \quad \rightarrow \quad \lambda_a + \partial_a \Lambda \quad (6.27)$$

$$\tilde{\lambda}_b \quad \rightarrow \quad \tilde{\lambda}_b + \partial_b \tilde{\Lambda} , \quad (6.28)$$

where $\Lambda(X)$ and $\tilde{\Lambda}(X)$ are local gauge parameters. We may therefore claim that the 3D theory induced on the boundary of 4D Maxwell theory should be a gauge theory of two, possibly coupled, gauge fields, which must satisfy the three constraints of Section 2.3 :

1. invariance under the gauge transformations

$$\delta_1 \lambda_a(X) = \partial_a \Lambda(X) \quad (6.29)$$

$$\delta_2 \tilde{\lambda}_a(X) = \partial_a \tilde{\Lambda}(X) ; \quad (6.30)$$

2. compatibility with the equal time KM algebra (6.18) ;
3. compatibility with the BC of the bulk .

Following this procedure a candidate shows up to be represented by the following 3D action

$$S_{3D} = \int d^3 X \left(-\frac{\tilde{\kappa}}{2} G_{ab} \tilde{G}^{ab} + \frac{\tilde{\kappa}}{2} \frac{\nu}{2\sigma} \tilde{G}_{ab} \tilde{G}^{ab} + m \epsilon^{abc} \tilde{\lambda}_a \partial_b \tilde{\lambda}_c \right) , \quad (6.31)$$

where

$$G_{ab} \equiv \partial_a \lambda_b - \partial_b \lambda_a \quad ; \quad \tilde{G}_{ab} \equiv \partial_a \tilde{\lambda}_b - \partial_b \tilde{\lambda}_a \quad ; \quad m \propto \frac{a_c}{3\rho} , \quad (6.32)$$

with a^c the trace of a coefficient of the boundary term (6.6), and whose EoM are

$$\partial_b \tilde{G}^{ab} = 0 \quad (6.33)$$

$$\partial_b G^{ab} + \frac{2m}{\tilde{\kappa}} \epsilon^{abc} \partial_b \tilde{\lambda}_c = 0. \quad (6.34)$$

As we anticipated, notice that in order to get a contact it has been necessary to set

$$a^{\alpha\beta} \propto \eta^{\alpha\beta} \quad ; \quad b^{\alpha\beta\gamma} = 0 \quad ; \quad c^{\alpha\beta} \propto \eta^{\alpha\beta}, \quad (6.35)$$

meaning that the boundary action term S_{bd} (6.6) *must* be covariant. That's exactly the opposite to what happens in TQFTs with boundary, where the request of covariance restricts the possible physical results, or even, as for the Chern-Simons case, prevents to get anything at all, and non-covariance is mandatory.

Until now we would have thus said that the most known and also physically relevant role played by boundaries concerned TQFTs, in particular in 3D and 4D. This fact constitutes a kind of interesting paradox : TQFTs, indeed, are characterized by global observables of geometrical type only, vanishing Hamiltonian, no energy-momentum tensor and lack of particle interpretation. The main point which should be stressed is therefore that 4D Maxwell theory shows a non trivial boundary dynamics, which thus is not peculiar to TQFT, contrary to what was usually believed. There are however similarities and differences with respect to TQFT :

- On the boundary of 4D Maxwell theory the broken Ward identities (6.9) and (6.10) are found, which identify two conserved currents (6.11) and (6.12). This reminds the physics of the surface states of Topological Insulators in 3D, which suggests that the 4D Maxwell theory might be seen as an effective bulk theory of the 3D Topological Insulators, alternative to the 4D topological BF models [34].
- A Kac-Moody algebra (6.18) with a central charge proportional to the inverse of the Maxwell coupling exists on the boundary, and the parameters appearing in (6.16) correspond to different central charges, as represented by (6.19), each identifying a different CFT. This is an important difference with respect to TQFTs, which are characterized by a one-to-one correspondence between bulk coupling constants and central charges. The relevant boundary algebra appears to be formed by the subset (6.16) of the total number of components of the bulk fields. An identical mechanism occurs in the topological twist of N=2 Super Yang-Mills Theories [79].
- The 3D action (6.31) induced by the 4D bulk Maxwell theory is unique and corresponds to a new theory, which has not been studied previously.
- such action describes two coupled photon-like vector fields, with a topological Chern-Simons term for one of them. We computed the propagators of the theory [1] which show that, despite the similarity with the 3D Maxwell-Chern-Simons theory, a mechanism of topological mass generation does not take place in this case.
- By computing the energy-momentum tensor of the theory, a non trivial physical content is revealed [1]. In particular, the coefficients appearing in the 3D action can be tuned in order to have a positive definite energy density.

- The holographic dictionary [44] might be improved by an additional entry involving the unitarity of the CFT found on the boundary of 4D Maxwell theory and the positivity of the energy density of its 3D holographic counterpart, represented by the action (6.31). In fact, asking that the 00-component of the energy-momentum tensor is positive, automatically implies that the central charge of the KM algebra (6.18) is positive as well, thus ensuring the unitarity of the corresponding CFT.

Each of these points is a proof of the relevance of studying non-topological QFTs with boundary : we have at least six reasons not to stop here, but to proceed to Chapter 7 and to a new non-topological theory with boundary.

Fracton phases of matter represent a novel paradigm both in condensed matter theory and high energy physics [54, 55]. Although they have originally been discovered in particular kinds of lattice models [50–53], fractons have been unveiled in many different systems and frameworks, ranging from elasticity [148–151], hydrodynamics [152–156] and quantum scars [157–159], to Quantum Field Theories [56, 160–169], curved space [170–173] and holography [174]. These phases are characterized by constrained dynamics, for which quasiparticle excitations are immobile or move in sub-dimensional spaces. This exotic behaviour can be encoded into conservation of multipole moment, the simplest example being the dipole. Indeed, typically fracton models are described in terms of a non-covariant higher-rank tensor theory which shares many similarities with Maxwell theory [160, 161, 175]. They are written in terms of a rank-2 symmetric tensor field $A_{ij}(x)$, whose conjugate momentum is referred to as an “electric-like” tensor field $E^{ij}(x)$, which plays a key role for the immobility constraint of fractons, which is recovered from a generalized Gauss law. These novel gauge theories are intrinsically non-relativistic, and many ingredients have been introduced by hand in order to implement the main characteristics of fractons, *i.e.* their restricted mobility. Examples of these inputs are, for instance, the Maxwell-like Hamiltonian and the Gauss law, introduced as an external constraint, and not derived from an action principle. As a consequence, terms appear with inhomogeneous numbers of derivatives, as remarked in [176], and all these *ad hoc* introductions are justified *a posteriori*, rather than deduced from first principles of QFT. Despite their intrinsic non-covariance, these models share remarkable similarities both with Maxwell theory (Gauss law, Hamiltonian, electric-like field...) [160] and LG (symmetric rank-2 field, gauge symmetry,...) [177], which however are fully covariant theories. Motivated by these similarities, in the following Sections a covariant 4D fracton gauge theory will be built [4, 5, 178], taking as unique ingredients locality, power counting and the covariant fracton symmetry $\delta A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda$, thus keeping strict the main spirit of this Thesis : everything must come from first principles of QFT. This procedure will make immediately apparent the correspondence with LG, and all the analogies with Maxwell theory will naturally come out. Embedding the ordinary non-covariant theory of fractons in a larger covariant theory will indeed lead to recover all the known results concerning fractons [160, 161, 175]. Therefore, the coherent theoretical framework of Section 7.2, and [4], will allow to apply standard QFT techniques to fractons. As a physically relevant case, in the context of this Thesis is the introduction of a flat spatial boundary in this newly built 4D covariant fracton model. We have indeed just seen, in Chapter 6, that on the boundary of Maxwell theory in 4D [1] a KM algebra is observed, and a non-trivial theory is induced on the lower-dimensional space. Fractons seem to share many relations with the electromagnetic theory, as we shall see, but this similarity is not the only motivation for studying fractons with boundary. It is in fact known that the topological θ -term [179], which is a boundary term, generates the well known Witten effect [180], relevant for instance in topological insulators [181]. For what concerns fractons, boundary

contributions have been introduced mainly as non-covariant Chern-Simons-like terms coming from a generalized topological-like θ -term in the bulk [175, 182, 183], inspired by the standard electromagnetic case. For instance in [175], as for the Maxwell case, a Witten-like effect is observed, for which, due to the presence of the fractonic θ -term, the “electric” charge density at the right hand side of the Gauss law acquires an additional “magnetic” contribution. Furthermore on the 3D boundary of [175] fractonic-like excitations seem to appear, in agreement with our results, as we shall see. Moreover, in [55, 184] it has been speculated that certain kinds of higher-order topological phases share some properties with boundary states of fracton quasiparticles. Another interesting example can be found in [185], where a similar non-covariant Chern-Simons-like term, built as a higher-spin generalization of the standard topological one, gives some insights in the context of dipolar behaviours of quantum Hall systems. Therefore, boundary effects might be important also in the framework of fracton theory, and the aim of this Chapter is to study the consequences of the presence of a flat boundary in the covariant theory of fractons, following the QFT approach pioneered by Symanzik in [10].

In order to do so, in this Chapter, after giving a brief overview about fractons and their main properties in Section 7.1, in Section 7.2 we will build the new covariant theory, which will be based on a QFT perspective. The gauge structure of the newly found model will then be studied in Section 7.3, and the propagators will be computed in the vectorial gauge. Finally in Section 7.4, having established that the theory is well defined, we will introduce a boundary and analyze its consequence as prescribed by our QFT approach.

7.1 WHAT ARE FRACTONS?

Fractons are quasiparticles with the defining property of having restricted mobility [50–52, 54–56, 160, 161, 175, 186, 187]. In particular “true” fractons cannot move at all, while other quasiparticles which can be traced back to fractons can move only in a sub-dimensional space, like “lineons”, which move on a line and “planons”, which move on a plane [55, 56]. The first observations of a fracton-like behaviour appeared in lattice spin models [50], and since then many developments followed [51–53, 163, 164, 188]. In particular, in recent years there has been a surge of interest which have come to the forefront of modern condensed matter theory [50–56, 160, 161, 170, 175, 187, 189–196], since these new kind of quasiparticles represent a new class of phases of matter, whose main property is that of being immobile in isolation, but may move by forming bound states. Fractons are found in a variety of physical settings, such as spin liquids [160] and elasticity theory [148, 197], and exhibit unusual phenomenology, such as gravitational physics and localization [177, 198–200]. Fractonic behaviours are to be connected with exotic global symmetries [52, 56, 195]. Systems with these symmetries challenge the common lore by which the low energy behaviour of every lattice system can be described by a continuum quantum field theory : these lattice constructions are usually not Lorentz invariant, and they present an unusual ground state degeneracy, infinite in the continuum limit. Significant efforts have been made to better understand their non-standard behaviours and to extend the known theoretical frameworks to include them [55, 56, 178, 187, 195, 196]. Moreover, this inability to move is the main

property shared by all fracton models, and ultimately can be of physical interest, for instance, in the development of quantum memories [51, 191, 201, 202]. Lattice models describing fractons fall into two classes, depending on their particle content : “type I”, the most representative of which is the X-cube model [52], has both fractons and 1,2-dimensional particles, while “type II”, of which the Haah’s code [51] is the prototypical example, describes fractons only. Fractons, like Linearized Gravity (LG), can be described by a gauge theory of a symmetric tensor field, which generalizes the ordinary Maxwell electromagnetism for a vector field [160]. This class of fracton theories, which can be related to the previous lattice models via a Higgs-like mechanism [162, 165], were first introduced to describe gravity-related phenomena [198–200, 203], and later were developed into the actual fracton theory [160, 161, 175, 186]. Written in terms of a rank-2 symmetric tensor field $A_{ij}(x)$ ($i, j \dots$ spatial indices), the typical starting point is a Maxwell-like Hamiltonian density

$$H = E^{ij} E_{ij} + B^{ij} B_{ij} \sim E^2 + B^2 , \quad (7.1.1)$$

where the “electric” field $E^{ij}(x)$ is the conjugate momentum of $A_{ij}(x)$ as in standard electromagnetism, and the “magnetic” field $B_{ij}(x)$ is defined as the gauge invariant object depending on the lowest possible number of derivatives of $A_{ij}(x)$ [175]. From these definitions, generalized Maxwell equations follow [160] : Faraday’s equation is a relation between $E_{ij}(x)$ as conjugate momentum and the time derivative of $B_{ij}(x)$, while Ampère law is a Hamilton’s equation for $E_{ij}(x)$. Finally, the usual Gauss theorem in this picture is not really a theorem, but, rather, is imposed as a constraint. The gauge transformation of $A_{ij}(x)$ is crucial since, besides determining $B_{ij}(x)$, it is strictly related to the Gauss-like constraint, which has a key role in implementing the restricted mobility of fractons [54, 160, 161, 175, 186, 187]. There are two possibilities [54, 55, 160, 161, 175, 187] :

- **scalar charge theory** : the Gauss constraint and the gauge transformation of $A_{ij}(x)$ are

$$\partial_i \partial_j E^{ij} = 0 \quad (7.1.2)$$

$$\delta A_{ij} = \partial_i \partial_j \Lambda . \quad (7.1.3)$$

In the presence of fractonic matter one can define a charge density operator $\rho(x)$ [187] and the Gauss constraint generalizes to [161]

$$\partial_i \partial_j E^{ij} = \rho . \quad (7.1.4)$$

The restricted mobility becomes evident, since (7.1.4) implies charge and dipole ($p^i(x) = x^i \rho(x)$) neutrality when integrated on an infinite volume V (or up to boundary terms) [54, 55, 160, 161, 177]

$$\int dV \partial_i \partial_j E^{ij} = \int dV \rho = 0 \quad (7.1.5)$$

$$- \int dV \partial_i E^{ik} = \int dV x^k \rho = \int dV p^k = 0 . \quad (7.1.6)$$

Eq. (7.1.6) states that single charges cannot move (fractons) due to dipole conservation, while dipoles do.

- **vector charge theory** : in this case the Gauss constraint involves one derivative instead of two, and the gauge transformation of $A_{ij}(x)$ is a non-covariant version of the diffeomorphisms transformation

$$\partial_i E^{ij} = 0 \quad (7.1.7)$$

$$\delta A_{ij} = \partial_i \Lambda_j + \partial_j \Lambda_i . \quad (7.1.8)$$

The nature of the charge density operator thus changes, becoming a vector $\rho^j(x)$ [187], while the “electric field” remains a symmetric rank-2 tensor and the generalized Gauss constraint is [161]

$$\partial_i E^{ij} = \rho^j , \quad (7.1.9)$$

which immediately implies (7.1.4) together with the conservation laws (7.1.5) and (7.1.6), but yields a further mobility constraint due to conservation of angular momentum

$$\int dV \epsilon_{0ijk} x^i \rho^j = - \int dV \epsilon_{0ijk} E^{ij} = 0 .$$

Hence, the vector charges of the system can move only along a line, which is the spatial direction related to the charged vector itself, thus being 1-dimensional particles (a.k.a. lineons).

Mobility can be further restricted in both scalar and vector charge models by adding tracelessness $E^i_i = 0$ of the “electric field” as an additional constraint. In these cases the models are referred to as “traceless (scalar/vector) fracton models”. In the scalar case the elementary charges can still be identified as fractons, since their motion is already maximally constrained by means of the generalized Gauss law, however now tracelessness also implies the conservation of a component of the quadrupole momentum, due to which the dipoles of the system are bound to move only on a plane transverse to their direction, thus becoming 2-dimensional particles (also called “planons”). On the contrary, in the vector charge model the quasiparticles can move in one dimension, and the tracelessness constraint on the electric-like field completely restricts their motion making them proper fractons. These models, properties and particle content can be summarized by the following table

The model	(Gauss) constraint	(integrated) conserved quantities	particle content
Scalar charge th.	$\partial_i \partial_j E^{ij} = \rho$	ρ charge $\vec{x}\rho$ dipole	charge = fracton dipole = free
Traceless scalar charge th.	$\partial_i \partial_j E^{ij} = \rho$ $E^i_i = 0$	ρ ; $\vec{x}\rho$ $x^2 \rho \sim$ quadrupole	charge = fracton dipole = planon
Vector charge th.	$\partial_i E^{ij} = \rho^j$	$\vec{\rho}$ charge (\sim dipole) $\vec{x} \wedge \vec{\rho}$ angular momentum	charge = lineon
Traceless vector charge th.	$\partial_i E^{ij} = \rho^j$ $E^i_i = 0$	$\vec{\rho}$; $\vec{x} \wedge \vec{\rho}$ $\vec{x} \cdot \vec{\rho} (\sim$ quadrupole); $(\vec{x} \cdot \vec{\rho})\vec{x} + \frac{1}{2}x^2 \vec{\rho}$	charge = fracton

Table 1: Fracton phenomenology.

In Maxwell theory the $A_0(x)$ component of the gauge field $A_\mu(x)$ is a multiplier enforcing the standard Gauss constraint $\vec{\nabla} \cdot \vec{E}(x) = 0$. Following this, in fracton models

the Gauss constraint (7.1.2) is implemented by introducing a Lagrange multiplier, as done for instance in [175], sometimes called $A_0(x)$ to enhance the Maxwell analogy. This multiplier seems to have no relation with the $A_{ij}(x)$ tensor field and in addition, due to this “by hand” implementation, the Lagrangian acquires an inhomogeneous number of (spatial) derivatives [175]. For these reasons, despite all the similarities we mentioned, while Maxwell theory has a natural covariant formulation, the above construction of fracton models appear to be intrinsically non-covariant. We shall see that the covariant formulation presented in the next Section will make all these ad hoc implementations come out naturally as QFT principles, and the nature and properties of fracton quasiparticles will emerge completely.

7.2 MAXWELL THEORY OF FRACTONS

In this Section we will show that the main results concerning fractons, in particular the existence of tensorial electric and magnetic fields, the Gauss constraints, the Maxwell-like Hamiltonian and the dipole response to “electromagnetic” fields through a “Lorentz force”, to cite a few, are indeed consequences of a *covariant* action for a symmetric rank-2 tensor field $A_{\mu\nu}(x)$, invariant under the covariant extension of the fracton transformation (7.1.3)

$$\delta_{fract} A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda, \quad (7.2.1)$$

which therefore plays, as usual in quantum field theory, a central role. We shall also show that from the gauge tensor field $A_{\mu\nu}(x)$ it is possible to construct a rank-3 tensor $F_{\mu\nu\rho}(x)$ which we may call the fracton field strength, invariant under (7.2.1) and satisfying a kind of geometrical Bianchi identity. Quite surprisingly, the fracton action can be written in terms of the fracton field strength as $\int F^2$, as the ordinary Maxwell theory, and all the above mentioned equations characterizing fractons are nothing else than the “Maxwell” equations, without need of introducing any external constraint or particular request, and therefore are just consequences of the covariant symmetry (7.2.1). Moreover, electric and magnetic tensor fields emerge naturally, and in terms of these the action and the energy density read, respectively, $\int (E^2 - B^2)$ and $(E^2 + B^2)$. Finally, the Lorentz force for fracton dipoles derived “by intuition” (*sic*) in [160] is here recovered as part of the conservation law for the stress-energy tensor. As a matter of fact, the covariant generalization described in this Section makes apparent that the fracton theory is, indeed, a direct extension of the standard electromagnetic theory which can be formulated covariantly according to the typical field theory chain

$$\begin{array}{ccccc} \text{symmetry} & \rightarrow & \text{action} & \rightarrow & \text{equations of motion} \\ \delta_{fract} A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda(x) & \rightarrow & -\frac{1}{6} \int d^4x F^{\mu\nu\rho} F_{\mu\nu\rho} & \rightarrow & \partial_\mu F^{\alpha\beta\mu} = 0, \end{array}$$

which really appears as a higher rank extension of Maxwell theory.

The relation between fractons and gravitons has been already remarked [54, 55, 177, 178]. From the field theory point of view this is evident from the covariant extension (7.2.1) of the fracton symmetry, which is a particular case of the stronger infinitesimal diffeomorphism transformation [178]

$$\delta_{diff} A_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu. \quad (7.2.2)$$

In practice, this means that, while the diff symmetry (7.2.2) uniquely defines the LG action, the most general action invariant under (7.2.1) is formed by two separately invariant terms : the LG action and the fracton action $\int F^2$, which is quite peculiar since, to our knowledge, this is the only case of a Lorentz invariant action which, although free and quadratic, shows a dimensionless constant which cannot be eliminated by a field redefinition, and which cannot be identified as a physical mass, like in topologically massive 3D gauge theory [70]. Coherently with this covariant picture, both fractons and LG can be given an electromagnetic formulation (the first, as discussed, in terms of tensors, while the second, known as gravitoelectromagnetism [105, 204, 205], involves vectors), but, as we shall see, they also share the ‘‘Gauss’’ constraint (7.1.2) (which is not an external constraint in our formalism) which underlies the fracton limited mobility property. As a first step towards the study of boundaries, the analogy with ordinary electromagnetism can be pushed further through the topological θ -term that can be added to the Maxwell Lagrangian (see for instance [179] and references therein). The role of an analogous boundary term has been studied in the case of fractons in [175] and, as for dyons, the result is that the ‘‘electric’’ charge gains an additional contribution related to a ‘‘magnetic’’ vector charge [54, 55, 175]. As the standard Witten effect has consequences in condensed matter physics, this higher rank version of the fracton θ -term might give interesting results in the context of higher order topological phases [184], which seem to be related to boundary effects in fractons, as we shall discuss in Section 7.4. The case of a local, instead of constant, $\theta(x)$ is relevant in axion models [206, 207], where Maxwell equations acquire additional contributions [208, 209]. In [205] a local $\theta(x)$ -term has been added to LG, with mainly two consequences : a correction to the Newtonian gravitational field and a Witten-like effect for gravitational dyons, in which ‘‘gravitipoles’’ [210] acquire mass. The results of [205] for LG suggest the possibility of generalizing what has been found in [175] for fractons, since both LG and fractons are described by a rank-2 symmetric tensor field.

There are of course, and fortunately, a few open questions, which deserve further efforts. The first is that we were not able to find a symmetry which separates fractons from gravitons. In other words : while the diff symmetry (7.2.2) uniquely defines the LG action, the fracton symmetry (7.2.1) gives two invariant functionals. One must necessarily buy gravitons, together with fractons. The way out in field theory is to recover the fracton action for vanishing LG constant, but, still, it would be more satisfying to pick up the fracton action by means of an additional symmetry. Moreover : the fracton symmetry (7.2.1) (but also the original (7.1.3)) is dimensionally problematic. In 4D the rank-2 tensor $A_{\mu\nu}(x)$ has mass dimension one, both in fracton and LG theory. Hence, to be dimensionally homogeneous, the fracton gauge transformation would require a scalar gauge parameter $\Lambda(x)$ with negative dimension

$$d = 4 \quad \Rightarrow \quad [A] = 1 \quad ; \quad \delta A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda \quad \Rightarrow \quad [\Lambda] = -1 , \quad (7.2.3)$$

but this would not be the case in 6D, which would be the most ‘‘natural’’ spacetime dimensions for fractons to live in

$$d = 6 \quad \Rightarrow \quad [A] = 2 \quad ; \quad \delta A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda \quad \Rightarrow \quad [\Lambda] = 0 . \quad (7.2.4)$$

As we shall see, this reflects also in the fact that, in 4D, the stress-energy tensor is not traceless, hence the theory is not scale invariant, differently from the classical Maxwell

theory. Instead, tracelessness is recovered in 6D.

The above construction of a covariant theory for fractons will be presented in this Section as follows :

7.2.1	Fractons and linearized gravity	81
7.2.2	Maxwell theory for fractons	85
7.2.3	Stress-energy tensor and fracton Lorentz force	89
7.2.4	θ -term	95
7.2.5	Final remarks	97

In particular Section 7.2.1 we derive the theory defined by the fracton symmetry (7.2.1), composed by two terms, fracton and LG. We then construct the rank-3 fracton field strength $F_{\mu\nu\rho}(x)$ and we show that both the fracton and the LG actions can be written in terms of this tensor, which satisfies an identity analogous to the Bianchi one. We then compute the canonical momentum $\Pi^{\alpha\beta}(x)$ associated to $A_{\alpha\beta}(x)$ and derive the field equations of motion. In Section 7.2.2 we consider the fracton theory only, obtained by putting the LG constant to zero. Without imposing any external constraint, we recover the main properties of the fractons simply from the equations of motion, which, written in terms of the electric and magnetic tensor fields, impressively reminds the ordinary Maxwell equations. In Section 7.2.3 we derive the stress-energy tensor and physically identify its components, which are the higher rank extensions of their Maxwell counterparts. We then write, interpret and discuss the stress-energy tensor conservation laws. We then add matter to the theory, represented by a symmetric rank-2 tensor coupled to $A_{\alpha\beta}(x)$, and extend the previously found results in presence of matter. The most important achievement of this Section is the expression of the Lorentz force which describes how dipole fractons respond to the electromagnetic tensor fields. This Lorentz force coincides with the one conjectured in [160]. Finally, in Section 7.2.4 we add to the fracton action the generalized θ -term, which, again, can be written both in terms of the electromagnetic tensor fields and of the fracton field strength, in complete analogy with Maxwell theory. We recover and generalize previous results obtained in the context of LG [205] and of fractons [175] giving to the θ -parameter a local dependence. Some final remarks can be found in Section 7.2.5.

7.2.1 *Fractons and linearized gravity*

The symmetry

We adopt the standard point of view of field theory, that is to consider the symmetry as the birth certificate of a theory. In our case, the symmetry, hereinafter “fracton” symmetry, is the covariant generalization of the extended electromagnetic transformation (7.1.3) invoked in [160, 161, 175, 186] for fractons, *i.e.*

$$\delta_{fract}A_{\mu\nu} = \partial_\mu\partial_\nu\Lambda, \quad (7.2.5)$$

where $A_{\mu\nu}(x)$ is a rank-2 symmetric tensor field and $\Lambda(x)$ a local scalar parameter. The fracton transformation (7.2.5) is obtained from the more general infinitesimal diffeomorphism transformation

$$\delta_{diff} A_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \quad (7.2.6)$$

for the particular choice of the gauge parameter $\Lambda_\mu(x) = \frac{1}{2} \partial_\mu \Lambda(x)$. The most general 4D action invariant under the fracton symmetry (7.2.5) is a linear combination of two invariant actions

$$S_{inv} = g_1 S_{fract} + g_2 S_{LG} , \quad (7.2.7)$$

where

$$S_{fract} = \int d^4x (\partial_\rho A_{\mu\nu} \partial^\rho A^{\mu\nu} - \partial_\rho A_{\mu\nu} \partial^\mu A^{\nu\rho}) \quad (7.2.8)$$

$$S_{LG} = \int d^4x (\partial_\mu A \partial^\mu A - \partial_\rho A_{\mu\nu} \partial^\rho A^{\mu\nu} - 2\partial_\mu A \partial_\nu A^{\mu\nu} + 2\partial_\rho A_{\mu\nu} \partial^\mu A^{\nu\rho}) , \quad (7.2.9)$$

g_1, g_2 are dimensionless constants, and $A \equiv \eta^{\mu\nu} A_{\mu\nu}$ is the trace of the tensor field. The action S_{LG} is readily recognized to be the linearized action for gravity [211], while S_{fract} is our candidate to be the covariant action for fractons, as we shall motivate. Hence, the space of 4D local integrated functionals invariant under the fracton symmetry (7.2.5) has dimension two, and one of the two constants g_1 and g_2 can be reabsorbed by a redefinition of $A_{\mu\nu}(x)$, so that we have the rather peculiar feature that the free quadratic theory defined by the action (7.2.7), hence by the fracton transformation (7.2.5), depends on one constant. To our knowledge, this is the only example of a free quadratic covariant theory depending on a constant which cannot be reabsorbed by a field redefinition, without being identified as a mass, like in 3D topologically massive gauge theories [70]. In particular we have that S_{fract} (7.2.8) and S_{LG} (7.2.9) are both invariant under the fracton transformation (7.2.5)

$$\delta_{fract} S_{LG} = \delta_{fract} S_{fract} = 0 , \quad (7.2.10)$$

but only the LG action (7.2.9) is invariant under the diff transformation (7.2.6)

$$\delta_{diff} S_{LG} = 0 , \quad (7.2.11)$$

while the fracton S_{fract} (7.2.8), hence the whole action S_{inv} (7.2.7), is not

$$\delta_{diff} S_{fract} = g_1 \int d^4x [2\partial^\mu \partial^\nu A_{\mu\nu} \partial^\rho \Lambda_\rho - \partial^2 A_{\mu\nu} (\partial^\mu \Lambda^\nu + \partial^\nu \Lambda^\mu)] \neq 0 . \quad (7.2.12)$$

The fracton field strength

The first step towards a Maxwell theory for fractons, which is the main purpose here, is the construction of the “building block” of the theory, namely the extension of the electromagnetic field strength $F_{\mu\nu}(x)$

$$\begin{aligned} A_\mu &\rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ A_{\mu\nu} &\rightarrow F_{\mu\nu\rho} = ? \end{aligned} \quad (7.2.13)$$

To this aim, we look for a rank-3 tensor built from the first derivative of the rank-2 tensor field $A_{\mu\nu}(x)$

$$F_{\mu\nu\rho} \equiv a_1 \partial_\mu A_{\nu\rho} + a_2 \partial_\rho A_{\mu\nu} + a_3 \partial_\nu A_{\mu\rho} , \quad (7.2.14)$$

where a_i are dimensionless constants. As the electromagnetic tensor $F_{\mu\nu}(x)$ is invariant under the ordinary gauge transformation $\delta_{gauge}A_\mu(x) = \partial_\mu\Lambda(x)$, in the same way we require that $F_{\mu\nu\rho}(x)$ is invariant under the fracton symmetry (7.2.5), which gives a constraint on the coefficients a_i

$$\delta_{fract}F_{\mu\nu\rho} = 0 \Rightarrow a_3 = -(a_1 + a_2), \quad (7.2.15)$$

so that

$$F_{\mu\nu\rho} = a_1\partial_\mu A_{\nu\rho} + a_2\partial_\rho A_{\mu\nu} - (a_1 + a_2)\partial_\nu A_{\mu\rho}. \quad (7.2.16)$$

As a consequence of its definition, the invariant tensor (7.2.16) has the properties listed in Table 2, compared to those of the Maxwell field strength $F_{\mu\nu}(x)$.

	fractons	Maxwell
invariance	$\delta_{fract}F_{\mu\nu\rho} = 0$	$\delta_{gauge}F_{\mu\nu} = 0$
cyclicity	$F_{\mu\nu\rho} + F_{\nu\rho\mu} + F_{\rho\mu\nu} = 0$	$F_{\mu\nu} + F_{\nu\mu} = 0$
Bianchi	$\epsilon_{\alpha\mu\nu\rho}\partial^\mu F^{\beta\nu\rho} = 0$	$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0$

Table 2: Properties of the fracton and Maxwell field strengths.

We remark that the fracton invariance of $F_{\mu\nu\rho}(x)$ (7.2.15) and the property which we called ‘‘cyclicity’’ in Table 2 are equivalent

$$\delta_{fract}F_{\mu\nu\rho} = 0 \Leftrightarrow F_{\mu\nu\rho} + F_{\nu\rho\mu} + F_{\rho\mu\nu} = 0 \Leftrightarrow a_1 + a_2 + a_3 = 0. \quad (7.2.17)$$

All the physically relevant quantities (like for instance the equations of motion and the conjugate momenta) are obtained by making functional derivatives with respect to $A_{\mu\nu}(x)$, which is a symmetric tensor field. With the aim of writing everything in terms of the tensor field strength $F_{\mu\nu\rho}(x)$, it is natural to ask that also this latter is symmetric by the change of two indices, for instance the first two

$$F_{\mu\nu\rho} = F_{\nu\mu\rho}, \quad (7.2.18)$$

which implies $a_2 = -2a_1$ ⁸. Therefore, after a rescaling of $A_{\mu\nu}(x)$, our fracton field strength is

$$F_{\mu\nu\rho} = F_{\nu\mu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\mu\rho} - 2\partial_\rho A_{\mu\nu}. \quad (7.2.19)$$

Rather surprisingly, the same symmetric tensor (7.2.19) appears as an unnumbered comment in the final part of a 1988 paper by Y.S. Wu and A. Zee [212] as a consequence of the covariant symmetry (7.2.5), but in a completely different context, since fractons were not even conceived yet.⁹ The actions (7.2.8) and (7.2.9) can be written in terms of the fracton field strength $F_{\mu\nu\rho}(x)$ (7.2.19) as

$$S_{fract} = \frac{1}{6} \int d^4x F^{\mu\nu\rho} F_{\mu\nu\rho} \quad (7.2.20)$$

$$S_{LG} = \int d^4x \left(\frac{1}{4} F_{\mu\nu}^\mu F_\rho^{\rho\nu} - \frac{1}{6} F^{\mu\nu\rho} F_{\mu\nu\rho} \right). \quad (7.2.21)$$

Notice that also the LG action (7.2.9) can be written in terms of the newly introduced tensor $F_{\mu\nu\rho}(x)$ (7.2.19). The fact that the fractonic component of the total action S_{inv} (7.2.7) turns out to be of the form $\int F^2$ tells us that we are on the right way to build a Maxwell theory of fractons, but the analogies are even more surprising in what follows.

⁸ We checked that this is indeed the case, *i.e.* $F_{\mu\nu\rho}(x) - F_{\nu\mu\rho}(x)$ is always ruled out.

⁹ We thank Giandomenico Palumbo for this remark.

The canonical momentum $\Pi^{\alpha\beta}(x)$

In the theory of the fracton quasiparticles an important role is played by the momentum canonically conjugated to $A_{\mu\nu}(x)$ [54, 160, 161, 175, 213]. From (7.2.7) we have

$$\Pi^{\alpha\beta} \equiv \frac{\partial \mathcal{L}_{inv}}{\partial (\partial_t A_{\alpha\beta})} = -g_1 F^{\alpha\beta 0} - g_2 \left[\eta^{\alpha\beta} F_\lambda^{\lambda 0} - \frac{1}{2} \left(\eta^{0\alpha} F_\lambda^{\lambda\beta} + \eta^{0\beta} F_\lambda^{\lambda\alpha} \right) - F^{\alpha\beta 0} \right], \quad (7.2.22)$$

whose components are

$$\Pi^{00} = 0 \quad (7.2.23)$$

$$\Pi^{i0} = -g_1 F^{i00} - \frac{1}{2} g_2 F_j^{ji} \quad (7.2.24)$$

$$\Pi^{ij} = -g_1 F^{ij0} + g_2 (F^{ij0} - \eta^{ij} F_k^{k0}). \quad (7.2.25)$$

From (7.2.23) we see that $A_{00}(x)$ is not a dynamical field for the whole theory (both fractons and LG). For what concerns LG alone, it is known [177, 211] that the components with a time index, $A_{00}(x)$ and $A_{0i}(x)$, have non-dynamical equations of motion, acting as Lagrange multipliers to enforce gauge constraints, in the same way as $A_0(x)$ acts as a Lagrange multiplier enforcing Gauss law in Maxwell theory. The physical degrees of freedom are contained in the spatial symmetric tensor $A_{ij}(x)$. We shall see that this property concerning LG holds for fracton theory too, which therefore remarkably shares close similarities with both LG and Maxwell theory. We finally notice that for a particular combination of g_1 and g_2 the trace of $\Pi^{\alpha\beta}$ vanishes

$$\eta_{\alpha\beta} \Pi^{\alpha\beta} = \Pi^\alpha_\alpha = \Pi^i_i = -(g_1 + 2g_2) F_\lambda^{\lambda 0} = 0 \quad \text{if } g_1 + 2g_2 = 0. \quad (7.2.26)$$

This corresponds to the fact that, as already remarked in [178] and as we shall see in the next Section 7.3, the theory defined by S_{inv} (7.2.7) at (7.2.26) does not depend on the trace of the tensor field $A_{\mu\nu}(x)$, further lowering the number of degrees of freedom.

The field equations of motion

As the fracton and LG actions (7.2.20) and (7.2.21), the field EoM can be written in terms of the fracton field strength $F_{\mu\nu\rho}(x)$ as well

$$\begin{aligned} \frac{\delta S_{inv}}{\delta A^{\alpha\beta}} &= g_1 [(\partial^\mu \partial_\alpha A_{\mu\beta} + \partial^\mu \partial_\beta A_{\mu\alpha}) - 2\partial^2 A_{\alpha\beta}] \\ &\quad + 2g_2 [\eta_{\alpha\beta} (\partial_\mu \partial_\nu A^{\mu\nu} - \partial^2 A) + \partial_\alpha \partial_\beta A + \partial^2 A_{\alpha\beta} - \partial^\mu (\partial_\alpha A_{\mu\beta} + \partial_\beta A_{\mu\alpha})] \\ &= g_1 \partial^\mu F_{\alpha\beta\mu} + g_2 \left[\eta_{\alpha\beta} \partial_\mu F_\nu^{\nu\mu} - \frac{1}{2} (\partial_\alpha F_{\mu\beta}^\mu + \partial_\beta F_{\mu\alpha}^\mu) - \partial^\mu F_{\alpha\beta\mu} \right] = 0, \end{aligned} \quad (7.2.27)$$

whose components are

- $\alpha = \beta = 0$

$$g_1 \partial_i F^{00i} - g_2 \partial_i (F_\lambda^{\lambda i} + F^{00i}) = 2\partial_i (-g_1 F^{i00} - \frac{1}{2} g_2 F_j^{ji}) = 2\partial_i \Pi^{i0} = 0, \quad (7.2.28)$$

where we used $F^{00i} = -2F^{i00}$ and the definition of the canonical momentum Π^{i0} (7.2.24).

- $\alpha = 0, \beta = i$

$$\begin{aligned} 0 &= g_1 \partial_\lambda F^{0i\lambda} - \frac{g_2}{2} (\partial^0 F_\lambda^{\lambda i} + \partial^i F_\lambda^{\lambda 0} + 2\partial_\lambda F^{0i\lambda}) \\ &= -\partial_0 \Pi^{i0} + g_1 \partial_j F^{0ij} - \frac{g_2}{2} (\partial^i F_\lambda^{\lambda 0} + 2\partial_j F^{0ij}) ; \end{aligned} \quad (7.2.29)$$

- $\alpha = i, \beta = j$

$$g_1 \partial^\mu F_{ij\mu} + g_2 [\eta_{ij} \partial_\mu F_\nu^{\nu\mu} - \frac{1}{2} (\partial_i F_{\mu j}^\mu + \partial_j F_{\mu i}^\mu) - \partial^\mu F_{ij\mu}] = 0 . \quad (7.2.30)$$

7.2.2 Maxwell theory for fractons

In this Section we treat the case $g_2 = 0$, and we shall recover the main features generally attributed to the fracton quasiparticles [54, 160, 161, 175], thus allowing us to justify the identification of S_{fract} (7.2.20) as the action for fractons.

Electric/magnetic tensor fields and “Maxwell” equations

As far as only fractons are considered, in [54, 160, 161, 175, 213] an electric tensor field $E^{ij}(x)$ is defined as spatial “canonical momentum” as follows

$$E_{ij} \propto -\partial_t A_{ij} + \partial_i \partial_j A_0 . \quad (7.2.31)$$

We would like to show here that $E_{ij}(x)$ (7.2.31) can indeed be derived from the action (7.2.20) in a way which also clarifies which is the origin of the scalar field $A_0(x)$ appearing in (7.2.31). In fact, $A_0(x)$ cannot be directly part of a canonical momentum unless in the Lagrangian weird terms with three derivatives are admitted [175]. The covariant extension (7.2.5) of the fracton symmetry has a central role in determining (7.2.31), without the need of any *ad-hoc* introduction. In fact, starting from the fracton transformation (7.2.5) one gets the action S_{inv} (7.2.7) from which the spatial canonical momentum (7.2.25) is derived. In the case where only fractons are present, namely $g_2 = 0$, the canonical momentum $\Pi^{ij}(x)|_{g_2=0}$ reads:

$$\Pi^{ij}|_{g_2=0} = -g_1 F^{ij0} = g_1 (2\partial^0 A^{ij} - \partial^j A^{0i} - \partial^i A^{0j}) , \quad (7.2.32)$$

which differs from (7.2.31). Nevertheless, the electric tensor field (7.2.31) can be indeed obtained from the spatial canonical momentum $\Pi^{ij}(x)|_{g_2=0}$ using the EoM (7.2.27) with $g_2 = 0$, *i.e.* those derived from the fracton action (7.2.20) alone, which closely remind the usual Maxwell equations

$$\partial^\mu F_{\alpha\beta\mu} = 0 . \quad (7.2.33)$$

In fact, taking (7.2.33) at $\alpha = \beta = 0$ (or, equivalently, (7.2.28) at $g_2 = 0$) we have

$$\partial^i F_{00i} = 2\partial^i (\partial_0 A_{0i} - \partial_i A_{00}) = 0 . \quad (7.2.34)$$

A particular solution is given by

$$A_{0\mu} = A_{\mu 0} \equiv \partial_\mu A_0 , \quad (7.2.35)$$

which introduces the missing scalar potential $A_0(x)$. What renders remarkable the solution (7.2.35), which is a direct consequence of our covariant approach, is that it leads to recover, up to a constant, the electric tensor field (7.2.31). In fact, using (7.2.35) in (7.2.32) we get

$$(\Pi^{ij}|_{g_2=0})|_{(7.2.35)} = 2g_1 (\partial^0 A^{ij} - \partial^i \partial^j A^0) \equiv E^{ij} , \quad (7.2.36)$$

which indeed coincides with the tensor electric field (7.2.31). Hence, finally, the answer to the question is the following : the electric tensor field $E^{ij}(x)$ (7.2.31) introduced in [160, 161, 175, 213] is defined as the canonical momentum Π_{ij} (7.2.32) of the fracton action S_{fract} (7.2.8) evaluated on the EoM (7.2.34). In addition to the properties listed in Table 2, which hold in general, the particular solution (7.2.35) implies also

$$F^{i00} = F^{0i0} = F^{00i} = 0 \quad (7.2.37)$$

$$F^{ij0} = -2F^{0ij} = -2F^{i0j} , \quad (7.2.38)$$

which hold for fractons only. As anticipated, because of (7.2.37), for the fracton theory $g_2 = 0$ we have an additional Hamiltonian constraint, besides (7.2.23)

$$(\Pi^{i0}|_{g_2=0})|_{(7.2.35)} = -g_1 F^{i00} = 0 , \quad (7.2.39)$$

which corresponds to the fact that, like in LG, also for fractons the degrees of freedom concern only the spatial components $A_{ij}(x)$. Moreover, again in surprising analogy with Maxwell theory where the electric field and the field strength are related by $E^i(x) = -F^{0i}(x)$, we have that

$$E^{ij} = -g_1 F^{ij0} = 2g_1 F^{0ij} = 2g_1 F^{i0j} , \quad (7.2.40)$$

which makes the electric tensor field an invariant quantity at sight. Taking (7.2.33) at $\alpha = 0$ and $\beta = i$ (or, equivalently, (7.2.29) at $g_2 = 0$) we have

$$\partial_\mu F^{0i\mu} = \partial_j F^{0ij} = -\frac{1}{2} \partial_j F^{ij0} = 0 , \quad (7.2.41)$$

which, using (7.2.40), writes

$$\partial_j E^{ij} = 0 , \quad (7.2.42)$$

which is the vacuum Gauss law for the electric tensor field (7.2.36). It is the tensorial extension of

$$\vec{\nabla} \cdot \vec{E} = 0 . \quad (7.2.43)$$

Eq. (7.2.42) trivially implies

$$\partial_i \partial_j E^{ij} = 0 , \quad (7.2.44)$$

which, together with (7.2.42), is crucial for the property of limited mobility characterizing the fracton quasiparticles [54, 55, 160, 161]. As we shall show in a moment, while the Gauss-like equation (7.2.42) holds only for the fracton action (7.2.20), an equation formally identical to the limited mobility equation (7.2.44) holds for the LG action S_{LG} (7.2.21), too. In fact, taking the divergence ∂_i of the whole EoM (7.2.29) and using (7.2.28) we have, at $g_1 = 0$, *i.e.* for LG only,

$$\partial_i \partial^i F_\lambda^{\lambda 0} + 2\partial_i \partial_j F^{ij0} = -\partial_i \partial_j \Pi^{ij}|_{g_1=0} = 0 , \quad (7.2.45)$$

where we used the cyclicity property in Table 2 of the tensor $F_{\mu\nu\rho}(x)$, which in particular implies

$$\partial_i \partial_j F^{0ij} = -\frac{1}{2} \partial_i \partial_j F^{ij0}. \quad (7.2.46)$$

The equation (7.2.45) is formally identical to its fracton counterpart (7.2.44), and its possible consequences on the limited mobility of the gravitational waves are worth to investigate and to interpret. Finally, taking (7.2.33) at $\alpha = i$ and $\beta = j$ (or, equivalently, (7.2.30) at $g_2 = 0$), we have

$$\partial_\mu F^{ij\mu} = \partial_0 F^{ij0} + \partial_k F^{ijk} = -\frac{1}{g_1} \partial_0 E^{ij} + \partial_k F^{ijk} = 0, \quad (7.2.47)$$

where we used the definition of the electric tensor field $E^{ij}(x)$ (7.2.40). The fracton EoM (7.2.47) suggests to define the magnetic tensor field, in analogy with the ordinary vector magnetic field $B_i(x) = \epsilon_{ijk} \partial^j A^k(x) = \frac{1}{2} \epsilon_{ijk} F^{jk}(x)$, as

$$B_i^j \equiv g \epsilon_{0ilk} \partial^l A^{jk} = \frac{g}{3} \epsilon_{0ikl} F^{jkl}, \quad (7.2.48)$$

where g is a constant to be suitably tuned. Its inverse is

$$F^{ijk} \equiv -\frac{1}{g} \left(\epsilon^{0ikl} B_l^j + \epsilon^{0jkl} B_l^i \right). \quad (7.2.49)$$

Again, thanks to the definition of the fracton field strength $F_{\mu\nu\rho}(x)$ the definition of the magnetic tensor field, as for the electric one (7.2.40), is explicitly invariant under the defining covariant fracton symmetry. The EoM (7.2.47) then can be written

$$-\frac{1}{g_1} \partial_0 E^{ij} - \frac{1}{g} \left(\epsilon^{0ikl} \partial_k B_l^j + \epsilon^{0jkl} \partial_k B_l^i \right) = 0, \quad (7.2.50)$$

which turns out to be completely analogous to the electromagnetic Ampère law of electromagnetism in vacuum

$$-\partial_t \vec{E} + \vec{\nabla} \times \vec{B} = 0, \quad (7.2.51)$$

of which (7.2.50) is the tensorial extension. It coincides with Eq.(26) in [160]. From the definition (7.2.48) we find that the magnetic tensor field is traceless

$$B_p^p = 0, \quad (7.2.52)$$

and satisfies

$$\partial^a B_a^p = 0, \quad (7.2.53)$$

which is analogous to the standard Maxwell equation

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (7.2.54)$$

and coincides with Pretko's second equation (38) in [160]. As in standard electromagnetism, the equation (7.2.53) is a geometric property, consequence of the definition of the magnetic tensor field $B_i^j(x)$ (7.2.48).

Let us now study which information comes from the "Bianchi" identity in Table 2

- $\alpha = 0, \beta = j$

$$\epsilon_{0iab} \partial^i F^{jab} = \frac{3}{g} \partial^i B_i^j = 0, \quad (7.2.55)$$

we therefore recover the tensor magnetic Gauss law (7.2.53).

- $\alpha = l, \beta = i$

$$\begin{aligned} 0 &= \epsilon_{l\mu\nu\rho} \partial^\mu F^{i\nu\rho} \\ &= \epsilon_{l\mu j 0} \partial^\mu F^{ij0} + \epsilon_{l\mu j k} \partial^\mu F^{ijk} + \epsilon_{l\mu 0 j} \partial^\mu F^{i0j} \\ &= \frac{3}{2} \epsilon_{lmj0} \partial^m F^{ij0} + \epsilon_{l0jk} \partial^0 F^{ijk} \\ &= \frac{3}{2g_1} \epsilon_{0lmj} \partial^m E^{ij} + \frac{3}{g} \partial_0 B_l^i, \end{aligned} \quad (7.2.56)$$

where we used (7.2.37), (7.2.38), (7.2.40) and the definition (7.2.48). Eq.(7.2.56) is new, and it is the tensorial extension of the Faraday equation of electromagnetism :

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0. \quad (7.2.57)$$

It coincides with Pretko's Eq.(36) in [160].

- $\alpha = \beta = 0$ and $\alpha = i, \beta = 0$ are trivial identities.

Summarizing, from the EoM (7.2.33) and the ‘‘Bianchi’’ identity in Table 2 we have the following strong analogy with classical electromagnetism :

Maxwell	Fractons	
$\vec{\nabla} \cdot \vec{E} = 0$	$\partial_j E^{ij} = 0$	(7.2.58)
$\vec{\nabla} \cdot \vec{B} = 0$	$\partial^a B_a^p = 0$	(7.2.59)
$\vec{\nabla} \times \vec{E} - \partial_t \vec{B} = 0$	$\epsilon_{0lmj} \partial^m E^{ij} - \frac{2g_1}{g} \partial^0 B_l^i = 0$	(7.2.60)
$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = 0$	$-\partial_0 E^{ij} - \frac{2g_1}{g} \frac{1}{2} \left(\epsilon^{0ikl} \partial_k B_l^j + \epsilon^{0jkl} \partial_k B_l^i \right) = 0.$	(7.2.61)

Setting

$$\frac{2g_1}{g} = -1 \quad (7.2.62)$$

the last two equations (7.2.60) and (7.2.61) are fully analogous to the corresponding ordinary Maxwell equations at the left hand side, and coincides with those introduced by Pretko in [160] from a completely different point of view, where actually it is written (7.2.44) rather than the more fundamental (7.2.42).

Fracton action in terms of electric and magnetic tensor fields

We have seen that the fracton action (7.2.8), originally written in terms of the field $A_{\mu\nu}(x)$, can be written in terms of the tensor $F_{\mu\nu\rho}(x)$ as (7.2.20). This makes apparent the strong analogy with the classical electromagnetic Maxwell theory, of which the fracton theory appears to be the higher rank generalization. This analogy is even more spectacular when the four equations (7.2.58), (7.2.59), (7.2.60) and (7.2.61) governing the theory are considered, which can be written in terms of the two electric and

magnetic tensor fields $E^{ij}(x)$ (7.2.36) and $B_i^j(x)$ (7.2.48). As in Maxwell theory, two equations, namely (7.2.58) and (7.2.61), are the EoM of the action (7.2.20), while the other two, (7.2.59) and (7.2.60), are consequences of the ‘‘Bianchi’’ identity written in Table 2 for the tensor $F_{\mu\nu\rho}(x)$, hence have a geometrical nature. The analogy with electromagnetism can be pushed further by noting that the fracton action (7.2.20) can be written in terms of the electric and magnetic tensor fields as follows :

$$\begin{aligned}
S_{fract} &= \frac{g_1}{6} \int d^4x F_{\mu\nu\rho} F^{\mu\nu\rho} \\
&= \frac{g_1}{6} \int d^4x \left(F^{0ij} F_{0ij} + F^{ij0} F_{ij0} + F^{i0j} F_{i0j} + F^{ijk} F_{ijk} \right) \\
&= \frac{g_1}{6} \int d^4x \left(\frac{3}{2} F^{ij0} F_{ij0} + F^{ijk} F_{ijk} \right) \\
&= \frac{g_1}{6} \int d^4x \left[-\frac{3}{2} \frac{1}{g_1^2} E^{ij} E_{ij} - \frac{2}{g^2} \epsilon^{0kmn} B_n^l \epsilon_{0kab} B_c^b (\delta_m^a \delta_l^c + \delta_l^a \delta_m^c) \right] \\
&= \frac{1}{2} \int d^4x \left(-\frac{1}{2g_1} E^{ij} E_{ij} + \frac{2g_1}{g^2} B_i^j B_j^i \right),
\end{aligned} \tag{7.2.63}$$

where, besides the definitions of the electric and magnetic tensor fields (7.2.36) and (7.2.48), we used the properties of $F_{\mu\nu\rho}(x)$ (7.2.37) and (7.2.38), and the tracelessness of the tensor magnetic field (7.2.52). The result (7.2.63) closely reminds the electromagnetic action, whose Lagrangian is proportional to $E^2 - B^2$, provided that

$$g^2 = 4g_1^2, \tag{7.2.64}$$

which is compatible with the previously found constraint (7.2.62), that we will assume from now on.

7.2.3 Stress-energy tensor and fracton Lorentz force

The energy-momentum tensor and conservation laws

The stress-energy tensor for the fracton action S_{fract} (7.2.20) is

$$\begin{aligned}
T_{\alpha\beta} &= -\frac{2}{\sqrt{-g}} \frac{\delta S_{fract}}{\delta g^{\alpha\beta}} \Big|_{g^{\alpha\beta}=\eta^{\alpha\beta}} \\
&= -\frac{g_1}{3\sqrt{-g}} \frac{\delta}{\delta g^{\alpha\beta}} \int d^4x \sqrt{-g} g^{\mu\lambda} g^{\nu\gamma} g^{\rho\sigma} F_{\lambda\gamma\sigma} F_{\mu\nu\rho} \Big|_{g^{\alpha\beta}=\eta^{\alpha\beta}} \\
&= \frac{g_1}{6} \eta_{\alpha\beta} F^2 - \frac{g_1}{3} \eta_{\alpha\gamma} \eta_{\beta\lambda} \left(2F^{\lambda\nu\rho} F_{\nu\rho}^\gamma + F^{\mu\nu\lambda} F_{\mu\nu}^\gamma \right).
\end{aligned} \tag{7.2.65}$$

Notice that taking the trace of $T_{\mu\nu}$, we have, in d -spacetime dimensions

$$T = \eta^{\alpha\beta} T_{\alpha\beta} = g_1 \frac{(d-6)}{6} F^2 \Big|_{d=4} = -\frac{g_1}{3} F^2, \tag{7.2.66}$$

which does not vanish in $d = 4$, differently from what happens in Maxwell theory. The tracelessness of the stress-energy tensor would be recovered in $d = 6$, which, as already remarked in the introductory part at the beginning of Section 7.2, seem to be the most natural, although unphysical, spacetime dimensions for fractons. The non-vanishing

of the trace of the fracton stress-energy tensor is the sign that the theory, already at classical level, is not scale invariant. This suggests the existence of an energy scale. Now, since tracelessness is eventually related to the masslessness of the photon, the fact that the trace (7.2.66) does not vanish might suggest the existence of a mass (as the typical energy scale) for the fractons, which can be introduced in a similar way as in LG [214, 215]. The components of the stress-energy tensor are physically interpretable as follows :

- $\alpha = \beta = 0$ gives the energy density $T_{00} = u$

$$\begin{aligned} T_{00} = u &= -\frac{g_1}{6}F^2 + \frac{g_1}{3}(2F^{0\mu\nu}F_{0\mu\nu} + F^{\mu\nu 0}F_{\mu\nu 0}) \\ &= \frac{1}{4g_1}(E^{ij}E_{ij} - B_i^j B_j^i) + \frac{g_1}{2}\left(-\frac{1}{g_1}\right)\left(\frac{1}{g_1}\right)E^{ij}E_{ij} \\ &= -\frac{1}{4g_1}(E^{ij}E_{ij} + B_i^j B_j^i), \end{aligned} \quad (7.2.67)$$

where (7.2.37), (7.2.38), (7.2.63), (7.2.40) and (7.2.64) have been used. Again, this expression is formally identical to the corresponding electromagnetic result $u \propto E^2 + B^2$. From the positivity constraint of the energy density u it must be

$$g_1 < 0, \quad (7.2.68)$$

and, from now on, we choose

$$g_1 = -1; \quad (7.2.69)$$

- $\alpha = 0, \beta = i$ gives the Poynting vector $T_{0i} = S_i$

$$\begin{aligned} T_{0i} = S_i &= \frac{1}{3}\eta_{i\lambda}(2F^{\lambda\nu\rho}F_{0\nu\rho} + F^{\mu\nu\lambda}F_{\mu\nu 0}) \\ &= \frac{1}{3}\eta_{ij}(2F^{jkl}F_{0kl} + F^{klj}F_{kl0}) \\ &= \frac{1}{6}\eta_{ij}E_{kl}\left(-2\epsilon^{0jlp}B_p^k - \epsilon^{0klp}B_p^j + \epsilon^{0kjp}B_p^l\right) \\ &= \frac{1}{2}\epsilon_{0ilp}E^{kl}B_p^i, \end{aligned} \quad (7.2.70)$$

which, as in Maxwell electromagnetism, is the vector product of the electric and magnetic tensor fields $\vec{S} \propto \vec{E} \times \vec{B}$;

- $\alpha = i, \beta = j$ gives the stress tensor $T_{ij} = \sigma_{ij}$

$$\begin{aligned} T_{ij} &= -\frac{1}{6}\eta_{ij}F^2 + \frac{1}{3}\eta_{jk}(2F^{k\mu\nu}F_{i\mu\nu} + F^{\mu\nu k}F_{\mu\nu i}) \\ &= -\frac{1}{6}\eta_{ij}F^2 - \eta_{jk}E^{ka}E_{ia} + \frac{1}{3}\eta_{jk}(2F^{kab}F_{iab} + F^{abk}F_{abi}) \\ &= -\frac{1}{6}\eta_{ij}F^2 - \eta_{jk}E^{ka}E_{ia} + \frac{1}{6}\eta_{jk}(2\delta_i^k B_a^b B_b^a - 2B_i^a B_a^k + 4B_a^k B_i^a - \epsilon^{0akp}\epsilon_{0biq}B_p^b B_q^a) \\ &= -\frac{1}{6}\eta_{ij}F^2 - \eta_{jk}E^{ka}E_{ia} + \frac{1}{2}\eta_{jk}(\delta_i^k B_a^b B_b^a - B_i^a B_a^k + B_a^k B_i^a) \\ &= \eta_{ij}T_{00} - \eta_{jk}\eta_{il}E^{ka}E_a^l - \frac{1}{2}\eta_{jk}\eta_{il}(B^{la}B_a^k - B_a^k B^{al}), \end{aligned} \quad (7.2.71)$$

where we used

$$\begin{aligned} 2F^{kab}F_{iab} &= -\frac{1}{2}\left(\epsilon^{0kbp}B_p^a + \epsilon^{0abp}B_p^k\right)\left(\epsilon_{0ibq}B_a^q + \epsilon_{0abq}B_b^q\right) \\ &= \frac{1}{2}\left(\delta_i^k B_a^b B_b^a - B_i^a B_a^k + 4B_a^k B_i^a\right) \end{aligned} \quad (7.2.72)$$

$$\begin{aligned} F^{abk}F_{abi} &= -\frac{1}{4}\left(\epsilon^{0akp}B_p^b + \epsilon^{0bkp}B_p^a\right)\left(\epsilon_{0aiq}B_b^q + \epsilon_{0biq}B_a^q\right) \\ &= \frac{1}{2}\left(\delta_i^k B_a^b B_b^a - B_i^a B_a^k - \epsilon^{0akp}\epsilon_{0biq}B_p^b B_a^q\right), \end{aligned} \quad (7.2.73)$$

and, from [216],

$$\epsilon^{0akp}\epsilon_{0biq} = -\delta_b^a(\delta_i^k \delta_q^p - \delta_i^p \delta_q^k) + \delta_i^a(\delta_b^k \delta_q^p - \delta_b^p \delta_q^k) - \delta_q^a(\delta_b^k \delta_i^p - \delta_b^p \delta_i^k), \quad (7.2.74)$$

so that

$$\epsilon^{0akp}\epsilon_{0biq}B_p^b B_a^q = -\delta_i^k B_a^b B_b^a + B_i^a B_a^k + B_a^k B_i^a, \quad (7.2.75)$$

because of the tracelessness of the magnetic tensor (7.2.52). Finally, we used also the fact that, because of (7.2.63) and (7.2.67) we have $T_{00} = -\frac{1}{6}F^2 + \frac{1}{2}B_a^b B_b^a$. As expected, the stress tensor is symmetric $i \leftrightarrow j$:

$$T_{ij} = \frac{1}{4}\eta_{ij}\left(E^{ab}E_{ab} + B_a^b B_b^a\right) - \eta^{ab}E_{ia}E_{jb} - \frac{1}{2}\eta^{ab}\left(B_{ia}B_{jb} - B_{aj}B_{bi}\right). \quad (7.2.76)$$

Once again, the analogy with Maxwell theory, for which the stress tensor is

$$\sigma_{ij} = \frac{1}{2}\eta_{ij}\left(E^2 + B^2\right) - E_i E_j - B_i B_j, \quad (7.2.77)$$

is impressive.

Let us now discuss the (on-shell) conservation of the stress-energy tensor

$$\partial^\nu T_{\mu\nu} = 0, \quad (7.2.78)$$

whose components are

- $\mu = 0$:

$$\begin{aligned} \partial^\nu T_{\nu 0} &= \partial^0 T_{00} + \partial^i T_{i0} \\ &= \partial^0 u + \partial^i S_i \\ &= -\frac{1}{2}\left[E_{ab}\partial_0 E^{ab} + B_a^b \partial_0 B_b^a - \epsilon_{0ilp}\partial^i\left(E^{kl}B_k^p\right)\right] \\ &= -\frac{1}{2}\left[\epsilon^{0akl}E_{ab}\partial_k B_l^b + \epsilon_{0amn}B_b^a \partial^m E^{bn} - \epsilon_{0ilp}\partial^i\left(E^{kl}B_k^p\right)\right] \\ &= \frac{1}{2}\epsilon_{0amn}\left[E^{ab}\partial^m B_b^n + B_b^n \partial^m E^{ab} - \partial^m\left(E^{ab}B_b^n\right)\right] \\ &= 0, \end{aligned} \quad (7.2.79)$$

where we used the EoM (7.2.50) and (7.2.56). The continuity equation is therefore verified on-shell

$$\partial^i S_i + \partial^0 u = 0. \quad (7.2.80)$$

- $\mu = i$:

$$\begin{aligned}
\partial^\nu T_{\nu i} &= \partial^0 T_{0i} + \partial^j T_{ji} \\
&= \partial^0 S_i + \partial^j \sigma_{ji} \\
&= -\frac{1}{2} \epsilon_{0imn} \partial_0 (E^{am} B_a^n) + \partial_i u - \eta_{il} \partial_k (E^{ka} E_a^l) - \eta_{il} \partial_k (B^{la} B_a^k - B_a^k B^{al}) \\
&= -\frac{1}{2} \epsilon_{0imn} \left\{ \left[\frac{1}{2} (\epsilon^{0akl} \partial_k B_l^m + \epsilon^{0mkl} \partial_k B_l^a) \right] B_a^n - \epsilon^{0nbc} \partial_b E_{ac} E^{am} \right\} + \\
&\quad + \partial_i u - \eta_{il} \partial_k (E^{ka} E_a^l) - \eta_{il} \partial_k (B^{la} B_a^k - B_a^k B^{al}) \\
&= -\frac{1}{4} \left[\epsilon_{0imn} \epsilon^{0akl} \partial_k B_l^m + (\delta_i^k \delta_n^l - \delta_n^k \delta_i^l) \partial_k B_l^a \right] B_a^n - \\
&\quad - \frac{1}{2} (\delta_i^b \delta_m^c - \delta_m^b \delta_i^c) E^{am} \partial_b E_{ac} + \partial_i u - \eta_{il} \partial_k (E^{ka} E_a^l) \\
&\quad - \eta_{il} \partial_k (B^{la} B_a^k - B_a^k B^{al}) \\
&= \frac{1}{4} \left[(3B_i^n \partial_m B_n^m + 2B_n^m \partial_m B_i^n) - 2E^{ab} \partial_b E_{ai} \right] \\
&\neq 0, \tag{7.2.81}
\end{aligned}$$

where (7.2.50), (7.2.56), (7.2.74) and (7.2.42) have been used. Differently from the continuity equation (7.2.80), the spatial components of the divergence of the stress-energy tensor do not vanish. Now, what should we expect actually ? If we think of the stress-energy tensor as the conserved current associated to the diffeomorphism symmetry, as the definition (7.2.65) suggests, it should not be conserved in a theory like the fracton one, which is not diffeomorphism invariant (7.2.12). On the other hand, we are facing here with a partial conservation of the stress-energy tensor, because its time component is indeed conserved, yielding the continuity equation (7.2.80), which relates the flux of the energy density to the divergence of the momentum density. The partial conservation of the stress-energy tensor might be explained by observing that the fracton symmetry (7.2.5) is indeed a diff transformation (7.2.6) with a particular choice of the vector diff parameter.

Fracton Lorentz force

It is interesting to study how the physics is modified if matter is introduced by means of a symmetric rank-2 tensor $J^{\mu\nu}(x) = J^{\nu\mu}(x)$ coupled to the fracton field $A_{\mu\nu}(x)$

$$S_{fract} \rightarrow S_{tot} = S_{fract} + S_J, \tag{7.2.82}$$

where S_{fract} is the pure fractonic action (7.2.8) (or (7.2.20)), and S_J is the matter action

$$S_J \equiv - \int d^4x J^{\mu\nu} A_{\mu\nu}. \tag{7.2.83}$$

The EoM (7.2.33) modifies as

$$\partial_\mu F^{\alpha\beta\mu} = -J^{\alpha\beta}. \tag{7.2.84}$$

We observe that, due to the cyclicity identity in Table 2, $J^{\alpha\beta}$ is conserved in the following sense

$$\partial_\alpha \partial_\beta J^{\alpha\beta} = 0. \tag{7.2.85}$$

The components of the EoM (7.2.84) are

- $\alpha = \beta = 0$:

$$\partial_i F^{00i} = -J^{00} = 0 , \quad (7.2.86)$$

which vanishes because of (7.2.37), consequence of (7.2.35). Hence, there is no coupling with $A_{00}(x)$, as expected, since it is not a dynamical degree of freedom of the theory, due to (7.2.23).

- $\alpha = 0, \beta = i$:

$$\partial_j F^{0ij} = -J^{0i} , \quad (7.2.87)$$

which, using (7.2.40), becomes

$$\partial_j E^{ij} = 2J^{i0} . \quad (7.2.88)$$

Taking the divergence of (7.2.88) we find the analogous of the Gauss law

$$\partial_i \partial_j E^{ij} = \rho , \quad (7.2.89)$$

where we defined the charge density

$$\rho \equiv 2\partial_i J^{i0} . \quad (7.2.90)$$

This equation plays a central role in [54, 55, 160, 161, 175], since it yields not only the charge neutrality condition, but also the vanishing of the total dipole moment. In fact, integrating (7.2.89), we get

$$\int dV \partial_i \partial_j E^{ij} = \int dV \rho = 0 , \quad (7.2.91)$$

which states that the total charge inside an infinite volume is zero. Moreover, from (7.2.89) we also have

$$\int dV x^k \partial_i \partial_j E^{ij} = \int dV x^k \rho = \int dV p^k = 0 , \quad (7.2.92)$$

according to which the dipole moment density, defined as

$$p^k = x^k \rho , \quad (7.2.93)$$

of an infinite volume vanishes.

- $\alpha = i, \beta = j$:

$$\partial_\mu F^{ij\mu} = -J^{ij} , \quad (7.2.94)$$

which, using (7.2.50), becomes

$$-\partial_0 E^{ij} + \frac{1}{2} \left(\epsilon^{0ikl} \partial_k B_l^j + \epsilon^{0jkl} \partial_k B_l^i \right) = J^{ij} . \quad (7.2.95)$$

Differentiating with $\partial_i \partial_j$, we have

$$\begin{aligned} 0 &= \partial_0 \partial_i \partial_j E^{ij} + \partial_i \partial_j J^{ij} \\ &= \partial_0 \rho + \partial_i \partial_j J^{ij} , \end{aligned} \quad (7.2.96)$$

where (7.2.89) has been used. It is a kind of continuity equation [160, 172, 213], which can also be obtained from the conservation equation (7.2.85)

$$0 = \partial_\alpha \partial_\beta J^{\alpha\beta} = 2\partial_0 \partial_i J^{0i} + \partial_i \partial_j J^{ij} = \partial_0 \rho + \partial_i \partial_j J^{ij}, \quad (7.2.97)$$

where we have used the definition of the density charge $\rho(x)$ (7.2.90) and the fact that $J^{00}(x) = 0$ (7.2.86).

It is also interesting to see how the (partial) conservation of the stress-energy tensor is modified by the presence of matter. The continuity equation (7.2.80) is modified as

$$\partial^\nu T_{\nu 0} = \partial^0 T_{00} + \partial^i T_{i0} = \partial^0 u + \partial^i S_i = E_{ab} J^{ab}, \quad (7.2.98)$$

while the spatial components of (7.2.81) acquire the term in the last row

$$\begin{aligned} \partial^\nu T_{\nu i} = \partial^0 T_{0i} + \partial^j T_{ji} = \partial^0 S_i + \partial^j \sigma_{ji} &= \frac{1}{4} \left[(3B_i^n \partial_m B_n^m + 2B_n^m \partial_m B_i^n) - 2E^{ab} \partial_b E_{ai} \right] \\ &+ \frac{1}{2} \epsilon_{0imn} J^{am} B_n^i - 2J^{a0} E_{ia}, \end{aligned} \quad (7.2.99)$$

The additional terms appearing in (7.2.98) and (7.2.99) can be easily interpreted if, again, we think to the standard Maxwell theory of electromagnetism, where the divergence of the stress-energy tensor in presence of matter involves the 4D ‘‘Lorentz force’’ per unit volume on matter f^μ :

$$\partial_\nu T^{\mu\nu} + f^\mu = 0. \quad (7.2.100)$$

At the right hand side of (7.2.98) appears

$$f^0 = E_{ab} J^{ab}, \quad (7.2.101)$$

which is the analogous of the electromagnetic power $\vec{E} \cdot \vec{J}$. The last term at the right hand side of (7.2.99)

$$f^i = 2J_{a0} E^{ia} - \frac{1}{2} \epsilon^{0imn} J_{am} B_n^i \quad (7.2.102)$$

can be traced back to the generalized Lorentz force on a dipole $p^i(x)$ moving with velocity v^i proposed in [160] once we take

$$J^{0i} \sim p^i, \quad (7.2.103)$$

and

$$J^{ij} \sim p^i v^j + p^j v^i. \quad (7.2.104)$$

The first identification (7.2.103) is compatible with (7.2.92), when (7.2.88) is taken into account, and the second relation (7.2.104) agrees with the microscopic lattice definition of the current of a dipole made in [160, 198]. What is remarkable is that we recover here as part of the conservation law of the stress-energy tensor the picture conjectured in [160]: the isolated electric monopoles of the theory described by the action (7.2.20) are fractons, which do not respond to the electromagnetic fields and, hence, do not move, due to the dipole conservation constraint (7.2.89). What we find here is that, instead, dipole motion, which preserves the global dipole moment, does respond to the electromagnetic field tensors $E^{ij}(x)$ and $B_i^j(x)$ according to (7.2.102), like a conventional charge particle responds to an ordinary electromagnetic field. Hence, from (7.2.102), we confirm the ‘‘intuition’’ proposed in [160] concerning the Lorentz force on a fracton dipole.

7.2.4 θ -term

In ordinary vector gauge field theory, it is known that a term can be added to the Maxwell action (or to its non-abelian extension, namely the Yang-Mills theory) : the so called θ -term, which has the form

$$S_\theta \sim \theta \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \sim \theta \int d^4x \vec{E} \cdot \vec{B}, \quad (7.2.105)$$

where θ is a constant parameter. The θ -term represented by (7.2.105) is topological, since it does not depend on the spacetime metric, and it is a total derivative, hence it does not contribute to the EoM. Nonetheless, the θ -term is relevant in several contexts, like axion electrodynamics, the Witten effect and the strong CP problem (see for instance [179, 206–209]). For what concerns the fracton theory, in [175] the θ -term has been generalized as $E_{ij}B^{ij}$, in analogy with the fractonic Hamiltonian density, assumed to be proportional to $(E_{ij}E^{ij} + B_{ij}B^{ij})$. Considering a compact tensor gauge field, which implies a “magnetic” monopole ($\partial_i B^{ij} = g^i \neq 0$), the introduction of the θ -term gives to the Gauss constraint an additional contribution related to the “magnetic” field. As for dyons in the Witten effect [180], the “electric” charge thus gains an additional contribution related to the “magnetic” vector charge [54, 55, 175]. On the other hand the possibility of a non-constant θ -term has not yet been investigated in the context of fractons. The motivation for such a generalization comes from TI, which are characterized by a step function θ -term, which switches between $\theta = 0$ outside the material and $\theta = \pi$ inside. Another example is given by axion models [206, 207], which describe a dynamical field coupled to photons via a local $\theta(x)$ -term. This generates modified Maxwell equations [208, 209] as follows

$$\vec{\nabla} \cdot \vec{E} = \rho - \vec{\nabla} \theta \cdot \vec{B} \quad (7.2.106)$$

$$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{J} + \partial_t \theta \vec{B} + \vec{\nabla} \theta \times \vec{E}, \quad (7.2.107)$$

where the additional terms contribute as an excess of charge ($\vec{\nabla} \theta \cdot \vec{B}$) and current ($\partial_t \theta \vec{B} + \vec{\nabla} \theta \times \vec{E}$) densities. Thus axion models are frequently used in the context of condensed matter and TI to mimic a non-constant θ -term, like for instance in [181]. As we are dealing with a rank-2 tensor theory, an interesting example is the case studied in [205] in the context of LG, where modified gravitoelectromagnetic [204] equations analogous to (7.2.106) and (7.2.107) are recovered. According to the formalism presented in this Section, based on the Maxwell-like construction of a consistent theory for fractons, the analogous of the θ -term should be the following

$$S_\theta = \frac{1}{9} \int d^4x \theta \epsilon_{\mu\nu\rho\sigma} F^{\lambda\mu\nu} F_\lambda^{\rho\sigma} = \int d^4x \theta \epsilon_{\mu\nu\rho\sigma} \partial^\mu A^{\lambda\nu} \partial^\rho A_\lambda^\sigma. \quad (7.2.108)$$

We shall see that this is indeed the case by comparing the consequences of adding this term to the action S_{fract} (7.2.8) to the known results concerning the θ -term in the theory of fractons [54, 55, 175] and of LG [205]. Notice that, differently from the standard θ -term introduced to solve the strong CP problem [179, 217], S_θ is not topological, due to the contraction of the λ -indices in (7.2.108). Moreover, here $\theta(x)$ is not constant, like in [205]. For instance, $\theta(x)$ might be the Heaviside step function, which would corre-

spond to introducing a boundary at $x = 0$, as we have seen in the previous Chapters [1–3, 12, 88]. The contribution of S_θ to the EoM is

$$\frac{\delta S_\theta}{\delta A^{\alpha\beta}} = -(\delta_\alpha^\gamma \delta_\beta^\sigma + \delta_\alpha^\sigma \delta_\beta^\gamma) \partial^\rho \theta \epsilon_{\mu\nu\rho\sigma} \eta_{\lambda\gamma} \partial^\mu A^{\lambda\nu}. \quad (7.2.109)$$

It is interesting to observe that this contribution is the same as the one that in [205] gives the θ -modified term of the gravitoelectromagnetic equations, where the electric and magnetic fields are vectors. The EoM (7.2.33) acquire an additional term

$$\frac{\delta S_{fract}}{\delta A^{\alpha\beta}} + \frac{\delta S_\theta}{\delta A^{\alpha\beta}} = 0, \quad (7.2.110)$$

whose components are

- $\alpha = \beta = 0$

$$\partial_i F^{00i} - 2\partial^k \theta \epsilon_{0ijk} \partial^i A^{0j} = 0, \quad (7.2.111)$$

which is still solved by $A^{0\mu} = \partial^\mu A^0$ (7.2.35) ;

- $\alpha = 0, \beta = i$

$$\partial_j E^{ij} - \partial^j \theta B_j^i = 0, \quad (7.2.112)$$

which is the tensorial extension of (7.2.106)

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \theta \cdot \vec{B}; \quad (7.2.113)$$

- $\alpha = i, \beta = j$

$$\delta_{ab}^{ij} \left[\partial_0 E^{ab} - \epsilon^{0akl} \partial_k B_l^b + \eta^{ac} \left(\epsilon^{0nmb} \partial_n \theta E_{mc} + \partial_0 \theta B_c^b \right) \right] = 0, \quad (7.2.114)$$

where we defined the symmetrized delta

$$\delta_{ij}^{ab} \equiv \frac{1}{2} (\delta_i^a \delta_j^b + \delta_i^b \delta_j^a), \quad (7.2.115)$$

which agrees with (7.2.107)

$$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \partial_t \theta \vec{B} + \vec{\nabla} \theta \times \vec{E}. \quad (7.2.116)$$

Therefore Eqs. (7.2.110) are generalized tensorial θ -modified Maxwell equations. In particular, as in the standard modified Maxwell equations [209], we can interpret the θ -dependent terms as an excess of charge and current densities:

$$\partial_j E^{ij} = \tilde{\rho}^i \quad ; \quad -\partial_0 E^{ij} + \frac{1}{2} \left(\epsilon^{0ikl} \partial_k B_l^j + \epsilon^{0jkl} \partial_k B_l^i \right) = \tilde{J}^{ij},$$

with

$$\tilde{\rho}^i \equiv \partial^j \theta B_j^i \quad ; \quad \tilde{J}^{ij} \equiv \delta_{ab}^{ij} \eta^{ac} \left(\epsilon^{0nmb} \partial_n \theta E_{mc} + \partial_0 \theta B_c^b \right).$$

We have a further confirmation that S_θ (7.2.105) is indeed the correct θ -term when we write it in terms of the electric and magnetic tensor fields (7.2.36) and (7.2.48)

$$S_\theta = -\frac{1}{3} \int d^4x \theta \eta_{lm} \epsilon_{0ijk} F^{il0} F^{mjk} = -\frac{1}{2} \int d^4x \theta E^{il} B_{il}, \quad (7.2.117)$$

which is the tensorial extension of the standard θ -term $S_\theta \sim \theta \int \vec{E} \cdot \vec{B}$ for constant θ [179].

7.2.5 Final remarks

In Section 7.1 we have observed that in the Literature the usual approach to deal with a theory of fractons, is to treat space and time separately, hence non covariantly. The standard way to proceed is to take the spatial Gauss constraint (7.1.2), written for a “tensor electric field” $E_{ij}(x)$, as the tool to realize the defining property of fractons, *i.e.* their limited mobility, by extending to fracton dipoles the usual conservation law holding for electric charges. This is usually achieved by introducing a “tensor”, instead of a vector, potential $A_{ij}(x)$, obeying the generalized *spatial* gauge transformation (7.1.3). As we explained, the tensor field $E_{ij}(x)$ was, somehow, defined as the *spatial* canonical momentum, as in (7.2.31). We say *somehow* because in the definition of $E_{ij}(x)$ a scalar field $A_0(x)$ appears, as a multiplier introduced by hand in order to enforce the Gauss constraint. Moreover, given the fracton limited mobility, it comes naturally to ask which is the generalization of the Lorentz force, and how the absence of motion could be compatible with the existence of an electromagnetic Lorentz force, and, above all, which is the elementary object on which such a force acts. Last but not least, in the Literature fractons are often seen in relation to gravity in a nontrivial way, starting from [177]. In this Section we adopted a covariant approach to the theory of fractons. This is not only a matter of formalism, but, rather, it allows to better understand the nature itself of these quasiparticles. The main contribution has thus been to show that, embedding the usual spatial and non-covariant theory of fractons in a more general *covariant gauge field theory*, everything goes to the right place naturally, without introducing by hand any external ingredient. Our unique and starting point, as usual in field theory, is the covariant transformation (7.2.1), from which the most general *covariant* invariant action (7.2.7) is derived. To cite a few new results following this approach, the relation with linearized gravity appears immediately, since the action (7.2.7) consists of two terms, one of which, namely (7.2.9), just describes linearized gravity. The theory of fractons as “emergent electromagnetism”, as often has been called, is evident from the beginning as well, once we defined the rank-three “electromagnetic” field strength (7.2.19) by means of which the fracton Lagrangian writes as F^2 , just like Maxwell theory (from which the title of the Section). According to our field-theoretical point of view, the Gauss constraint is not an external constraint anymore, but turns out to be one of the equations of motion, the others formally coinciding with the Maxwell equations, which is mostly interesting, in our opinion. Finally, studying the conservation of the stress-energy tensor, we recovered the Lorentz force (7.2.102), as it was correctly guessed in [160], from which we see that, as one might expect, the force acts on fracton dipoles, and not on isolated charges, thus preserving the absence of mobility for isolated fractons. Here, we remark a subtle point which concerns the stress-energy tensor of the fracton action. In Section 7.2.3 we computed the stress-energy tensor, defined as

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}}, \quad (7.2.118)$$

and we show that it has the correct components, which are exactly the higher rank generalizations of the Maxwell energy density ($T_{00}(x)$), of the Poynting vector ($T_{0i}(x)$) and of the stress tensor ($T_{ij}(x)$). The time component is conserved on shell, *i.e.* $\partial_\mu T^{\mu 0}(x) = 0$, and gives the continuity equation relating the energy density to the Poynting vector. So far so good. But the space component of the stress-energy tensor conservation

law is not exactly conserved. We find a breaking term which might be interpreted as follows. The stress-energy tensor (7.2.118) is the conserved current associated to the infinitesimal diff invariance (7.2.2) [105], which is not a symmetry of the theory defined by (7.2.1). Hence, the stress-energy tensor (7.2.118) should not be conserved. Nonetheless, the fracton transformation (7.2.1) is a particular case of the general diff transformation (7.2.2). Hence, it is not that unexpected that the stress-energy tensor is *almost* conserved. A further confirmation that $T_{\alpha\beta}(x)$ (7.2.118) is the correct one is that, when matter is added, quite remarkably the conservation equation gives exactly the Lorentz force for fracton dipoles that has been conjectured in [160].

7.3 GAUGE FOR TWO

The aim here is to study the theory of fractons from the field theoretical point of view where, again, peculiar and unusual features appear. In our approach, presented in the previous Section, fractons are described by a gauge field theory involving a symmetric tensor field $A_{\mu\nu}(x)$ transforming as

$$\delta A_{\mu\nu} = \partial_\mu \partial_\nu \Lambda , \quad (7.3.1)$$

where $\Lambda(x)$ is a scalar local gauge parameter. The transformation (7.3.1) is a particular case of the infinitesimal diffeomorphisms

$$\delta_{diff} A_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu , \quad (7.3.2)$$

when the vector gauge parameter $\Lambda_\mu(x)$ is the derivative of a scalar field

$$\Lambda_\mu = \frac{1}{2} \partial_\mu \Lambda . \quad (7.3.3)$$

The most general action S_{inv} (7.2.7) invariant under the fracton symmetry (7.3.1) is the sum of two terms, one of which, (7.2.9), can be immediately identified as LG [105, 211]. This term is to be expected since the infinitesimal diffeomorphisms (7.3.2), which is the defining symmetry of LG, embeds the fracton symmetry (7.3.1) through (7.3.3). We have shown in the previous Section that the other term (7.2.8) describes a theory with pure fractonic features [4]: from this term one can recover all the properties characterizing the so called “scalar charge theory” of fractons [160], including Maxwell-like equations of which the Gauss law is at the origin of the limited mobility property that defines fractons [54, 55, 160, 161, 189]. Since the transformation (7.3.1) is a particular case of the more general infinitesimal diffeomorphism (7.3.2), which is a gauge transformation [105, 211, 215, 218], the theory of the symmetric tensor field $A_{\mu\nu}(x)$ defined by the symmetry (7.3.1) is a *gauge* field theory as well. Evidence of this appears when trying to compute the propagator from the quadratic action S_{inv} (7.2.7), which, similarly to the electromagnetic Maxwell theory, leads to a non invertible matrix. Therefore a gauge fixing term must be added. For any other gauge field theory, this procedure goes smoothly thanks to the Faddeev-Popov procedure [71], which focuses on the gauge parameter, and in particular on its tensorial character. In (7.3.1) the gauge parameter is a scalar, and this would require a scalar gauge fixing condition. The most general covariant one is

$$\partial^\mu \partial^\nu A_{\mu\nu} + \kappa \partial^2 A = 0 , \quad (7.3.4)$$

where κ is a constant gauge fixing parameter. This is analogous to the covariant Lorentz condition for the vector field $A_\mu(x)$

$$\partial^\mu A_\mu = 0 . \quad (7.3.5)$$

The standard situation in gauge field theory is that of having a gauge field, represented by a p -tensor field, and a $(p-1)$ -tensorial gauge parameter. This is the case of all known gauge theories. We already mentioned Maxwell theory, or its non-abelian counterpart, the Yang-Mills theory, but this is also true for higher rank theories, like the topological BF theories in any dimensions [14]. Linearized Gravity, described by a symmetric tensor field and by the symmetry (7.3.2), does not escape this rule. In this case, the commonly used gauge fixing condition is vectorial [211, 214, 219]

$$\partial^\nu A_{\mu\nu} + \kappa \partial_\mu A = 0 . \quad (7.3.6)$$

Due to the presence of the parameter κ , the gauge fixing conditions (7.3.4) and (7.3.6) represent a class of covariant gauges. For instance, in (7.3.6) the particular case $\kappa = -\frac{1}{2}$ corresponds to the “harmonic” gauge fixing [105, 218]. The theory we are considering here is, again, quite peculiar. Not only because, as we have seen, it displays a “non-coupling” constant, but also because it is defined by the gauge transformation (7.3.1), which associates a scalar gauge parameter to a rank-2 symmetric tensor field. Concerning the gauge fixing procedure, one might therefore look at both sides of (7.3.1), each of which opens a different path. Looking at the right hand side of (7.3.1), one sees a scalar gauge parameter and this leads to adopt the scalar gauge condition (7.3.4). This standard way has been followed in [178], where the propagators have been derived and the degrees of freedom have been studied. The scalar gauge condition (7.3.4) has two important drawbacks. The first is that the Landau gauge $\xi = 0$ turns out to be mandatory. The theory seems not to be defined outside this gauge. Now, it is true that physical results should not depend on the gauge choice, but, still, being forced to a unique choice is unpleasant, and it would be much preferable to find all the physical results in a generic gauge and to show that they do not, indeed, depend on a particular choice. The second reason is that the scalar gauge condition (7.3.4) does not allow to reach the limit of pure LG, which necessarily needs the vector gauge condition (7.3.2). This results in a singularity both in the propagators of the theory and in the degrees of freedom, which indeed has been found in [178]: the theory with the scalar gauge condition (7.3.4) is not defined in the limit of pure LG. The alternative, which we consider in this Section, is to focus on the left hand side of (7.3.1), where the same symmetric rank-2 tensor field of LG appears, and decide to adopt the same vectorial gauge condition (7.3.6) as LG. In doing so, two questions should be answered :

1. is the vector gauge condition a good gauge fixing, or, equivalently, do the propagators exist in the pure fractonic limit, possibly without being forced to choose a particular gauge ?
2. in gauge field theory the gauge fixing condition serves to eliminate the redundant degrees of freedom which render infinite the Green functions’ generating functional $Z[J]$

$$Z[J] = \int [dh_{\mu\nu}] e^{iS_{inv} + \int J^{\mu\nu} A_{\mu\nu}} . \quad (7.3.7)$$

Do the fact of imposing four (vector) gauge conditions instead of one (scalar) affect the number of physical degrees of freedom of the whole theory S_{inv} (7.2.7) ?

The above are legitimate and well posed questions and, naively, one might answer positively to both. To the first simply because four conditions are more than one, and one expects that they are more than enough to invert the gauge fixed action to find the propagators; to the second for the same reason : four conditions are more than one, and hence the degrees of freedom which are eliminated are too much and differ from those “killed” by the scalar gauge choice (7.3.4). We shall see that the propagators are not singular in the limits of pure fractons or pure LG and that the number of physical degrees of freedom is the same for the two gauge fixing choices (7.3.4) and (7.3.6), which therefore are equivalent, with the advantage that the vectorial choice (7.3.6) does not constrain us to the Landau gauge and allows to easily recover LG.

The analysis of the gauge structure of the covariant fracton theory presented in this Section is organized as follows :

7.3.1	The model	100
7.3.2	Propagators	101
7.3.3	Degrees of freedom	105
7.3.4	Final remarks	109

In Section 7.3.1, starting from the symmetry (7.3.1), the action of the theory is introduced, which consists of two terms: LG and a fractonic term. The vector gauge condition (7.3.6) is realized by adding a gauge fixing term to the action. In Section 7.3.2 the propagators are computed, and the singularities are studied, which correspond to particular phases of the theory. In Section 7.3.3 we study the degrees of freedom, and we verify that their counting coincides with the known one [178], without the drawback of being confined to the Landau gauge, which reassures of the fact that the number of degrees of freedom does not depend on a particular choice and that the alternative vectorial gauge fixing condition (7.3.6) is, indeed, a good one. In Section 7.3.4 we discuss our results.

7.3.1 *The model*

Let us consider the 4D theory of a symmetric tensor field $A_{\mu\nu}(x)$ which transforms as (7.3.1). The choice (7.3.3) is motivated by the fact that the theory defined by the symmetry (7.3.1) describes the so called “fractons”. The most general action invariant under (7.3.1) was found in Section 7.2 to be

$$S_{inv}(g_1, g_2) = g_1 S_{fract} + g_2 S_{LG} , \tag{7.3.8}$$

where

$$S_{fract} = \int d^4x (A^{\mu\nu} \partial^\rho \partial_\mu A_{\rho\nu} - A_{\mu\nu} \partial^2 A^{\mu\nu}) \tag{7.3.9}$$

$$S_{LG} = \int d^4x (-A \partial^2 A + A_{\mu\nu} \partial^2 A^{\mu\nu} + 2A \partial_\mu \partial_\nu A^{\mu\nu} - 2A^{\mu\nu} \partial^\rho \partial_\mu A_{\nu\rho}) \tag{7.3.10}$$

and $A(x)$ is the Minkowskian trace of $A_{\mu\nu}(x)$

$$A = \eta^{\mu\nu} A_{\mu\nu} . \tag{7.3.11}$$

The action (7.3.8) appears to be the linear combination of two terms, which we recognize to be the action S_{LG} of LG [105, 211] and the pure covariant fractonic action previously described [4, 178]. The actions S_{LG} and S_{fract} are separately invariant under (7.3.1)

$$\delta S_{LG} = \delta S_{fract} = 0, \quad (7.3.12)$$

and $g_{1,2}$ are constants, on which we will come back in a moment. Notice that while the space of 4D local integrated functionals invariant under (7.3.1) is the linear combination (7.3.8) of two elements, the infinitesimal diffeomorphism symmetry (7.3.2) uniquely determines one functional only : the LG action S_{LG} (7.3.10)

$$\delta_{diff} S_{LG} = 0, \quad (7.3.13)$$

under which the fractonic action S_{fract} (7.3.9) is not invariant, as we have seen in (7.2.12). In other words, the ‘‘fractonic’’ symmetry (7.3.1) is less constraining than the diffeomorphism transformation (7.3.2), of which it is a particular case. The action (7.3.8) actually depends on one constant only, because of the possibility of redefining the gauge field by a multiplicative constant c without affecting the physical content of the theory: $A_{\mu\nu} \rightarrow cA_{\mu\nu}$. Nevertheless, we will keep both g_1 and g_2 , in order to track the contributions of the gravitational ($g_1 \rightarrow 0$) and the fractonic ($g_2 \rightarrow 0$) parts. Concerning the gauge fixing, the standard way to realize the condition (7.3.6) is to add to the invariant action (7.3.8) the gauge fixing term

$$S_{gf}(\xi, \kappa) = -\frac{1}{2\xi} \int d^4x (\partial^\nu A_{\mu\nu} + \kappa \partial_\mu A)^2, \quad (7.3.14)$$

as it has been done in [214, 215, 218, 219] for LG alone. In a fully equivalent way, (7.3.14) can be linearized by means of a Lagrange multiplier $b^\mu(x)$, also known as Nakanishi-Lautrup field [107, 108] :

$$S_{gf}(\xi, \kappa) = \int d^4x \left[b^\mu (\partial^\nu A_{\mu\nu} + \kappa \partial_\mu A) + \frac{\xi}{2} b_\mu b^\mu \right]. \quad (7.3.15)$$

In $S_{gf}(\xi, \kappa)$ two gauge fixing parameters appear : ξ and κ . The first - ξ - governs the type of gauge fixing. For instance $\xi = 0$ and $\xi = 1$ are respectively the Landau and Feynman gauges. The second - κ - tunes the type of gauge fixing picked up by ξ . For instance, the Landau gauge in LG corresponds to a class of gauge choices, and it is realized by $\xi = 0$ and generic κ .

7.3.2 Propagators

To find the propagators of the theory, we write the gauge fixed action

$$S(g_1, g_2; \xi, \kappa) = S_{inv}(g_1, g_2) + S_{gf}(\xi, \kappa) \quad (7.3.16)$$

in momentum space :

$$\int d^4p \begin{pmatrix} \tilde{A}^{\mu\nu}(p) & \tilde{b}^\gamma(p) \end{pmatrix} \begin{pmatrix} \Omega_{\mu\nu, \alpha\beta}(p) & \Lambda_{\mu\nu, \lambda}^*(p) \\ \Lambda_{\gamma, \alpha\beta}(p) & H_{\gamma\lambda}(p) \end{pmatrix} \begin{pmatrix} \tilde{A}^{\alpha\beta}(-p) \\ \tilde{b}^\lambda(-p) \end{pmatrix}, \quad (7.3.17)$$

where $\Omega_{\mu\nu, \alpha\beta}(p)$, $\Lambda_{\mu\nu, \lambda}^*(p)$ and $H_{\gamma\lambda}(p)$ are p -dependent tensor operators. The propagators of the theory are obtained by inverting the operator matrix appearing in (7.3.17).

In order to do this, it is useful to write $\Omega_{\mu\nu,\alpha\beta}(p)$, $\Lambda_{\alpha\beta,\mu}(p)$ and $H_{\mu\alpha}(p)$ on the corresponding tensorial basis, as follows

$$\Omega_{\mu\nu,\alpha\beta} = t A_{\mu\nu,\alpha\beta} + u B_{\mu\nu,\alpha\beta} + v C_{\mu\nu,\alpha\beta} + z D_{\mu\nu,\alpha\beta} + w E_{\mu\nu,\alpha\beta} \quad (7.3.18)$$

$$\Lambda_{\alpha\beta,\mu} = -\frac{i}{2} [f (d_{\alpha\mu} p_\beta + d_{\beta\mu} p_\alpha) + g d_{\alpha\beta} p_\mu + l e_{\alpha\beta} p_\mu] \quad (7.3.19)$$

$$H_{\mu\alpha} = r d_{\mu\alpha} + s e_{\mu\alpha}, \quad (7.3.20)$$

where $e_{\mu\nu}(p)$ and $d_{\mu\nu}(p)$ are transverse and longitudinal projectors, respectively,

$$e_{\mu\nu} = \frac{p_\mu p_\nu}{p^2} \quad ; \quad d_{\mu\nu} = \eta_{\mu\nu} - e_{\mu\nu}, \quad (7.3.21)$$

and the rank-4 tensor $\Omega_{\mu\nu,\alpha\beta}(p)$ (7.3.18) is expanded on a basis of operators

$$X_{\mu\nu,\alpha\beta} \equiv (A, B, C, D, E)_{\mu\nu,\alpha\beta} \quad (7.3.22)$$

which can be found in Appendix A.2, together with their properties. The coefficients appearing in (7.3.18), (7.3.19) and (7.3.20) are found to be

$$t = (g_1 + 2g_2)p^2 \quad ; \quad u = 0 \quad ; \quad v = (g_1 - g_2)p^2 \quad ; \quad z = \frac{1}{2}g_1 p^2 \quad ; \quad w = 0 \quad (7.3.23)$$

$$f = \frac{1}{2} \quad ; \quad g = \kappa \quad ; \quad l = 1 + \kappa \quad ; \quad r = \frac{\xi}{2} \quad ; \quad s = \frac{\xi}{2}. \quad (7.3.24)$$

The propagators of the theory are organized in a matrix of tensor operators as well

$$\begin{pmatrix} \hat{G}^{\alpha\beta,\rho\sigma} & \hat{G}^{\alpha\beta,\tau} \\ \hat{G}^{*\lambda,\rho\sigma} & \hat{G}^{\lambda\tau} \end{pmatrix}, \quad (7.3.25)$$

where

$$\hat{G}_{\alpha\beta,\rho\sigma} = \langle \tilde{A}_{\alpha\beta} \tilde{A}_{\rho\sigma} \rangle = \hat{t} A_{\alpha\beta,\rho\sigma} + \hat{u} B_{\alpha\beta,\rho\sigma} + \hat{v} C_{\alpha\beta,\rho\sigma} + \hat{z} D_{\alpha\beta,\rho\sigma} + \hat{w} E_{\alpha\beta,\rho\sigma} \quad (7.3.26)$$

$$\hat{G}_{\alpha\beta,\rho} = \langle \tilde{A}_{\alpha\beta} \tilde{b}_\rho \rangle = i \left[\hat{f} (d_{\alpha\rho} p_\beta + d_{\beta\rho} p_\alpha) + \hat{g} d_{\alpha\beta} p_\rho + \hat{l} e_{\alpha\beta} p_\rho \right] \quad (7.3.27)$$

$$\hat{G}_{\alpha\rho} = \langle \tilde{b}_\alpha \tilde{b}_\rho \rangle = \hat{r} d_{\alpha\rho} + \hat{s} e_{\alpha\rho}, \quad (7.3.28)$$

and the set of coefficients

$$\{\hat{t}, \hat{u}, \hat{v}, \hat{z}, \hat{w}, \hat{f}, \hat{g}, \hat{l}, \hat{r}, \hat{s}\} \quad (7.3.29)$$

are determined by the request that the matrix of propagators (7.3.25) satisfies

$$\begin{pmatrix} \Omega_{\mu\nu,\alpha\beta} & \Lambda_{\mu\nu,\lambda}^* \\ \Lambda_{\gamma,\alpha\beta} & H_{\gamma\lambda} \end{pmatrix} \begin{pmatrix} \hat{G}^{\alpha\beta,\rho\sigma} & \hat{G}^{\alpha\beta,\tau} \\ \hat{G}^{*\lambda,\rho\sigma} & \hat{G}^{\lambda\tau} \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{\mu\nu}^{\rho\sigma} & 0 \\ 0 & \delta_\gamma^\tau \end{pmatrix}, \quad (7.3.30)$$

where $\mathcal{I}_{\mu\nu,\alpha\beta}$ is the rank-4 tensor identity

$$\mathcal{I}_{\mu\nu,\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}). \quad (7.3.31)$$

In Appendix B.2.1 we show that the four tensor equations (7.3.30) are solved by

$$\hat{t} = \frac{(4\kappa + 1)}{(\kappa + 1)(g_1 + 2g_2)p^2} \quad ; \quad \hat{u} = \frac{\kappa(4\kappa + 1) - 2\xi(g_1 + 2g_2)}{(\kappa + 1)^2(g_1 + 2g_2)p^2} \quad (7.3.32)$$

$$\hat{v} = \frac{1}{(g_1 - g_2)p^2} \quad ; \quad \hat{z} = \frac{4\xi}{(2\xi g_1 - 1)p^2} \quad (7.3.33)$$

$$\hat{w} = \frac{-4\kappa}{(\kappa + 1)(g_1 + 2g_2)p^2} \quad ; \quad \hat{f} = \frac{-2}{(2\xi g_1 - 1)p^2} \quad (7.3.34)$$

$$\hat{g} = 0 \quad ; \quad \hat{l} = \frac{2}{(\kappa + 1)p^2} \quad (7.3.35)$$

$$\hat{r} = \frac{4g_1}{(2\xi g_1 - 1)} \quad ; \quad \hat{s} = 0. \quad (7.3.36)$$

Now we can answer the first of our questions, concerning the fractonic limit : the vector gauge fixing condition (7.3.6) is a good one not only for LG ($g_1 = 0$), but also for the pure fractonic case ($g_2 = 0$), as expected. On the other hand, we see that the propagators are singular in four cases :

$$g_1 + 2g_2 = 0 \quad (7.3.37)$$

$$g_1 - g_2 = 0 \quad (7.3.38)$$

$$2\xi g_1 - 1 = 0 \quad (7.3.39)$$

$$\kappa + 1 = 0. \quad (7.3.40)$$

For what concerns the singularity (7.3.40), it simply implies that the “secondary” gauge fixing parameter κ , which tunes the gauge fixing choice of the “primary” parameter ξ in S_{gf} (7.3.15), should be

$$\kappa \neq -1, \quad (7.3.41)$$

as it happens also in LG [214, 215, 218, 219]. The remaining singularities involve the action parameters g_1 and g_2 and the gauge fixing parameter ξ . The poles in the propagators give us information on the structure of the theory. For instance, they might signal the presence of masses, possibly not standard, as in the topologically massive Maxwell-Chern-Simons theory [70]. Or they might indicate the presence of phase transitions in the theory, like in QCD [220, 221], or in the sigma model [222, 223], or in the fracton theory itself [178]. Therefore, the singularities appearing in the propagators of the theory S_{inv} should be treated separately and with care.

- $g_1 = g_2$

In this case, after a field redefinition, the invariant action (7.3.8) is

$$S_{inv}(g_1, g_2)|_{g_1=g_2} = \int d^4p (\tilde{A}p^2\tilde{A} - 2\tilde{A}p_\mu p_\nu \tilde{A}^{\mu\nu} + \tilde{A}^{\mu\nu} p^\rho p_\mu \tilde{A}_{\nu\rho}). \quad (7.3.42)$$

Notice that defining

$$\tilde{A}^\rho \equiv p^\rho \tilde{A} - p_\nu \tilde{A}^{\rho\nu}, \quad (7.3.43)$$

the action (7.3.42) trivializes into

$$S_{inv}(g_1, g_2)|_{g_1=g_2} = - \int d^4p \tilde{A}^\rho \tilde{A}_\rho, \quad (7.3.44)$$

which does not contain any kinetic term. Hence, the singularity at $g_1 = g_2$ is explained as a point where the theory trivializes and does not propagate, and this case will be excluded from now on.

- $g_1 + 2g_2 = 0$

In this case the invariant action (7.3.8), after a field redefinition, reads

$$S_{inv}(g_1, g_2)|_{g_1+2g_2=0} = \int d^4p \left(\tilde{A}p^2 \tilde{A} - 3\tilde{A}_{\mu\nu}p^2 \tilde{A}^{\mu\nu} - 2\tilde{A}p_\mu p_\nu \tilde{A}^{\mu\nu} + 4\tilde{A}^{\mu\nu}p^\rho p_\mu \tilde{A}_{\nu\rho} \right). \quad (7.3.45)$$

We see that with this choice the action does not depend on the trace $\tilde{A}(p)$. In fact, defining

$$\bar{A}_{\mu\nu}(p) \equiv \tilde{A}_{\mu\nu}(p) - \frac{1}{4}\eta_{\mu\nu}\tilde{A}(p), \quad (7.3.46)$$

with

$$\bar{A}(p) = 0, \quad (7.3.47)$$

the action (7.3.45) can be written in terms of $\bar{A}_{\mu\nu}(p)$ only :

$$S_{inv}(g_1, g_2)|_{g_1+2g_2=0} = \int d^4p \left(-3\bar{A}_{\mu\nu}p^2 \bar{A}^{\mu\nu} + 4\bar{A}^{\mu\nu}p^\rho p_\mu \bar{A}_{\nu\rho} \right). \quad (7.3.48)$$

Hence, in this case the theory is traceless, and the singularity at the point $g_1 + 2g_2 = 0$ indicates a change in the counting of the degrees of freedom, as we shall explicitly show. The gauge fixing term (7.3.15) does not depend on the trace $A(x)$ anymore and, hence, on the gauge fixing parameter κ . In momentum space it reads

$$S_{gf}(\xi) = \int d^4p \left(-i\tilde{b}_\mu p_\nu \bar{A}^{\mu\nu} + \frac{\xi}{2}\tilde{b}_\mu \tilde{b}^\mu \right). \quad (7.3.49)$$

The propagators of the traceless theory are well defined, and the coefficients, computed in Appendix B.2.2, are

$$\hat{t} = -\frac{1}{3p^2} \quad \hat{u} = \frac{2\xi}{(2\xi - 1)p^2} \quad (7.3.50)$$

$$\hat{v} = -\frac{1}{3p^2} \quad \hat{z} = \frac{-4\xi}{(4\xi + 1)p^2} \quad (7.3.51)$$

$$\hat{w} = 0 \quad \hat{f} = \frac{2}{(4\xi + 1)p^2} \quad (7.3.52)$$

$$\hat{g} = 0 \quad \hat{l} = \frac{-2}{(2\xi - 1)p^2} \quad (7.3.53)$$

$$\hat{r} = \frac{8}{(4\xi + 1)} \quad \hat{s} = \frac{4}{(2\xi - 1)}. \quad (7.3.54)$$

From the above coefficients we see that the particular values of the primary gauge parameter $\xi = -1/4$ and $\xi = 1/2$ should be excluded.

- $2\xi g_1 - 1 = 0$

We first notice that this singularity is not present in the pure LG case $g_1 = 0$, as it is readily seen from the coefficients \hat{z} (7.3.33), \hat{f} (7.3.34) and \hat{r} (7.3.36) of the propagators. Then, we remark that g_1 is a physical parameter, hence cannot depend on ξ , which is a gauge, unphysical, parameter. This means that $2\xi g_1 - 1 = 0$ should be interpreted as a singularity in ξ as a function of g_1 , and not *viceversa*. In other words, we shall not exclude values of g_1 in order to admit a particular

gauge, but, rather, the singularity must be read as a condition on the gauge fixing parameter ξ :

$$\xi \neq \frac{1}{2g_1} . \quad (7.3.55)$$

We consider the gauge fixing term (7.3.14), before the the introduction of the Lagrange multiplier $b^\mu(x)$. At $2\xi g_1 - 1 = 0$ the gauge fixed action is

$$S(g_1, g_2; \xi, \kappa)|_{2\xi g_1 - 1 = 0} = \int d^4p \left[(g_2 - g_1 \kappa^2) \tilde{A} p^2 \tilde{A} - (g_2 - g_1) \tilde{A}_{\mu\nu} p^2 \tilde{A}^{\mu\nu} + \right. \\ \left. - 2(g_2 + g_1 \kappa) \tilde{A} p_\mu p_\nu \tilde{A}^{\mu\nu} + 2(g_2 - g_1) \tilde{A}^{\mu\nu} p_\mu p^\rho \tilde{A}_{\nu\rho} \right] . \quad (7.3.56)$$

Using the general results of Appendix B.2.1 it is easy to verify that this theory does not have propagators. As a remark, if we choose also $\kappa + 1 = 0$, we find a curious result

$$S(g_1, g_2; \xi, \kappa)|_{2\xi g_1 - 1 = 0, \kappa + 1 = 0} = (g_2 - g_1) S_{LG} . \quad (7.3.57)$$

This means that with the particular gauge choice which involves both the singularities in the two gauge parameters ξ and κ , the gauge fixing procedure fails in choosing one representative for each gauge orbit, which is what the gauge fixing is supposed to do. In fact, according to (7.3.57), in this particular gauge, the fracton contribution disappears, and the gauge fixed action coincides with S_{LG} alone, which still needs to be gauge fixed. It also appears the fact which we already know that for $g_1 = g_2$, which is the trivial, non propagating case already considered, the action vanishes. We thus showed that for $\xi = 1/2g_1$ and $\kappa = -1$ the gauge fixed action $S(g_1, g_2; \xi, \kappa)$ coincides with the invariant, not gauge fixed, action S_{LG} .

7.3.3 Degrees of freedom

The counting of the degrees of freedom of the theory described by the action S_{inv} (7.3.8) is a crucial point. This is true in general, but for this model it is even more true. What we already know from [178] is that, if we adopt the standard scalar gauge choice (7.3.4), which is the natural one when dealing with a gauge transformation depending on a scalar parameter, the degrees of freedom turn out to be six, as in LG alone (five in the traceless case), as if the fractonic contribution S_{fract} (7.3.9) was not present. However, in [178] the choice of the Landau gauge appears to be mandatory, which is rather unpleasant, although the physical results should not depend on the gauge choice. Still, it would be preferable to avoid such a restriction, which instead seems unavoidable in the scalar gauge. This, together with the fact that LG cannot be reached as a limit, leads us to conclude that the scalar gauge choice is not that natural, as the direct application of the Faddeev-Popov procedure would suggest. The aim is therefore to see whether the alternative and, at first sight, exotic choice of the *vector* gauge condition (7.3.6) is an acceptable, and possibly better, one. In the previous Subsection we passed the first test: we have seen that the vector gauge condition leads to well defined propagators, with a pole which corresponds to the traceless theory, in accordance to the scalar case [178]. In this Subsection, instead, we face the trickier point of the counting of the

degrees of freedom. Not only we should recover the known result, but, and more important, we should show that the number of degrees of freedom does not depend on the gauge choice, which was impossible with the scalar gauge condition (7.3.4) frozen in the Landau gauge. This fact is not obvious, since the role of the gauge fixing is to eliminate the redundant degrees of freedom, in order to render finite the path integral $Z[J]$ (7.3.7), and the justified fear is that the four conditions represented by the vector choice (7.3.6) might lead us to underestimate the degrees of freedom with respect to the unique scalar condition (7.3.4). The usual way to count the degrees of freedom is to look for the constraints deriving from the equations of motion of the gauge fixed action

$$S(g_1, g_2; \xi, \kappa) = S_{inv}(g_1, g_2) + S_{gf}(\xi, \kappa), \quad (7.3.58)$$

where the invariant action $S_{inv}(g_1, g_2)$ and the gauge fixing term $S_{gf}(\xi, \kappa)$ are given by (7.3.8) and (7.3.15), respectively. In momentum space, we get

$$\begin{aligned} \frac{\delta S}{\delta \tilde{A}^{\mu\nu}} &= 2g_2 \eta_{\mu\nu} p^2 \tilde{A} + 2(g_1 - g_2) p^2 \tilde{A}_{\mu\nu} - 2g_2 \eta_{\mu\nu} p_\alpha p_\beta \tilde{A}^{\alpha\beta} - 2g_2 p_\mu p_\nu \tilde{A} \\ &\quad + (2g_2 - g_1) p^\alpha (p_\mu \tilde{A}_{\alpha\nu} + p_\nu \tilde{A}_{\alpha\mu}) + \frac{i}{2} (p_\nu \tilde{b}_\mu + p_\mu \tilde{b}_\nu) + i\kappa \eta_{\mu\nu} p_\alpha \tilde{b}^\alpha = 0 \end{aligned} \quad (7.3.59)$$

$$\frac{\delta S}{\delta \tilde{b}^\mu} = -i p^\alpha \tilde{A}_{\alpha\mu} - i\kappa p_\mu \tilde{A} + \xi \tilde{b}_\mu = 0. \quad (7.3.60)$$

If our task was just to count the degrees of freedom, given that they must not depend on the gauge choice, we would immediately find the result by choosing $\xi = \kappa = 0$ in S_{gf} (7.3.15), which belongs to the class of Landau gauges. The \tilde{b}^μ -equation of motion (7.3.60) gives

$$p^\alpha \tilde{A}_{\alpha\mu} = 0, \quad (7.3.61)$$

which are the four constraints needed to recover the six degrees of freedom (five in the traceless case) which we expect for the symmetric rank-2 tensor field $A_{\mu\nu}(x)$. But we want more, that is to show that this number does not depend on the gauge choice, which would render the vector gauge condition a good one under any respect. Achieved that, the vector choice would be even preferable to the scalar one, since the Landau gauge would not be the only possibility, and LG could be obtained as a limit. Hence we proceed without choosing a particular gauge, and we saturate (7.3.59) with $\eta^{\mu\nu}$, $e^{\mu\nu}(p)$ (7.3.21) and p^μ :

$$\eta^{\mu\nu} \frac{\delta S}{\delta \tilde{A}^{\mu\nu}} = 2(g_1 + 2g_2) (p^2 \tilde{A} - p_\alpha p_\beta \tilde{A}^{\alpha\beta}) + i(1 + 4\kappa) p_\alpha \tilde{b}^\alpha = 0 \quad (7.3.62)$$

$$e^{\mu\nu} \frac{\delta S}{\delta \tilde{A}^{\mu\nu}} = i(1 + \kappa) p_\alpha \tilde{b}^\alpha = 0 \quad (7.3.63)$$

$$p^\nu \frac{\delta S}{\delta \tilde{A}^{\mu\nu}} = 2g_1 p^\alpha (p^2 \tilde{A}_{\alpha\mu} - p_\mu p^\beta \tilde{A}_{\alpha\beta}) + i p^2 \tilde{b}_\mu + i(1 + 2\kappa) p_\mu p_\alpha \tilde{b}^\alpha = 0. \quad (7.3.64)$$

Multiplying (7.3.60) by p^μ , we get

$$p^\mu \frac{\delta S}{\delta \tilde{b}^\mu} = i p_\alpha p_\beta \tilde{A}^{\alpha\beta} + i\kappa p^2 \tilde{A} - \xi p_\alpha \tilde{b}^\alpha = 0. \quad (7.3.65)$$

We separately study the generic case $g_1 + 2g_2 \neq 0$ and the traceless case $g_1 + 2g_2 = 0$.

Case $g_1 + 2g_2 \neq 0$

From (7.3.63), remembering that $\kappa + 1 \neq 0$, we get the condition

$$p_\alpha \tilde{b}^\alpha = 0, \quad (7.3.66)$$

which, plugged in (7.3.60), (7.3.62), (7.3.64) and (7.3.65), yields

$$\xi \tilde{b}_\mu = i \left(p^\alpha \tilde{A}_{\alpha\mu} + \kappa p_\mu \tilde{A} \right) \quad (7.3.67)$$

$$(g_1 + 2g_2) \left(p^2 \tilde{A} - p_\alpha p_\beta \tilde{A}^{\alpha\beta} \right) = 0 \quad (7.3.68)$$

$$2g_1 p^\alpha \left(p^2 \tilde{A}_{\alpha\mu} - p_\mu p^\beta \tilde{A}_{\alpha\beta} \right) + i p^2 \tilde{b}_\mu = 0 \quad (7.3.69)$$

$$i p_\alpha p_\beta \tilde{A}^{\alpha\beta} + i \kappa p^2 \tilde{A} = 0. \quad (7.3.70)$$

Now, since we are outside the critical point $g_1 + 2g_2 = 0$ (7.3.37), from (7.3.68) and (7.3.70) we have

$$p^2 \tilde{A} = p_\alpha p_\beta \tilde{A}^{\alpha\beta} \quad (7.3.71)$$

$$-\kappa p^2 \tilde{A} = p_\alpha p_\beta \tilde{A}^{\alpha\beta}, \quad (7.3.72)$$

that is

$$(1 + \kappa) p^2 \tilde{A} = 0 \quad \Rightarrow \quad p^2 \tilde{A} = 0, \quad (7.3.73)$$

hence

$$p_\alpha p_\beta \tilde{A}^{\alpha\beta} = 0. \quad (7.3.74)$$

Notice that the conditions (7.3.73) and (7.3.74) are the ones holding for LG alone ($g_1 = 0$) [214, 219]. It appears, therefore, that the fracton contribution ($g_2 = 0$) (7.3.9) to the total invariant action (7.3.8) is irrelevant as far as the degrees of freedom are concerned, which is an unexpected result. Nonetheless we can directly check this result. Substituting the conditions (7.3.73) and (7.3.74) into (7.3.69), we find

$$-i p^2 \tilde{b}_\mu = 2g_1 p^2 p^\alpha \tilde{A}_{\alpha\mu}. \quad (7.3.75)$$

Using (7.3.67) and (7.3.73), assuming $\xi \neq 0$, *i.e.* excluding for the moment the Landau gauge, equation (7.3.75) becomes

$$(2g_1 \xi - 1) p^2 p^\alpha \tilde{A}_{\alpha\mu} = 0. \quad (7.3.76)$$

We previously studied the case $2g_1 \xi - 1 = 0$, which we now exclude. This means that

$$p^2 p^\alpha \tilde{A}_{\alpha\mu} = 0 \quad (7.3.77)$$

and, from (7.3.75),

$$p^2 \tilde{b}_\mu = 0. \quad (7.3.78)$$

We now define

$$\tilde{J}_\alpha \equiv p^\beta \tilde{A}_{\alpha\beta}, \quad (7.3.79)$$

which, because of (7.3.74), is a conserved current

$$p^\alpha \tilde{J}_\alpha = 0. \quad (7.3.80)$$

The solution of (7.3.80) is

$$\tilde{J}_\alpha = \epsilon_{\alpha\mu\nu\rho} p^\mu \tilde{B}^{\nu\rho}, \quad (7.3.81)$$

where $\tilde{B}^{\nu\rho}$ is a generic antisymmetric tensor. Plugging (7.3.79) in (7.3.77), we have

$$p^2 \tilde{J}_\alpha = p^2 \epsilon_{\alpha\mu\nu\rho} p^\mu \tilde{B}^{\nu\rho} = 0, \quad (7.3.82)$$

and therefore

$$p^2 \tilde{B}_{\rho\lambda} = p_\rho \tilde{B}_\lambda - p_\lambda \tilde{B}_\rho. \quad (7.3.83)$$

From (7.3.81) we then deduce that the current J_α vanishes

$$\tilde{J}_\alpha = p^\beta \tilde{A}_{\alpha\beta} = 0. \quad (7.3.84)$$

We now come back to the Landau gauge $\xi = 0$, which has been excluded in achieving the above result. We now show that (7.3.84) holds also in this case. The \tilde{b}^μ -equation of motion in the Landau gauge is

$$p^\alpha \tilde{A}_{\alpha\mu} = -\kappa p_\mu \tilde{A}. \quad (7.3.85)$$

The degrees of freedom must not depend on the gauge choice. We therefore choose $\kappa = 0$ and we get

$$p^\alpha \tilde{A}_{\alpha\mu} = 0, \quad (7.3.86)$$

which represents four constraints on the 4D symmetric tensor field $\tilde{A}_{\mu\nu}(p)$. Hence the number of degrees of freedom are six, at least if $g_1 + 2g_2 \neq 0$. This coincides with the number of degrees of freedom of LG alone [214, 219]. We have seen that $g_1 + 2g_2 = 0$ corresponds to the traceless case, which is interesting and will be treated separately.

Case $g_1 + 2g_2 = 0$

As we have seen in Section 7.3.2, this case corresponds to the traceless theory. The gauge fixed action can be written in terms of the traceless field $\bar{A}_{\mu\nu}(x)$ (7.3.46), with $\kappa = 0$, and, in momentum space, it reads

$$S(g_1, g_2; \xi, \kappa)|_{g_1+2g_2=0, \kappa=0} = \int d^4p \left(-3\bar{A}_{\mu\nu} p^2 \bar{A}^{\mu\nu} + 4\bar{A}^{\mu\nu} p_\mu p^\rho \bar{A}_{\nu\rho} - i\tilde{b}_\mu p_\nu \bar{A}^{\mu\nu} + \frac{\xi}{2} \tilde{b}_\mu \tilde{b}^\mu \right), \quad (7.3.87)$$

whose equations of motion are

$$\begin{aligned} \frac{\delta S}{\delta \bar{A}^{\mu\nu}} &= -6p^2 \bar{A}_{\mu\nu} + 4p^\alpha (p_\mu \bar{A}_{\alpha\nu} + p_\nu \bar{A}_{\alpha\mu}) - 2\eta_{\mu\nu} p^\alpha p^\beta \bar{A}_{\alpha\beta} \\ &\quad + \frac{i}{2} (p_\nu \tilde{b}_\mu + p_\mu \tilde{b}_\nu) - \frac{i}{4} \eta_{\mu\nu} p_\alpha \tilde{b}^\alpha = 0 \end{aligned} \quad (7.3.88)$$

$$\frac{\delta S}{\delta \tilde{b}^\mu} = -ip^\nu \bar{A}_{\mu\nu} + \xi \tilde{b}_\mu = 0, \quad (7.3.89)$$

where we remind that, for a traceless, symmetric tensor field

$$\frac{\delta \bar{A}^{\alpha\beta}}{\delta \bar{A}^{\mu\nu}} = \frac{1}{2} \left(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\nu^\alpha \delta_\mu^\beta \right) - \frac{1}{4} \eta_{\mu\nu} \eta^{\alpha\beta}. \quad (7.3.90)$$

Saturating (7.3.88) with $e^{\mu\nu}(p)$ and p^ν , we get

$$e^{\mu\nu} \frac{\delta S}{\delta \bar{A}^{\mu\nu}} = ip_\alpha \tilde{b}^\alpha = 0 \quad (7.3.91)$$

$$p^\nu \frac{\delta S}{\delta \bar{A}^{\mu\nu}} = -2p^2 p^\alpha \bar{A}_{\alpha\mu} + 2p_\mu p_\alpha p_\beta \bar{A}^{\alpha\beta} + \frac{i}{2} p^2 \tilde{b}_\mu - \frac{i}{4} p_\mu p_\alpha \tilde{b}^\alpha = 0. \quad (7.3.92)$$

From saturating (7.3.89) with p^μ , and (7.3.91) we have

$$p_\alpha p_\beta \bar{A}^{\alpha\beta} = 0. \quad (7.3.93)$$

If $\xi \neq 0$, the Lagrange multiplier $\tilde{b}_\mu(p)$ can be obtained from (7.3.89) and, plugged in (7.3.92), using (7.3.93), we get

$$(4\xi + 1)p^2 p^\alpha \bar{A}_{\alpha\mu} = 0. \quad (7.3.94)$$

We have already excluded the gauge choice $4\xi + 1 = 0$, which is the propagator singularity $2\xi g_1 - 1 = 0$ at $g_1 + 2g_2 = 0$, hence

$$p^2 p^\alpha \bar{A}_{\alpha\mu} = 0. \quad (7.3.95)$$

Now, using the same argument of Section 7.3.3, involving $\tilde{J}_\alpha(p)$ (7.3.79) and $\tilde{B}^{\nu\rho}(p)$ (7.3.81), Eqs. (7.3.93) and (7.3.95) imply the four constraints

$$p^\alpha \bar{A}_{\alpha\mu} = 0. \quad (7.3.96)$$

On the other hand, when $\xi = 0$, *i.e.* in the Landau gauge, the equation of motion of the Lagrange multiplier (7.3.89) directly gives the constraint (7.3.96). Hence, in all cases we have four constraints on a traceless rank-2 symmetric tensor. Therefore, when $g_1 + 2g_2 = 0$, the degrees of freedom are five.

The results for the different values of the action constants g_1 and g_2 and of the gauge fixing parameters ξ and κ are summarized in Table 3.

g_1, g_2	vectorial gauge fixing		scalar gauge fixing	
	degrees of freedom	forbidden gauges	degrees of freedom	forbidden gauges
$g_1 \neq g_2 \neq 0, g_1 + 2g_2 \neq 0$	6	$\xi = \frac{1}{2g_1}, \kappa = -1$	6	$\xi \neq 0$
$g_1 = 0$ (LG)	6	$\kappa = -1$	not defined	
$g_1 + 2g_2 = 0$	5	$\xi = \{\frac{1}{2}, -\frac{1}{4}\}$	5	$\xi \neq 0$
$g_1 = g_2$	trivial			

Table 3: Summary of results and comparison with the scalar case

7.3.4 Final remarks

The aim of this Section was to find a well defined gauge fixed theory, not constrained to a particular gauge choice, and where both limiting cases, fractonic and linearized gravity, could be reached smoothly. The idea is simply to look at the left hand side of (7.3.1) and take the vectorial gauge fixing choice (7.3.6), which, again, is the most

general covariant one and is the same of linearized gravity. It is not obvious at all that it could work. First, what does it mean that “it works” ? Besides the possibility of getting both fractons and gravitons as a limit, we asked two minimal requirements: that propagators are defined in a generic gauge, and that the counting of the degrees of freedom of the theory coincides with the one found in [178], but without referring to a particular gauge choice. The result is that both requests have been achieved. We have now a covariant gauge fixed theory of fractons and linearized gravity, which has six degrees of freedom, or five, since for a particular combination of fractons and gravitons the theory is traceless. All the results are gauge independent, therefore the vector gauge fixing (7.3.6) seems to be a better choice than the standard scalar one (7.3.4).

7.4 FRACTONS WITH BOUNDARY

We now have a 4D covariant theory that describes fracton quasiparticles, whose gauge structure is peculiar but well defined. We are thus able to apply the QFT procedure of Chapter 2 to study the consequences of the introduction of a flat boundary, with the aim of investigating whether an induced 3D theory exists and, in that case, which is its physical meaning. Tightly related is the question of the existence of an algebraic structure on the boundary. The theory we are dealing with is not topological, and it was a common belief that only topological field theories show non-trivial boundary physics, until the study of the Maxwell case. Moreover, we have seen that when a boundary is introduced in a QFT, the gauge symmetry plays a fundamental role, since it is the breaking of gauge invariance caused by the presence of the boundary that gives rise to an algebraic structure on the boundary which “holographically” induces a lower-dimensional gauge theory. The fracton symmetry (7.2.5) is unusual, due to the presence of a double derivative, and considering a boundary on such a model *a priori* may have a non-trivial outcome. On the other hand it would not be the first case of a non-topological QFT exhibiting an induced theory on the boundary. In fact, as we mentioned, this also happens in the case of Maxwell theory in 3D [49] and 4D [1], and we know from Section 7.2 that fracton models share many similarities with the electromagnetic theory [4, 160]. Additionally, it has been shown that a fractonic θ -term, which is a pure boundary term when θ is constant, gives rise to a 3D Chern-Simons-like term and a generalized Witten effect [175], with important consequences in condensed matter systems [55, 184]. A non-covariant Chern-Simons-like term of that kind was also studied in [185], where the higher-spin formalism is associated to dipolar behaviours in the context of Hall systems. In order to find these answers, we proceed as follows

7.4.1	The model with boundary	111
7.4.2	The induced 3D theory	115
7.4.3	The bulk and the boundary : holographic contact	119
7.4.4	Physical interpretation of the 3D theory	122
7.4.5	Discrete symmetries : parity and time reversal	126
7.4.6	Summary and discussion	127

In particular in Section 7.4.1 the boundary is introduced in the invariant action (7.2.7) described in Section 7.2 together with the gauge fixing and the most general boundary term. From the total action the EoM and the most general BC are computed. Because of the presence of the boundary, the Ward identities of the theory are broken, and this allows to identify the boundary DoF, represented by two traceless symmetric rank-2 tensor fields. Moreover, the broken Ward identities give rise to an algebraic structure, which can be identified as a generalized KM algebra, which, in Section 7.4.2, we interpret as canonical commutators of a 3D action. In Section 7.4.3 the bulk/boundary correspondence is obtained by requiring that the EoM of the induced 3D action are compatible with the BC of the 4D bulk theory. This can be achieved by suitably tuning the parameters appearing in the BC and in the 3D action. To physically interpret the 3D theory we found, in Section 7.4.4 we study its EoM, which appear to be Gauss and Ampère-like laws for the boundary tensor fields, exactly as in the “ordinary” fracton theory. In particular our boundary theory can be identified with a traceless fracton model. In Section 7.4.5 we analyze the effect of taking into account discrete symmetries such as parity (\mathcal{P}) and time reversal (\mathcal{T}). Finally in Section 7.4.6 we discuss our results.

7.4.1 The model with boundary

The action

We introduce the boundary in the invariant action S_{inv} (7.2.7) by means of a Heaviside step function in the action :

$$S_{bulk} = \int d^4x \theta(x^3) \left\{ \frac{g_1}{6} F^{\mu\nu\rho} F_{\mu\nu\rho} + g_2 \left(\frac{1}{4} F_{\mu\nu}^\mu F_{\rho}^{\rho\nu} - \frac{1}{6} F^{\mu\nu\rho} F_{\mu\nu\rho} \right) \right\}. \quad (7.4.1)$$

Notice that in what follows we cannot just set $g_1 = 0$ and restrict our results to LG alone, because S_{LG} (7.2.9) is uniquely defined by the infinitesimal diffeomorphism transformation (7.2.6), and not by its subset (7.2.5). The transformations (7.2.6) and (7.2.5) differs in two aspects : the first, (7.2.6), depends on a vector gauge parameter, while (7.2.5) has a scalar gauge parameter, hence the former is more restrictive. Secondly, (7.2.6) and (7.2.5) depend on one and two derivatives respectively. As we have seen, this results in a mismatch in the mass dimensions. In fact, since from the action (7.4.1) we have $[A_{\mu\nu}] = 1$, due to the double derivatives in (7.2.5) it must be $[\Lambda] = -1$, which is an exotic dimension assignment for the scalar gauge parameter. Moreover, on the x^3 -boundary, the field $A_{\mu\nu}(x)$ and its ∂_3 -derivative must be treated as independent fields [1, 49, 145, 147], like we have seen for the Maxwell case of Chapter 6. Hence, on the boundary we define

$$\tilde{A}_{\mu\nu} \equiv \partial_3 A_{\mu\nu}|_{x^3=0}, \quad (7.4.2)$$

with $[\tilde{A}_{\mu\nu}] = 2$. We add to the invariant action S_{inv} (7.2.7) the gauge-fixing term

$$S_{gf} = \int d^4x \theta(x^3) b^\mu A_{\mu 3}, \quad (7.4.3)$$

which is of a vectorial type, as discussed in the previous Section 7.3, where $b^\mu(x)$ is a Nakanishi-Lautrup Lagrange multiplier [107, 108] implementing the axial gauge condition

$$A_{\mu 3} = 0. \quad (7.4.4)$$

As a consequence of the fact that the field and its ∂_3 -derivative on the boundary are independent quantities, together with the usual external field $J^{ab}(x)$ coupled to $A_{ab}(x)$, it is necessary to couple a source $\tilde{J}^{ab}(x)$ also to the ∂_3 -derivative of $A_{ab}(x)$ on the boundary. The external source term is then

$$S_J = \int d^4x \left[\theta(x^3) J^{ab} A_{ab} + \delta(x^3) \tilde{J}^{ab} \tilde{A}_{ab} \right]. \quad (7.4.5)$$

The presence of a boundary in a QFT naturally rises the question of which BC should be assigned to the quantum fields and/or their derivatives. A possible way is to impose them by hand, but one should worry about the dependence of the results on the particular choice. We have seen that this arbitrariness affecting QFTs with boundary has been elegantly solved by Symanzik in his pioneering paper [10], where a scalar QFT with boundary was considered. According to Symanzik's approach, the BC are not imposed by hand, but are determined by the theory itself. This is achieved by adding a boundary term to the action, as the most general one, satisfying the requests of locality, power counting and 3D Lorentz invariance. The BC are then determined from the EoM, modified by the boundary term

$$S_{bd} = \int d^4x \delta(x^3) \left[\xi_0 A_{ab} A^{ab} + \xi_1 \tilde{A}_{ab} A^{ab} + \xi_2 \epsilon^{abc} A_{ai} \partial_b A_c^i + \xi_3 A^2 + \xi_4 \tilde{A} A \right], \quad (7.4.6)$$

where, due to the gauge condition (7.4.4),

$$A \equiv \eta^{\mu\nu} A_{\mu\nu} = \eta^{ab} A_{ab} \quad ; \quad \tilde{A} \equiv \eta^{\mu\nu} \tilde{A}_{\mu\nu} = \eta^{ab} \tilde{A}_{ab}, \quad (7.4.7)$$

and ξ_i are constant parameters, whose mass dimensions are

$$[\xi_0] = [\xi_3] = 1 \quad ; \quad [\xi_1] = [\xi_2] = [\xi_4] = 0. \quad (7.4.8)$$

Notice that the general QFT requirements which constrain S_{bd} imply the presence of the Chern-Simons-like ξ_2 -term, which can be traced back to the covariant fractonic θ -term (7.2.108). In fact, this latter is

$$S_\theta = \int d^4x \theta(x^3) \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_{\nu\lambda} \partial_\rho A_\sigma^\lambda, \quad (7.4.9)$$

which, integrating by parts, reduces to

$$\int d^3X \epsilon^{abc} A_{a\lambda} \partial_b A_c^\lambda, \quad (7.4.10)$$

which, on the gauge condition (7.4.4), coincides with the ξ_2 -term in (7.4.6). The total action is then

$$S_{tot} = S_{bulk} + S_{gf} + S_J + S_{bd}. \quad (7.4.11)$$

Equations of motion and boundary conditions

The EoM for $A_{\alpha\beta}(x)$ and $\tilde{A}_{\alpha\beta}(x)$ are

$$\begin{aligned} \frac{\delta S_{tot}}{\delta A_{\alpha\beta}} = & \theta(x^3) \left\{ (g_1 - g_2) \partial_\mu F^{\alpha\beta\mu} + g_2 \left[\eta^{\alpha\beta} \partial_\mu F_\nu^{\nu\mu} - \frac{1}{2} \left(\partial^\alpha F_\mu^{\mu\beta} + \partial^\beta F_\mu^{\mu\alpha} \right) \right] + \right. \\ & \left. + \delta_a^\alpha \delta_b^\beta J^{ab} + \frac{1}{2} (b^\alpha \delta_3^\beta + b^\beta \delta_3^\alpha) \right\} + \\ & + \delta(x^3) \left\{ (g_1 - g_2) F^{\alpha\beta 3} + g_2 \left[\eta^{\alpha\beta} F_\mu^{\mu 3} - \frac{1}{2} \left(\eta^{\alpha 3} F_\mu^{\mu\beta} + \eta^{\beta 3} F_\mu^{\mu\alpha} \right) \right] + \right. \\ & \left. + \delta_a^\alpha \delta_b^\beta \left[2\xi_0 A^{ab} + \xi_1 \tilde{A}^{ab} + \xi_2 (\epsilon^{aij} \partial_i A_j^b + \epsilon^{bij} \partial_i A_j^a) + 2\xi_3 \eta^{ab} A + \xi_4 \eta^{ab} \tilde{A} \right] \right\} = 0, \end{aligned} \quad (7.4.12)$$

and

$$\begin{aligned} \frac{\delta S_{tot}}{\delta \partial_3 A_{\alpha\beta}} &= \theta(x^3) \left\{ (g_2 - g_1) F^{\alpha\beta 3} - g_2 \left[\eta^{\alpha\beta} F_\mu^{\mu 3} - \frac{1}{2} \left(\eta^{\alpha 3} F_\mu^{\mu\beta} + \eta^{\beta 3} F_\mu^{\mu\alpha} \right) \right] \right\} \\ &+ \delta(x^3) \delta_a^\alpha \delta_b^\beta \left\{ \tilde{J}^{ab} + \xi_1 A^{ab} + \xi_4 \eta^{ab} A \right\} = 0. \end{aligned} \quad (7.4.13)$$

The most general BC are obtained by applying $\lim_{\epsilon \rightarrow 0} \int_0^\epsilon dx^3$ to the EoM. From (7.4.12) we get

$$\begin{aligned} &\left\{ (g_1 - g_2) F^{\alpha\beta 3} + g_2 \left[\eta^{\alpha\beta} F_\mu^{\mu 3} - \frac{1}{2} \left(\eta^{\alpha 3} F_\mu^{\mu\beta} + \eta^{\beta 3} F_\mu^{\mu\alpha} \right) \right] + \right. \\ &\left. + \delta_a^\alpha \delta_b^\beta \left[2\xi_0 A^{ab} + \xi_1 \tilde{A}^{ab} + \xi_2 (\epsilon^{aij} \partial_i A_j^b + \epsilon^{bij} \partial_i A_j^a) + 2\xi_3 \eta^{ab} A + \xi_4 \eta^{ab} \tilde{A} \right] \right\}_{x^3=0} = 0. \end{aligned} \quad (7.4.14)$$

We observe that

- $\alpha = \beta = 3$ is trivially realized ;

- $\alpha = 3, \beta = b$:

$$g_2 \left(\partial^b A - \partial_a A^{ab} \right)_{x^3=0} = g_2 F_\mu^{\mu b} |_{x^3=0} = 0 ; \quad (7.4.15)$$

- $\alpha = a, \beta = b$:

$$\begin{aligned} &\left[2\xi_0 A^{ab} + 2 \left(g_2 - g_1 + \frac{\xi_1}{2} \right) \tilde{A}^{ab} + \xi_2 \left(\epsilon^{aij} \partial_i A_j^b + \epsilon^{bij} \partial_i A_j^a \right) + \right. \\ &\left. + 2\xi_3 \eta^{ab} A + 2 \left(\frac{\xi_4}{2} - g_2 \right) \eta^{ab} \tilde{A} \right]_{x^3=0} = 0. \end{aligned} \quad (7.4.16)$$

Going on-shell, *i.e.* at vanishing external sources $\tilde{J}(x) = 0$, taking the $\lim_{\epsilon \rightarrow 0} \int_0^\epsilon dx^3$ of the EoM (7.4.13) we get

$$\delta_a^\alpha \delta_b^\beta \left(\xi_1 A^{ab} + \xi_4 \eta^{ab} A \right)_{x^3=0} = 0, \quad (7.4.17)$$

and again we observe that

- $\alpha = 3, \beta$ free, is trivially realized ;

- $\alpha = a, \beta = b$:

$$\left(\xi_1 A^{ab} + \xi_4 \eta^{ab} A \right)_{x^3=0} = 0. \quad (7.4.18)$$

Ward identities and boundary degrees of freedom

The EoM (7.4.12) yields the following integrated Ward identity

$$\begin{aligned} 0 &= \int dx^3 \partial_a \partial_b \frac{\delta S_{tot}}{\delta A_{ab}} \\ &= \int dx^3 \theta(x^3) \left\{ (g_1 - g_2) \partial_a \partial_b \partial_3 F^{ab3} + g_2 \partial_a \partial^a \partial_3 F_\mu^{\mu 3} + \partial_a \partial_b J^{ab} \right\}, \end{aligned} \quad (7.4.19)$$

where we used the BC (7.4.16) and the cyclic property of $F_{\mu\nu\rho}(x)$ in Table 2. Integrating by parts we get

$$\int dx^3 \theta(x^3) \partial_i \partial_j J^{ij} = 2(g_2 - g_1) \partial_i \partial_j \tilde{A}^{ij} - 2g_2 \partial_i \partial^i \tilde{A} |_{x^3=0}. \quad (7.4.20)$$

Analogously, from the EoM (7.4.13) we find

$$\begin{aligned} 0 &= \int dx^3 \partial_a \partial_b \frac{\delta S_{tot}}{\delta \partial_3 A_{ab}} \\ &= - \int dx^3 \theta(x^3) \partial_a \partial_b \left[2(g_2 - g_1) \partial^3 A^{ab} - 2g_2 \eta^{ab} \partial^3 A \right] + \partial_a \partial_b \tilde{J}^{ab} \Big|_{x^3=0}, \end{aligned} \quad (7.4.21)$$

where we used the BC (7.4.18). Integrating by parts

$$\partial_i \partial_j \tilde{J}^{ij} \Big|_{x^3=0} = -2(g_2 - g_1) \partial_i \partial_j A^{ij} + 2g_2 \partial_i \partial^i A \Big|_{x^3=0}. \quad (7.4.22)$$

Notice that the second Ward identity (7.4.22), associated to the $\tilde{A}_{ab}(x)$ field on the boundary, is local and not integrated as the first (7.4.20). The two Ward identities (7.4.20) and (7.4.22) are analogous to those characterizing Maxwell theory with boundary both in 3D and 4D [1, 49], which we have seen in Chapter 6 in Eqs. (6.9) and (6.10). At vanishing external source $J^{ab}(x) = 0$, the Ward identity (7.4.20) gives

$$\partial_i \partial_j \left[(g_2 - g_1) \tilde{A}^{ij} - g_2 \eta^{ij} \tilde{A} \right] \Big|_{x^3=0} = 0. \quad (7.4.23)$$

In [5, 178] it has been shown that, when $g_1 = g_2$ in the invariant action S_{inv} (7.2.7), it is possible to redefine the components of $A_{\mu\nu}(x)$ in such a way that the theory has no kinetic term, hence it is not dynamical. Therefore in what follows we shall exclude the trivial case

$$g_1 = g_2. \quad (7.4.24)$$

For $g_1 \neq g_2$, and $g_{1,2} \neq 0$, (7.4.23) implies

$$\partial_i \partial_j \tilde{A}^{ij}(X) = 0 \quad (7.4.25)$$

$$\square \tilde{A}(X) = 0. \quad (7.4.26)$$

Eq.(7.4.25) is solved as follows [224, 225]

$$\partial_i \left(\partial_j \tilde{A}^{ij} \right) = 0 \quad \Rightarrow \quad \partial_j \tilde{A}^{ij} = \epsilon^{imn} \partial_m C_n, \quad (7.4.27)$$

where $C_n(X)$ is a generic 3D vector field. Eq.(7.4.27), in turn, gives

$$\partial_j \left(\tilde{A}^{ij} - \epsilon^{ijn} C_n \right) = 0 \quad \Rightarrow \quad \tilde{A}^{ij} - \epsilon^{ijn} C_n = 2\epsilon^{jab} \partial_a \tilde{a}_b^i, \quad (7.4.28)$$

where $\tilde{a}_{ij}(X)$ is a generic rank-2 tensor field. On the other hand, $\tilde{A}^{ij}(x)$ is symmetric, hence $C_n = 0$ and we have

$$\tilde{A}^{ij}(X) \equiv \epsilon^{iab} \partial_a \tilde{a}_b^j(X) + \epsilon^{jab} \partial_a \tilde{a}_b^i(X). \quad (7.4.29)$$

The tensor field $\tilde{a}_{ij}(X)$ represents the DoF on the boundary, with $[\tilde{a}_{ij}] = 1$. Moreover, since $\tilde{A}_{ij}(x) = \tilde{A}_{ji}(x)$ has six independent components, the boundary field $\tilde{a}_{ij}(X)$ must be symmetric as well

$$\tilde{a}_{ij} = \tilde{a}_{ji}, \quad (7.4.30)$$

in order that the boundary DoF does not exceed the number of components of its bulk ancestor $\tilde{A}_{ij}(x)$. The solution (7.4.29) is traceless

$$\tilde{A}(X) \Big|_{(7.4.29)} = 0, \quad (7.4.31)$$

so that the condition (7.4.26) is automatically satisfied. Analogously, from the local Ward identity (7.4.22) we have, at vanishing external source $\tilde{J}^{ab}(x) = 0$

$$\partial_i \partial_j [(g_2 - g_1) A^{ij} - g_2 \eta^{ij} A]_{x^3=0} = 0, \quad (7.4.32)$$

whose solution is

$$A^{ij}(X) \equiv \epsilon^{iab} \partial_a a_b^j(X) + \epsilon^{jab} \partial_a a_b^i(X), \quad (7.4.33)$$

where $a_{ij}(X) = a_{ji}(X)$ are DoF on the boundary with $[a_{ij}] = 0$. Let us now consider the two broken Ward identities (7.4.20) and (7.4.22) and make functional derivatives with respect to $J^{mn}(X')$ and $\tilde{J}^{mn}(X')$. Referring to Appendix B.3.1.a) for the details, we obtain the following equal time commutation relations

$$\left[\Delta \tilde{A}(X), A_{MN}(X') \right]_{x^0=x'^0} = +i \partial_M \partial_N \{ \delta(x^1 - x'^1) \delta(x^2 - x'^2) \} \quad (7.4.34)$$

$$\left[\Delta A(X), \tilde{A}_{MN}(X') \right]_{x^0=x'^0} = -i \partial_M \partial_N \{ \delta(x^1 - x'^1) \delta(x^2 - x'^2) \}, \quad (7.4.35)$$

where

$$\Delta \tilde{A} \equiv 2(g_1 - g_2) \left(\partial_j \tilde{A}^{0j} + \partial_A \tilde{A}^{0A} \right) + 2g_2 \partial^0 \tilde{A} \quad (7.4.36)$$

$$\Delta A \equiv 2(g_1 - g_2) \left(\partial_j A^{0j} + \partial_A A^{0A} \right) + 2g_2 \partial^0 A. \quad (7.4.37)$$

Importantly, the commutation relations (7.4.34) and (7.4.35) resemble the generalized U(1) KM algebra derived in [184] for a 3D non-chiral bosonic theory that lives on the boundary of a 4D dipolar fracton theory. Hence, the theory described by the action S_{inv} (7.2.7) has an algebraic structure on the boundary that is different from that of topological field theories [2, 3, 11–13, 77, 87, 88, 142] and Maxwell theory [1, 49, 226]. The reason for that lies in the structure of the fracton symmetry (7.2.5), characterized by two derivatives, which prevents the presence, at the right hand side of (7.4.34) and (7.4.35), of the central charge term $\partial\delta$, typical of usual KM algebras. This leads us to guess that a conserved current algebra might exist on the boundary of LG, whose defining symmetry (7.2.6) depends on one derivative only. This would be in agreement with the conjecture concerning the existence of a KM algebra in LG mentioned in [227]. We shall discuss this situation (LG with boundary) separately, in Chapter 8.

7.4.2 The induced 3D theory

Canonical variables

We look for the transformations of the boundary fields $a_{ij}(X)$, $\tilde{a}_{ij}(X)$ which preserve the solutions (7.4.29) and (7.4.33). The most general ones are

$$\delta a_{mn} = \eta_{mn} \phi + \partial_m \xi_n + \partial_n \xi_m + \partial_m \partial_n \lambda \quad ; \quad \delta \tilde{a}_{mn} = 0 \quad (7.4.38)$$

$$\tilde{\delta} \tilde{a}_{mn} = \eta_{mn} \tilde{\phi} + \partial_m \tilde{\xi}_n + \partial_n \tilde{\xi}_m + \partial_m \partial_n \tilde{\lambda} \quad ; \quad \tilde{\delta} a_{mn} = 0, \quad (7.4.39)$$

where $\lambda(X)$, $\tilde{\lambda}(X)$, $\phi(X)$, $\tilde{\phi}(X)$, $\xi(X)$ and $\tilde{\xi}(X)$ are generic local parameters. The solutions $A_{ij}(X)$ (7.4.29) and $\tilde{A}_{ij}(X)$ (7.4.33) remain unchanged, *i.e.* $\delta A_{ij} = \tilde{\delta} \tilde{A}_{ij} = 0$, if $\xi_m = \partial_m \xi^l$, $\tilde{\xi}_m = \partial_m \tilde{\xi}^l$, so that (7.4.38) and (7.4.39) reduce to

$$\delta a_{mn} = \eta_{mn} \phi + \partial_m \partial_n \lambda \quad ; \quad \delta \tilde{a}_{mn} = 0 \quad (7.4.40)$$

$$\tilde{\delta} \tilde{a}_{mn} = \eta_{mn} \tilde{\phi} + \partial_m \partial_n \tilde{\lambda} \quad ; \quad \tilde{\delta} a_{mn} = 0. \quad (7.4.41)$$

We decompose the boundary fields $a_{ij}(X)$ and $\tilde{a}_{ij}(X)$ in terms of their trace and traceless contributions, *i.e.*

$$a_{ij} = \alpha_{ij} + \frac{1}{3}\eta_{ij}a \quad (7.4.42)$$

$$\tilde{a}_{ij} = \tilde{\alpha}_{ij} + \frac{1}{3}\eta_{ij}\tilde{a}, \quad (7.4.43)$$

where $a \equiv \eta^{ij}a_{ij}$, $\tilde{a} \equiv \eta^{ij}\tilde{a}_{ij}$, and $\alpha_{ij}(X)$, $\tilde{\alpha}_{ij}(X)$ are symmetric traceless fields

$$\eta^{ij}\alpha_{ij} = \eta^{ij}\tilde{\alpha}_{ij} = 0, \quad (7.4.44)$$

which transform as

$$\delta\alpha_{mn} = \partial_m\partial_n\lambda - \frac{1}{3}\eta_{mn}\partial^2\lambda \quad ; \quad \delta\tilde{\alpha}_{mn} = 0 \quad (7.4.45)$$

$$\tilde{\delta}\tilde{\alpha}_{mn} = \partial_m\partial_n\tilde{\lambda} - \frac{1}{3}\eta_{mn}\partial^2\tilde{\lambda} \quad ; \quad \tilde{\delta}\alpha_{mn} = 0. \quad (7.4.46)$$

The solutions (7.4.29) and (7.4.33) depend only on the traceless components

$$\tilde{A}^{ij}(X) \equiv \epsilon^{iab}\partial_a\tilde{\alpha}_b{}^j(X) + \epsilon^{jab}\partial_a\tilde{\alpha}_b{}^i(X) \quad (7.4.47)$$

$$A^{ij}(X) \equiv \epsilon^{iab}\partial_a\alpha_b{}^j(X) + \epsilon^{jab}\partial_a\alpha_b{}^i(X),$$

and the trace contributions disappear. The DoF of the boundary theory are then described by the rank-2 traceless tensor fields $\alpha_{ij}(X)$ and $\tilde{\alpha}_{ij}(X)$. This is consistent with the fact that, as we showed, on the boundary the bulk fields $A_{ij}(x)$ and $\tilde{A}_{ij}(x)$ have only five components, exactly as the 3D boundary fields $\alpha_{ij}(X)$ and $\tilde{\alpha}_{ij}(X)$, which are symmetric and traceless. The solutions (7.4.47) highly simplify the definitions of $\Delta A(X)$ (7.4.37) and $\Delta\tilde{A}(X)$ (7.4.36) :

$$\Delta A|_{(7.4.33)} = 4(g_1 - g_2)\epsilon^{0MN}\partial_M\partial^A\alpha_{NA} \quad (7.4.48)$$

$$\Delta\tilde{A}|_{(7.4.29)} = 4(g_1 - g_2)\epsilon^{0MN}\partial_M\partial^A\tilde{\alpha}_{NA}, \quad (7.4.49)$$

where we observe that only spatial derivatives appear. Considering the following combinations of the commutators (7.4.34) and (7.4.35) and their traces

$$\left[\Delta\tilde{A}, A'_{\text{DF}} - \frac{1}{2}\eta_{\text{DF}}\eta^{\text{MN}}A'_{\text{MN}} \right] = \frac{i}{2}(\delta_{\text{D}}^{\text{M}}\delta_{\text{F}}^{\text{N}} + \delta_{\text{D}}^{\text{N}}\delta_{\text{F}}^{\text{M}} - \eta^{\text{MN}}\eta_{\text{DF}})\partial_{\text{M}}\partial_{\text{N}}\delta^{(2)}(X - X') \quad (7.4.50)$$

$$\left[\Delta A, \tilde{A}'_{\text{DF}} - \frac{1}{2}\eta_{\text{DF}}\eta^{\text{MN}}\tilde{A}'_{\text{MN}} \right] = -\frac{i}{2}(\delta_{\text{D}}^{\text{M}}\delta_{\text{F}}^{\text{N}} + \delta_{\text{D}}^{\text{N}}\delta_{\text{F}}^{\text{M}} - \eta^{\text{MN}}\eta_{\text{DF}})\partial_{\text{M}}\partial_{\text{N}}\delta^{(2)}(X - X'), \quad (7.4.51)$$

and using the solutions (7.4.47), we can identify the following two canonical commutation relations at the boundary (the details can be found in Appendix B.3.1.b))

$$[q_{\text{AB}}, p'^{\text{CD}}] = \frac{i}{2}(\delta_{\text{A}}^{\text{C}}\delta_{\text{B}}^{\text{D}} + \delta_{\text{A}}^{\text{D}}\delta_{\text{B}}^{\text{C}} - \eta^{\text{CD}}\eta_{\text{AB}})\delta^{(2)}(X - X') \quad (7.4.52)$$

$$[\tilde{q}_{\text{AB}}, \tilde{p}'^{\text{CD}}] = \frac{i}{2}(\delta_{\text{A}}^{\text{C}}\delta_{\text{B}}^{\text{D}} + \delta_{\text{A}}^{\text{D}}\delta_{\text{B}}^{\text{C}} - \eta^{\text{CD}}\eta_{\text{AB}})\delta^{(2)}(X - X'), \quad (7.4.53)$$

where

$$q_{\text{AB}} \equiv \alpha_{\text{AB}} - \frac{1}{2}\eta_{\text{AB}}\alpha_{\text{M}}^{\text{M}} \quad (7.4.54)$$

$$p^{\text{CD}} \equiv 2g_{12}\left(\tilde{f}^{\text{CD}0} - \frac{1}{2}\eta^{\text{CD}}\tilde{f}_a{}^{a0}\right) \quad (7.4.55)$$

$$\tilde{q}_{\text{AB}} \equiv \tilde{\alpha}_{\text{AB}} - \frac{1}{2}\eta_{\text{AB}}\tilde{\alpha}_{\text{M}}^{\text{M}} \quad (7.4.56)$$

$$\tilde{p}^{\text{CD}} \equiv -2g_{12}\left(f^{\text{CD}0} - \frac{1}{2}\eta^{\text{CD}}f_a{}^{a0}\right), \quad (7.4.57)$$

and

$$g_{12} \equiv 2(g_1 - g_2). \quad (7.4.58)$$

In analogy to $F_{\mu\nu\rho}(x)$ (7.2.19), $f_{abc}(X)$ and $\tilde{f}_{abc}(X)$ are defined as

$$\tilde{f}_{abc} \equiv \partial_a \tilde{\alpha}_{bc} + \partial_b \tilde{\alpha}_{ac} - 2\partial_c \tilde{\alpha}_{ab} \quad (7.4.59)$$

$$f_{abc} \equiv \partial_a \alpha_{bc} + \partial_b \alpha_{ac} - 2\partial_c \alpha_{ab}. \quad (7.4.60)$$

It is interesting to notice that a canonical commutator similar to those we found in (7.4.52) and (7.4.53), appears in [213], in the context of the traceless fracton models [160, 161]. The aim of [213] is to build a non-abelian model for fractons in 2+1 dimensions. To do so the abelian traceless theory needs to be defined first. As for any fracton theory [54, 55, 160, 161], the “electric field” $E_{\mathbb{J}}(x)$ is the conjugate momentum of $A_{\mathbb{J}}(x)$, from which the commutator holds

$$[E^{\mathbb{J}}, A_{\mathbb{M}\mathbb{N}}] = i(\delta_{\mathbb{M}}^{\mathbb{I}} \delta_{\mathbb{N}}^{\mathbb{I}} + \delta_{\mathbb{M}}^{\mathbb{I}} \delta_{\mathbb{N}}^{\mathbb{I}}). \quad (7.4.61)$$

After that, the scalar Gauss constraints is imposed, together with a tracelessness condition :

$$\partial_i \partial_j E^{\mathbb{J}} = \rho \quad ; \quad E_i^i = 0, \quad (7.4.62)$$

which imply three conservation equations :

$$\int \rho = const \quad ; \quad \int \vec{x} \rho = const \quad ; \quad \int x^2 \rho = const, \quad (7.4.63)$$

of charge, dipole and a component of the quadrupole, respectively. The main characteristic of fracton theories, *i.e.* the limited mobility, is here translated into the fact that single charges cannot move, while dipole bound states can only move along their transverse direction. The constraints (7.4.62) imply that the tensor field $A_{\mathbb{J}}(x)$ transforms exactly as (7.4.40) and (7.4.41), which is a remarkable check of our reasoning. However, while in our case it is natural to identify the DoF of the theory with the traceless fields $\alpha_{ij}(X)$ and $\tilde{\alpha}_{ij}(X)$, in [213] the tracelessness condition is imposed as a kind of gauge fixing, while for us it comes from the solutions (7.4.47). As a consequence, the definition (7.4.61) of the canonical commutator is no longer valid (since $A_i^i = E_i^i = 0$ would not commute), and the commutator for the traceless theory of fractons is defined as Dirac brackets [228], which turns out to be identical to ours (7.4.52) and (7.4.53).

The most general 3D action

The action of the 3D boundary theory is constructed as the most general local integrated functional of the traceless rank-2 symmetric tensor fields $\alpha_{ij}(X)$ and $\tilde{\alpha}_{ij}(X)$ compatible with

- power-counting $[\alpha] = 0$, $[\tilde{\alpha}] = 1$;
- symmetry $\delta S = \tilde{\delta} S = 0$, where δ and $\tilde{\delta}$ are defined in (7.4.45) and (7.4.46) ;
- canonical variables identified in (7.4.52) and (7.4.53) : $\frac{\partial \mathcal{L}_{kin}}{\partial \dot{q}} = p$.

In Appendix B.3.2 we show that the most general 3D action satisfying these three requests is

$$S_{3D} = \int d^3 X \left(-\frac{2}{3} g_{12} \varphi_{abc} \tilde{\varphi}^{abc} + \omega_5 \epsilon^{abc} \tilde{\alpha}_a^d \tilde{\varphi}_{dbc} \right), \quad (7.4.64)$$

where we defined

$$\begin{aligned} \varphi_{abc} = \varphi_{bac} &\equiv f_{abc} + \frac{1}{4} \left(-2\eta_{ab} f_{dc}^d + \eta_{bc} f_{da}^d + \eta_{ac} f_{db}^d \right) \\ &= -2\partial_c \alpha_{ab} + \partial_a \alpha_{bc} + \partial_b \alpha_{ac} - \eta_{ab} \partial^d \alpha_{dc} + \frac{1}{2} \eta_{bc} \partial^d \alpha_{da} + \frac{1}{2} \eta_{ac} \partial^d \alpha_{db}, \end{aligned} \quad (7.4.65)$$

and, analogously, $\tilde{\varphi}_{abc}(X)$ in terms of $\tilde{\alpha}_{ab}(X)$, with the following properties of cyclicity and “tracelessness”

$$\varphi_{abc} + \varphi_{cab} + \varphi_{bca} = 0 = \tilde{\varphi}_{abc} + \tilde{\varphi}_{cab} + \tilde{\varphi}_{bca} \quad (7.4.66)$$

$$\eta^{ab} \varphi_{abc} = \eta^{bc} \varphi_{abc} = \eta^{ab} \tilde{\varphi}_{abc} = \eta^{bc} \tilde{\varphi}_{abc} = 0. \quad (7.4.67)$$

The ω_5 term in (7.4.64) looks like a Chern-Simons term, and the similarity is even more evident if we explicit the $\tilde{\alpha}_{ab}(X)$ dependence, since $\epsilon^{abc} \tilde{\alpha}_a^d \tilde{\varphi}_{dbc} \propto \epsilon^{abc} \tilde{\alpha}_a^d \partial_b \tilde{\alpha}_{cd}$. Intriguingly, this Chern-Simons-like term resembles the massless limit of 3D self-dual massive gravity [229, 230]. This theory contains a 3D Fierz-Pauli mass term that breaks the gauge invariance [214, 215]. However, it has been shown [231] that it is dual to linearized topologically massive gravity [232], which is gauge invariant and contains the Chern-Simons-like term together with the linearized 3D Einstein-Hilbert action. These two equivalent theories were originally proposed as a viable way to describe a single propagating massive graviton in 3D in contrast with the standard Einstein-Hilbert theory, which is topological in 3D and does not support any propagating spin-2 particle. In our case, the linearized Einstein-Hilbert term is replaced by the tensorial Maxwell-like term such that our boundary action still supports a propagating “graviton”. Notice that a non-covariant version of our Chern-Simons-like term has been also considered in [185, 233] in the context of fractional quantum Hall effect and chiral fractons. However, these non-covariant field theories, that can be seen as dual one to each other, do not take into account any tensorial Maxwell-like terms. Notice also that if $\omega_5 = 0$ it is possible to decouple the fields. In fact, by defining

$$\alpha_{ab}^\pm \equiv \sqrt{M} \alpha_{ab} \pm \frac{1}{\sqrt{M}} \tilde{\alpha}_{ab} \quad \Rightarrow \quad \varphi_{abc}^\pm = \sqrt{M} \varphi_{abc} \pm \frac{1}{\sqrt{M}} \tilde{\varphi}_{abc}, \quad (7.4.68)$$

where M is a parameter with mass dimension $[M] = 1$ and $[\alpha^\pm] = \frac{1}{2}$, the 3D action (7.4.64) becomes

$$\begin{aligned} S_{3D} = \int d^3 X \left[\frac{g_{12}}{6} \left(\varphi_{abc}^- \varphi^{-abc} - \varphi_{abc}^+ \varphi^{+abc} \right) + \right. \\ \left. + \frac{M\omega_5}{4} \left(\alpha_a^{+d} \epsilon^{abc} \varphi_{dbc}^+ - 2\alpha_a^{+d} \epsilon^{abc} \varphi_{dbc}^- + \alpha_a^{-d} \epsilon^{abc} \varphi_{dbc}^- \right) \right], \end{aligned} \quad (7.4.69)$$

which for $\omega_5 = 0$ decouples:

$$S_{3D}[\alpha^+, \alpha^-, \omega_5 = 0] = S_{3D}^+[\alpha^+] + S_{3D}^-[\alpha^-] \quad (7.4.70)$$

with

$$S_{3D}^\pm \equiv \mp \frac{g_{12}}{6} \int d^3 X \varphi_{abc}^\pm \varphi^{\pm abc}. \quad (7.4.71)$$

As we shall show in Section 7.4.5, this second case keeps the \mathcal{T} -invariance of the boundary in agreement with the \mathcal{T} -symmetry of the bulk action (7.4.1).

7.4.3 The bulk and the boundary : holographic contact

Once the most general 3D action (7.4.64) has been derived, we have to establish the “holographic contact” between this induced 3D theory and the 4D theory S_{tot} (7.4.11). This is accomplished by requiring that the EoM of the 3D theory coincide with the BC (7.4.15), (7.4.16), (7.4.18) of the 4D theory. To do so we have at our disposal the ξ_i parameters appearing in S_{bd} (7.4.6) and ω_5 in S_{3D} (7.4.64). The EoM of S_{3D} are

$$\frac{\delta S_{3D}}{\delta \alpha_{mn}} = -2g_{12} \partial_a \tilde{\varphi}^{mna} = 0, \quad (7.4.72)$$

where we used the cyclic property (7.4.66), and

$$\frac{\delta S_{3D}}{\delta \tilde{\alpha}_{mn}} = -2g_{12} \partial_a \varphi^{mna} + \omega_5 \left(\epsilon^{mab} \tilde{\varphi}_{ab}^n + \epsilon^{nab} \tilde{\varphi}_{ab}^m \right) = 0. \quad (7.4.73)$$

We now consider the BC of the bulk theory (7.4.15), (7.4.16) and (7.4.18), which we write in terms of the solutions (7.4.47) and of the definitions of $\varphi_{abc}(X)$, $\tilde{\varphi}_{abc}(X)$ (7.4.65)

$$\frac{1}{3} \partial^a \left(\epsilon_{aij} \varphi_b^{ij} + \epsilon_{bij} \varphi_a^{ij} \right) = 0 \quad (7.4.74)$$

$$\frac{2}{3} \xi_0 \left(\epsilon^{aij} \varphi_{ij}^b + \epsilon^{bij} \varphi_{ij}^a \right) + \frac{1}{3} (\xi_1 - g_{12}) \left(\epsilon^{aij} \tilde{\varphi}_{ij}^b + \epsilon^{bij} \tilde{\varphi}_{ij}^a \right) - 2\xi_2 \partial_c \varphi^{abc} = 0 \quad (7.4.75)$$

$$\frac{\xi_1}{3} \left(\epsilon^{aij} \varphi_{ij}^b + \epsilon^{bij} \varphi_{ij}^a \right) = 0. \quad (7.4.76)$$

The contact is governed by two coefficients : ξ_1 , which appears in S_{bd} (7.4.6), and ω_5 in the action S_{3D} (7.4.64). The first - ξ_1 - is relevant because it determines the existence of the BC (7.4.76), the second - ω_5 - decouples the EoM of the boundary fields $\alpha_{ab}(X)$, $\tilde{\alpha}_{ab}(X)$, *i.e.* eliminates the Chern-Simons-like term from the action (7.4.64). Additionally, we remark that $\tilde{\alpha}_{ij}(X)$ appears only in the BC (7.4.75), coupled to $(\xi_1 - g_{12})$, which should not vanish, otherwise no contact is possible. To summarize, the constraints on the coefficients, up to now, are

$$g_1 \neq 0 \quad ; \quad g_{12} \neq 0 \quad ; \quad \xi_1 \neq g_{12}. \quad (7.4.77)$$

Therefore, depending on ξ_1 and ω_5 , we distinguish the following cases :

- $\xi_1 \neq 0$, $\omega_5 \neq 0$: using (7.4.76) in (7.4.75) (or setting $\xi_0 = 0$), we have

$$\frac{1}{3} (\xi_1 - g_{12}) \left(\epsilon^{aij} \tilde{\varphi}_{ij}^b + \epsilon^{bij} \tilde{\varphi}_{ij}^a \right) - 2\xi_2 \partial_c \varphi^{abc} = 0, \quad (7.4.78)$$

which coincides with the 3D EoM (7.4.73)

$$-2g_{12} \partial_a \varphi^{mna} + \omega_5 \left(\epsilon^{mab} \tilde{\varphi}_{ab}^n + \epsilon^{nab} \tilde{\varphi}_{ab}^m \right) = 0 \quad (7.4.79)$$

if

$$\omega_5 = \frac{1}{3} (\xi_1 - g_{12}) \neq 0 \quad ; \quad \xi_2 = g_{12} \neq 0 \quad ; \quad \xi_1 \neq 0. \quad (7.4.80)$$

Notice that setting $\omega_5 = 0$ would imply $\xi_1 = g_{12}$, which we excluded in (7.4.77). Up to a numerical coefficient, we now consider the following symmetric combination of the curl of the BC (7.4.76)

$$\begin{aligned} 0 &= \epsilon_{mac} \partial^c (7.4.76)_b^a + \epsilon_{bac} \partial^c (7.4.76)_m^a \\ &= 6 \partial^i \varphi_{bmi} , \end{aligned} \quad (7.4.81)$$

where we used the properties of tracelessness (7.4.67) and cyclicity (7.4.66) of $\varphi_{abc}(X)$. We then use this result in the BC (7.4.78), which becomes

$$\frac{1}{3} (\xi_1 - g_{12}) \left(\epsilon^{aij} \tilde{\varphi}_{ij}^b + \epsilon^{bij} \tilde{\varphi}_{ij}^a \right) = 0 , \quad (7.4.82)$$

of which we compute again the curl

$$0 = \epsilon_{mac} \partial^c (7.4.82)_b^a + \epsilon_{bac} \partial^c (7.4.82)_m^a = 2 (\xi_1 - g_{12}) \partial^i \tilde{\varphi}_{bmi} , \quad (7.4.83)$$

which finally coincides with the 3D EoM (7.4.72)

$$-2g_{12} \partial_a \tilde{\varphi}^{mna} = 0 . \quad (7.4.84)$$

Notice that this second contact is obtained without the need of any additional constraint on the parameters, we just need (7.4.80). Taking into account (7.4.80), the 3D action (7.4.64) becomes

$$S_{3D} = \frac{1}{3} \int d^3 X \left[-2g_{12} \varphi_{abc} \tilde{\varphi}^{abc} + (\xi_1 - g_{12}) \tilde{\alpha}_a^d \epsilon^{abc} \tilde{\varphi}_{dbc} \right] , \quad (7.4.85)$$

while the boundary term (7.4.6) now is

$$S_{bd} = \int d^4 x \delta(x^3) \left[\xi_0 A_{ab} A^{ab} + \xi_1 \tilde{A}_{ab} A^{ab} + g_{12} \epsilon^{abc} A_{ai} \partial_b A_c^i + \xi_3 (A_a^a)^2 + \xi_4 \tilde{A}_a^a A_b^b \right] , \quad (7.4.86)$$

where the coefficients ξ_0 , ξ_3 and ξ_4 are free and can for instance be set to zero, while $\xi_1 \neq \{0, g_{12}\}$.

- $\xi_1 \neq 0$, $\omega_5 = 0$: the EoM of the 3D theory (7.4.72) and (7.4.73) are

$$\partial_a \varphi^{mna} = 0 \quad ; \quad \partial_a \tilde{\varphi}^{mna} = 0 , \quad (7.4.87)$$

while, ignoring the first BC (7.4.74), which is automatically solved by the third one (7.4.76), and using (7.4.76) in (7.4.75), the remaining BC are

$$\frac{1}{3} (\xi_1 - g_{12}) \left(\epsilon^{aij} \tilde{\varphi}_{ij}^b + \epsilon^{bij} \tilde{\varphi}_{ij}^a \right) - 2\xi_2 \partial_c \varphi^{abc} = 0 \quad (7.4.88)$$

$$\frac{\xi_1}{3} \left(\epsilon^{aij} \varphi_{ij}^b + \epsilon^{bij} \varphi_{ij}^a \right) = 0 . \quad (7.4.89)$$

As in (7.4.81), we can again compute

$$0 = \epsilon_{mac} \partial^c (7.4.89)_b^a + \epsilon_{bac} \partial^c (7.4.89)_m^a = 2\xi_1 \partial^i \varphi_{bmi} \quad (7.4.90)$$

which coincides with the first EoM of (7.4.87). If we use this result (7.4.90) in (7.4.88) (analogous to setting $\xi_2 = 0$) and consider the same combination as

(7.4.90), we obtain the second EoM of (7.4.87) and thus we get the second matching between bulk and boundary. In that case the 3D action (7.4.64) is

$$S_{3D} = -\frac{2g_{12}}{3} \int d^3X \varphi_{abc} \tilde{\varphi}^{abc}, \quad (7.4.91)$$

while the boundary term S_{bd} (7.4.6) becomes

$$S_{bd} = \int d^4x \delta(x^3) \left[\xi_0 A_{ab} A^{ab} + \xi_1 \tilde{A}_{ab} A^{ab} + \xi_2 \epsilon^{abc} A_{ai} \partial_b A_c^i + \xi_3 (A_a^a)^2 + \xi_4 \tilde{A}_a^a A_b^b \right], \quad (7.4.92)$$

where the coefficients ξ_0 , ξ_2 , ξ_3 and ξ_4 are free, *i.e.* do not contribute to the contact between the bulk and the boundary and can be set to zero without loss of generality, provided that $\xi_1 \neq \{0, g_{12}\}$. Therefore a second holographic contact is possible if

$$\omega_5 = 0 \quad ; \quad \xi_1 \neq \{0, g_{12}\}. \quad (7.4.93)$$

This second result has a relevant consequence : $\omega_5 = 0$ allows to decouple the action, as seen in (7.4.70).

We observe that the holographic contacts obtained in (7.4.80) and (7.4.93) affect the boundary action S_{bd} (7.4.6) in different ways, in particular in the first case (7.4.80) the number of free parameters from five reduces to three, while in the second case (7.4.93) it goes to four. Finally, if $\xi_1 = 0$, no complete matching between BC and 3D EoM is possible, in fact setting $\xi_1 = 0$, one of the BC (7.4.76) disappears. We are left with

$$\frac{1}{3} \partial^a \left(\epsilon_{aij} \varphi_b^{ij} + \epsilon_{bij} \varphi_a^{ij} \right) = 0 \quad (7.4.94)$$

$$\frac{2}{3} \xi_0 \left(\epsilon^{aij} \varphi_{ij}^b + \epsilon^{bij} \varphi_{ij}^a \right) - \frac{g_{12}}{3} \left(\epsilon^{aij} \tilde{\varphi}_{ij}^b + \epsilon^{bij} \tilde{\varphi}_{ij}^a \right) - 2\xi_2 \partial_c \varphi^{abc} = 0. \quad (7.4.95)$$

For what concerns the parameter ω_5 in (7.4.64), two cases are possible

- $\omega_5 \neq 0$: the BC (7.4.95) coincides with the 3D EoM (7.4.73)

$$-2g_{12} \partial_a \varphi^{mna} + \omega_5 \left(\epsilon^{mab} \tilde{\varphi}_{ab}^n + \epsilon^{nab} \tilde{\varphi}_{ab}^m \right) = 0 \quad (7.4.96)$$

if

$$\omega_5 = -\frac{g_{12}}{3} \neq 0 \quad ; \quad \xi_2 = g_{12} \neq 0 \quad ; \quad \xi_0 = \xi_1 = 0, \quad (7.4.97)$$

however it is not possible to establish a link with the other EoM (7.4.72).

- $\omega_5 = 0$: the EoM of the 3D boundary theory are given by (7.4.87). A matching is possible with the BC (7.4.95) only if $\xi_2 = 0$, and if we compute

$$0 = \epsilon_{mac} \partial^c (7.4.95)_b^a + \epsilon_{bac} \partial^c (7.4.95)_m^a = 4\xi_0 \partial^i \varphi_{bmi} - 2g_{12} \partial^i \tilde{\varphi}_{bmi}, \quad (7.4.98)$$

which coincides with a combination of the two EoM of the boundary. We see that also in this case a complete holographic contact between 3D EoM and bulk BC, is not possible.

This enforces the fact that the ξ_1 -term in S_{bd} (7.4.6) plays a key role in the holographic contact. To summarize

	BC-EoM matching
$\xi_1 \neq 0, \omega_5 \neq 0$	$\omega_5 = \frac{1}{3}(\xi_1 - g_{12}) ; \xi_2 = g_{12}$
$\xi_1 \neq 0, \omega_5 = 0$	$\omega_5 = 0$
$\xi_1 = 0$	no contact

Table 4: Holographic contacts.

7.4.4 Physical interpretation of the 3D theory

To understand the physical content of the 3D theory described by the action S_{3D} (7.4.64), we study its EoM. The first EoM (7.4.72) for $m = n = 0$ gives

$$0 = \partial_A \tilde{\varphi}^{00A} = -2\partial_A \partial^A \tilde{\alpha}^{00} + 2\partial_A \partial^0 \tilde{\alpha}^{A0} + \partial_A \partial_b \tilde{\alpha}^{Ab}. \quad (7.4.99)$$

Taking the ∂_N -derivative of (7.4.72) for $m = 0, n = N$, we get

$$0 = \partial_N \partial_a \tilde{\varphi}^{0Na} = -\frac{1}{4g_{12}} \partial_A \partial_N p^{AN}, \quad (7.4.100)$$

where we used (7.4.99), the cyclicity property (7.4.66) and the definition of conjugate momentum in terms of $\tilde{\varphi}_{abc}(X)$ (7.4.65)

$$p^{MN} = 2g_{12} \tilde{\varphi}^{MN0}. \quad (7.4.101)$$

We see that (7.4.100) is a Gauss-like equation, analogous to the one related to the traceless scalar charge model of fractons in vacuum [160, 161], which is at the base of the limited mobility property. We therefore realize that the induced 3D theory shows fractonic properties. To analyze the second EoM (7.4.73), we first compute the conjugate momentum of $\tilde{\alpha}_{ab}(X)$:

$$\tilde{p}^{MN} = \frac{\partial \mathcal{L}_{3D}}{\partial \dot{\tilde{\alpha}}_{MN}} = 2g_{12} \varphi^{MN0} - \frac{3}{2} \omega_5 (\epsilon^{0AM} \tilde{\alpha}_A^N + \epsilon^{0AN} \tilde{\alpha}_A^M). \quad (7.4.102)$$

The EoM for $\tilde{\alpha}_{ab}(X)$ (7.4.73) at $m = n = 0$ is

$$\partial_A \varphi^{00A} = \frac{3\omega_5}{g_{12}} \epsilon^{0AB} \partial_A \tilde{\alpha}_B^0, \quad (7.4.103)$$

and, as in the previous case, taking the ∂_N -derivative of (7.4.73) at $m = 0, n = N$ we have

$$\begin{aligned} 0 &= -2g_{12} \partial_a \partial_N \varphi^{0Na} + 3\omega_5 \partial_N (\epsilon^{0AB} \partial_A \tilde{\alpha}_B^N + \epsilon^{Nab} \partial_a \tilde{\alpha}_b^0) \\ &= \frac{1}{2} \partial_M \partial_N \tilde{p}^{MN} - \frac{3}{2} \omega_5 \epsilon^{0AM} \partial_M \partial_N \tilde{\alpha}_A^N, \end{aligned} \quad (7.4.104)$$

where we used the cyclic property of $\varphi_{abc}(X)$ (7.4.66) and (7.4.103). Here again we find a Gauss-like equation for the traceless scalar charge theory of fractons [160, 161], but with a matter contribution at the right hand side

$$\partial_M \partial_N \tilde{p}^{MN} = \tilde{\rho}_5 = \omega_5 \tilde{\rho}, \quad (7.4.105)$$

where

$$\tilde{\rho} \equiv 3\epsilon^{0AM}\partial_M\partial_N\tilde{\alpha}_A^N \quad (7.4.106)$$

plays the role of charge. This gives an interesting interpretation of the Chern-Simons-like term in the induced 3D action (7.4.64) as “internal” matter. Notice that this term coincides with the charge identified by Pretko in [175], where a non-covariant Chern-Simons-like term is studied. In that case the Chern-Simons term comes from a non-covariant fractonic θ -term in the bulk, and it is written in terms of a spatial traceless tensor. The charge $\tilde{\rho}(X)$ (7.4.106) comes from a constraint generated by a Lagrange multiplier that is inherited by the Chern-Simons action from the definition of the θ -term. We observe that this $\tilde{\rho}(X)$ charge implies, by definition, a dipole conservation. The 3D theory (7.4.64) depends on two fields $\alpha_{ij}(X)$ and $\tilde{\alpha}_{ij}(X)$, hence the conjugate momenta are two as well (7.4.101) and (7.4.102), which in fracton models play the role of “electric” fields :

$$E^{MN} \equiv p^{MN} = 2g_{12}\tilde{\varphi}^{MN0} \quad (7.4.107)$$

$$\tilde{E}^{MN} \equiv \tilde{p}^{MN} = 2g_{12}\varphi^{MN0} - \frac{3}{2}\omega_5 (\epsilon^{0AM}\tilde{\alpha}_A^N + \epsilon^{0AN}\tilde{\alpha}_A^M) , \quad (7.4.108)$$

which satisfy the Gauss equations (7.4.100) and (7.4.105), which we write

$$\partial_M\partial_N E^{MN} = 0 \quad (7.4.109)$$

$$\partial_M\partial_N \tilde{E}^{MN} = \tilde{\rho}_5 . \quad (7.4.110)$$

Concerning the corresponding “magnetic” fields for this theory, inspired by the ordinary 4D electromagnetism, where $B_i \propto \epsilon_{0ijk}F^{jk}$, it is natural to define

$$B^M \equiv g\epsilon_{0AB}\tilde{\varphi}^{MAB} \quad ; \quad B_M \equiv -g\epsilon^{0AB}\tilde{\varphi}_{MAB} \quad (7.4.111)$$

$$\tilde{B}^M \equiv \tilde{g}\epsilon_{0AB}\varphi^{MAB} \quad ; \quad \tilde{B}_M \equiv -\tilde{g}\epsilon^{0AB}\varphi_{MAB} . \quad (7.4.112)$$

Notice that, while in 4D fracton theories both electric and magnetic fields are rank-2 tensors [4], in our 3D case, the electric field is still a tensor, while the magnetic field is a vector. Moreover, the definitions (7.4.111) and (7.4.112) are consistent with the fact that in ordinary 3D electromagnetism the electric field $\vec{E}(x)$ is a vector, while the magnetic field is a pseudo-scalar $B = \epsilon^{0ij}F_{ij}$ [234]. As we shall see, our guess (7.4.111) and (7.4.112) will be confirmed by a consistent physical interpretation of a fractonic “magnetic-like” behaviour. In terms of $\alpha_{ab}(X)$, $\tilde{\alpha}_{ab}(X)$ the magnetic fields read

$$B^M = -3g\epsilon_{0AB} \left(\partial^B\tilde{\alpha}^{MA} + \frac{1}{2}\eta^{MA}\partial_d\tilde{\alpha}^{Bd} \right) \quad ; \quad \tilde{B}^M = -3\tilde{g}\epsilon_{0AB} \left(\partial^B\alpha^{MA} + \frac{1}{2}\eta^{MA}\partial_d\alpha^{Bd} \right) , \quad (7.4.113)$$

which imply

$$\tilde{\varphi}^{ABC} = -\frac{1}{3g} (\epsilon^{0AC}B^B + \epsilon^{0BC}B^A) \quad (7.4.114)$$

$$\varphi^{ABC} = -\frac{1}{3\tilde{g}} (\epsilon^{0AC}\tilde{B}^B + \epsilon^{0BC}\tilde{B}^A) . \quad (7.4.115)$$

Due to tracelessness property (7.4.67), we get

$$\varphi_A^{AB} = \varphi^{00B} = -\frac{2}{3\tilde{g}}\epsilon^{0AB}\tilde{B}_A \quad ; \quad \tilde{\varphi}_A^{AB} = \tilde{\varphi}^{00B} = -\frac{2}{3g}\epsilon^{0AB}B_A . \quad (7.4.116)$$

Notice that

$$\partial_{\mathbf{M}} B^{\mathbf{M}} = -\frac{3}{2} g \epsilon_{0\mathbf{AB}} \partial^{\mathbf{A}} (\partial_0 \tilde{\alpha}^{0\mathbf{B}} - \partial_{\mathbf{C}} \tilde{\alpha}^{\mathbf{BC}}) \neq 0 \quad (7.4.117)$$

$$\partial_{\mathbf{M}} \tilde{B}^{\mathbf{M}} = -\frac{3}{2} \tilde{g} \epsilon_{0\mathbf{AB}} \partial^{\mathbf{A}} (\partial_0 \alpha^{0\mathbf{B}} - \partial_{\mathbf{C}} \alpha^{\mathbf{BC}}) \neq 0, \quad (7.4.118)$$

which would suggest the presence, in the 3D theory (7.4.64), of a fractonic “magnetic”-like vortex. Consistently with the fact of having non-vanishing divergences of the magnetic vector fields, we find also a broken Bianchi identity, which also suggests the presence of a kind of magnetic fracton vortex. This would imply that a part of our fracton fields give rise to 2D fracton vortex defects that represent a lower-dimensional version of the 3D fracton magnetic monopole proposed in [175]. In fact, we have

$$\epsilon^{mbc} \partial_m \varphi_{abc} = -\frac{3}{2} \epsilon_{mac} \partial^m \partial_d \alpha^{cd} \neq 0 \quad ; \quad \epsilon^{mbc} \partial_m \tilde{\varphi}_{abc} = -\frac{3}{2} \epsilon_{mac} \partial^m \partial_d \tilde{\alpha}^{cd} \neq 0, \quad (7.4.119)$$

which for $a = 0$ give the non-vanishing divergences (7.4.117) and (7.4.118). Setting instead $a = \mathbf{A}$ we find

$$\frac{1}{g} \partial_0 B_{\mathbf{A}} + \frac{1}{2g_{12}} \epsilon^{0\mathbf{MB}} \partial_{\mathbf{M}} E_{\mathbf{AB}} = \frac{3}{2} \epsilon_{m\mathbf{Ab}} \partial^m \partial_d \tilde{\alpha}^{bd} + \epsilon^{\mathbf{MOB}} \partial_{\mathbf{M}} \tilde{\varphi}_{0\mathbf{AB}} \neq 0 \quad (7.4.120)$$

$$\frac{1}{\tilde{g}} \partial_0 \tilde{B}_{\mathbf{A}} + \frac{1}{2g_{12}} \epsilon^{0\mathbf{MB}} \partial_{\mathbf{M}} \tilde{E}_{\mathbf{AB}} = \frac{9}{4} \frac{\omega_5}{g_{12}} \partial^{\mathbf{M}} \tilde{\alpha}_{\mathbf{MA}} + \frac{3}{2} \epsilon_{m\mathbf{Ab}} \partial^m \partial_d \alpha^{bd} + \epsilon^{\mathbf{MOB}} \partial_{\mathbf{M}} \varphi_{0\mathbf{AB}} \neq 0 \quad (7.4.121)$$

which have non-vanishing right hand sides. Nonetheless, we have two scalar identities for $\varphi_{abc}(X)$ and $\tilde{\varphi}_{abc}(X)$:

$$\epsilon^{mbc} \partial_m \partial^a \varphi_{abc} = -\frac{3}{2} \epsilon_{mac} \partial^m \partial^a \partial_d \alpha^{cd} = 0 \quad (7.4.122)$$

$$\epsilon^{mbc} \partial_m \partial^a \tilde{\varphi}_{abc} = -\frac{3}{2} \epsilon_{mac} \partial^m \partial^a \partial_d \tilde{\alpha}^{cd} = 0. \quad (7.4.123)$$

Going back to the 3D EoM, we consider (7.4.72) at $m = \mathbf{M}$, $n = \mathbf{N}$

$$\partial_0 \tilde{\varphi}^{\mathbf{MNO}} + \partial_{\mathbf{A}} \tilde{\varphi}^{\mathbf{MNA}} = \frac{1}{2g_{12}} \partial_t E^{\mathbf{MN}} - \frac{1}{3g} \partial_{\mathbf{A}} (\epsilon^{0\mathbf{MA}} B^{\mathbf{N}} + \epsilon^{0\mathbf{NA}} B^{\mathbf{M}}), \quad (7.4.124)$$

where we used the definitions (7.4.107), (7.4.114). We thus have

$$\partial_t E^{\mathbf{MN}} - \frac{2g_{12}}{3g} \partial_{\mathbf{A}} (\epsilon^{0\mathbf{MA}} B^{\mathbf{N}} + \epsilon^{0\mathbf{NA}} B^{\mathbf{M}}) = 0, \quad (7.4.125)$$

which, remarkably, coincides with the traceless analog of the Ampère equation of 3D fractons identified in Eq.(16) of [148], where our (7.4.125) is obtained as an EoM from a 3D Maxwell-like Hamiltonian defined *ad hoc*. Differently from our case, the 3D fracton theory in [148] is not traceless and, in particular, $E_{\mathbf{A}}^{\mathbf{A}} \neq 0$. The aim of [148] is to study the so called fracton-elasticity duality and, more specifically, the analog of our (7.4.125) is used to investigate the effect of creation of defects as consequence of longitudinal motion of dipoles, which in the traceless fracton theory is not present, since dipoles only move along their transverse direction. We keep considering the EoM (7.4.73) at $m = \mathbf{M}$, $n = \mathbf{N}$:

$$\begin{aligned} 0 &= -2g_{12} \partial_a \varphi^{\mathbf{MNa}} + \omega_5 \left(\epsilon^{\mathbf{Mab}} \tilde{\varphi}_{ab}^{\mathbf{N}} + \epsilon^{\mathbf{Nab}} \tilde{\varphi}_{ab}^{\mathbf{M}} \right) \\ &= -\partial_0 \tilde{E}^{\mathbf{MN}} + \frac{3}{2} \omega_5 \partial_0 (\epsilon^{0\mathbf{BM}} \tilde{\alpha}_{\mathbf{B}}^{\mathbf{N}} + \epsilon^{0\mathbf{BN}} \tilde{\alpha}_{\mathbf{B}}^{\mathbf{M}}) + \frac{2g_{12}}{3\tilde{g}} \partial_{\mathbf{A}} (\epsilon^{0\mathbf{MA}} \tilde{B}^{\mathbf{N}} + \epsilon^{0\mathbf{NA}} \tilde{B}^{\mathbf{M}}) + \\ &\quad + 3\omega_5 \partial_{\mathbf{B}} (\epsilon^{0\mathbf{MB}} \tilde{\alpha}_0^{\mathbf{N}} + \epsilon^{0\mathbf{NB}} \tilde{\alpha}_0^{\mathbf{M}}), \end{aligned} \quad (7.4.126)$$

where we used the definitions (7.4.108), (7.4.115). We have

$$\partial_t \tilde{E}^{MN} - \frac{2g_{12}}{3\tilde{g}} \partial_A (\epsilon^{0MA} \tilde{B}^N + \epsilon^{0NA} \tilde{B}^M) = \tilde{\mathcal{J}}_5^{MN} \equiv \omega_5 \tilde{\mathcal{J}}^{MN}, \quad (7.4.127)$$

which is analogous to the Ampère equation, in presence of a tensorial current, again related to the Chern-Simons-like term in (7.4.64), which behaves as a matter term

$$\tilde{\mathcal{J}}^{MN} \equiv 3 \left[\frac{1}{2} \partial_0 (\epsilon^{0BM} \tilde{\alpha}_B^N + \epsilon^{0BN} \tilde{\alpha}_B^M) + \partial_B (\epsilon^{0MB} \tilde{\alpha}_0^N + \epsilon^{0NB} \tilde{\alpha}_0^M) \right]. \quad (7.4.128)$$

By computing $\partial_M \partial_N$ of (7.4.127), we also get

$$\partial_t \tilde{\rho}_5 = \partial_M \partial_N \tilde{\mathcal{J}}_5^{MN}, \quad (7.4.129)$$

where we used the Gauss-like equation (7.4.110). Eq.(7.4.129) represents a continuity equation typical of scalar fracton theories [160, 161] if $\omega_5 \neq 0$

$$\partial_t \tilde{\rho} - \partial_M \partial_N \tilde{\mathcal{J}}^{MN} = 0. \quad (7.4.130)$$

In $\tilde{\mathcal{J}}^{MN}(X)$ (7.4.128), the contribution associated to the time derivative coincides with the one defined by Pretko (Eq.(118) of [175]) as “generalized Hall response”. As in our case, it is derived from a Chern-Simons-like term seen as a boundary contribution originated by a fractonic θ -term in the bulk. In particular, it comes from the dynamical part of the action. From the action S_{3D} (7.4.64) we can identify both the current and the Ampère-like equation (7.4.127), to which the current (7.4.128) contributes. Moreover, as already mentioned for (7.4.125), this second equation (7.4.127) is compatible with the traceful version identified in [148] in the context of an analysis of 3D fracton-elasticity duality. Since $E_M^M = \tilde{E}_M^M = 0$, by computing the trace of the Ampère-like equations (7.4.125) and (7.4.127), we find

$$\epsilon^{0MN} \partial_M B_N = 0 \quad (7.4.131)$$

$$\epsilon^{0MN} \partial_M \tilde{B}_N = -\frac{3}{4} \frac{\tilde{g}}{g_{12}} \tilde{\mathcal{J}}_5^M = -\frac{9}{2} \frac{g\omega_5}{g_{12}} \epsilon^{0MN} \partial_M \tilde{\alpha}_{0N}, \quad (7.4.132)$$

which consistently coincide with the EoM for $m = n = 0$ (7.4.99), (7.4.103) previously found, *i.e.*

$$\partial_A \tilde{\varphi}^{00A} = 0 \quad ; \quad \partial_A \varphi^{00A} = \frac{3\omega_5}{g_{12}} \epsilon^{0AB} \partial_A \tilde{\alpha}_B^0, \quad (7.4.133)$$

due to (7.4.116). The EoM of the 3D boundary theory may be interpreted as a traceless tensorial extension of the standard 3D Maxwell equations [234], as summarized in Table 5

		Maxwell	Boundary of LG and fractons
fields	electric, magnetic	\vec{E}, B	$E^{AB}, B^A; \tilde{E}^{AB}, \tilde{B}^A$
Gauss	in vacuum	$\vec{\nabla} \cdot \vec{E} = 0$	$\partial_A \partial_B E^{AB} = 0$
	with matter	$\vec{\nabla} \cdot \vec{E} = \rho$	$\partial_A \partial_B \tilde{E}^{AB} = \tilde{\rho}_5$
Ampère	in vacuum	$\partial_t \vec{E} - \vec{\nabla}_\perp B = 0$	$\partial_t E^{MN} - \frac{2g_{12}}{3g} \partial_A (\epsilon^{0MA} B^N + \epsilon^{0NA} B^M) = 0$
	with matter	$\partial_t \vec{E} - \vec{\nabla}_\perp B = \vec{J}$	$\partial_t \tilde{E}^{MN} - \frac{2g_{12}}{3\tilde{g}} \partial_A (\epsilon^{0MA} \tilde{B}^N + \epsilon^{0NA} \tilde{B}^M) = \tilde{\mathcal{J}}_5^{MN}$

Table 5: Comparison between EoM and 3D Maxwell.

where $\vec{\nabla}_\perp B \equiv \epsilon^{0ij}\partial_j B$, and the results are consistent with what can be found in the fracton literature [148, 160, 161, 175]. We thus recovered, as EoM, the Gauss constraints related to the mobility of the traceless fracton theory in 3D [160, 161], where the Chern-Simons-like term contributes as a matter term through $\tilde{\rho}(X)$ (7.4.106), also identified by Pretko in [175]. This Chern-Simons-like term plays the role of matter contribution also in the fractonic Ampère equation (7.4.127), as a current $\tilde{\mathcal{J}}^{MN}(X)$ (7.4.128). Here again the term is in accordance with the literature, and in particular with what has been defined as “generalized Hall response” in [175]. The Ampère equations (7.4.125) and (7.4.127), to which the current $\tilde{\mathcal{J}}^{MN}(X)$ belongs, can be traced back to fracton theories as well, and, more specifically, they have the same structure as the fractonic Ampère equation used in [148] to study a duality between the theory of fractons and the theory of elasticity. However the one considered in [148] refers to the traceful theory of fractons, whose aim is to study defects as consequence of longitudinal motion of dipoles, which in the traceless fracton theory is not present since dipoles only move along their transverse direction [160, 161]. Therefore it would be interesting to understand if a fracton-elasticity duality also exists for the traceless model. We also notice that in [233] a charge $\tilde{\rho}(x)$, a current $\tilde{J}_{AB}(x)$, and a continuity equation as (7.4.130) are identified from a Chern-Simons-like theory with torsion $T(x)$, *i.e.* $\int d^3x \epsilon^{\mu\nu\rho} e_\mu^A T_{\nu\rho}^A$. In particular, the model coincides with the one proposed in [175] for the non-covariant Chern-Simons-like action for a specific choice of vielbein $e_\mu^A(x)$, and under the condition of “area-preserving diffeomorphisms”, which seems to be strictly related to fracton models, as also studied in [213]. This intriguing role of torsion in 3D chiral fractons is extended to 4D fractons of Section 7.2, where it has been shown that a linearized topological term with torsion [205] gives rise to the fracton θ -term [175].

7.4.5 Discrete symmetries : parity and time reversal

As extensively shown in the recent literature concerning non-perturbative aspects of quantum field theories, discrete symmetries play a central role in the identification of global anomalies and anomaly inflow, which are related to topological obstructions and impose strong constraints on the renormalization group flows, massive boundary states, quantum dualities and the vacua of quantum field theories [235–241]. Moreover, the anomaly inflow has been also extended to certain non-covariant fracton models [182]. Thus, here we analyze some discrete symmetries, such as \mathcal{T} and \mathcal{P} in the context of the induced theory we just derived. In fact, we can further constrain the induced 3D action by requiring a matching between the discrete symmetries in the bulk and on the boundary. In particular, under \mathcal{P} and \mathcal{T} the bulk fields transform as follows

$$\mathcal{T}\{A_{00}, A_{0A}, A_{AB}\} = \{A_{00}, -A_{0A}, A_{AB}\} \quad (7.4.134)$$

$$\mathcal{T}\{\tilde{A}_{00}, \tilde{A}_{0A}, \tilde{A}_{AB}\} = \{\tilde{A}_{00}, -\tilde{A}_{0A}, \tilde{A}_{AB}\}$$

$$\mathcal{P}\{A_{00}, A_{0A}, A_{AB}\} = \{A_{00}, -A_{0A}, A_{AB}\} \quad (7.4.135)$$

$$\mathcal{P}\{\tilde{A}_{00}, \tilde{A}_{0A}, \tilde{A}_{AB}\} = \{-\tilde{A}_{00}, \tilde{A}_{0A}, -\tilde{A}_{AB}\}$$

$$\mathcal{T}A = A \quad ; \quad \mathcal{T}\tilde{A} = \tilde{A} \quad ; \quad \mathcal{P}A = A \quad ; \quad \mathcal{P}\tilde{A} = -\tilde{A}. \quad (7.4.136)$$

The bulk action (7.4.1) is invariant under \mathcal{T} . Instead, due to the presence of the boundary $x^3 = 0$, the action is no longer \mathcal{P} -invariant. We now consider the boundary term (7.4.6), and distinguish between space and time indices

$$S_{bd} = \int d^4x \delta(x^3) \left[\xi_0 (A_{00}A^{00} + 2A_{0A}A^{0A} + A_{AB}A^{AB}) + \xi_1 (\tilde{A}_{00}A^{00} + 2\tilde{A}_{0A}A^{0A} + \tilde{A}_{AB}A^{AB}) + \xi_2 \epsilon^{0AB} (A_{0i}\partial_A A_B^i - A_{Ai}\partial_0 A_B^i + A_{Ai}\partial_B A_0^i) + \xi_3 A^2 + \xi_4 \tilde{A}A \right]. \quad (7.4.137)$$

We then observe that

- $\mathcal{P}S_{bd} = S_{bd}$ if $\xi_1 = \xi_4 = 0$;
- $\mathcal{T}S_{bd} = S_{bd}$ if $\xi_2 = 0$;
- $\mathcal{TP}S_{bd} = S_{bd}$ if $\xi_1 = \xi_2 = \xi_4 = 0$.

Under these considerations, we can update Table 4 of holographic contacts with the discrete symmetries allowed on the boundary term S_{bd}

	BC-EoM matching	Discrete symmetries of S_{bd}
$\xi_1 \neq 0, \omega_5 \neq 0$	$\omega_5 = \frac{1}{3}(\xi_1 - g_{12}) ; \xi_2 = g_{12}$	No
$\xi_1 \neq 0, \omega_5 = 0$	$\omega_5 = 0$	\mathcal{T}

Table 6: Holographic contacts, constraints and possible symmetries on S_{bd} (7.4.6).

From Table 6 we see that imposing \mathcal{P} on S_{bd} does not lead to a holographic contact, since the ξ_1 term, crucial for the existence of the induced 3D action, is not \mathcal{P} -invariant. We also highlight a relation between $\omega_5 \leftrightarrow \xi_2 \leftrightarrow \mathcal{T}$, in fact \mathcal{T} symmetry is possible only when $\xi_2 = 0$, which is allowed only in the second holographic contact (7.4.93), *i.e.* when ω_5 is set to zero as well (in the first case (7.4.80) the parameter ξ_2 is constrained by (7.4.77)). Therefore the Chern-Simons-like terms must be absent both in S_{bd} (7.4.6) and in S_{3D} (7.4.64), in order to have \mathcal{T} symmetry preserved on the boundary.

7.4.6 Summary and discussion

The questions that were raised at the beginning of this analysis in Section 7.4, have thus found their answers. Indeed an algebraic structure on the boundary does exist, as a consequence of the breaking of the Ward identities, and it can be interpreted as a generalization of the standard $U(1)$ KM algebra, characterized by a double derivative, as it appears for instance also in [184], further stressing the suspected relations between fractons and higher order topological insulators. From the two broken Ward identities, the boundary DoF of the induced theory are identified as two symmetric traceless rank-2 tensors $\alpha_{ij}(X)$ and $\tilde{\alpha}_{ij}(X)$. It is worth to remark that on the boundary some DoF disappear, since the boundary tensor fields turn out to be traceless. This might be due to the presence of a hidden symmetry, a guess that should be further investigated. The procedure to recover the induced theory leads to the action S_{3D} (7.4.64), which is composed of a term similar to a higher-rank Maxwell contribution, written in terms of traceless rank-3 field strengths, which mixes both fields $\alpha_{ij}(X)$ and $\tilde{\alpha}_{ij}(X)$, with a coefficient depending on the bulk constants g_1 and g_2 , and a Chern-Simons-like term

for $\tilde{\alpha}_{ij}(X)$ with a free coefficient. Concerning the physical interpretation of our 3D induced theory S_{3D} (7.4.64), this can be identified with the “traceless scalar charge” model of fractons [160, 161, 213]. In fact the transformations of the boundary fields, the canonical commutators, the traceless conjugate momenta, *i.e.* the “electric fields”, coincide with what appears in the literature. This claim is confirmed also by the EoM of the 3D induced theory, from which two Gauss-like laws are derived, which imply the defining property of the fracton quasiparticles, *i.e.* their limited mobility. Thus, one of the main results of this Section is that a non-standard covariant 3D traceless fracton theory turns out to be holographically induced from a 4D ordinary traceful covariant fracton theory. This claim gets even stronger confirmation from other components of the EoM, which can be identified with the Ampère-like equations of fractons [148], further stressing the relation of fracton models with Maxwell theory. Concerning this analogy, we remark a close resemblance of our 3D action S_{3D} (7.4.64) with Maxwell-Chern-Simons theory [70], of which it appears to be a kind of spin-two generalization. A similar observation can also be found in [231] in the context of self-dual massive gravity, where an identical covariant Chern-Simons term appears, and whose relation with our 3D model is worth to be further investigated. However, differently from the standard Maxwell-Chern-Simons theory, here all the coefficients are dimensionless, hence no topological mass can be identified. Therefore to better analyze this analogy, the study of the propagators would be helpful. Notice also that the Chern-Simons coefficient is free, thus it can be switched off. The choice of keeping the Chern-Simons-like term or not is relevant for the physical interpretation of the model: by switching it off, the 3D action S_{3D} (7.4.64) can be decoupled into two Maxwell-like terms, and the boundary theory is compatible with \mathcal{T} -symmetry, which characterizes the phenomenology involved. For instance, we have seen in Chapters 4 and 5 that the physics on the boundary of the topological BF models [34, 77, 142, 242] is identified with the effective description of the edge states of Topological Insulators, where \mathcal{T} is preserved both on the bulk and on the boundary. On the other hand, keeping the Chern-Simons-like term, *i.e.* relaxing the \mathcal{T} constraint, the EoM get a matter contribution. In particular the Chern-Simons-like term plays the role of fractonic charge $\tilde{\rho}(X)$ (7.4.106) and current $\tilde{J}_\Pi(X)$ (7.4.128) in two of the Maxwell-like equations, in accordance with [175]. Some final physical remarks are in order. Differently from the standard electromagnetic theory and the 4D traceful fracton model, here the magnetic-like vectors $B_A(X)$ and $\tilde{B}_A(X)$ do not have zero divergence, nor a Bianchi identity exists for the traceless rank-3 field strengths $\varphi_{abc}(X)$, $\tilde{\varphi}_{abc}(X)$, which suggests the presence of fractonic 3D vortices. Additionally, 3D fracton models are known to be related to the elasticity theory of topological defects through a duality [148]. For instance the traceful Ampère-like equation can be seen as describing the motion of these defects. Under this respect, it would be interesting to understand if and how our traceless boundary theory can be related to topological defects. Finally, there seems to be an interesting possible interpretation of the fractonic Chern-Simons-like term as associated to torsion contributions, as in [233], which also would be worth to further analyze.

In this Chapter, we consider the 4D theory of LG with a planar boundary, motivated by the guessed relation between KM algebras [26, 27] and LG [227, 243, 244], providing a precise interpretation of the physical quantity characterizing KM algebras – its central charge. According to the “general rule”, the central charge should be related to the LG coupling constant, normally set to one by a redefinition of the field, which in LG is a symmetric rank-2 tensor field. Therefore the positivity of the central charge could also give us a way to determine the sign of the LG “coupling” constant, which in this case cannot be inferred from an energy constraint. Apart from the KM algebra of conserved current, the introduction of a boundary yields more. The presence of a boundary in the 3D Chern-Simons theory induces a 2D theory, the Tomonaga-Luttinger theory of a chiral scalar field [87, 115, 245, 246], which we may refer to as a “holographic” reduction of the 3D bulk theory. Similar dimensional reductions – lower dimensional theories induced by the presence of a boundary in higher-dimensional ones – can be performed in other theories, both topological and non-topological. For instance we have seen in the previous Chapter the case of the fractonic theory, whose symmetry is a particular case of the diffeomorphisms, from which we have observed that fractons always come together with LG. Thus the theory with boundary studied in Section 7.4 comprises both fractonic contribution and LG, but without allowing for the limit of $g_1 = 0$ *i.e.* pure LG, which is not contained in that analysis. The reason is that, as explained, LG is a theory defined by a more general symmetry (the diff symmetry), which we know from Chapter 2 plays an important role in the search for the induced boundary theory. Indeed it is the breaking of the symmetry due to the presence of the boundary that leads to the algebra and the degrees of freedom on the lower dimensional theory. If the symmetry changes, the boundary theory may change as well. Therefore here we will apply the same mechanism of the previous Chapters to the case of pure LG to find the holographically induced 3D theory of 4D LG with a planar boundary. In particular in Section 8.1 we will first briefly review the theory of LG and its bulk physical content, and in Section 8.2 we will proceed with the introduction of a planar boundary.

8.1 THE THEORY OF GRAVITATIONAL WAVES

We will therefore consider the theory of a symmetric tensor field $h_{\mu\nu}(x)$ which transforms under diffeomorphisms

$$\delta_{diff} h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu . \quad (8.1.1)$$

As we mentioned previously, this theory contains the theory of fractons of Chapter 7 through a reduction of the diff transformation (8.1.1) into the longitudinal one (7.2.1), but the physical content is quite different, hence the difference in the notation : the symmetric rank-2 tensor field $A_{\mu\nu}(x)$ of the previous Chapter was of fractonic nature, here we will refer to $h_{\mu\nu}(x)$ as the rank-2 symmetric tensor field of LG. We are stepping from the microscopic world of condensed matter and immobile quasiparticles, to the

linearized theory of gravity describing gravitational waves. Indeed this theory can be either derived from a pure QFT point of view as the theory invariant under the diff symmetry (8.1.1), as a perturbation theory in General Relativity, where the gauge field $h_{\mu\nu}(x)$ is a perturbation around the flat Minkowskian metric. In the first case, *i.e.* QFT perspective, the transformation (8.1.1) immediately gives the invariant LG action we already mentioned in the previous Chapters

$$S_{LG} = \int d^4x (\partial_\mu h \partial^\mu h - \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} - 2\partial_\mu h \partial_\nu h^{\mu\nu} + 2\partial_\rho h_{\mu\nu} \partial^\mu h^{\nu\rho}) , \quad (8.1.2)$$

which describes a theory of a massless symmetric rank-2 tensor field $h_{\mu\nu}(x)$ in a flat spacetime. It is relevant to mention here an important generalization in which this gauge field is given a mass by breaking gauge invariance. This is known as the Fierz-Pauli action [211, 214, 215, 219]

$$S_{FP} = \int d^4x [m^2(h^2 - h_{\mu\nu}h^{\mu\nu}) + \partial_\mu h \partial^\mu h - \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} - 2\partial_\mu h \partial_\nu h^{\mu\nu} + 2\partial_\rho h_{\mu\nu} \partial^\mu h^{\nu\rho}] , \quad (8.1.3)$$

which describes a single massive spin 2 degree of freedom of mass m . This model is of particular interest nowadays since it seems that observations of gravitational waves, which we will show to be the physics related to the LG action (8.1.2), seem to indicate that gravitons are not massless indeed. Instead, starting from the the Einstein-Hilbert action of General Relativity

$$S_{EH} = \int d^4x \sqrt{-g} R , \quad (8.1.4)$$

one can consider a weak field limit, in which the metric of the spacetime $g_{\mu\nu}(x)$ is a perturbation around flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} . \quad (8.1.5)$$

Under this assumption translational invariance in curved spacetime, which is encoded in the Lie derivative \mathcal{L}_ξ [105], generates the diff transformation (8.1.1) through linearization :

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \xrightarrow{(8.1.5)} \delta_{diff} h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu . \quad (8.1.6)$$

At the same time the Einstein's EoM

$$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 , \quad (8.1.7)$$

matches the EoM recovered from the LG action (8.1.2) in the weak-field limit (8.1.5), for which at first order in the perturbation the Ricci tensor and scalar are respectively

$$R_{\mu\nu} \sim \frac{1}{2} \left(\partial_\mu \partial^\lambda h_{\lambda\nu} + \partial_\nu \partial^\lambda h_{\lambda\mu} - \square h_{\mu\nu} - \partial_\mu \partial_\nu h \right) ; \quad R \sim \partial^\mu \partial^\nu h_{\mu\nu} - \square h , \quad (8.1.8)$$

thus recovering

$$\mathcal{G}_{\mu\nu} \sim \partial_\mu \partial^\lambda h_{\nu\lambda} + \partial_\nu \partial^\lambda h_{\mu\lambda} - \square h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} + \eta_{\mu\nu} \square h = 0 . \quad (8.1.9)$$

In order to analyze the physical content of this equation the first step is to write the gauge field $h_{\mu\nu}(x)$ in terms of the irreducible representation of rotations [105], as we have also done in Section 7.4 in (B.3.49)-(B.3.51). Therefore we have

$$h_{00} = -2\phi \quad ; \quad h_{0i} = w_i \quad ; \quad h^k_k = -6\psi \quad ; \quad h_{ij} = 2(s_{ij} - \eta_{ij}\psi), \quad (8.1.10)$$

thus we have two scalars - $\phi(x), \psi(x)$ -, a vector - $w_i(x)$ -, and a traceless symmetric spatial tensor - $s_{ij}(x)$ -. One then has to fix a gauge which gives four conditions on the model, due to the nature of the gauge parameter of the symmetry. The typical (non-QFT) way to proceed [105] is thus to impose the so called "transverse gauge"

$$\partial^i w_i = 0 \quad ; \quad \partial^i s_{ij} = 0. \quad (8.1.11)$$

Clearly the same results can be obtained also by means of other choices, however this is the best one to quickly recover the physical content. Indeed, from this choice and the decomposition (8.1.10), the EoM (8.1.9) can be analyzed in terms of its time and space components as follows

$$\mathcal{G}_{00} = 2\nabla^2\psi = 0 \quad \Rightarrow \quad \psi = 0 \quad (8.1.12)$$

$$\mathcal{G}_{0i} = 2\partial_0\partial_i\cancel{\psi} - \frac{1}{2}\nabla^2 w_i = 0 \quad \Rightarrow \quad w_i = 0 \quad (8.1.13)$$

$$\mathcal{G}_{ij} = (\eta_{ij}\nabla^2 - \partial_i\partial_j)(\phi - \cancel{\psi}) + 2\eta_{ij}\partial_0^2\cancel{\psi} + \partial_0(\partial_i w_j + \partial_j w_i) - \square s_{ij} = 0 \quad (8.1.14)$$

where (8.1.12) and (8.1.13) give Poisson's equations, whose smooth solutions are only given by vanishing fields [105]. By computing the trace of the last equation and the fact that $\psi = w_i = 0$, we get another Poisson's equation for $\phi(x)$, which thus must be zero as well. We therefore get

$$\square s_{ij} = 0, \quad (8.1.15)$$

which is a wave equation for a traceless symmetric field. This is the core of gravitational waves. In terms of the gauge field $h_{\mu\nu}(x)$, the above conditions (Poisson's solutions and transverse gauge (8.1.11)), are usually (misleadingly) translated into the so called "transverse-traceless" ($\tau\tau$) gauge. We say misleading because this does not only contain a gauge, but also solutions to the EoM and, additionally, the results could be recovered also in different gauges. Anyway, the gauge field in the " $\tau\tau$ -gauge" $h_{\mu\nu}^{\tau\tau}(x)$ satisfy the following relations

$$h_{0\mu}^{\tau\tau} = 0 \quad (8.1.16)$$

$$\eta^{\mu\nu} h_{\mu\nu}^{\tau\tau} = 0 \quad (\text{traceless}) \quad (8.1.17)$$

$$\partial^\mu h_{\mu\nu}^{\tau\tau} = 0 \quad (\text{transverse}) \quad (8.1.18)$$

$$\square h_{\mu\nu}^{\tau\tau} = 0. \quad (8.1.19)$$

From (8.1.19) we can find plane waves solution

$$h_{\mu\nu}^{\tau\tau} = c_{\mu\nu} e^{ik_\lambda x^\lambda}, \quad (8.1.20)$$

with $c_{\mu\nu}$ being purely spatial and traceless (*i.e.* $c_{0\mu} = 0 = \eta^{\mu\nu} c_{\mu\nu}$), and k_λ is a lightlike vector, meaning that the gravitational waves move at the speed of light (unless some kind of mass term can be introduced in the theory, like for instance through the Fierz-Pauli term (8.1.3)). By choosing an appropriate reference frame, for example by fixing

an axis along the direction of propagation $k^\mu = (\omega, 0, 0, \omega)$, and considering the transversality equation (8.1.18), one finally get that $h_{\mu\nu}^{\text{TT}}(x)$ is represented by the following matrix

$$h_{\mu\nu}^{\text{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (8.1.21)$$

which means that the gravitational wave has two polarizations which oscillates transversely to its path [105]. These oscillations causes stretches of the space, which can be measured and are the bases around which experiments on detection of gravitational waves are built [105].

8.2 LINEARIZED GRAVITY WITH BOUNDARY

The analysis of this Section on the effects of the presence of a boundary in the diff invariant theory of LG is organized as follows.

8.2.1	The bulk model	132
8.2.2	The boundary	138
8.2.3	Contact between bulk and boundary	142
8.2.4	Summary of results	145

Specifically in Section 8.2.1, we consider the 4D LG theory with a planar boundary, implemented using a Heaviside theta distribution in the bulk action. We derive the most general BC following a method introduced by Symanzik, without imposing them by hand. The presence of the boundary breaks the invariance under diffeomorphisms, resulting in a breaking of the diffeomorphism Ward identity. From this, conserved currents and their KM algebra are derived using standard QFT methods. The central charge appears to be proportional to the inverse of the LG “coupling” constant, as in topological field theories. Section 8.2.2 focuses on the identification of the holographically induced 3D theory. By first solving the current conservation equation, we find the 3D degrees of freedom and determine the most general transformations that preserve their invariance. Remarkably, we discover that these transformations are diffeomorphisms, which is not obvious. Thus, diffeomorphism invariance emerges as a consequence of our procedure rather than a mere requirement. Once we have the quantum fields with their transformations, we arrive at the most general 3D action, satisfying the additional QFT requirements of locality and power counting. Using this, in Section 8.2.3 we establish the holographic connection, yielding two solutions for the induced 3D theory. Our results are summarized in Section 8.2.4.

8.2.1 *The bulk model*

Linearized Gravity is the theory of a rank-2 symmetric tensor field $h_{\mu\nu}(x)$ on a flat Minkowskian background. We introduce a planar boundary at $x^3 = 0$ in the LG action

by means of a Heaviside step function, which confines the model on a half space with single-sided planar boundary

$$S_{bulk} = \lambda \int d^4x \theta(x^3) \left(\frac{1}{4} F_{\mu\nu}^\mu F_{\rho}^{\rho\nu} - \frac{1}{6} F^{\mu\nu\rho} F_{\mu\nu\rho} \right), \quad (8.2.1)$$

where $F^{\mu\nu\rho}(x)$ is the rank-3 tensor field strength associated to the tensor field $h_{\mu\nu}(x)$ introduced in Section 7.2 [4, 6]

$$F_{\mu\nu\rho} = F_{\nu\rho\mu} = \partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - 2\partial_\rho h_{\mu\nu}, \quad (8.2.2)$$

satisfying the following properties

$$F_{\mu\nu\rho} + F_{\nu\rho\mu} + F_{\rho\mu\nu} = 0 \quad (8.2.3)$$

$$\epsilon_{\alpha\mu\nu\rho} \partial^\mu F^{\beta\nu\rho} = 0. \quad (8.2.4)$$

Notice that this field strength, defined in the context of Section 7.2, is invariant under the longitudinal diff (fracton) symmetry (7.2.1), but it is not invariant under the more general diff transformation (8.1.1). The constant λ in (8.2.1) could be reabsorbed through a redefinition of $h_{\mu\nu}(x)$, however we maintain it in order to keep track of the bulk contributions. Due to the presence of the boundary at $x^3 = 0$ the x^3 -derivative of the gauge field at $x^3 = 0$ must be considered as independent from $h_{\mu\nu}(x)$ [147], thus we define

$$\tilde{h}_{\mu\nu} \equiv \partial_3 h_{\mu\nu}|_{x^3=0}, \quad (8.2.5)$$

and the fields have the mass dimensions

$$[h_{\mu\nu}] = 1 \quad ; \quad [\tilde{h}_{\mu\nu}] = 2. \quad (8.2.6)$$

As already discussed, the LG theory (8.2.1) in the absence of a boundary is invariant under the infinitesimal diffeomorphism transformation

$$\delta_{diff} h_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu, \quad (8.2.7)$$

where $\Lambda_\mu(x)$ is a local vector parameter. LG is a gauge field theory, for which a gauge should be fixed. We choose the axial gauge, as customary in presence of a boundary

$$h_{\mu 3} = 0. \quad (8.2.8)$$

This can be realized by means of a vector Lagrange multiplier $b^\mu(x)$ through the gauge-fixing term

$$S_{gf} = \int d^4x \theta(x^3) b^\mu h_{\mu 3}. \quad (8.2.9)$$

Moreover, the following source term is needed

$$S_J = \int d^4x \left[\theta(x^3) J^{ab} h_{ab} + \delta(x^3) \tilde{J}^{ab} \tilde{h}_{ab} \right], \quad (8.2.10)$$

where, together with the external field $J^{ab}(x)$ associated to the tensor gauge field $h_{ab}(x)$, on the boundary an additional external source $\tilde{J}^{ab}(x)$ is coupled to $\tilde{h}_{ab}(x)$ (8.2.5). The presence of a boundary requires BC. We follow Symanzik's approach [10] previously discussed, by adding to the action a boundary term constrained only by power-counting

and locality. This modifies the EoM by a boundary contribution, and the BC are then obtained by means of a simple variational principle. For our model the boundary term is

$$S_{bd} = \int d^4x \delta(x^3) \left[\xi_0 h_{ab} h^{ab} + \xi_1 \tilde{h}_{ab} h^{ab} + \xi_2 \epsilon^{abc} h_{ai} \partial_b h_c^i + \xi_3 h^2 + \xi_4 \tilde{h} h \right], \quad (8.2.11)$$

where

$$h \equiv \eta^{\mu\nu} h_{\mu\nu} \quad ; \quad \tilde{h} \equiv \eta^{\mu\nu} \tilde{h}_{\mu\nu}, \quad (8.2.12)$$

and the ξ_i , $i = \{0, \dots, 4\}$, are constant parameters with mass dimensions

$$[\xi_0] = [\xi_3] = 1 \quad ; \quad [\xi_1] = [\xi_2] = [\xi_4] = 0. \quad (8.2.13)$$

Notice that, due to the gauge-fixing condition (8.2.8), the nontrivial part of the trace $h(x)$ is $\eta^{ab} h_{ab}(x)$. The full action of the model finally is

$$S_{tot} = S_{bulk} + S_{gf} + S_J + S_{bd}. \quad (8.2.14)$$

Equations of motion and boundary conditions

Besides the EoM of the Lagrange multiplier $b^\mu(x)$, which implements the axial gauge condition (8.2.8)

$$\frac{\delta S_{tot}}{\delta b^\mu} = h_{\mu 3} = 0, \quad (8.2.15)$$

the EoM of the gauge field $h_{\alpha\beta}(x)$ and its ∂_3 -derivative $\tilde{h}_{\alpha\beta}(x)$ are, respectively

$$\begin{aligned} \frac{\delta S_{tot}}{\delta h_{\alpha\beta}} = & \theta(x^3) \left\{ \lambda \left[\eta^{\alpha\beta} \partial_\mu F_\nu^{\nu\mu} - \partial_\mu F^{\alpha\beta\mu} - \frac{1}{2} \left(\partial^\alpha F_\mu^{\mu\beta} + \partial^\beta F_\mu^{\mu\alpha} \right) \right] + \delta_a^\alpha \delta_b^\beta J^{ab} \right. \\ & + \frac{1}{2} (b^\alpha \delta_3^\beta + b^\beta \delta_3^\alpha) \left. \right\} + \delta(x^3) \left\{ \lambda \left[\eta^{\alpha\beta} F_\mu^{\mu 3} - F^{\alpha\beta 3} - \frac{1}{2} \left(\eta^{\alpha 3} F_\mu^{\mu\beta} + \eta^{\beta 3} F_\mu^{\mu\alpha} \right) \right] \right. \\ & + \delta_a^\alpha \delta_b^\beta \left[2\xi_0 h^{ab} + \xi_1 \tilde{h}^{ab} + \xi_2 (\epsilon^{aij} \partial_i h_j^b + \epsilon^{bij} \partial_i h_j^a) + 2\xi_3 \eta^{ab} h + \xi_4 \eta^{ab} \tilde{h} \right] \left. \right\} = 0, \end{aligned} \quad (8.2.16)$$

and

$$\begin{aligned} \frac{\delta S_{tot}}{\delta \partial_3 h_{\alpha\beta}} = & \lambda \theta(x^3) \left[F^{\alpha\beta 3} - \eta^{\alpha\beta} F_\mu^{\mu 3} + \frac{1}{2} \left(\eta^{\alpha 3} F_\mu^{\mu\beta} + \eta^{\beta 3} F_\mu^{\mu\alpha} \right) \right] + \\ & + \delta(x^3) \delta_a^\alpha \delta_b^\beta \left\{ \tilde{J}^{ab} + \xi_1 h^{ab} + \xi_4 \eta^{ab} h \right\} = 0. \end{aligned} \quad (8.2.17)$$

The BC come from a variational principle applied on the EoM as $\lim_{\epsilon \rightarrow 0} \int_0^\epsilon dx^3$ (EoM), which corresponds to putting equal to zero the $\delta(x^3)$ contribution of the EoM (8.2.16) and (8.2.17). From $\lim_{\epsilon \rightarrow 0} \int_0^\epsilon dx^3$ (8.2.16) we have

$$\begin{aligned} & \left\{ \lambda \left[-F^{\alpha\beta 3} + \eta^{\alpha\beta} F_\mu^{\mu 3} - \frac{1}{2} \left(\eta^{\alpha 3} F_\mu^{\mu\beta} + \eta^{\beta 3} F_\mu^{\mu\alpha} \right) \right] + \right. \\ & + \delta_a^\alpha \delta_b^\beta \left[2\xi_0 h^{ab} + \xi_1 \tilde{h}^{ab} + \xi_2 (\epsilon^{aij} \partial_i h_j^b + \epsilon^{bij} \partial_i h_j^a) + 2\xi_3 \eta^{ab} h + \xi_4 \eta^{ab} \tilde{h} \right] \left. \right\}_{x^3=0} = 0. \end{aligned} \quad (8.2.18)$$

The nontrivial components are

- $\alpha = 3, \beta = b$:

$$\lambda \left(\partial^b h - \partial_a h^{ab} \right)_{x^3=0} = \lambda F_\mu^{\mu b} |_{x^3=0} = 0. \quad (8.2.19)$$

- $\alpha = a, \beta = b$:

$$\left[2\xi_0 h^{ab} + (2\lambda + \xi_1) \tilde{h}^{ab} + \xi_2 (\epsilon^{aij} \partial_i h_j^b + \epsilon^{bij} \partial_i h_j^a) + 2\xi_3 \eta^{ab} h + (\xi_4 - 2\lambda) \eta^{ab} \tilde{h} \right]_{x^3=0} = 0. \quad (8.2.20)$$

Taking $\lim_{\epsilon \rightarrow 0} \int_0^\epsilon dx^3$ (8.2.17) and going on-shell ($\tilde{J} = 0$), we get

$$\delta_a^\alpha \delta_b^\beta \left(\xi_1 h^{ab} + \xi_4 \eta^{ab} h \right)_{x^3=0} = 0, \quad (8.2.21)$$

whose nonvanishing components are $\alpha = a, \beta = b$, which give

$$\left(\xi_1 h^{ab} + \xi_4 \eta^{ab} h \right)_{x^3=0} = 0. \quad (8.2.22)$$

Notice that from (8.2.19) and taking ∂_a -derivative of (8.2.22) we have the following constraint on the boundary parameters

$$\xi_1 = -\xi_4. \quad (8.2.23)$$

To summarize, the most general BC on the planar boundary $x^3 = 0$ are the following

$$\partial^b h - \partial_a h^{ab} = 0 \quad (8.2.24)$$

$$\xi_1 \left(h^{ab} - \eta^{ab} h \right) = 0 \quad (8.2.25)$$

$$2\xi_0 h^{ab} + (2\lambda + \xi_1) (\tilde{h}^{ab} - \eta^{ab} \tilde{h}) + \xi_2 (\epsilon^{aij} \partial_i h_j^b + \epsilon^{bij} \partial_i h_j^a) + 2\xi_3 \eta^{ab} h = 0. \quad (8.2.26)$$

The BC (8.2.24) is universal, in the sense that it does not depend on S_{bd} (8.2.11). It represents the conservation of a current on the boundary

$$\partial_a K^{ab} = 0, \quad (8.2.27)$$

with

$$K^{ab} \equiv h^{ab} - \eta^{ab} h. \quad (8.2.28)$$

On the other hand, we remark that if $\xi_1 = 0$, the BC are given by (8.2.24) and (8.2.26). If instead $\xi_1 \neq 0$, (8.2.25) implies (8.2.24) and the BC are given by (8.2.25) and (8.2.26).

Ward identities

From the EoM for $h_{\mu\nu}(x)$ (8.2.16) we get

$$0 = \int dx^3 \partial_a \frac{\delta S_{tot}}{\delta h_{ab}} = 2\lambda \left(\partial^b \tilde{h} - \partial_a \tilde{h}^{ab} \right)_{x^3=0} + \int dx^3 \theta(x^3) \partial_a J^{ab}, \quad (8.2.29)$$

where we used the BC (8.2.18). We thus obtain the following Ward identity

$$\int dx^3 \theta(x^3) \partial_a J^{ab} = -2\lambda \left(\partial^b \tilde{h} - \partial_a \tilde{h}^{ab} \right)_{x^3=0}, \quad (8.2.30)$$

which is broken on the boundary $x^3 = 0$. In the same way, from the EoM of $\tilde{h}_{\mu\nu}(x)$ (8.2.17), we find

$$0 = \int dx^3 \partial_a \frac{\delta S_{tot}}{\delta \partial_3 h_{ab}} = \partial_a \tilde{J}^{ab} |_{x^3=0} - 2\lambda \left(-\partial_a h^{ab} + \partial^b h \right)_{x^3=0}, \quad (8.2.31)$$

which represents a local Ward identity, broken by the boundary

$$\partial_a \tilde{J}^{ab}|_{x^3=0} = 2\lambda \left(-\partial_a h^{ab} + \partial^b h \right)_{x^3=0}. \quad (8.2.32)$$

Notice that the r.h.s. describes the conservation on the boundary of the current $K^{ab}(X)$ (8.2.28), previously found as the BC (8.2.24), hence we may write

$$\partial_a \tilde{J}^{ab}|_{x^3=0} = 0. \quad (8.2.33)$$

Going on-shell ($J = \tilde{J} = 0$), the broken Ward identity (8.2.30) yields

$$\left(\partial^b \tilde{h} - \partial_a \tilde{h}^{ab} \right)_{x^3=0} = 0, \quad (8.2.34)$$

which, again, is a current conservation equation

$$\partial_a \tilde{K}^{ab} = 0, \quad (8.2.35)$$

with

$$\tilde{K}^{ab} \equiv \tilde{h}^{ab} - \eta^{ab} \tilde{h}. \quad (8.2.36)$$

Hence, the presence of a planar boundary in LG theory has as a consequence the presence of conserved currents, which consist of the particular combinations (8.2.28) and (8.2.36).

Kac-Moody Algebra

By computing the functional derivative with respect to $J^{mn}(x')$ of the broken Ward identity (8.2.30), *i.e.* $\frac{\delta}{\delta J^{mn}(x')}$ (8.2.30) :

$$\int_0^\infty dx^3 \partial_a \left(\frac{\delta_m^a \delta_n^b + \delta_n^a \delta_m^b}{2} \delta^{(4)}(x - x') \right) = -2\lambda \left(\eta_{ac} \partial^b - \delta_c^b \partial_a \right) \frac{\delta Z_c[J, \tilde{J}]}{\delta \tilde{J}_{ac} \delta J^{mn}}, \quad (8.2.37)$$

we get the commutation relations

$$\frac{1}{2} \left(\delta_m^a \delta_n^b + \delta_n^a \delta_m^b \right) \partial_a \delta^{(3)}(X - X') = -2i\lambda \left(\eta_{ac} \eta^{b0} - \delta_c^b \delta_a^0 \right) \left[\tilde{h}^{ac}, h'_{mn} \right] \delta(x^0 - x'^0), \quad (8.2.38)$$

where we used the on-shell constraint (8.2.34). By setting

- $b = 0$ we have

$$\left(\delta_m^a \delta_n^0 + \delta_n^a \delta_m^0 \right) \partial_a \delta^{(3)}(X - X') = 4i\lambda \left[\tilde{h}_D^D, h'_{mn} \right] \delta(x^0 - x'^0), \quad (8.2.39)$$

from which, integrating over time,

- $m = n = 0$ gives

$$\left[\tilde{h}_D^D, h'_{00} \right]_{x^0=x'^0} = 0. \quad (8.2.40)$$

- $m = 0, n = N$ we get

$$\left[\tilde{h}_D^D, h'_{0N} \right]_{x^0=x'^0} = -\frac{i}{4\lambda} \partial_N \delta^{(2)}(X - X'). \quad (8.2.41)$$

This can be identified as a KM algebraic structure [26, 27] with central charge

$$c = -\frac{1}{4\lambda}, \quad (8.2.42)$$

which implies

$$\lambda < 0, \quad (8.2.43)$$

because of the positivity of central charge of KM algebras [73, 74].

– $m = \mathbf{M}$, $n = \mathbf{N}$ gives

$$\left[\tilde{h}_{\mathbf{D}}^{\mathbf{D}}, h'_{\mathbf{MN}} \right]_{x^0=x'^0} = 0. \quad (8.2.44)$$

• $b = \mathbf{B}$

$$(\delta_m^a \delta_n^{\mathbf{B}} + \delta_n^a \delta_m^{\mathbf{B}}) \partial_a \delta^{(3)}(X - X') = 4i\lambda \left[\tilde{h}^{0\mathbf{B}}, h'_{mn} \right] \delta(x^0 - x'^0). \quad (8.2.45)$$

– $m = n = 0$

$$\left[\tilde{h}^{0\mathbf{B}}, h'_{00} \right]_{x^0=x'^0} = 0. \quad (8.2.46)$$

– $m = 0$, $n = \mathbf{N}$

$$\left[\tilde{h}^{0\mathbf{B}}, h'_{0\mathbf{N}} \right]_{x^0=x'^0} = 0. \quad (8.2.47)$$

– $m = \mathbf{M}$, $n = \mathbf{N}$

$$\left[\tilde{h}^{0\mathbf{B}}, h'_{\mathbf{MN}} \right]_{x^0=x'^0} = -\frac{i}{4\lambda} (\delta_{\mathbf{M}}^{\mathbf{A}} \delta_{\mathbf{N}}^{\mathbf{B}} + \delta_{\mathbf{N}}^{\mathbf{A}} \delta_{\mathbf{M}}^{\mathbf{B}}) \partial_{\mathbf{A}} \delta^{(2)}(X - X'). \quad (8.2.48)$$

Here again a KM algebraic structure is observed with the same central charge c (8.2.42).

We now compute the functional derivative of the broken Ward identity (8.2.30) with respect to $\tilde{J}^{mn}(x')$, *i.e.* $\frac{\delta}{\delta \tilde{J}^{mn}(x')}$ (8.2.30) :

$$0 = -2i\lambda \left(\eta_{ac} \eta^{b0} - \delta_c^b \delta_a^0 \right) \left[\tilde{h}^{ac}, \tilde{h}'_{mn} \right]_{x^0=x'^0}, \quad (8.2.49)$$

where we used the on-shell constraint (8.2.34) and integrated over time. In particular we have at

• $b = 0$

$$\left[\tilde{h}_{\mathbf{D}}^{\mathbf{D}}, \tilde{h}'_{mn} \right]_{x^0=x'^0} = 0; \quad (8.2.50)$$

• $b = \mathbf{B}$

$$\left[\tilde{h}^{0\mathbf{B}}, \tilde{h}'_{mn} \right]_{x^0=x'^0} = 0. \quad (8.2.51)$$

Summarizing, from the integrated Ward identity (8.2.30) we get the semidirect sum of KM algebras with the same central charge

$$\left[\tilde{h}_{\mathbf{D}}^{\mathbf{D}}, h'_{0\mathbf{N}} \right] = -\frac{i}{4\lambda} \partial_{\mathbf{N}} \delta^{(2)}(X - X') \quad (8.2.52)$$

$$\left[\tilde{h}^{0\mathbf{B}}, h'_{\mathbf{MN}} \right] = -\frac{i}{4\lambda} (\delta_{\mathbf{M}}^{\mathbf{A}} \delta_{\mathbf{N}}^{\mathbf{B}} + \delta_{\mathbf{N}}^{\mathbf{A}} \delta_{\mathbf{M}}^{\mathbf{B}}) \partial_{\mathbf{A}} \delta^{(2)}(X - X'). \quad (8.2.53)$$

The above KM algebraic structure has a physical meaning when expressed in terms of the conserved currents $K^{ab}(X)$ (8.2.28) and $\tilde{K}^{ab}(X)$ (8.2.36), which are expressed in terms of the tensor fields $h^{ab}(X)$, $\tilde{h}^{ab}(X)$ and their traces. In fact, as a consequence of (8.2.52) and (8.2.53) we find that $K^{ab}(X)$ and $\tilde{K}^{ab}(X)$ form a KM algebra with central charge (8.2.42) whose non vanishing components are

$$\left[\tilde{K}^{00}, K'_{0M} \right] = -\frac{i}{4\lambda} \partial_M \delta^{(2)}(X - X') \quad (8.2.54)$$

$$\left[\tilde{K}^{0B}, K'_{mn} \right] = -\frac{i}{4\lambda} (\delta_m^A \delta_n^B + \delta_n^A \delta_m^B - 2\eta^{AB} \eta_{mn}) \partial_A \delta^{(2)}(X - X'). \quad (8.2.55)$$

The existence of a KM algebraic structure for conserved currents on the boundary of 4D LG confirms the guess made in [227] as a particularly interesting possibility in connection with Weinberg's soft graviton theorems [247–249].

8.2.2 The boundary

The degrees of freedom

The presence of a 3D boundary in the 4D theory described by the action S_{tot} (8.2.14) induces a 3D theory, whose field content is determined by the solution of the on-shell broken Ward identity (8.2.30)

$$\partial_a \left(\tilde{h}^{ab} - \eta^{ab} \tilde{h} \right)_{x^3=0} = 0 \quad (8.2.56)$$

and of the BC (8.2.24)

$$\partial_a \left(h^{ab} - \eta^{ab} h \right)_{x^3=0} = 0. \quad (8.2.57)$$

Let us consider first (8.2.56). Define

$$\tilde{C}^{ab} \equiv \tilde{h}^{ab} - \eta^{ab} \tilde{h}, \quad (8.2.58)$$

whose trace is

$$\tilde{C} = \eta_{ab} \tilde{C}^{ab} = -2\tilde{h}. \quad (8.2.59)$$

Eq. (8.2.56) then reads

$$\partial_a \tilde{C}^{ab} = 0. \quad (8.2.60)$$

In order to find the most general solution, let us parametrize the symmetric tensor $\tilde{C}^{ab}(X)$ as follows

$$\tilde{C}^{ab} = \frac{1}{2} \left(\epsilon^{amn} \partial_m \tilde{\Sigma}_n^b + \epsilon^{bmn} \partial_m \tilde{\Sigma}_n^a \right). \quad (8.2.61)$$

Because of (8.2.60) it must be

$$\epsilon^{bmn} \partial_m \partial_a \tilde{\Sigma}_n^a = 0, \quad (8.2.62)$$

which is solved by

$$\tilde{\Sigma}_n^a = \epsilon^{acd} \partial_c \tilde{\sigma}_{nd} + \partial_n \phi^a, \quad (8.2.63)$$

but we observe that the $\phi^a(X)$ contribution trivializes $\tilde{C}^{ab}(X)$ (8.2.61). Hence

$$\tilde{\Sigma}_n^a = \epsilon^{acd} \partial_c \tilde{\sigma}_{nd}. \quad (8.2.64)$$

In terms of this result, $\tilde{C}^{ab}(X)$ (8.2.61) solves (8.2.60), and reads

$$\tilde{C}^{ab} = \epsilon^{bmn} \epsilon^{acd} \partial_m \partial_c \tilde{\sigma}_{nd}, \quad (8.2.65)$$

with $\tilde{\sigma}_{ab}(X) = \tilde{\sigma}_{ba}(X)$ as a consequence of the fact that $\tilde{C}^{ab}(X)$ is symmetric $\tilde{C}^{ab}(X) = \tilde{C}^{ba}(X)$, and with $[\tilde{\sigma}] = 0$. Thus the general solution for $\tilde{h}^{ab}(X)$ is

$$\begin{aligned} \tilde{h}^{ab} &= \tilde{C}^{ab} - \frac{1}{2} \eta^{ab} \tilde{C} \\ &= -\frac{1}{2} \eta^{ab} (\partial_m \partial^m \tilde{\sigma}_n^n - \partial^m \partial^n \tilde{\sigma}_{mn}) + \partial_m \partial^m \tilde{\sigma}^{ab} + \partial^a \partial^b \tilde{\sigma}_n^n - \partial_c (\partial^b \tilde{\sigma}^{ac} + \partial^a \tilde{\sigma}^{bc}). \end{aligned} \quad (8.2.66)$$

The Eq.(8.2.57) for $h^{ab}(x)$ has the same structure as (8.2.56), therefore the solution has the same form (8.2.66). We finally get

$$\tilde{h}^{ab} = \epsilon^{bmn} \epsilon^{acd} \partial_m \partial_c \tilde{\sigma}_{nd} + \frac{1}{2} \eta^{ab} (\partial_m \partial^m \tilde{\sigma}_n^n - \partial^m \partial^n \tilde{\sigma}_{mn}) \quad (8.2.67)$$

$$= -\frac{1}{2} \eta^{ab} (\partial_m \partial^m \tilde{\sigma}_n^n - \partial^m \partial^n \tilde{\sigma}_{mn}) + \partial_m \partial^m \tilde{\sigma}^{ab} + \partial^a \partial^b \tilde{\sigma}_n^n - \partial_c (\partial^b \tilde{\sigma}^{ac} + \partial^a \tilde{\sigma}^{bc})$$

$$h^{ab} = \epsilon^{bmn} \epsilon^{acd} \partial_m \partial_c \sigma_{nd} + \frac{1}{2} \eta^{ab} (\partial_m \partial^m \sigma_n^n - \partial^m \partial^n \sigma_{mn}) \quad (8.2.68)$$

$$= -\frac{1}{2} \eta^{ab} (\partial_m \partial^m \sigma_n^n - \partial^m \partial^n \sigma_{mn}) + \partial_m \partial^m \sigma^{ab} + \partial^a \partial^b \sigma_n^n - \partial_c (\partial^b \sigma^{ac} + \partial^a \sigma^{bc}),$$

which means that the fields of the induced 3D theory are identified as the rank-2 symmetric tensors $\sigma^{ab}(X)$ and $\tilde{\sigma}^{ab}(X)$. Moreover, these solutions are invariant under the following transformations of the boundary fields $\sigma_{ab}(X)$, $\tilde{\sigma}_{ab}(X)$

$$\delta \tilde{h}_{ab} = 0 \quad \Leftrightarrow \quad \delta \tilde{\sigma}_{mn} = \partial_m \tilde{\xi}_n + \partial_n \tilde{\xi}_m \quad (8.2.69)$$

$$\delta h_{ab} = 0 \quad \Leftrightarrow \quad \delta \sigma_{mn} = \partial_m \xi_n + \partial_n \xi_m, \quad (8.2.70)$$

which remarkably means that the induced boundary theory must be invariant under infinitesimal diffeomorphisms, which therefore is a consequence of the general method we followed to introduce a boundary in LG, without need of requiring it explicitly.

Most general 3D action

As a consequence of the solutions $\tilde{h}_{ab}(x)$ (8.2.67) and $h_{ab}(x)$ (8.2.68) and of their mass dimensions (8.2.6), the boundary fields $\sigma_{ab}(X)$ and $\tilde{\sigma}_{ab}(X)$ should have mass dimensions $[\sigma] = -1$ and $[\tilde{\sigma}] = 0$. However, in 3D the canonical choices for the mass dimensions of the tensor fields are two :

1. $[\sigma] = [\tilde{\sigma}] = 1$, which can be realized by rescaling as follows

$$\tilde{\sigma} \rightarrow \tilde{M}^{-1} \tilde{\sigma} \quad ; \quad \sigma \rightarrow M^{-2} \sigma. \quad (8.2.71)$$

However in this case power-counting and locality constrain the action to the following Chern-Simons/BF-like action [14],

$$S = \int d^3x \epsilon^{abc} \left(a_1 \sigma_{ad} \partial_b \sigma_c^d + a_2 \tilde{\sigma}_{ad} \partial_b \sigma_c^d + a_3 \tilde{\sigma}_{ad} \partial_b \tilde{\sigma}_c^d \right) \quad (8.2.72)$$

which is not invariant under the diffeomorphism transformations δ (8.2.70) and $\tilde{\delta}$ (8.2.69) :

$$\delta S = \int d^3x \epsilon^{abc} (2a_1 \sigma_{ad} + a_2 \tilde{\sigma}_{ad}) \partial_b \partial^d \xi_c \quad (8.2.73)$$

$$\tilde{\delta} S = \int d^3x \epsilon^{abc} (a_2 \sigma_{ad} + 2a_3 \tilde{\sigma}_{ad}) \partial_b \partial^d \tilde{\xi}_c, \quad (8.2.74)$$

which indeed vanish only at the trivial case ($a_1 = a_2 = a_3 = 0$). Thus we must discard this possibility.

2. $[\sigma] = [\tilde{\sigma}] = \frac{1}{2}$, achieved by rescaling

$$\tilde{\sigma} \rightarrow \tilde{M}^{-\frac{1}{2}} \tilde{\sigma} \quad ; \quad \sigma \rightarrow M^{-\frac{3}{2}} \sigma, \quad (8.2.75)$$

which, instead, leads to a nontrivial solution, as we shall see in what follows.

The most general action invariant under the infinitesimal diffeomorphisms $\tilde{\delta}$ (8.2.69) and δ (8.2.70) has the following structure

$$S_{3D}[\sigma, \tilde{\sigma}] = \kappa S_{LG}[\sigma] + \tilde{\kappa} \tilde{S}_{LG}[\tilde{\sigma}] + \kappa_m S_{mix}[\sigma, \tilde{\sigma}], \quad (8.2.76)$$

where κ , $\tilde{\kappa}$, κ_m are dimensionless constants, and $S_{LG}[\sigma]$ and $\tilde{S}_{LG}[\tilde{\sigma}]$ are LG contributions analogous to (8.2.1), written in terms of the boundary tensor field $\sigma_{ab}(X)$ and $\tilde{\sigma}_{ab}(X)$, respectively

$$S_{LG} = \int d^3x \left(\frac{1}{4} f_{ac}^a f_b^{bc} - \frac{1}{6} f^{abc} f_{abc} \right) \quad (8.2.77)$$

$$\tilde{S}_{LG} = \int d^3x \left(\frac{1}{4} \tilde{f}_{ac}^a \tilde{f}_b^{bc} - \frac{1}{6} \tilde{f}^{abc} \tilde{f}_{abc} \right), \quad (8.2.78)$$

with

$$f_{abc} = f_{bac} = \partial_a \sigma_{bc} + \partial_b \sigma_{ac} - 2\partial_c \sigma_{ab} \quad (8.2.79)$$

$$\tilde{f}_{abc} = \tilde{f}_{bac} = \partial_a \tilde{\sigma}_{bc} + \partial_b \tilde{\sigma}_{ac} - 2\partial_c \tilde{\sigma}_{ab}, \quad (8.2.80)$$

satisfying the cyclicity property

$$f^{abc} + f^{bca} + f^{cab} = 0 \quad (8.2.81)$$

$$\tilde{f}^{abc} + \tilde{f}^{bca} + \tilde{f}^{cab} = 0. \quad (8.2.82)$$

Notice that no Chern-Simons or BF contributions like in (8.2.72) are allowed as a consequence of the diffeomorphism invariances $\delta S_{3D} = \tilde{\delta} S_{3D} = 0$. The S_{mix} term in (8.2.76) is the most general one depending on both $\sigma_{ab}(X)$ and $\tilde{\sigma}_{ab}(X)$, compatible with power-counting and the invariances $\delta S_{mix} = \tilde{\delta} S_{mix} = 0$. Excluding again Chern-Simons/BF-like contributions, which are not invariant under diffeomorphisms, we have

$$S_{mix} = \int d^3x \left\{ a_0 \partial_a \sigma \partial^a \tilde{\sigma} + a_1 \partial_c \sigma_{ab} \partial^c \tilde{\sigma}^{ab} + a_2 \partial_a \sigma \partial_b \tilde{\sigma}^{ab} + a_3 \partial_a \tilde{\sigma} \partial_b \sigma^{ab} + a_4 \partial_c \sigma_{ab} \partial^a \tilde{\sigma}^{bc} \right\}. \quad (8.2.83)$$

Under the diff transformation δ (8.2.70) we get

$$\delta S_{mix} = - \int d^3x \left\{ \tilde{\sigma}^{ab} [(2a_1 + a_4) \partial_a \partial^2 \xi_b + (2a_2 + a_4) \partial_a \partial_b \partial_m \xi^m] + 2\tilde{\sigma} (a_0 + a_3) \partial_m \partial^2 \xi^m \right\}, \quad (8.2.84)$$

which means that the action is invariant, *i.e.* $\delta S_{mix} = 0$, if

$$\begin{aligned} a_3 &= -a_0 \\ a_4 &= -2a_1 \\ a_2 &= a_1 . \end{aligned} \quad (8.2.85)$$

The δ -invariant S_{mix} action term is

$$S_{mix} = \int d^3x \left\{ a_0 \partial_a \sigma \partial^a \tilde{\sigma} + a_1 \partial_c \sigma_{ab} \partial^c \tilde{\sigma}^{ab} + a_1 \partial_a \sigma \partial_b \tilde{\sigma}^{ab} - a_0 \partial_a \tilde{\sigma} \partial_b \sigma^{ab} - 2a_1 \partial_c \sigma_{ab} \partial^a \tilde{\sigma}^{bc} \right\} . \quad (8.2.86)$$

Requiring now invariance under $\tilde{\delta}$ (8.2.69), we get

$$\tilde{\delta} S_{mix} = 0 = 2 \int d^3x \left\{ \sigma_{ab} (a_0 + a_1) \partial^a \partial^b \partial^m \tilde{\xi}_m - \sigma (a_0 + a_1) \partial_m \partial^2 \tilde{\xi}^m \right\} , \quad (8.2.87)$$

hence it must be

$$a_1 = -a_0 . \quad (8.2.88)$$

Therefore the mixed action term (8.2.86) invariant under both $\tilde{\delta}$ (8.2.69) and δ (8.2.70) is

$$S_{mix} = a_0 \int d^3x \left\{ \partial_a \sigma \partial^a \tilde{\sigma} - \partial_c \sigma_{ab} \partial^c \tilde{\sigma}^{ab} - \partial_a \sigma \partial_b \tilde{\sigma}^{ab} - \partial_a \tilde{\sigma} \partial_b \sigma^{ab} + 2 \partial_c \sigma_{ab} \partial^a \tilde{\sigma}^{bc} \right\} . \quad (8.2.89)$$

After reabsorbing the a_0 parameter into κ_m in (8.2.76), we observe that using the definitions of $f^{abc}(X)$ (8.2.79) and $\tilde{f}^{abc}(X)$ (8.2.80) S_{mix} (8.2.89) can be written as

$$S_{mix} = \int d^3x \left(\frac{1}{4} f_{ac}^a \tilde{f}_b^{bc} - \frac{1}{6} f^{abc} \tilde{f}_{abc} \right) . \quad (8.2.90)$$

The most general invariant action therefore is

$$\begin{aligned} S_{3D} &= \kappa S_{LG} + \tilde{\kappa} \tilde{S}_{LG} + \kappa_m S_{mix} \\ &= \int d^3x \left\{ \kappa \left(\frac{1}{4} f_{ac}^a f_b^{bc} - \frac{1}{6} f^{abc} f_{abc} \right) + \tilde{\kappa} \left(\frac{1}{4} \tilde{f}_{ac}^a \tilde{f}_b^{bc} - \frac{1}{6} \tilde{f}^{abc} \tilde{f}_{abc} \right) + \right. \\ &\quad \left. + \kappa_m \left(\frac{1}{4} f_{ac}^a \tilde{f}_b^{bc} - \frac{1}{6} f^{abc} \tilde{f}_{abc} \right) \right\} . \end{aligned} \quad (8.2.91)$$

We finally observe that S_{mix} (8.2.90) can be written as

$$S_{mix} = \int d^3x \epsilon^{abc} \epsilon^{def} \sigma_{ad} \partial_b \partial_e \tilde{\sigma}_{cf} , \quad (8.2.92)$$

hence, replacing $\tilde{\sigma}_{ab}(X)$ with $\sigma_{ab}(X)$ we have an alternative way to write the 3D LG action

$$S_{LG} = \int d^3x (\epsilon^{abc} \partial_b \sigma_{am}) (\epsilon^{pnm} \partial_n \sigma_{pc}) , \quad (8.2.93)$$

whose EoM are

$$\epsilon^{ap_1 p_2} \epsilon^{bp_3 p_4} \partial_{p_1} \partial_{p_3} h_{p_2 p_4} = 0 , \quad (8.2.94)$$

which are those of LG written in an alternative and more compact way. A similar expression holds for 4D LG, whose EoM can be written as

$$\epsilon^{\mu\alpha_1 \alpha_2 \alpha_3} \epsilon^{\nu\alpha_4 \alpha_5 \alpha_6} \eta_{\alpha_3 \alpha_6} \partial_{\alpha_1} \partial_{\alpha_4} h_{\alpha_2 \alpha_5} = 0 . \quad (8.2.95)$$

Equations of motion of the 3D induced theory

From the cyclicity property of $f^{abc}(X)$ and $\tilde{f}^{abc}(X)$ (8.2.81), (8.2.82), we find the following EoM for the boundary fields $\sigma_{ab}(X)$ and $\tilde{\sigma}_{ab}(X)$

$$\begin{aligned} \frac{\delta S_{3D}}{\delta \sigma_{mn}} = & \kappa \left[-\partial_a f^{mna} + \eta^{mn} \partial_a f_b^{ba} - \frac{1}{2} \left(\partial^m f_b^{bn} + \partial^n f_b^{bm} \right) \right] + \\ & + \frac{\kappa_m}{2} \left[-\partial_a \tilde{f}^{mna} + \eta^{mn} \partial_a \tilde{f}_b^{ba} - \frac{1}{2} \left(\partial^m \tilde{f}_b^{bn} + \partial^n \tilde{f}_b^{bm} \right) \right] = 0 \end{aligned} \quad (8.2.96)$$

$$\begin{aligned} \frac{\delta S_{3D}}{\delta \tilde{\sigma}_{mn}} = & \tilde{\kappa} \left[-\partial_a \tilde{f}^{mna} + \eta^{mn} \partial_a \tilde{f}_b^{ba} - \frac{1}{2} \left(\partial^m \tilde{f}_b^{bn} + \partial^n \tilde{f}_b^{bm} \right) \right] + \\ & + \frac{\kappa_m}{2} \left[-\partial_a f^{mna} + \eta^{mn} \partial_a f_b^{ba} - \frac{1}{2} \left(\partial^m f_b^{bn} + \partial^n f_b^{bm} \right) \right] = 0. \end{aligned} \quad (8.2.97)$$

8.2.3 Contact between bulk and boundary

It is possible to make a holographic contact between the 4D bulk theory described by the action S_{tot} (8.2.14) and the induced 3D theory whose action is S_{3D} (8.2.91) by requiring that the EoM (8.2.96) and (8.2.97) derived from S_{3D} coincide with the BC (8.2.24), (8.2.25) and (8.2.26) we found for the 4D bulk theory. This can be achieved by suitably fine tuning the ξ_i parameters appearing in S_{bd} (8.2.11), and κ , $\tilde{\kappa}$, κ_m in S_{3D} (8.2.91). The first step is to write the BC (8.2.24), (8.2.25) and (8.2.26) in terms of the boundary fields $\sigma_{ab}(X)$ and $\tilde{\sigma}_{ab}(X)$ through the solutions (8.2.67) and (8.2.68). The BC (8.2.24) is the defining equation for $h_{ab}(X)$ on the boundary (8.2.68), thus the contact is automatically satisfied. Concerning (8.2.25), using (8.2.68) we have, on $x^3 = 0$

$$0 = h^{ab} - \eta^{ab} h = M^{-\frac{3}{2}} \left[\partial^2 \sigma^{ab} + \partial^a \partial^b \sigma - \partial_c \left(\partial^a \sigma^{bc} + \partial^b \sigma^{ac} \right) + \eta^{ab} \left(\partial^c \partial^d \sigma_{cd} - \partial^2 \sigma \right) \right] \quad (8.2.98)$$

where M is the rescaling factor of $\sigma_{ab}(X)$ introduced in (8.2.75). This can also be written as

$$H^{mn} \equiv (h^{mn} - \eta^{mn} h) |_{(8.2.68)} = \frac{M^{-\frac{3}{2}}}{2} \left[-\partial_a f^{mna} + \eta^{mn} \partial_a f_b^{ba} - \frac{1}{2} \left(\partial^m f_b^{bn} + \partial^n f_b^{bm} \right) \right]. \quad (8.2.99)$$

Analogously

$$\tilde{H}^{mn} \equiv (\tilde{h}^{mn} - \eta^{mn} \tilde{h}) |_{(8.2.68)} = \frac{\tilde{M}^{-\frac{1}{2}}}{2} \left[-\partial_a \tilde{f}^{mna} + \eta^{mn} \partial_a \tilde{f}_b^{ba} - \frac{1}{2} \left(\partial^m \tilde{f}_b^{bn} + \partial^n \tilde{f}_b^{bm} \right) \right], \quad (8.2.100)$$

where \tilde{M} is the rescaling factor of $\tilde{\sigma}_{ab}(X)$ introduced in (8.2.75). We introduced H^{ab} and \tilde{H}^{ab} so that the contact between the bulk BC and the boundary EoM will be more evident, as we shall see. Indeed the 3D EoM (8.2.96) and (8.2.97) can be written as linear combination of (8.2.99) and (8.2.100)

$$\alpha H^{mn} + \beta \tilde{H}^{mn} = 0. \quad (8.2.101)$$

Explicitly we have

$$(8.2.96) = 2\kappa M^{\frac{3}{2}} H^{ab} + \kappa_m \tilde{M}^{\frac{1}{2}} \tilde{H}^{ab} = 0 \quad (8.2.102)$$

$$(8.2.97) = \kappa_m M^{\frac{3}{2}} H^{ab} + 2\tilde{\kappa} \tilde{M}^{\frac{1}{2}} \tilde{H}^{ab} = 0. \quad (8.2.103)$$

The BC (8.2.25) can be written as

$$\xi_1 H^{ab} = 0, \quad (8.2.104)$$

while the BC (8.2.26) cannot be written as (8.2.101)

$$2\xi_0 H^{ab} + (2\lambda + \xi_1)\tilde{H}^{ab} + \xi_2(\epsilon^{aij}\partial_i h_j^b + \epsilon^{bij}\partial_i h_j^a) + 2(\xi_3 + \xi_0)\eta^{ab}h = 0, \quad (8.2.105)$$

unless

$$\xi_3 = -\xi_0 \quad ; \quad \xi_2 = 0, \quad (8.2.106)$$

in which case the BC (8.2.26) becomes

$$2\xi_0 H^{ab} + (2\lambda + \xi_1)\tilde{H}^{ab} = 0, \quad (8.2.107)$$

recalling that λ is the coefficient of the bulk action S_{bulk} (8.2.1). Now that both EoM and BC have a similar structure, we can *holographically* match them by tuning their parameters so that (EoM) \leftrightarrow (BC). Keeping in mind that the ξ_1 parameter defines two situations

- $\xi_1 = 0$: one BC (8.2.107)
- $\xi_1 \neq 0$: two BC (8.2.104) and (8.2.107),

let us look at the first case.

1. $\xi_1 = 0$: the only BC is, after a multiplication by $\frac{1}{2\lambda}$ (remember that $\lambda \neq 0$, being the coupling constant of the bulk)

$$\frac{\xi_0}{\lambda} H^{ab} + \tilde{H}^{ab} = 0. \quad (8.2.108)$$

In the same way we have seen that the EoM (8.2.96) and (8.2.97) can be written as (8.2.102), (8.2.103)

$$2\mu \frac{\kappa}{\kappa_m} H^{ab} + \tilde{H}^{ab} = 0 \quad (8.2.109)$$

$$\frac{\mu}{2} \frac{\kappa_m}{\tilde{\kappa}} H^{ab} + \tilde{H}^{ab} = 0, \quad (8.2.110)$$

where $\mu \equiv \sqrt{\frac{M^3}{M}}$ with $[\mu] = 1$. They both match with the BC (8.2.108) if

$$\frac{\xi_0}{\lambda} = 2\mu \frac{\kappa}{\kappa_m} = \frac{\mu}{2} \frac{\kappa_m}{\tilde{\kappa}} \quad \Rightarrow \quad \kappa_m^2 = 4\kappa\tilde{\kappa}, \quad \kappa\tilde{\kappa} > 0. \quad (8.2.111)$$

The implication on the 3D action (8.2.91) is that the following redefinition of the fields is possible

$$\rho_{ab} \equiv \sqrt{\kappa}\sigma_{ab} \pm \sqrt{\tilde{\kappa}}\tilde{\sigma}_{ab} \quad ; \quad \Phi_{abc} \equiv \sqrt{\kappa}f_{abc} \pm \sqrt{\tilde{\kappa}}\tilde{f}_{abc}, \quad (8.2.112)$$

such that the action only depends on one field as

$$\begin{aligned} S_{3D} &= \frac{1}{6} \int d^3x \left(\frac{1}{\sqrt{2}} \eta^{ab} \Phi_m^{mc} - \Phi^{abc} \right) \left(\frac{1}{\sqrt{2}} \eta_{ab} \Phi_{nc}^n + \Phi_{abc} \right) \\ &= \int d^3x \left(\frac{1}{4} \Phi_m^{mc} \Phi_{nc}^n - \frac{1}{6} \Phi^{abc} \Phi_{abc} \right) \\ &= S_{LG}[\rho], \end{aligned} \quad (8.2.113)$$

which is LG in 3D. The sign \pm in (8.2.112) depends on the sign of κ_m as a consequence of the contact (8.2.111) for which $\kappa_m = \pm 2\sqrt{\kappa\tilde{\kappa}}$. Notice that if $\xi_1 = 0$, S_{bd} (8.2.11) becomes

$$S_{bd} = \xi_0 \int d^4x \delta(x^3) \left(h_{ab} h^{ab} - h^2 \right), \quad (8.2.114)$$

i.e. the boundary action S_{bd} (8.2.11) does not depend on the ∂_3 -derivative of the gauge field anymore. We recognize in S_{bd} the Fierz-Pauli mass term (8.1.3) [211, 214, 215, 218, 219], which renders the relation with LG even more remarkable. This allows to interpret ξ_0 as a Fierz-Pauli mass for the tensor field $h_{ab}(X)$ on the boundary $x^3 = 0$.

2. $\xi_1 \neq 0$: the BC are the following

$$H^{ab} = 0 \quad (8.2.115)$$

$$2\xi_0 \cancel{H^{ab}} + (2\lambda + \xi_1) \tilde{H}^{ab} = 0. \quad (8.2.116)$$

We have to distinguish between two cases: $2\lambda + \xi_1 = 0$ and $2\lambda + \xi_1 \neq 0$. For $\xi_1 = -2\lambda$ we are left with the BC (8.2.115) only, which depends on $h_{ab}(X)$, hence on $\sigma_{ab}(X)$ through the solution (8.2.68). To have a contact, we have to switch off the $\tilde{\sigma}_{ab}(X)$ dependence in the 3D action (8.2.91) (and in the EoM (8.2.102), (8.2.103)) by putting $\kappa_m = \tilde{\kappa} = 0$. The induced theory in this case is LG for the field $\sigma_{ab}(X)$

$$S_{3D} = \kappa S_{LG} = \kappa \int d^3x \left(\frac{1}{4} f_{ac}^a f_b^{bc} - \frac{1}{6} f^{abc} f_{abc} \right), \quad (8.2.117)$$

where we can reabsorb the κ parameter through a redefinition of the field $\sigma_{ab}(X)$. The boundary action term S_{bd} (8.2.11) is

$$S_{bd} = \int d^4x \delta(x^3) \left[\xi_0 (h_{ab} h^{ab} - h^2) - 2\lambda (\tilde{h}_{ab} h^{ab} - \tilde{h}h) \right]. \quad (8.2.118)$$

Notice that also in this case the ξ_0 parameter plays the role of a Fierz-Pauli mass for $h_{ab}(X)$ on the boundary. If instead $\xi_1 \neq \{-2\lambda, 0\}$

$$H^{ab} = 0 \quad (8.2.119)$$

$$\tilde{H}^{ab} = 0. \quad (8.2.120)$$

Looking at the 3D boundary-side (EoM) we can use the EoM (8.2.109)

$$\tilde{H}^{ab} = -2\mu \frac{\kappa}{\kappa_m} H^{ab} \quad (8.2.121)$$

in the EoM (8.2.110), which becomes

$$\frac{\mu}{2} \left(\frac{\kappa_m^2 - 4\kappa\tilde{\kappa}}{\tilde{\kappa}\kappa_m} \right) H^{ab} = 0 \quad ; \quad \tilde{\kappa}, \kappa_m \neq 0. \quad (8.2.122)$$

Now we notice that if $\kappa_m^2 - 4\kappa\tilde{\kappa} = 0$ the EoM (8.2.122) becomes trivial, and we only have one EoM, which is (8.2.121), which can never match the two BC (8.2.115) and (8.2.116) at the same time. Indeed this case ($\kappa_m^2 - 4\kappa\tilde{\kappa} = 0$) allows a contact only if we look at the BC in the form (8.2.104) and (8.2.107) and set $\xi_1 = 0$, which

coincide with Case 1 (8.2.111). Therefore $\kappa_m^2 - 4\kappa\tilde{\kappa} = 0 \Leftrightarrow \xi_1 = 0$. Considering $\kappa_m^2 - 4\kappa\tilde{\kappa} \neq 0$ we can use the second EoM (8.2.122) back into the first one (8.2.121) and get

$$H^{ab} = 0 \quad (8.2.123)$$

$$\tilde{H}^{ab} = 0 \quad ; \quad \kappa_m^2 - 4\kappa\tilde{\kappa} \neq 0, \tilde{\kappa}, \kappa_m \neq 0, \quad (8.2.124)$$

which matches exactly the BC (8.2.119) and (8.2.120). Thus the holographic contact is possible for $\xi_1 \neq \{-2\lambda, 0\}$, $\kappa_m \neq \{0, 2\sqrt{\kappa\tilde{\kappa}}\}$ and $\tilde{\kappa} \neq 0$. Again ξ_0 does not affect the contact and can be interpreted as a Fierz-Pauli mass.

We summarize our results in the following Table 7

S_{bd} parameters	Constraints	$S_{bd} =$	$S_{3D} =$
$\xi_1 = 0, \xi_0$ free	$\kappa = \frac{\xi_0}{2\mu\lambda}\kappa_m$; $\tilde{\kappa} = \frac{\mu\lambda}{2\xi_0}\kappa_m$; $\kappa_m^2 = 4\kappa\tilde{\kappa}$	$S_{bd}[h]$	$S_{LG}[\rho]$
$\xi_1 = -2\lambda, \xi_0$ free	κ free; $\tilde{\kappa} = 0$; $\kappa_m = 0$	$S_{bd}[h, \tilde{h}]$	$S_{LG}[\sigma]$
$\xi_1 \neq \{-2\lambda, 0\}, \xi_0$ free	κ free; $\tilde{\kappa} \neq 0$; $\kappa_m \neq \{0, 2\sqrt{\kappa\tilde{\kappa}}\}$	$S_{bd}[h, \tilde{h}]$	$\kappa S_{LG} + \tilde{\kappa} \tilde{S}_{LG} + \kappa_m S_{mix}$

Table 7: Scheme of the contacts between bulk (BC) and boundary (EoM) with constraints on the parameters of S_{bd} (8.2.11) and of S_{3D} (8.2.91).

As we see from Table 7, depending on the value of the ξ_1 parameter of S_{bd} (8.2.11), we found two possibilities for the 3D theory induced by the presence of a planar boundary on the 4D LG theory

- $\xi_1 = \{0, -2\lambda\}$ the 3D induced theory is LG for one symmetric rank-2 tensor field

$$S_{3D} = S_{LG} . \quad (8.2.125)$$

The corresponding 4D S_{bd} (8.2.114) is a Fierz-Pauli mass term for the 4D tensor field $h_{ab}(X)$ whose mass parameter is ξ_0 .

- $\xi_1 \neq \{0, -2\lambda\}$ the 3D induced action depends on two rank-2 symmetric tensor fields $\sigma_{ab}(X)$ and $\tilde{\sigma}_{ab}(X)$. After a field redefinition, it reads

$$S_{3D}[\sigma, \tilde{\sigma}] = S_{LG}[\sigma] + \tilde{S}_{LG}[\tilde{\sigma}] + k S_{mix}[\sigma, \tilde{\sigma}] , \quad (8.2.126)$$

where k is a constant which cannot be reabsorbed.

In both cases, the 4D boundary term S_{bd} (8.2.11) contains a Fierz-Pauli mass term for the bulk tensor field $h_{ab}(X)$, whose mass parameter is ξ_0 .

8.2.4 Summary of results

In this Section we studied the effect of the presence of a planar boundary on 4D LG, realized by means of a Heaviside step function in the action (8.2.1). Following the QFT approach presented in Chapter 2 we observed that the presence of the boundary breaks the invariance under diffeomorphisms, which are the symmetry transformations of LG. Correspondingly, the Ward identity which describes the invariance under diffeomorphisms (8.2.30) acquires a breaking, which is crucial, because from it the main information of the theory might be derived, namely the fields content, the symmetry

transformations and the boundary algebra. We wrote “might” because it is not obvious that this can always be done. In fact, we stressed throughout this whole Thesis that this seemed to work for all TQFTs, where non trivial boundary dynamics has been first observed, and for a long time this property has been believed to be peculiar of these kind of theories. But we have seen that more recently, similar results have been found in non-topological field theories, like Maxwell theory [1], and this motivated the boundary investigations of this Thesis for more general theories, like we did in this Section for LG. A first remarkable result is that on the boundary we found two conserved currents (8.2.28) and (8.2.36) which form the algebraic structure (8.2.54) and (8.2.55) of the KM type, whose central charge is proportional to the inverse of the LG “coupling” constant (8.2.42). This confirms what has been guessed in [227], where it was suspected the existence, in 4D LG, of a KM algebra as a particularly interesting possibility in connection with Weinberg’s soft graviton theorems [247–249]. Since the central charge of a KM algebra must be positive, this is mostly useful to determine the sign of the overall LG action, which otherwise should be determined by imposing that the energy density, that is the 00-component of the energy-momentum tensor, is positive, which in gravity is a known tricky issue [105, 250]. Moreover, we were able to solve the on-shell Ward identity (8.2.56) and the universal BC (8.2.57) getting (8.2.67) and (8.2.68), which allowed us to express, on the boundary, the 4D bulk fields $h^{ab}(X)$ and $\tilde{h}^{ab}(X)$ in terms of 3D fields which are the degrees of freedom of the induced 3D theory. We found that these latter, like their 4D ancestors, are rank-2 symmetric tensor fields : $\sigma^{ab}(X)$ and $\tilde{\sigma}^{ab}(X)$. This, as LG shows, seems to be peculiar of non topological QFTs. Indeed what is usually found in TQFTs is that the fields living on the $d - 1$ -dimensional boundary are tensors of lower rank with respect their d -dimensional counterpart : from rank-2 tensors one finds vectors in the topological 4D BF theory [88] and the boundary reduction of the gauge field in Chern-Simons theory gives scalars, as we have seen in Chapters 4 and 5. Here, instead, the 3D boundary fields are rank-2 symmetric tensor fields as those of 4D LG. And, quite interestingly, the transformation which keeps invariant the definition of the boundary fields turns out to be the diffeomorphisms (8.2.69) and (8.2.70), which therefore are a consequence of the introduction of the boundary, rather than an *a priori* request. Given the dynamical fields and the symmetry transformations, requiring locality and power counting allowed us to find the most general 3D action S_{3D} (8.2.91), which consists of three terms. Each term being invariant by its own, S_{3D} depends on three constants which we do not reduced by redefining the 3D fields as we could, but we fixed them by establishing a “holographic” contact as our last step. This has been realized by requiring that the EoM of the 3D action S_{3D} coincide with the BC of the 4D theory. To do that, we had at our disposal the 4 parameters on which S_{bd} (8.2.11) depends and the three constants in S_{3D} (8.2.91). As an outcome of this tuning, we found two possibilities, depending on the value of one particular parameter appearing in S_{bd} : S_{3D} describes either LG for one single tensor field (8.2.125), or the action (8.2.126), containing two decoupled LG terms for the boundary tensor fields $\sigma^{ab}(X)$ and $\tilde{\sigma}^{ab}(X)$ and one term which mixes them. As a last, but probably not least, fact, we remark that in any case the S_{bd} action term which governs the holographic contact contains a mass term (8.2.114) for the bulk tensor field $h_{ab}(x)$ of the particular Fierz-Pauli type [211, 214, 215, 218, 219], with a free parameter ξ_0 which we can interpret as a mass.

Part IV

FINAL REMARKS

CONCLUDING REMARKS

We started with the idea of an analysis of boundary effects on theories both standard and not, within a formal QFT approach. We ended up doing this and more. Indeed the formal requirements of QFTs, together with phenomenological inputs, also pushed us into the development of a new, well defined, QFT for fracton quasiparticles [4]. This was surely necessary in order to study boundary effects on fracton models in a precise and QFT-oriented way, but the fact of having a proper QFT for such models means even more, since one is now able to delve into the secrets of fractons without boundary as well, and without the necessity of invoking “intuitions” as it is explicitly admitted in the seminal papers [160, 161], but just by applying straightforwardly QFT first principles. All that would not have happened if it were not for phenomenological inputs, which made us look with interest into the theory of fractons, together with the desire to only trust first principles and that “high flexibility and perfection of the QFT formalism” that motivated Symanzik. With this spirit we were thus able to build a new theory, but also to expand the dictionary of QFTs with boundary : we observed the effect of a metric in TQFTs to have a non-trivial effect on the edge states, but we also showed that the bulk-boundary paradigm is much wider than expected, being extensible also to non-TQFTs. We have remarked many times that dealing with boundaries may have surprising outcomes, and this just by thinking about the standard, textbook example of TQFTs with boundary in flat spacetime, from which one starts with an unphysical theory and end up talking about experimental observation on the edge of Hall systems. But, as we shall see in more details in what follows, the results presented in this Thesis concerns an even wider variety of areas of physics : condensed matter (Fractional Quantum Hall Effect, Topological Insulators, higher order Topological Insulators, Quantum Anomalous Hall Effect, fracton phases...), elasticity, gravity (massive, self dual, torsional effects...) all these are words that have appeared when discussing about the results we obtained on the edge of either TQFTs, fractons or Linearized Gravity with boundary.

Final overview on the specific results

The results achieved in this Thesis could be quickly summarized by saying that on the boundary of TQFTs on a generic manifold in curved spacetime Hall systems with accelerated edge modes are found, a 3D traceless fracton model can be recovered as induced by a traceful scalar charge theory of fractons in the bulk, and finally a 3D LG-like theory exists on the edge of 4D LG itself. But there is much more to say about the results obtained from the works collected in this Thesis, both dealing with boundary effects and not. In particular

TQFTS IN CURVED SPACETIMES : even though the metric does not affect the bulk of the TQFT, by definition of a topological theory, a memory of it is kept by the boundary, from which we observed that the metric does intervene in the local observables of the

lower-dimensional theory, namely in the chiral velocities. The dictionary has thus been updated as follows

Flat TQFT with boundary	\Rightarrow	constant chiral edge velocities
Curved TQFT with boundary	\Rightarrow	local chiral edge velocities .

This allows for a formal theoretical explanation of the observed accelerated edge modes of some condensed matter systems, which could not be justified in terms of the standard flat bulk-boundary approach, but only through an *ad hoc* introduction coming from phenomenological requirements. Moreover, with our results we can make a contact between these phenomenological requirements (confining potential...) and the QFT procedure : the induced metric $\gamma(x)$ play the role of the phenomenological potential $V(x)$ which makes the edge velocities acquire spacetime dependence *i.e.* $v = v(x)$. This same thing happen, also in the flat case, when thinking about the Chern-Simons coupling constant κ , which represents the filling factor ν of the phenomenological model on the boundary. These results can be found in [2] and [3].

NEW QFT FOR FRACTONS : built from a QFT symmetry-based approach, this new *covariant* theory for fractons allowed to recover all the results of the Literature from first principles of QFT, and more. Indeed we observed

- an immediate relation with LG. This claim can be found in the fracton Literature [177, 198–200], but it is immediately apparent from the field theoretical point of view. This relation appears as an additional term in the action with its own “coupling”. The fact that a quadratic, Lorentz and gauge invariant theory depends on one unavoidable constant is quite uncommon, if not unique. It is not even clear how to call this constant, since it cannot be a “coupling” constant, being the theory S_{inv} (7.2.7) free and non interacting, nor a mass, being dimensionless. This peculiarity originates from the fact that the space of functionals invariant under the covariant fractonic transformation (7.2.5) has dimension two, instead of one as it commonly happens. To our knowledge, the only exception is given by the 3D Maxwell-Chern-Simons theory [70], which depends on one “true” constant as well, but in that case the constant can be identified as a mass. We might say that gravitons may exist alone, while fractons necessarily come with gravitons, lacking, up to now, a symmetry which uniquely determines them.
- the presence of an energy momentum tensor which has strong analogies with the standard electromagnetic one. The possibility of computing it as a variation of the metric was allowed only by the fact that we had a covariant theory, and confirmed, through the computation of the “Lorentz” force, that the theory concerned fully mobile dipoles, as one would expect from a fracton model. We however observe that such energy-momentum tensor is not conserved, which, again, could be a confirmation of the theory we are dealing with : standard non-covariant fracton models display broken translational symmetry, as a consequence of dipole conservation. In our covariant theory this can be encoded in the non-conservation of the energy-momentum tensor (its spatial components in particular). From a formal point of view one can also notice that this is in agreement with Weinberg-Witten theorem [251] when keeping in mind that here we have massless spin two objects

Theorem 2 : A theory that allows the construction of a conserved Lorentz covariant energy-momentum tensor $T^{\mu\nu}$ for which $\int d^3x T^{\mu\nu}$ is the energy-momentum four-vector cannot contain massless particles of spin $j > 1$.

Additionally, another interesting observation on that regard is that the stress-energy tensor is not traceless, hence the theory is not scale invariant, differently from the classical Maxwell theory. Instead, tracelessness is recovered in 6D. Had we dealt with a field theory exercise, it would have been better to face the problem in 6D, where the gauge field has mass dimensions $[A_{\mu\nu}] = 2$ and, consequently, the gauge parameter would have been given vanishing dimensions, as usual in gauge field theory. But the most general invariant action (7.2.7) consists of two terms : S_{LG} (7.2.9) and S_{fract} (7.2.8) which are respectively the actions for linearized gravity and for fractons, both physically relevant in 4D.

- every physical quantity is invariant at sight. The definition of the invariant rank-3 field strength $F_{\mu\nu\rho}(x)$ in terms of which the electric and magnetic tensor fields $E^{ij}(x)$ (7.2.40) and $B^{ij}(x)$ (7.2.48) of fractons can be written, makes them immediately invariant quantities under the fracton symmetry (7.2.5), allowing for them to be identifiable as the physical objects of the theory.
- a nontrivial boundary physics. As a consequence of the peculiarity of the fractonic symmetry, a “generalized” Kac-Moody algebra is recovered on the boundary, which matches exactly the one observed in the context of higher order Topological Insulators, thus adding a checkmark on the always suspected relation between these materials and edge states of fracton models [55, 184]. The boundary theory display a curious phenomenon for which the edge DoF turn out to be traceless, and the physical model appears to be related to the traceless scalar charge theory of fractons where a higher rank Chern-Simons contribution plays the role of matter, in particular as the charge in the Gauss constraint and of current in the Ampère law. This means that the particle content of the boundary theory involves fractons as the immobile charges, and dipoles as one-dimensional particles (lineons). A broken Faraday equation also suggest the possible presence of fractonic vortices.

These results are collected in [4], [5] and [6].

LINEARIZED GRAVITY : we observe the emergence of a standard KM algebra on the boundary which thus makes us able to give a confirmation on speculations about its existence on the boundary of LG. This example highlights the drastic consequences of considering a different symmetry in the context of a theory with boundary. Indeed, in comparison with the results on the boundary of the theory of fractons and LG of Section 7.4, defined by invariance under longitudinal diffeomorphisms (a.k.a. fracton symmetry), the more general symmetry of diffeomorphisms, defining LG alone, makes evident how results on the boundary are strictly related to the symmetry involved. This fact can be better expressed by the following Table, which compares the results in the two cases

	Fractons and LG ($g_{1,2} \neq 0$)	LG ($g_1 = 0$)
Defining symmetry	Longitudinal diffs $\partial_\mu \partial_\nu \Lambda$	diffs $\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu$
Boundary algebra	Generalized KM	Standard KM
Boundary DoF	α_{mn} symmetric, traceless	σ_{mn} symmetric
3D theory	Maxwell-Chern-Simons-like	3D LG
Induced physics	Traceless fracton phases	3D Linearized gravity-like

Table 8: comparison between the theory of fractons and LG with boundary and of LG alone with boundary.

It is interesting to notice that it seems that the pure LG case is the first situation where a theory with boundary does not involve results in the context of condensed matter. At least at first sight. These results were obtained in [7].

Concerning the Future

We conclude the Thesis with some remarks on future prospects which we think will be possible thanks to the results achieved here. From Chapter 5, *i.e.* TQFTs with boundary in curved spacetime, the first thing to mention is that as a consequence of the necessity of an update of the dictionary of TQFTs with boundary for explaining the observed local velocities of edge states of FQHE, we are now also able to predict the existence of local velocities of edge modes in other Hall systems, like, in particular, TI. Another interesting aspect to analyze from this new dictionary, is the relation between the induced metric on the boundary of TQFTs, and the confining/interaction potential responsible in the phenomenological models for such local velocities. For instance one could think of a perturbation expansion of the induced metric $\gamma_{ij}(x) = \eta_{ij} + h_{ij}(x)$, as one does in LG, isolating the spacetime dependent part (which would be related to the metric perturbation $h_{ij}(x)$) from the standard, constant, contribution, and see how this non-constant, metric perturbation can encode the *ad hoc* potential typically introduced in the models. That would complete the “duality” of the dictionary : we knew the relation between the coupling and the filling factor $\kappa \leftrightarrow \nu$, and now we also add the relation between the metric and the potential $h_{ij} \leftrightarrow V$. Concerning Part III, the fact of having found a non-trivial physics on the boundary of non-TQFTs, on its own, is already a striking result which should enhance the interest in investigating other non-TQFTs with boundary. Even more because of the richness in the physical content of those analyzed in the present work. Indeed from the study of both fracton and LG theories with boundary, many questions and insights emerge. For instance on the boundary of the theory of fractons described by the action (7.4.1) a 3D traceless scalar charge theory arise, which is a new 3D fracton model. The tracelessness of the theory is an unexpected consequence of the DoF induced by the boundary, thus a consequence of the fracton symmetry as a breaking from a Ward identity. We see in fact that a different, more general, symmetry such as the one defining LG (*i.e.* diffs) gives traceful DoF, meaning that maybe these disappearance of trace dependence is encoded in something hidden, which still has to be looked for. Concerning this 3D theory, it is known [148] that fractons in 3D possess a duality with the theory of elasticity, as an analog higher-rank version of the particle-vortex duality [90, 91], therefore

the possibility that the 3D action recovered on the boundary shares a form of such higher spin duality is highly nontrivial and worth to be investigated. It will also be interesting to further analyze the relations which both the pure theory of LG (8.2.1) and the fracton one (7.4.1) seem to have with a variety of models of gravity and massive gravity, on the boundary. Indeed we have seen that the Symanzik boundary term (8.2.11), necessary in LG, needs to contain a Fierz-Pauli mass term (8.1.3) in order to induce a lower-dimensional physics. Moreover, the Chern-Simons-like contribution contained in the 3D traceless theory (7.4.64) induced by fractons, strongly reminds a gravity model known as “self-dual massive gravity”, and the full action appears as a higher rank Maxwell-Chern-Simons term. However it is interesting to notice that the coupling seems to be dimensionless, which means that to better understand the theory and relations with the above-mentioned models the gauge structure and propagators of the theory will have to be computed. But in this Thesis results does not only involve boundaries. Indeed also the covariant QFT of fractons is a novelty, and one of great interest. In fact, having now a new and well defined theory for these quasiparticles opens the doors at many other QFT-oriented computations and physical analysis. The first that comes to mind is to apply the same approach of the four-dimensional bulk theory to a three-dimensional one, with the same spirit of building everything from first principles like symmetry, power-counting and covariance, and from this to look at the duality between fractons and elasticity from the QFT side. In this context 3D fracton models also seem to share relations with theories with torsion [233], and since fractons and LG are strongly intertwined (remind that no symmetry principle allow us to isolate fractons from gravity!), this fact makes very promizing the idea of investigating torsional effect from a theory of gravity, in order to see if fractons may come out of it and thus better understand their nature. This could also help us with the task of finding a way (symmetry principle,...) to isolate fractons from LG without the need of switching off any coefficient by hand. Sharing so many similarities with Maxwell theory, a final remark concerns the possibility of promoting the fracton theory to a non-abelian one, and the possible quantization of the action (7.2.7). As it is well known, the quantization of LG is a long standing issue. As far as we know, a quantum field theory of fractons has not been achieved yet. In view of this, the covariant formulation adopted in this Thesis should be quite suitable, especially because the fractonic part (7.2.8) of the action (7.2.7) impressively reminds the electromagnetic Maxwell theory. Hence, one might think about a kind of “fracton QED”, where matter is coupled to fractons. Under this respect, the vector gauge fixing studied in Section 7.3 can be very useful, not being restricted to the Landau gauge.

APPENDIX

SOME TECHNICAL REMARKS

A.1 DERIVATIVE OF THE HEAVISIDE STEP DISTRIBUTION IN CURVED SPACETIME

The scalar Dirac delta distribution $\delta^{(n)}(x-x')$ and the corresponding density $\tilde{\delta}^{(n)}(x-x')$ are related by [113, 252]

$$\delta^{(n)}(x-x') = \frac{\tilde{\delta}^{(n)}(x-x')}{\sqrt{-g}}, \quad (\text{A.1.1})$$

with $g \equiv g(x)$, acting on a test function $f(x)$ as follows

$$\int d^n x \sqrt{-g} \delta^{(n)}(x-x') f(x) = \int d^n x \tilde{\delta}^{(n)}(x-x') f(x) = f(x'). \quad (\text{A.1.2})$$

Hence, the functional derivative of a generic dual vector field $V_\mu(x)$ is the (1,1) tensor

$$\frac{\delta V_\mu(x)}{\delta V_\nu(x')} = \delta_\mu^\nu \delta^{(n)}(x-x'). \quad (\text{A.1.3})$$

The Heaviside theta distribution is used to describe a boundary. Its general form is $\theta(f(x))$, where $f(x) = f(t, r, \theta) = 0$ describes the equation of the hypersurface $\partial\mathcal{M}$ of the manifold \mathcal{M} , *i.e.*

$$\theta(f(x)) = \begin{cases} 1, & \text{if } x \in \mathcal{M} \\ 0, & \text{if } x \notin \mathcal{M}. \end{cases} \quad (\text{A.1.4})$$

For a generic manifold \mathcal{M} , Stokes' theorem states that

$$\int_{\mathcal{M}} d^n x \sqrt{-g} \nabla_\mu V^\mu = \int_{\partial\mathcal{M}} d^{n-1} y \sqrt{-\gamma} e_\mu V^\mu, \quad (\text{A.1.5})$$

where e_μ is the unit vector normal to the boundary $\partial\mathcal{M}$ described by the equation $f(x) = 0$, *i.e.*

$$e_\mu = -\frac{\partial_\mu f}{\sqrt{g^{\mu\nu} \partial_\mu f \partial_\nu f}}, \quad (\text{A.1.6})$$

and γ_{ij} is the induced metric on $\partial\mathcal{M}$. The presence of the boundary $f(x) = 0$ in the l.h.s. of (A.1.5) can be implemented through the introduction of the step distribution (A.1.4):

$$\begin{aligned} \int_{\mathcal{M}} d^n x \sqrt{|g|} \nabla_\mu V^\mu &= \int d^n x \sqrt{|g|} \theta(f(x)) \nabla_\mu V^\mu \\ &= - \int d^n x \sqrt{|g|} \nabla_\mu \theta(f(x)) V^\mu. \end{aligned} \quad (\text{A.1.7})$$

The integration on the r.h.s. of (A.1.7) is performed over all spacetime (*i.e.* boundary at infinity), then, when integrating by parts on the second line, the boundary term vanishes. We can also observe, from (A.1.7), that the step function is a scalar quantity. Identifying the r.h.s. of (A.1.5) and (A.1.7) we get

$$- \int d^n x \sqrt{|g|} \nabla_\mu \theta(f(x)) V^\mu = \int_{\partial\mathcal{M}} d^{n-1} x \sqrt{|\gamma|} e_\mu V^\mu. \quad (\text{A.1.8})$$

In (A.1.8) we can identify $\nabla_\mu \theta(f(x))$ as a scalar Dirac delta $\delta_{\partial\mathcal{M}}$ ¹⁰

$$-e_\mu \delta_{\partial\mathcal{M}} \equiv \nabla_\mu \theta(f(x)). \quad (\text{A.1.9})$$

To see it explicitly and deduce how this distribution acts, we insert (A.1.9) back in (A.1.8)

$$\int d^n x \sqrt{|g|} \delta_{\partial\mathcal{M}} e_\mu V^\mu = \int_{\partial\mathcal{M}} d^{n-1} x \sqrt{|\gamma|} e_\mu V^\mu. \quad (\text{A.1.10})$$

Considering a constant, radial boundary

$$f(x) = R - r \quad \Rightarrow \quad e_\mu = \frac{\delta_\mu^r}{\sqrt{g^{rr}}} = \delta_\mu^r \frac{\sqrt{|g|}}{\sqrt{|\gamma|}}, \quad (\text{A.1.11})$$

where we used [254] :

$$\sqrt{g^{rr}} = \frac{\sqrt{-\gamma}}{\sqrt{-g}}, \quad (\text{A.1.12})$$

we have

$$\delta_{\partial\mathcal{M}} = \frac{\sqrt{|\gamma|}}{\sqrt{|g|}} \delta(f(x)) = \frac{\sqrt{|\gamma|}}{\sqrt{|g|}} \delta(R - r). \quad (\text{A.1.13})$$

Therefore, using (A.1.11), the derivative of the step function (A.1.9) simplifies to

$$\nabla_\mu \theta(R - r) = \partial_\mu \theta(f(x)) = -\delta_\mu^r \delta(r - R). \quad (\text{A.1.14})$$

A.2 BASIS FOR THE Ω -TENSORS

In the momentum space gauge fixed action $S(g_1, g_2; \xi, \kappa)$ (7.3.16) the kinetic operator $\Omega_{\mu\nu, \alpha\beta}(p)$ displays the following symmetries

$$\Omega_{\mu\nu, \alpha\beta}(p) = \Omega_{\nu\mu, \alpha\beta}(p) = \Omega_{\mu\nu, \beta\alpha}(p) = \Omega_{\alpha\beta, \mu\nu}(p). \quad (\text{A.2.1})$$

It can be expanded on a basis formed by a set of five rank-4 tensors, collectively denoted $X_{\mu\nu, \alpha\beta}(p)$ [178, 214, 215, 218, 219]

$$X_{\mu\nu, \alpha\beta} \equiv (A, B, C, D, E)_{\mu\nu, \alpha\beta} \quad (\text{A.2.2})$$

with the same symmetry properties (A.2.1). Explicitly, the X -tensors read [178, 214, 215, 218, 219, 255]

$$A_{\mu\nu, \alpha\beta} = \frac{d_{\mu\nu} d_{\alpha\beta}}{3} \quad (\text{A.2.3})$$

$$B_{\mu\nu, \alpha\beta} = e_{\mu\nu} e_{\alpha\beta} \quad (\text{A.2.4})$$

$$C_{\mu\nu, \alpha\beta} = \frac{1}{2} \left(d_{\mu\alpha} d_{\nu\beta} + d_{\mu\beta} d_{\nu\alpha} - \frac{2}{3} d_{\mu\nu} d_{\alpha\beta} \right) \quad (\text{A.2.5})$$

$$D_{\mu\nu, \alpha\beta} = \frac{1}{2} (d_{\mu\alpha} e_{\nu\beta} + d_{\mu\beta} e_{\nu\alpha} + e_{\mu\alpha} d_{\nu\beta} + e_{\mu\beta} d_{\nu\alpha}) \quad (\text{A.2.6})$$

$$E_{\mu\nu, \alpha\beta} = \frac{\eta_{\mu\nu} \eta_{\alpha\beta}}{4}, \quad (\text{A.2.7})$$

¹⁰ We adopted the notation of [253], p.13 Eq.(4.10).

and $e_{\mu\nu}(p)$ and $d_{\mu\nu}(p)$ are the transverse and longitudinal projectors (7.3.21), which are idempotent and orthogonal

$$e_{\mu\lambda}e^\lambda{}_\nu = e_{\mu\nu}, \quad d_{\mu\lambda}d^\lambda{}_\nu = d_{\mu\nu}, \quad e_{\mu\lambda}d^\lambda{}_\nu = 0. \quad (\text{A.2.8})$$

The X -tensors have the following properties:

- decomposition of the rank-4 tensor identity $\mathcal{I}_{\mu\nu,\alpha\beta}$:

$$A_{\mu\nu,\alpha\beta} + B_{\mu\nu,\alpha\beta} + C_{\mu\nu,\alpha\beta} + D_{\mu\nu,\alpha\beta} = \mathcal{I}_{\mu\nu,\alpha\beta} \quad (\text{A.2.9})$$

$$\mathcal{I}_{\mu\nu,\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \quad (\text{A.2.10})$$

- idempotency :

$$X_{\mu\nu}{}^{\rho\sigma}X_{\rho\sigma,\alpha\beta} = X_{\mu\nu,\alpha\beta}; \quad (\text{A.2.11})$$

- orthogonality of A, B, C and D :

$$X_{\mu\nu,\alpha\beta}X'^{\alpha\beta}{}_{\rho\sigma} = 0 \text{ if } (X, X') \neq E \text{ and } X \neq X'; \quad (\text{A.2.12})$$

- contractions with E :

$$A_{\mu\nu,\alpha\beta}E^{\alpha\beta}{}_{\rho\sigma} = \frac{d_{\mu\nu}\eta_{\rho\sigma}}{4} \quad (\text{A.2.13})$$

$$B_{\mu\nu,\alpha\beta}E^{\alpha\beta}{}_{\rho\sigma} = \frac{e_{\mu\nu}\eta_{\rho\sigma}}{4} \quad (\text{A.2.14})$$

$$C_{\mu\nu,\alpha\beta}E^{\alpha\beta}{}_{\rho\sigma} = D_{\mu\nu,\alpha\beta}E^{\alpha\beta}{}_{\rho\sigma} = 0. \quad (\text{A.2.15})$$

DETAILED CALCULATIONS

B.1 SOLUTIONS OF THE BOUNDARY CONDITIONS FOR THE BF MODEL

The BC for the 3D bulk theory are given by (5.2.14):

$$\Lambda^{IJ} X_J|_{r=R} = 0, \quad (\text{B.1.1})$$

where $I, J = \{i; j\} = \{0, 2; 0, 2\}$,

$$\Lambda^{IJ} \equiv \begin{pmatrix} \alpha^{ij} & \zeta^{ij} - \kappa \epsilon^{i1j} \\ \zeta^{ji} & \beta^{ij} \end{pmatrix} = \begin{pmatrix} \alpha^{00} & \alpha^{02} & \zeta^{00} & \zeta^{02} - \hat{\kappa} \\ \alpha^{20} & \alpha^{22} & \zeta^{20} + \hat{\kappa} & \zeta^{22} \\ \zeta^{00} & \zeta^{20} & \beta^{00} & \beta^{02} \\ \zeta^{02} & \zeta^{22} & \beta^{02} & \beta^{22} \end{pmatrix}, \quad (\text{B.1.2})$$

and

$$X_J \equiv \begin{pmatrix} A_j \\ B_j \end{pmatrix}, \quad (\text{B.1.3})$$

and $\hat{\kappa}$ is given by (5.2.17). Being the linear system of equations (5.2.14) homogeneous, three of the four components X_J can be written in terms of the fourth, provided that

$$\det \Lambda = 0. \quad (\text{B.1.4})$$

The choice we make is

$$B_\theta(X) = -l_1 B_t(X) \quad (\text{B.1.5})$$

$$A_\theta(X) = -l_2 B_t(X) \quad (\text{B.1.6})$$

$$A_t(X) = -l_3 B_t(X), \quad (\text{B.1.7})$$

with

$$l_1 \equiv -\frac{-\alpha^{00}(\beta^{00}\zeta^{22} - \beta^{02}\zeta^{20}) + \alpha^{02}(\beta^{00}\zeta^{02} - \beta^{02}\zeta^{00}) + \zeta^{00} \det \zeta}{\alpha^{00}(\beta^{02}\zeta^{22} - \beta^{22}\zeta^{20}) + \alpha^{02}(\beta^{22}\zeta^{00} - \beta^{02}\zeta^{02}) - (\zeta^{02} - \hat{\kappa}) \det \zeta} \quad (\text{B.1.8})$$

$$l_2 \equiv -\frac{\alpha^{00} \det \beta - \beta^{00}\zeta^{02}(\zeta^{02} - \hat{\kappa}) - \hat{\kappa}\beta^{02}\zeta^{00} + 2\beta^{02}\zeta^{00}\zeta^{02} - \beta^{22}(\zeta^{00})^2}{\alpha^{00}(\beta^{02}\zeta^{22} - \beta^{22}\zeta^{20}) + \alpha^{02}(\beta^{22}\zeta^{00} - \beta^{02}\zeta^{02}) - (\zeta^{02} - \hat{\kappa}) \det \zeta} \quad (\text{B.1.9})$$

$$l_3 \equiv -\frac{-\alpha^{02} \det \beta + (\zeta^{02} - \hat{\kappa})(\beta^{00}\zeta^{22} - \beta^{02}\zeta^{20}) + \zeta^{00}(\beta^{22}\zeta^{20} - \beta^{02}\zeta^{22})}{\alpha^{00}(\beta^{02}\zeta^{22} - \beta^{22}\zeta^{20}) + \alpha^{02}(\beta^{22}\zeta^{00} - \beta^{02}\zeta^{02}) - (\zeta^{02} - \hat{\kappa}) \det \zeta}, \quad (\text{B.1.10})$$

where the request of non-Dirichelet solutions implies the requirements $l_i \neq 0$ and $l_i^{-1} \neq 0$. We remind that these coefficients are local, depending on the induced metric

determinant and/or components from the bulk parameters. In the case of \mathcal{T} -invariant S_{bd} (5.2.129) discussed in Section 5.2.4, the coefficients (5.2.136) are given by

$$l_1|_{(5.2.128)} = -\gamma^{t\theta} \frac{\hat{\beta}\alpha^{00}\hat{\zeta}^{20} + \hat{\alpha}\beta^{00}\hat{\zeta}^{02}}{-\alpha^{00}\beta^{22}\hat{\zeta}^{20} + \hat{\zeta}^{02} \left[\hat{\zeta}^{20}(\hat{\zeta}^{02} - \hat{\kappa}) - \hat{\alpha}\hat{\beta}(\gamma^{t\theta})^2 \right]} \quad (\text{B.1.11})$$

$$l_2|_{(5.2.128)} = -\frac{\alpha^{00} \det \beta - \beta^{00}\hat{\zeta}^{02}(\hat{\zeta}^{02} - \hat{\kappa})}{-\alpha^{00}\beta^{22}\hat{\zeta}^{20} + \hat{\zeta}^{02} \left[\hat{\zeta}^{20}(\hat{\zeta}^{02} - \hat{\kappa}) - \hat{\alpha}\hat{\beta}(\gamma^{t\theta})^2 \right]} \quad (\text{B.1.12})$$

$$l_3|_{(5.2.128)} = -\gamma^{t\theta} \frac{-\hat{\alpha} \det \beta - \hat{\beta}\hat{\zeta}^{20}(\hat{\zeta}^{02} - \hat{\kappa})}{-\alpha^{00}\beta^{22}\hat{\zeta}^{20} + \hat{\zeta}^{02} \left[\hat{\zeta}^{20}(\hat{\zeta}^{02} - \hat{\kappa}) - \hat{\alpha}\hat{\beta}(\gamma^{t\theta})^2 \right]} . \quad (\text{B.1.13})$$

B.2 CALCULATION OF THE FRACTON PROPAGATORS

B.2.1 $g_1 + 2g_2 \neq 0$

The matrix equation (7.3.30) yields

$$\Omega_{\mu\nu,\alpha\beta}\hat{G}^{\alpha\beta,\rho\sigma} + \Lambda_{\mu\nu,\lambda}^*\hat{G}^{*\lambda,\rho\sigma} = \mathcal{I}_{\mu\nu}^{\rho\sigma} \quad (\text{B.2.1})$$

$$\Omega_{\mu\nu,\alpha\beta}\hat{G}^{\alpha\beta,\tau} + \Lambda_{\mu\nu,\lambda}^*\hat{G}^{\lambda\tau} = 0 \quad (\text{B.2.2})$$

$$\Lambda_{\gamma,\alpha\beta}\hat{G}^{\alpha\beta,\rho\sigma} + H_{\gamma\lambda}\hat{G}^{*\lambda,\rho\sigma} = 0 \quad (\text{B.2.3})$$

$$\Lambda_{\gamma,\alpha\beta}\hat{G}^{\alpha\beta,\tau} + H_{\gamma\lambda}\hat{G}^{\lambda\tau} = \delta_\gamma^\tau . \quad (\text{B.2.4})$$

The first equation (B.2.1), using the expansions (7.3.18), (7.3.19), (7.3.26) and the properties of the X -basis listed in Appendix A.2, gives a system of six equations (remember that our aim is to find the set of coefficients of the propagators (7.3.29))

$$4t\hat{t} + 3t\hat{w} - 4v\hat{v} + 6p^2g\hat{g} = 0 \quad (\text{B.2.5})$$

$$t\hat{w} + 2p^2g\hat{l} = 0 \quad (\text{B.2.6})$$

$$p^2\hat{l} = 2 \quad (\text{B.2.7})$$

$$p^2l\hat{g} = 0 \quad (\text{B.2.8})$$

$$v\hat{v} = 1 \quad (\text{B.2.9})$$

$$z\hat{z} + p^2f\hat{f} = 1 . \quad (\text{B.2.10})$$

In the same way, from (B.2.2), (B.2.3) and (B.2.4) we get three more sets of equations

$$2z\hat{f} + f\hat{r} = 0 \quad (\text{B.2.11})$$

$$4t\hat{g} + 2g\hat{s} = 0 \quad (\text{B.2.12})$$

$$l\hat{s} = 0 \quad (\text{B.2.13})$$

$$f\hat{z} + 2r\hat{f} = 0 \quad (\text{B.2.14})$$

$$4g\hat{t} + 3g\hat{w} + l\hat{w} + 8s\hat{g} = 0 \quad (\text{B.2.15})$$

$$4l\hat{u} + 3g\hat{w} + l\hat{w} + 8s\hat{l} = 0 \quad (\text{B.2.16})$$

$$p^2f\hat{f} + r\hat{r} = 1 \quad (\text{B.2.17})$$

$$3p^2g\hat{g} + p^2\hat{l} + 2s\hat{s} = 2 . \quad (\text{B.2.18})$$

The above systems of equations are easily solved and, using the coefficients of the kinetic term (7.3.23) and (7.3.24), we finally get

$$\hat{t} = \frac{(4\kappa + 1)}{(\kappa + 1)(g_1 + 2g_2)p^2} \quad ; \quad \hat{u} = \frac{\kappa(4\kappa + 1) - 2\xi(g_1 + 2g_2)}{(\kappa + 1)^2(g_1 + 2g_2)p^2} \quad (\text{B.2.19})$$

$$\hat{v} = \frac{1}{(g_1 - g_2)p^2} \quad ; \quad \hat{z} = \frac{4\xi}{(2\xi g_1 - 1)p^2} \quad (\text{B.2.20})$$

$$\hat{w} = \frac{-4\kappa}{(\kappa + 1)(g_1 + 2g_2)p^2} \quad ; \quad \hat{f} = \frac{-2}{(2\xi g_1 - 1)p^2} \quad (\text{B.2.21})$$

$$\hat{g} = 0 \quad ; \quad \hat{l} = \frac{2}{(\kappa + 1)p^2} \quad (\text{B.2.22})$$

$$\hat{r} = \frac{4g_1}{(2\xi g_1 - 1)} \quad ; \quad \hat{s} = 0. \quad (\text{B.2.23})$$

B.2.2 $g_1 + 2g_2 = 0$

The action of the gauge fixed theory, after a field redefinition and setting $\kappa = 0$ because this is the traceless case, is

$$S(g_1, g_2; \xi, \kappa)|_{g_1+2g_2=0; \kappa=0} = S_{inv}(g_1, g_2)|_{g_1+2g_2=0} + S_{gf}(\xi), \quad (\text{B.2.24})$$

where the invariant action $S_{inv}(g_1, g_2)|_{g_1+2g_2=0}$ and the gauge fixing term $S_{gf}(\xi)$ are given by (7.3.45) and (7.3.49), respectively. It can be written in the form (7.3.17), and the coefficients in $\Omega_{\mu\nu, \alpha\beta}$ (7.3.18), $\Lambda_{\alpha\beta, \mu}$ (7.3.19) and $H_{\mu\alpha}$ (7.3.20) are

$$t = -3p^2 \quad ; \quad u = p^2 \quad ; \quad v = -3p^2 \quad ; \quad z = -p^2 \quad ; \quad w = 0 \quad (\text{B.2.25})$$

$$f = \frac{1}{2} \quad ; \quad g = 0 \quad ; \quad l = 1 \quad ; \quad r = \frac{\xi}{2} \quad ; \quad s = \frac{\xi}{2}. \quad (\text{B.2.26})$$

Following the same steps of the general case, we find the system of equations for the coefficients of the propagators

$$v\hat{v} = 1 \quad ; \quad z\hat{z} + p^2 f\hat{f} = 1 \quad (\text{B.2.27})$$

$$p^2 f\hat{f} + r\hat{r} = 1 \quad ; \quad 2z\hat{f} + f\hat{r} = 0 \quad (\text{B.2.28})$$

$$f\hat{z} + 2r\hat{f} = 0 \quad ; \quad p^2 \hat{l} + 2s\hat{s} = 2 \quad (\text{B.2.29})$$

$$4t\hat{g} = 0 \quad ; \quad 4u\hat{l} + 2\hat{s} = 0 \quad (\text{B.2.30})$$

$$4t\hat{t} + 3t\hat{w} = 4 \quad ; \quad \hat{w} + 8s\hat{g} = 0 \quad (\text{B.2.31})$$

$$u\hat{w} + 2p^2 \hat{g} = 0 \quad ; \quad 4u\hat{u} + u\hat{w} + 2p^2 \hat{l} = 4 \quad (\text{B.2.32})$$

$$t\hat{w} = 0 \quad ; \quad 4\hat{u} + \hat{w} + 8s\hat{l} = 0, \quad (\text{B.2.33})$$

which is simpler than the general case. In particular, since $t(p) \neq 0$ in (B.2.25), from (B.2.30) and (B.2.31) we immediately get $\hat{g} = \hat{w} = 0$. The solutions are therefore easily found

$$\hat{t} = -\frac{1}{3p^2} \quad ; \quad \hat{u} = \frac{2\xi}{(2\xi - 1)p^2} \quad (\text{B.2.34})$$

$$\hat{v} = -\frac{1}{3p^2} \quad ; \quad \hat{z} = \frac{-4\xi}{(4\xi + 1)p^2} \quad (\text{B.2.35})$$

$$\hat{w} = 0 \qquad \hat{f} = \frac{2}{(4\xi + 1)p^2} \qquad (\text{B.2.36})$$

$$\hat{g} = 0 \qquad \hat{l} = \frac{-2}{(2\xi - 1)p^2} \qquad (\text{B.2.37})$$

$$\hat{r} = \frac{8}{(4\xi + 1)} \qquad \hat{s} = \frac{4}{(2\xi - 1)}. \qquad (\text{B.2.38})$$

B.3 FRACTONS WITH BOUNDARY : DETAILED CALCULATIONS

B.3.1 Commutators

a) The bulk : generalized Kaç-Moody algebra

Considering the first Ward identity (7.4.20)

$$\int dx^3 \theta(x^3) \partial_i \partial_j J^{ij} = 2(g_2 - g_1) \partial_i \partial_j \tilde{A}^{ij} - 2g_2 \partial_i \partial^i \tilde{A} |_{x^3=0}, \qquad (\text{B.3.1})$$

we compute

$\frac{\delta}{\delta J^{mn}(x')}$ (B.3.1) :

$$\begin{aligned} \partial_m \partial_n \delta^{(3)}(X - X') &= 2(g_2 - g_1) \partial_i \partial_j \frac{\delta^2 Z_c[J, \tilde{J}]}{\delta J^{mn}(X') \tilde{J}_{ij}(X)} - 2g_2 \eta_{ij} \partial_a \partial^a \frac{\delta^2 Z_c[J, \tilde{J}]}{\delta J^{mn}(X') \tilde{J}_{ij}(X)} \\ &= 2i \left[(g_2 - g_1) \delta_i^k \delta_j^l - g_2 \eta^{kl} \eta_{ij} \right] \partial_k \partial_l \langle T(A_{mn}(X') \tilde{A}^{ij}(X)) \rangle \\ &= 2i \left[(g_2 - g_1) \delta_i^k \delta_j^l - g_2 \eta^{kl} \eta_{ij} \right] \langle \overline{T(A_{mn}(X') \partial_k \partial_l \tilde{A}^{ij})} \rangle + \\ &\quad + 2i \left[(g_2 - g_1) \delta_i^k \delta_j^l - g_2 \eta^{kl} \eta_{ij} \right] \left\{ \left[\partial_l \tilde{A}^{ij}(X), A_{mn}(X') \right] \delta_k^0 \delta(x^0 - x'^0) + \right. \\ &\quad \left. + \partial_k \left(\left[\tilde{A}^{ij}(X), A_{mn}(X') \right] \delta_l^0 \delta(x^0 - x'^0) \right) \right\} \\ &= 2i \left[(g_2 - g_1) \left(\partial_j \tilde{A}^{0j} + \partial_A \tilde{A}^{0A} \right) - g_2 \partial^0 \tilde{A}, A'_{mn} \right] \delta(x^0 - x'^0) + \\ &\quad + 2i \partial_0 \left\{ \left[(g_2 - g_1) \tilde{A}^{00} + g_2 \tilde{A}, A'_{mn} \right] \delta(x^0 - x'^0) \right\}, \end{aligned} \qquad (\text{B.3.2})$$

where we used the conserved current equation (7.4.23). Integrating over dx^0 , we finally get to the following equal time commutators

$$\left[\Delta \tilde{A}(X), A_{0n}(X') \right]_{x^0=x'^0} = 0 \qquad (\text{B.3.3})$$

$$\left[\Delta \tilde{A}(X), A_{MN}(X') \right]_{x^0=x'^0} = i \partial_M \partial_N \{ \delta(x^1 - x'^1) \delta(x^2 - x'^2) \}, \qquad (\text{B.3.4})$$

where we defined

$$\Delta \tilde{A} \equiv 2(g_1 - g_2) \left(\partial_j \tilde{A}^{0j} + \partial_A \tilde{A}^{0A} \right) + 2g_2 \partial^0 \tilde{A}. \qquad (\text{B.3.5})$$

In the same way, we now compute $\frac{\delta}{\delta \tilde{J}^{mn}(x')}$ (B.3.1) :

$$\begin{aligned} 0 &= 2(g_2 - g_1) \partial_i \partial_j \frac{\delta^2 Z_c[J, \tilde{J}]}{\delta \tilde{J}^{mn}(X') \tilde{J}_{ij}(X)} - 2g_2 \eta_{ij} \partial_a \partial^a \frac{\delta^2 Z_c[J, \tilde{J}]}{\delta \tilde{J}^{mn}(X') \tilde{J}_{ij}(X)} \\ &= 2i \left[(g_2 - g_1) \left(\partial_j \tilde{A}^{0j} + \partial_A \tilde{A}^{0A} \right) - g_2 \partial^0 \tilde{A}, \tilde{A}'_{mn} \right] \delta(x^0 - x'^0) + \\ &\quad + 2i \partial_0 \left\{ \left[(g_2 - g_1) \tilde{A}^{00} + g_2 \tilde{A}, \tilde{A}'_{mn} \right] \delta(x^0 - x'^0) \right\}, \end{aligned} \qquad (\text{B.3.6})$$

where we used again the conserved current equation (7.4.23). By integrating over time and using the definition (B.3.5), we find the following equal time commutator

$$\left[\Delta \tilde{A}(X), \tilde{A}_{mn}(X') \right]_{x^0=x'^0} = 0. \quad (\text{B.3.7})$$

Taking the second broken Ward identity (7.4.22)

$$\partial_i \partial_j \tilde{J}^{ij} |_{x^3=0} = -2(g_2 - g_1) \partial_i \partial_j A^{ij} + 2g_2 \partial_i \partial^i A |_{x^3=0}. \quad (\text{B.3.8})$$

we compute $\frac{\delta}{\delta J^{mn}(x')}$ (B.3.8) :

$$\begin{aligned} 0 &= -2(g_2 - g_1) \partial_i \partial_j \frac{\delta^2 Z_c[J, \tilde{J}]}{\delta J^{mn}(X') J_{ij}(X)} + 2g_2 \eta_{ij} \partial_a \partial^a \frac{\delta^2 Z_c[J, \tilde{J}]}{\delta J^{mn}(X') J_{ij}(X)} \\ &= -2i \left[(g_2 - g_1) (\partial_j A^{0j} + \partial_A A^{0A}) - g_2 \partial^0 A, A'_{mn} \right] \delta(x^0 - x'^0) + \\ &\quad - 2i \partial_0 \left\{ [(g_2 - g_1) A^{00} + g_2 A, A'_{mn}] \delta(x^0 - x'^0) \right\}, \end{aligned} \quad (\text{B.3.9})$$

where we used (7.4.32). Integrating over dx^0 we find the equal time commutator

$$\left[\Delta A(X), A_{mn}(X') \right]_{x^0=x'^0} = 0, \quad (\text{B.3.10})$$

where $\Delta A(X)$ is defined as (B.3.5)

$$\Delta A \equiv 2(g_1 - g_2) (\partial_j A^{0j} + \partial_A A^{0A}) + 2g_2 \partial^0 A. \quad (\text{B.3.11})$$

We finally compute $\frac{\delta}{\delta \tilde{J}^{mn}(x')}$ (B.3.8) :

$$\begin{aligned} \partial_m \partial_n \delta^{(3)}(X - X') &= -2(g_2 - g_1) \partial_i \partial_j \frac{\delta^2 Z_c[J, \tilde{J}]}{\delta \tilde{J}^{mn}(X') J_{ij}(X)} + 2g_2 \eta_{ij} \partial_a \partial^a \frac{\delta^2 Z_c[J, \tilde{J}]}{\delta \tilde{J}^{mn}(X') J_{ij}(X)} \\ &= -2i \left[(g_2 - g_1) (\partial_j A^{0j} + \partial_A A^{0A}) - g_2 \partial^0 A, \tilde{A}'_{mn} \right] \delta(x^0 - x'^0) + \\ &\quad - 2i \partial_0 \left\{ [(g_2 - g_1) A^{00} + g_2 A, \tilde{A}'_{mn}] \delta(x^0 - x'^0) \right\}, \end{aligned} \quad (\text{B.3.12})$$

where we used (7.4.32) and from which, integrating over dx^0 , we find

$$\left[\Delta A(X), \tilde{A}_{0n}(X') \right]_{x^0=x'^0} = 0 \quad (\text{B.3.13})$$

$$\left[-\Delta A(X), \tilde{A}_{MN}(X') \right]_{x^0=x'^0} = i \partial_M \partial_N \{ \delta(x^1 - x'^1) \delta(x^2 - x'^2) \}. \quad (\text{B.3.14})$$

b) *The boundary : canonical commutators*

We take the commutator (7.4.34) and its trace, in the following equal time combination

$$\left[\Delta \tilde{A}, A'_{\text{DF}} - \frac{1}{2} \eta_{\text{DF}} \eta^{\text{MN}} A'_{\text{MN}} \right] = \frac{i}{2} (\delta_{\text{D}}^{\text{M}} \delta_{\text{F}}^{\text{N}} + \delta_{\text{D}}^{\text{N}} \delta_{\text{F}}^{\text{M}} - \eta^{\text{MN}} \eta_{\text{DF}}) \partial_{\text{M}} \partial_{\text{N}} \delta^{(2)}(X - X'). \quad (\text{B.3.15})$$

In terms of the solutions on the boundary (7.4.47) we have

$$A_{\text{DF}}(X) |_{(7.4.33)} = \epsilon_{\text{Dab}} \partial^a \alpha_{\text{F}}^b(X) + \epsilon_{\text{Fab}} \partial^a \alpha_{\text{D}}^b(X) \quad (\text{B.3.16})$$

$$\eta^{\text{MN}} A_{\text{MN}} |_{(7.4.33)} = 2\epsilon_{0\text{BC}} \partial^{\text{B}} \alpha^{0\text{C}} = A_{00}(X) |_{(7.4.33)} \quad (\text{B.3.17})$$

$$\Delta \tilde{A} |_{(7.4.29)} = \partial_{\text{M}} \partial_{\text{N}} \left[g_{12} (\epsilon^{\text{0MB}} \tilde{\alpha}_{\text{B}}^{\text{N}} + \epsilon^{\text{0NB}} \tilde{\alpha}_{\text{B}}^{\text{M}}) \right], \quad (\text{B.3.18})$$

where $g_{12} \equiv 2(g_1 - g_2)$. As a consequence of the tracelessness of $A_{ij}(X)$, we can use (B.3.17) and write the commutator (B.3.3) for $n = 0$ as follows

$$\left[\Delta \tilde{A}(X)|_{(7.4.29)}, A_{00}(X')|_{(7.4.33)} \right] = \left[\Delta \tilde{A}(X)|_{(7.4.29)}, \eta^{MN} A_{MN}(X')|_{(7.4.33)} \right] = 0, \quad (\text{B.3.19})$$

then, using (B.3.16), (B.3.18) and (B.3.19), the commutator (B.3.15) becomes

$$\begin{aligned} \frac{i}{2} \partial_M \partial_N \{ \dots \}_{\text{DF}}^{\text{MN}} &= \left[\Delta \tilde{A}(X)|_{(7.4.29)}, A_{\text{DF}}(X')|_{(7.4.33)} \right] \\ &= \partial_M \partial_N \left[g_{12} (\epsilon^{0\text{MB}} \tilde{\alpha}_B^{\text{N}}(X) + \epsilon^{0\text{NB}} \tilde{\alpha}_B^{\text{M}}(X)), \epsilon_{\text{Dab}} \partial^a \alpha_{\text{F}}^b(X') + \epsilon_{\text{Fab}} \partial^a \alpha_{\text{D}}^b(X') \right], \end{aligned} \quad (\text{B.3.20})$$

from which we can identify the following canonical commutation relation

$$[Q^{\text{MN}}, P'_{\text{DF}}] = i \left(\frac{\delta_{\text{D}}^{\text{M}} \delta_{\text{F}}^{\text{N}} + \delta_{\text{D}}^{\text{N}} \delta_{\text{F}}^{\text{M}}}{2} - \frac{1}{2} \eta^{\text{MN}} \eta_{\text{DF}} \right) \delta^{(2)}(X - X'), \quad (\text{B.3.21})$$

with

$$Q^{\text{MN}} = Q^{\text{NM}} \equiv \epsilon^{0\text{MB}} \tilde{\alpha}_B^{\text{N}} + \epsilon^{0\text{NB}} \tilde{\alpha}_B^{\text{M}} \quad (\text{B.3.22})$$

$$P_{\text{DF}} = P_{\text{FD}} \equiv g_{12} (\epsilon_{\text{Dab}} \partial^a \alpha_{\text{F}}^b + \epsilon_{\text{Fab}} \partial^a \alpha_{\text{D}}^b) \quad (\text{B.3.23})$$

and $Q_{\text{M}}^{\text{M}} = P_{\text{M}}^{\text{M}} = 0$. We can go further, multiplying both right and left hand sides of (B.3.21) by $\epsilon_{0\text{MA}} \epsilon^{0\text{FK}} = \delta_{\text{M}}^{\text{K}} \delta_{\text{A}}^{\text{F}} - \delta_{\text{M}}^{\text{F}} \delta_{\text{A}}^{\text{K}}$

$$[\epsilon_{0\text{MA}} Q^{\text{MN}}, \epsilon^{0\text{FK}} P'_{\text{DF}}] = i (\delta_{\text{M}}^{\text{K}} \delta_{\text{A}}^{\text{F}} - \delta_{\text{M}}^{\text{F}} \delta_{\text{A}}^{\text{K}}) \left(\frac{\delta_{\text{D}}^{\text{M}} \delta_{\text{F}}^{\text{N}} + \delta_{\text{D}}^{\text{N}} \delta_{\text{F}}^{\text{M}}}{2} - \frac{1}{2} \eta^{\text{MN}} \eta_{\text{DF}} \right) \delta^{(2)}(X - X'), \quad (\text{B.3.24})$$

for which

$$\begin{aligned} \epsilon_{0\text{MA}} Q^{\text{MN}} &= \epsilon_{0\text{MA}} (\epsilon^{0\text{MB}} \tilde{\alpha}_B^{\text{N}} + \epsilon^{0\text{NB}} \tilde{\alpha}_B^{\text{M}}) \\ &= -2 \tilde{\alpha}_A^{\text{N}} + \delta_A^{\text{N}} \tilde{\alpha}_M^{\text{M}} \end{aligned} \quad (\text{B.3.25})$$

which is the traceless spatial part of $\tilde{\alpha}_{mn}(X)$. Then

$$\begin{aligned} \epsilon^{0\text{FK}} P_{\text{DF}} &= g_{12} \left[(\delta_a^0 \delta_b^{\text{K}} - \delta_b^0 \delta_a^{\text{K}}) \partial^a \alpha_{\text{D}}^b + \epsilon^{0\text{FK}} \epsilon_{0\text{DA}} \partial^{\text{A}} \alpha_{\text{F}}^0 - \epsilon^{0\text{FK}} \epsilon_{0\text{DB}} \partial^0 \alpha_{\text{F}}^{\text{B}} \right] \\ &= g_{12} \left[2(\partial^0 \alpha_{\text{D}}^{\text{K}} - \partial^{\text{K}} \alpha_{\text{D}}^0) + \delta_{\text{D}}^{\text{K}} (\partial^{\text{A}} \alpha_{\text{A}}^0 - \partial^0 \alpha_{\text{B}}^{\text{B}}) \right]. \end{aligned} \quad (\text{B.3.26})$$

Finally, at the right hand side we have

$$(\delta_{\text{M}}^{\text{K}} \delta_{\text{A}}^{\text{F}} - \delta_{\text{M}}^{\text{F}} \delta_{\text{A}}^{\text{K}}) \left(\frac{\delta_{\text{D}}^{\text{M}} \delta_{\text{F}}^{\text{N}} + \delta_{\text{D}}^{\text{N}} \delta_{\text{F}}^{\text{M}}}{2} - \frac{1}{2} \eta^{\text{MN}} \eta_{\text{DF}} \right) = \frac{1}{2} (\delta_{\text{A}}^{\text{N}} \delta_{\text{D}}^{\text{K}} - \delta_{\text{D}}^{\text{N}} \delta_{\text{A}}^{\text{K}} - \eta_{\text{AD}} \eta^{\text{KN}}). \quad (\text{B.3.27})$$

By properly raising and lowering the indices with $\eta^{\text{DL}} \eta_{\text{NB}}$, we finally get

$$4g_{12} \left[\tilde{\alpha}_{\text{AB}} - \eta_{\text{AB}} \tilde{\alpha}_{\text{M}}^{\text{M}}, (\partial^0 \alpha^{\text{KL}} - \partial^{\text{K}} \alpha^{\text{0L}})' + \eta^{\text{KL}} (\partial_{\text{F}} \alpha^{\text{0F}} - \partial^0 \alpha_{\text{F}}^{\text{F}})' \right] = \frac{i}{2} (\delta_{\text{A}}^{\text{K}} \delta_{\text{B}}^{\text{L}} + \delta_{\text{A}}^{\text{L}} \delta_{\text{B}}^{\text{K}} - \eta^{\text{KL}} \eta_{\text{AB}}) \delta^{(2)}(X - X'), \quad (\text{B.3.28})$$

where the primed quantities depend on X' . At the right hand side we have the index symmetry $\text{A} \leftrightarrow \text{B}$ and $\text{C} \leftrightarrow \text{D}$, while at the left hand side the symmetry is only for $\text{A} \leftrightarrow \text{B}$. We thus symmetrize the result as follows

$$\frac{1}{2} \left(\left[\dots_{\text{AB}}, \dots^{\text{CD}} \right] + \left[\dots_{\text{AB}}, \dots^{\text{DC}} \right] \right) = \frac{i}{2} (\delta_{\text{A}}^{\text{C}} \delta_{\text{B}}^{\text{D}} + \delta_{\text{A}}^{\text{D}} \delta_{\text{B}}^{\text{C}} - \eta^{\text{CD}} \eta_{\text{AB}}) \delta^{(2)}(X - X'), \quad (\text{B.3.29})$$

obtaining

$$\left[\tilde{\alpha}_{AB} - \frac{1}{2} \eta_{AB} \tilde{\alpha}_M^M, -g_{12} (2f'^{CD0} - \eta^{CD} f_a^{a0}) \right] = \frac{i}{2} (\delta_A^C \delta_B^D + \delta_A^D \delta_B^C - \eta^{CD} \eta_{AB}) \delta^{(2)}(X - X'), \quad (\text{B.3.30})$$

where $f_{abc}(X)$ is analogous to $F_{\mu\nu\rho}(x)$ (7.2.19), but referred to $\alpha_{ab}(X)$, *i.e.*

$$f_{abc} \equiv \partial_a \alpha_{bc} + \partial_b \alpha_{ac} - 2\partial_c \alpha_{ab}. \quad (\text{B.3.31})$$

From (B.3.30) we can identify the new canonical variables as

$$\tilde{q}_{AB} \equiv \tilde{\alpha}_{AB} - \frac{1}{2} \eta_{AB} \tilde{\alpha}_M^M \quad (\text{B.3.32})$$

$$\tilde{p}^{CD} \equiv -2g_{12} \left(f'^{CD0} - \frac{1}{2} \eta^{CD} f_a^{a0} \right) = -2g_{12} [(\partial^c \alpha^{0D} + \partial^D \alpha^{0c} - 2\partial^0 \alpha^{CD}) - \eta^{CD} \partial_a \alpha^{0a}]. \quad (\text{B.3.33})$$

Not surprisingly, starting from the commutator (7.4.35), and proceeding as we just did, we land on an analogous result (up to a sign) with $\alpha \leftrightarrow \tilde{\alpha}$ switched, *i.e.* we get

$$\left[\alpha_{AB} - \frac{1}{2} \eta_{AB} \alpha_M^M, g_{12} (2\tilde{f}'^{CD0} - \eta^{CD} \tilde{f}_a^{a0}) \right] = \frac{i}{2} (\delta_A^C \delta_B^D + \delta_A^D \delta_B^C - \eta^{CD} \eta_{AB}) \delta^{(2)}(X - X'), \quad (\text{B.3.34})$$

where $\tilde{f}_{abc}(X)$ refers to $\tilde{\alpha}_{ab}(X)$

$$\tilde{f}_{abc} \equiv \partial_a \tilde{\alpha}_{bc} + \partial_b \tilde{\alpha}_{ac} - 2\partial_c \tilde{\alpha}_{ab}. \quad (\text{B.3.35})$$

From (B.3.34) we can identify another set of canonical variables

$$q_{AB} \equiv \alpha_{AB} - \frac{1}{2} \eta_{AB} \alpha_M^M \quad (\text{B.3.36})$$

$$p^{CD} \equiv 2g_{12} \left(\tilde{f}'^{CD0} - \frac{1}{2} \eta^{CD} \tilde{f}_a^{a0} \right) = 2g_{12} [(\partial^c \tilde{\alpha}^{0D} + \partial^D \tilde{\alpha}^{0c} - 2\partial^0 \tilde{\alpha}^{CD}) - \eta^{CD} \partial_a \tilde{\alpha}^{0a}]. \quad (\text{B.3.37})$$

We observe that the canonical variables $\tilde{q}_{AB}(X)$, $q_{AB}(X)$ in (B.3.30) and (B.3.34), depend on the traceless spatial part of the fields $\tilde{\alpha}_{ij}(X)$ and $\alpha_{ij}(X)$.

B.3.2 The most general action

The most general action of the 3D boundary theory must be compatible with

- power-counting $[\alpha] = 0$, $[\tilde{\alpha}] = 1$;
- symmetry $\delta S = \tilde{\delta} S = 0$, where δ and $\tilde{\delta}$ are defined in (7.4.45) and (7.4.46) ;
- canonical variables, *i.e.* $\frac{\partial \mathcal{L}^{kin}}{\partial \dot{q}} = p$, identified in (7.4.52) and (7.4.53).

From the first two requests we have that the most general action must be

$$\begin{aligned} S_{3D} &= \int d^3 X \left\{ \omega_1 \left(\partial_c \alpha_{ab} \partial^c \alpha^{ab} - \frac{3}{2} \partial_c \alpha_{ab} \partial^a \alpha^{bc} \right) + \omega_2 \left(\partial_c \alpha_{ab} \partial^c \tilde{\alpha}^{ab} - \frac{3}{2} \partial_c \alpha_{ab} \partial^a \tilde{\alpha}^{bc} \right) - \right. \\ &\quad \left. - 3\omega_3 \alpha_a^d \epsilon^{abc} \partial_b \alpha_{cd} - 3\omega_4 \tilde{\alpha}_a^d \epsilon^{abc} \partial_b \alpha_{cd} - 3\omega_5 \tilde{\alpha}_a^d \epsilon^{abc} \partial_b \tilde{\alpha}_{cd} \right\} \quad (\text{B.3.38}) \\ &= \int d^3 X \left\{ \frac{\omega_1}{6} \varphi_{abc} \varphi^{abc} + \frac{\omega_2}{6} \varphi_{abc} \tilde{\varphi}^{abc} + \omega_3 \alpha_a^d \epsilon^{abc} \varphi_{dbc} + \omega_4 \alpha_a^d \epsilon^{abc} \tilde{\varphi}_{dbc} + \omega_5 \tilde{\alpha}_a^d \epsilon^{abc} \tilde{\varphi}_{dbc} \right\} \end{aligned}$$

where ω_i are constants, $[\omega_2] = [\omega_5] = 0$, $[\omega_1] = [\omega_4] = 1$, $[\omega_3] = 2$, and we defined the tensor

$$\begin{aligned}\varphi_{abc} &\equiv f_{abc} + \frac{1}{4} \left(-2\eta_{ab}f_{dc}^d + \eta_{bc}f_{da}^d + \eta_{ac}f_{db}^d \right) \\ &= -2\partial_c\alpha_{ab} + \partial_a\alpha_{bc} + \partial_b\alpha_{ac} - \eta_{ab}\partial^d\alpha_{dc} + \frac{1}{2}\eta_{bc}\partial^d\alpha_{da} + \frac{1}{2}\eta_{ac}\partial^d\alpha_{db},\end{aligned}\quad (\text{B.3.39})$$

and its analog $\tilde{\varphi}_{abc}(X)$ with respect to $\tilde{\alpha}_{ab}(X)$, with the following properties

$$\varphi_{abc} = \varphi_{bac} \quad ; \quad \tilde{\varphi}_{abc} = \tilde{\varphi}_{bac} \quad (\text{B.3.40})$$

$$\varphi_{abc} + \varphi_{cab} + \varphi_{bca} = 0 = \tilde{\varphi}_{abc} + \tilde{\varphi}_{cab} + \tilde{\varphi}_{bca} \quad (\text{B.3.41})$$

$$\delta\varphi_{abc} = \tilde{\delta}\tilde{\varphi}_{abc} = 0 \quad (\text{B.3.42})$$

$$\eta^{ab}\varphi_{abc} = \eta^{bc}\varphi_{abc} = \eta^{ab}\tilde{\varphi}_{abc} = \eta^{bc}\tilde{\varphi}_{abc} = 0. \quad (\text{B.3.43})$$

Notice that

$$\tilde{\varphi}^{\text{MN}0} = \tilde{f}^{\text{MN}0} - \frac{1}{2}\eta^{\text{MN}}\tilde{f}_a^{a0} = \frac{1}{2g_{12}}p^{\text{MN}} \quad (\text{B.3.44})$$

$$\varphi^{\text{MN}0} = f^{\text{MN}0} - \frac{1}{2}\eta^{\text{MN}}f_a^{a0} = -\frac{1}{2g_{12}}\tilde{p}^{\text{MN}}. \quad (\text{B.3.45})$$

We rewrite the fields $\alpha_{ab}(X)$ and $\tilde{\alpha}_{ab}(X)$ according to the representation of the rotation group, as follows

$$\alpha_{00} = -4\psi = \alpha_A^A \quad (\text{B.3.46})$$

$$\alpha_{0A} = v_A \quad (\text{B.3.47})$$

$$\alpha_{AB} = 2s_{AB} - 2\eta_{AB}\psi \quad ; \quad s_{AB} \equiv \frac{1}{2} \left(\alpha_{AB} - \frac{1}{2}\eta_{AB}\alpha_D^D \right) = \frac{1}{2}q_{AB} \quad (\text{B.3.48})$$

and

$$\tilde{\alpha}_{00} = -4\tilde{\psi} = \tilde{\alpha}_A^A \quad (\text{B.3.49})$$

$$\tilde{\alpha}_{0A} = \tilde{v}_A \quad (\text{B.3.50})$$

$$\tilde{\alpha}_{AB} = 2\tilde{s}_{AB} - 2\eta_{AB}\tilde{\psi} \quad ; \quad \tilde{s}_{AB} \equiv \frac{1}{2} \left(\tilde{\alpha}_{AB} - \frac{1}{2}\eta_{AB}\tilde{\alpha}_D^D \right) = \frac{1}{2}\tilde{q}_{AB}, \quad (\text{B.3.51})$$

in terms of which the commutators (7.4.52) and (7.4.53) become

$$[2s_{AB}, 2g_{12}\tilde{\varphi}'^{\text{CD}0}] = \frac{i}{2} (\delta_A^C\delta_B^D + \delta_A^D\delta_B^C - \eta_{AB}\eta^{\text{CD}}) \delta^{(2)}(X - X') \quad (\text{B.3.52})$$

$$[2\tilde{s}_{AB}, -2g_{12}\varphi'^{\text{CD}0}] = \frac{i}{2} (\delta_A^C\delta_B^D + \delta_A^D\delta_B^C - \eta_{AB}\eta^{\text{CD}}) \delta^{(2)}(X - X'). \quad (\text{B.3.53})$$

Compatibility of the action S_{3D} (B.3.38) with the first commutator (B.3.52), *i.e.*

$$\frac{\partial\mathcal{L}_{3D}}{\partial\dot{q}_{AB}} = p^{AB}, \quad (\text{B.3.54})$$

requires

$$\omega_1 = \omega_3 = \omega_4 = 0, \quad (\text{B.3.55})$$

and ω_5 is left free. Finally, distinguishing between time and space indices in the term of (B.3.38) which has ω_2 as coefficient, we have

$$\varphi_{abc}\tilde{\varphi}^{abc} = \frac{3}{2}\varphi_{00A}\tilde{\varphi}^{00A} + 2\varphi_{0AB}\tilde{\varphi}^{0AB} + \varphi_{AB0}\tilde{\varphi}^{AB0} + \varphi_{ABC}\tilde{\varphi}^{ABC}, \quad (\text{B.3.56})$$

where we observed that $\varphi_{A00} = -\frac{1}{2}\varphi_{00A}$ (and the same for $\tilde{\varphi}_{A00}$). Additionally, by using (B.3.46), (B.3.47) and (B.3.48), we have

$$\varphi_{00A} = 6\partial_A\psi + \partial_0v_A + \partial^Bq_{AB} \quad (\text{B.3.57})$$

$$\varphi_{AB0} = -2\partial_0q_{AB} + \partial_Av_B + \partial_Bv_A - \eta_{AB}\partial^Dv_D \quad (\text{B.3.58})$$

$$\varphi_{0AB} = -2\partial_Bv_A + \partial_Av_B + \partial_0q_{AB} + \frac{1}{2}\eta_{AB}\partial^Dv_D \quad (\text{B.3.59})$$

$$\begin{aligned} \varphi_{ABC} = & -2\partial_Cq_{AB} + \partial_Bq_{AC} + \partial_Aq_{BC} + 6\eta_{AB}\partial_C\psi - 3\eta_{BC}\partial_A\psi - 3\eta_{AC}\partial_B\psi - \\ & - \partial^0(\eta_{AB}v_C - \frac{1}{2}\eta_{AC}v_B - \frac{1}{2}\eta_{BC}v_A) - \partial^D(\eta_{AB}q_{DC} - \frac{1}{2}\eta_{AC}q_{DB} - \frac{1}{2}\eta_{BC}q_{DA}) . \end{aligned} \quad (\text{B.3.60})$$

The only terms containing \dot{q} contributions are related to φ_{AB0} and φ_{0AB} , *i.e.* in the 3D Lagrangian they only appear in $\frac{\omega_2}{6}(2\varphi_{0AB}\tilde{\varphi}^{0AB} + \varphi_{AB0}\tilde{\varphi}^{AB0})$. Keeping in mind this, consistently with its definition, for a traceless tensor in d dimensions we have

$$\frac{\partial s_{\mu\nu}}{\partial s_{\alpha\beta}} = \frac{\delta_\mu^\alpha\delta_\nu^\beta + \delta_\nu^\alpha\delta_\mu^\beta}{2} - \frac{1}{d}\eta^{\alpha\beta}\eta_{\mu\nu} , \quad (\text{B.3.61})$$

the compatibility condition (B.3.54) implies

$$\begin{aligned} 2g_{12}\tilde{\varphi}^{AB0} &= \frac{\partial\mathcal{L}_{3D}}{\partial\dot{q}_{AB}} \\ &= \frac{\omega_2}{6} \left(2\frac{\partial\varphi_{0MN}}{\partial\dot{q}_{AB}}\tilde{\varphi}^{0MN} + \frac{\partial\varphi_{MNO}}{\partial\dot{q}_{AB}}\tilde{\varphi}^{MNO} \right) \\ &= \frac{\omega_2}{6} (-2\tilde{\varphi}^{AB0} + \tilde{\varphi}^{0AB} + \tilde{\varphi}^{0BA}) \\ &= -\frac{\omega_2}{2}\tilde{\varphi}^{AB0} , \end{aligned} \quad (\text{B.3.62})$$

due to the cyclicity of $\tilde{\varphi}_{abc}(X)$ (B.3.41). We thus find

$$\omega_2 = -4g_{12} , \quad (\text{B.3.63})$$

from which the 3D action (B.3.38) becomes

$$\begin{aligned} S_{3D} &= \int d^3X \left[-4g_{12} \left(\partial_c\alpha_{ab}\partial^c\tilde{\alpha}^{ab} - \frac{3}{2}\partial_c\alpha_{ab}\partial^a\tilde{\alpha}^{bc} \right) + 3\omega_5\tilde{\alpha}_a^d\epsilon^{abc}\partial_b\tilde{\alpha}_{cd} \right] \\ &= \int d^3X \left[-g_{12} \left(\frac{2}{3}f_{abc}\tilde{f}^{abc} - \frac{1}{2}f_a^{ab}\tilde{f}_{cb}^c \right) + \omega_5\tilde{\alpha}_a^d\epsilon^{abc}\tilde{f}_{dbc} \right] \\ &= \int d^3X \left(-\frac{2}{3}g_{12}\varphi_{abc}\tilde{\varphi}^{abc} + \omega_5\tilde{\alpha}_a^d\epsilon^{abc}\tilde{\varphi}_{dbc} \right) . \end{aligned} \quad (\text{B.3.64})$$

The second commutator (B.3.53) yields the same result, with $\alpha \leftrightarrow \tilde{\alpha}$. The two choices are alternative and equivalent, in fact by choosing the first (B.3.52) we have $q \sim \alpha$, $p \sim \tilde{\alpha}$, while the second (B.3.53) corresponds to $\tilde{q} \sim \tilde{\alpha}$, $\tilde{p} \sim \alpha$. Something similar also happens in Maxwell theory [1].

BIBLIOGRAPHY

- [1] Erica Bertolini and Nicola Maggiore. «Holographic Projection of Electromagnetic Maxwell Theory». *Symmetry* 12.7 (2020), p. 1134. doi: [10 . 3390 / sym12071134](https://doi.org/10.3390/sym12071134). arXiv: [2006.14902](https://arxiv.org/abs/2006.14902) [hep-th].
- [2] Erica Bertolini, Giulio Gambuti, and Nicola Maggiore. «Notes from the bulk: Metric dependence of the edge states of Chern-Simons theory». *Phys. Rev. D* 104.10 (2021), p. 105011. doi: [10 . 1103 / PhysRevD . 104 . 105011](https://doi.org/10.1103/PhysRevD.104.105011). arXiv: [2110 . 13203](https://arxiv.org/abs/2110.13203) [hep-th].
- [3] Erica Bertolini, Filippo Fecit, and Nicola Maggiore. «Topological BF Description of 2D Accelerated Chiral Edge Modes». *Symmetry* 14.4 (2022), p. 675. doi: [10 . 3390/sym14040675](https://doi.org/10.3390/sym14040675). arXiv: [2203.13520](https://arxiv.org/abs/2203.13520) [hep-th].
- [4] Erica Bertolini and Nicola Maggiore. «Maxwell theory of fractons». *Phys. Rev. D* 106.12 (2022), p. 125008. doi: [10 . 1103/PhysRevD . 106 . 125008](https://doi.org/10.1103/PhysRevD.106.125008). arXiv: [2209 . 01485](https://arxiv.org/abs/2209.01485) [hep-th].
- [5] Erica Bertolini, Alberto Blasi, Andrea Damonte, and Nicola Maggiore. «Gauging Fractons and Linearized Gravity». *Symmetry* 15.4 (2023), p. 945. doi: [10 . 3390/sym15040945](https://doi.org/10.3390/sym15040945). arXiv: [2304.10789](https://arxiv.org/abs/2304.10789) [hep-th].
- [6] Erica Bertolini, Nicola Maggiore, and Giandomenico Palumbo. «Covariant fracton gauge theory with boundary». *Phys. Rev. D* 108.2 (2023), p. 025009. doi: [10.1103/PhysRevD.108.025009](https://doi.org/10.1103/PhysRevD.108.025009). arXiv: [2306.13883](https://arxiv.org/abs/2306.13883) [hep-th].
- [7] Erica Bertolini and Nicola Maggiore. «Theory of a symmetric tensor field with boundary: Kac-Moody algebras in linearized gravity». *Phys. Rev. D* 108.10 (2023), p. 105012. doi: [10 . 1103 / PhysRevD . 108 . 105012](https://doi.org/10.1103/PhysRevD.108.105012). arXiv: [2310 . 20303](https://arxiv.org/abs/2310.20303) [hep-th].
- [8] H. B. G. Casimir. «On the attraction between two perfectly conducting plates». *Indag. Math.* 10.4 (1948), pp. 261–263.
- [9] S. K. Lamoreaux. «Demonstration of the Casimir force in the 0.6 to 6 micrometers range». *Phys. Rev. Lett.* 78 (1997). [Erratum: *Phys.Rev.Lett.* 81, 5475–5476 (1998)], pp. 5–8. doi: [10.1103/PhysRevLett.78.5](https://doi.org/10.1103/PhysRevLett.78.5).
- [10] K. Symanzik. «Schrodinger Representation and Casimir Effect in Renormalizable Quantum Field Theory». *Nucl. Phys. B* 190 (1981), pp. 1–44. doi: [10.1016/0550-3213\(81\)90482-X](https://doi.org/10.1016/0550-3213(81)90482-X).
- [11] Alberto Blasi, Dario Ferraro, Nicola Maggiore, Nicodemo Magnoli, and Maura Sassetti. «Symanzik’s Method Applied To The Fractional Quantum Hall Edge States». *Annalen Phys.* 17 (2008), pp. 885–896. doi: [10 . 1002 / andp . 200810323](https://doi.org/10.1002/andp.200810323). arXiv: [0804.0164](https://arxiv.org/abs/0804.0164) [hep-th].
- [12] Andrea Amoretti, Alessandro Braggio, Giacomo Caruso, Nicola Maggiore, and Nicodemo Magnoli. «3+1D Massless Weyl spinors from bosonic scalar-tensor duality». *Adv. High Energy Phys.* 2014 (2014), p. 635286. doi: [10 . 1155 / 2014 / 635286](https://doi.org/10.1155/2014/635286). arXiv: [1308.6674](https://arxiv.org/abs/1308.6674) [hep-th].

- [13] Andrea Amoretti, Alberto Blasi, Giacomo Caruso, Nicola Maggiore, and Nicodemo Magnoli. «Duality and Dimensional Reduction of 5D BF Theory». *Eur. Phys. J. C* 73.6 (2013), p. 2461. doi: [10.1140/epjc/s10052-013-2461-3](https://doi.org/10.1140/epjc/s10052-013-2461-3). arXiv: [1301.3688 \[hep-th\]](https://arxiv.org/abs/1301.3688).
- [14] Danny Birmingham, Matthias Blau, Mark Rakowski, and George Thompson. «Topological field theory». *Phys. Rept.* 209 (1991), pp. 129–340. doi: [10.1016/0370-1573\(91\)90117-5](https://doi.org/10.1016/0370-1573(91)90117-5).
- [15] Gregory W. Moore and Nathan Seiberg. «Taming the Conformal Zoo». *Phys. Lett. B* 220 (1989), pp. 422–430. doi: [10.1016/0370-2693\(89\)90897-6](https://doi.org/10.1016/0370-2693(89)90897-6).
- [16] Xiao-Gang Wen. «Theory of the edge states in fractional quantum Hall effects». *Int. J. Mod. Phys. B* 6 (1992), pp. 1711–1762. doi: [10.1142/S0217979292000840](https://doi.org/10.1142/S0217979292000840).
- [17] Michael Stone. «Edge Waves in the Quantum Hall Effect». *Annals Phys.* 207 (1991), pp. 38–52. doi: [10.1016/0003-4916\(91\)90177-A](https://doi.org/10.1016/0003-4916(91)90177-A).
- [18] J. Frohlich, Ali H. Chamseddine, F. Gabbiani, T. Kerler, C. King, P. A. Marchetti, U. M. Studer, and E. Thiran. «The Fractional quantum Hall effect, Chern-Simons theory, and integral lattices» (Sept. 1994).
- [19] Samuel Bieri and Jürg Frohlich. «Physical principles underlying the quantum Hall effect». *Comptes Rendus Physique* 12 (2011), pp. 332–346. doi: [10.1016/j.crhy.2011.02.001](https://doi.org/10.1016/j.crhy.2011.02.001). arXiv: [1006.0457 \[cond-mat.mes-hall\]](https://arxiv.org/abs/1006.0457).
- [20] Jürg Fröhlich. «Chiral Anomaly, Topological Field Theory, and Novel States of Matter». *Rev. Math. Phys.* 30.06 (2018), p. 1840007. doi: [10.1142/9789813233867_0013](https://doi.org/10.1142/9789813233867_0013). arXiv: [1802.01385 \[cond-mat.mes-hall\]](https://arxiv.org/abs/1802.01385).
- [21] M. Z. Hasan and C. L. Kane. «Topological Insulators». *Rev. Mod. Phys.* 82 (2010), p. 3045. doi: [10.1103/RevModPhys.82.3045](https://doi.org/10.1103/RevModPhys.82.3045). arXiv: [1002.3895 \[cond-mat.mes-hall\]](https://arxiv.org/abs/1002.3895).
- [22] Xiao Liang Qi and Shou Cheng Zhang. «Topological insulators and superconductors». *Rev. Mod. Phys.* 83.4 (2011), pp. 1057–1110. doi: [10.1103/RevModPhys.83.1057](https://doi.org/10.1103/RevModPhys.83.1057). arXiv: [1008.2026 \[cond-mat.mes-hall\]](https://arxiv.org/abs/1008.2026).
- [23] M. Zahid Hasan and Joel E. Moore. «Three-Dimensional Topological Insulators». *Ann. Rev. Condensed Matter Phys.* 2 (2011), pp. 55–78. doi: [10.1146/annurev-conmatphys-062910-140432](https://doi.org/10.1146/annurev-conmatphys-062910-140432). arXiv: [1011.5462 \[cond-mat.str-el\]](https://arxiv.org/abs/1011.5462).
- [24] Gregory W. Moore. «The birth of topological insulators». *Nature* 464 (2010), pp. 194–198. doi: [10.1038/nature08916](https://doi.org/10.1038/nature08916).
- [25] Shou-Cheng Zhang. «The Chern-Simons-Landau-Ginzburg theory of the fractional quantum Hall effect». *Int. J. Mod. Phys. B* 6 (1992), pp. 25–58. doi: [10.1142/S0217979292000037](https://doi.org/10.1142/S0217979292000037).
- [26] V. G. Kac. «Simple graded algebras of finite growth». *Funct. Anal. Appl.* 1 (1967), p. 328.
- [27] R. V. Moody. «Lie Algebras associated with generalized Cartan matrices». *Bull. Am. Math. Soc.* 73 (1967), pp. 217–221. doi: [10.1090/S0002-9904-1967-11688-4](https://doi.org/10.1090/S0002-9904-1967-11688-4).

- [28] A. Blasi and R. Collina. «Chern-Simons and two-dimensional conformal field theories in a covariant Lagrangian approach». *Phys. Lett. B* 243 (1990), pp. 99–104. DOI: [10.1016/0370-2693\(90\)90963-7](https://doi.org/10.1016/0370-2693(90)90963-7).
- [29] Stephane Emery and Olivier Piguet. «Chern-Simons theory in the axial gauge: Manifold with boundary». *Helv. Phys. Acta* 64 (1991), pp. 1256–1270.
- [30] A. P. Balachandran, G. Bimonte, K. S. Gupta, and A. Stern. «Conformal edge currents in Chern-Simons theories». *Int. J. Mod. Phys. A* 7 (1992), pp. 4655–4670. DOI: [10.1142/S0217751X92002106](https://doi.org/10.1142/S0217751X92002106). arXiv: [hep-th/9110072](https://arxiv.org/abs/hep-th/9110072).
- [31] Gary T. Horowitz. «Exactly Soluble Diffeomorphism Invariant Theories». *Commun. Math. Phys.* 125 (1989), p. 417. DOI: [10.1007/BF01218410](https://doi.org/10.1007/BF01218410).
- [32] A. Karlhede and M. Rocek. «Topological Quantum Field Theories in Arbitrary Dimensions». *Phys. Lett. B* 224 (1989), pp. 58–60. DOI: [10.1016/0370-2693\(89\)91050-2](https://doi.org/10.1016/0370-2693(89)91050-2).
- [33] Alberto Blasi, Nicola Maggiore, and Michele Montobbio. «Noncommutative two dimensional BF model». *Nucl. Phys. B* 740 (2006), pp. 281–296. DOI: [10.1016/j.nuclphysb.2006.01.028](https://doi.org/10.1016/j.nuclphysb.2006.01.028). arXiv: [hep-th/0512006](https://arxiv.org/abs/hep-th/0512006).
- [34] Gil Young Cho and Joel E. Moore. «Topological BF field theory description of topological insulators». *Annals Phys.* 326 (2011), pp. 1515–1535. DOI: [10.1016/j.aop.2010.12.011](https://doi.org/10.1016/j.aop.2010.12.011). arXiv: [1011.3485 \[cond-mat.str-el\]](https://arxiv.org/abs/1011.3485).
- [35] A. P. Balachandran and P. Teotonio-Sobrinho. «The Edge states of the BF system and the London equations». *Int. J. Mod. Phys. A* 8 (1993), pp. 723–752. DOI: [10.1142/S0217751X9300028X](https://doi.org/10.1142/S0217751X9300028X). arXiv: [hep-th/9205116](https://arxiv.org/abs/hep-th/9205116).
- [36] Xiao Chen, Apoorv Tiwari, and Shinsei Ryu. «Bulk-boundary correspondence in (3+1)-dimensional topological phases». *Phys. Rev. B* 94.4 (2016). [Addendum: *Phys.Rev.B* 94, 079903 (2016)], p. 045113. DOI: [10.1103/PhysRevB.94.045113](https://doi.org/10.1103/PhysRevB.94.045113). arXiv: [1509.04266 \[cond-mat.str-el\]](https://arxiv.org/abs/1509.04266).
- [37] Marc Geiller and Puttarak Jai-akson. «Extended actions, dynamics of edge modes, and entanglement entropy». *JHEP* 09 (2020), p. 134. DOI: [10.1007/JHEP09\(2020\)134](https://doi.org/10.1007/JHEP09(2020)134). arXiv: [1912.06025 \[hep-th\]](https://arxiv.org/abs/1912.06025).
- [38] Apoorv Tiwari, Xiao Chen, Ken Shiozaki, and Shinsei Ryu. «Bosonic topological phases of matter: Bulk-boundary correspondence, symmetry protected topological invariants, and gauging». *Phys. Rev. B* 97.24 (2018), p. 245133. DOI: [10.1103/PhysRevB.97.245133](https://doi.org/10.1103/PhysRevB.97.245133). arXiv: [1710.04730 \[cond-mat.str-el\]](https://arxiv.org/abs/1710.04730).
- [39] Edward Witten. «Anti-de Sitter space and holography». *Adv. Theor. Math. Phys.* 2 (1998), pp. 253–291. DOI: [10.4310/ATMP.1998.v2.n2.a2](https://doi.org/10.4310/ATMP.1998.v2.n2.a2). arXiv: [hep-th/9802150](https://arxiv.org/abs/hep-th/9802150).
- [40] Igor R. Klebanov. «TASI lectures: Introduction to the AdS / CFT correspondence». *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 99): Strings, Branes, and Gravity*. Sept. 2000, pp. 615–650. DOI: [10.1142/9789812799630_0007](https://doi.org/10.1142/9789812799630_0007). arXiv: [hep-th/0009139](https://arxiv.org/abs/hep-th/0009139).
- [41] Joseph Polchinski. «Introduction to Gauge/Gravity Duality». *Theoretical Advanced Study Institute in Elementary Particle Physics: String theory and its Applications: From meV to the Planck Scale*. Oct. 2010, pp. 3–46. DOI: [10.1142/9789814350525_0001](https://doi.org/10.1142/9789814350525_0001). arXiv: [1010.6134 \[hep-th\]](https://arxiv.org/abs/1010.6134).

- [42] Sean A. Hartnoll. «Lectures on holographic methods for condensed matter physics». *Class. Quant. Grav.* 26 (2009). Ed. by A. M. Uranga, p. 224002. DOI: [10.1088/0264-9381/26/22/224002](https://doi.org/10.1088/0264-9381/26/22/224002). arXiv: [0903.3246](https://arxiv.org/abs/0903.3246) [hep-th].
- [43] Christopher P. Herzog. «Lectures on Holographic Superfluidity and Superconductivity». *J. Phys. A* 42 (2009), p. 343001. DOI: [10.1088/1751-8113/42/34/343001](https://doi.org/10.1088/1751-8113/42/34/343001). arXiv: [0904.1975](https://arxiv.org/abs/0904.1975) [hep-th].
- [44] Jan Zaanen, Ya-Wen Sun, Yan Liu, and Koenraad Schalm. *Holographic Duality in Condensed Matter Physics*. Cambridge Univ. Press, 2015. ISBN: 978-1-107-08008-9.
- [45] Subir Sachdev. «Condensed Matter and AdS/CFT». *Lect. Notes Phys.* 828 (2011), pp. 273–311. DOI: [10.1007/978-3-642-04864-7_9](https://doi.org/10.1007/978-3-642-04864-7_9). arXiv: [1002.2947](https://arxiv.org/abs/1002.2947) [hep-th].
- [46] J. McGreevy. *Holography with and without gravity*. Lectures held at the “2013 Arnold Sommerfeld School on Gauge-gravity duality and condensed matter physics”. URL: https://www.theorie.physik.uni-muenchen.de/activities/schools/archiv/2013_asc_school/videos_ads_cmt/mcgreevy/index.html.
- [47] Andrea Amoretti, Alessandro Braggio, Giacomo Caruso, Nicola Maggiore, and Nicodemo Magnoli. «Holography in flat spacetime: 4D theories and electromagnetic duality on the border». *JHEP* 04 (2014), p. 142. DOI: [10.1007/JHEP04\(2014\)142](https://doi.org/10.1007/JHEP04(2014)142). arXiv: [1401.7101](https://arxiv.org/abs/1401.7101) [hep-th].
- [48] E. Bocquillon, V. Freulon, J.-M Berroir, P. Degiovanni, B. Plaças, A. Cavanna, Y. Jin, and G. Fève. «Separation of neutral and charge modes in one-dimensional chiral edge channels». *Nature Communications* 4.1 (2013). DOI: [10.1038/ncomms2788](https://doi.org/10.1038/ncomms2788). URL: <https://doi.org/10.1038/ncomms2788>.
- [49] Nicola Maggiore. «Conserved chiral currents on the boundary of 3D Maxwell theory». *J. Phys. A* 52.11 (2019), p. 115401. DOI: [10.1088/1751-8121/ab045a](https://doi.org/10.1088/1751-8121/ab045a). arXiv: [1902.01901](https://arxiv.org/abs/1902.01901) [hep-th].
- [50] Claudio Chamon. «Quantum Glassiness». *Phys. Rev. Lett.* 94.4 (2005), p. 040402. DOI: [10.1103/physrevlett.94.040402](https://doi.org/10.1103/physrevlett.94.040402). arXiv: [cond-mat/0404182](https://arxiv.org/abs/cond-mat/0404182).
- [51] Jeongwan Haah. «Local stabilizer codes in three dimensions without string logical operators». *Phys. Rev. A* 83.4 (2011), p. 042330. DOI: [10.1103/physreva.83.042330](https://doi.org/10.1103/physreva.83.042330). arXiv: [1101.1962](https://arxiv.org/abs/1101.1962) [quant-ph].
- [52] Sagar Vijay, Jeongwan Haah, and Liang Fu. «Fracton Topological Order, Generalized Lattice Gauge Theory and Duality». *Phys. Rev. B* 94.23 (2016), p. 235157. DOI: [10.1103/PhysRevB.94.235157](https://doi.org/10.1103/PhysRevB.94.235157). arXiv: [1603.04442](https://arxiv.org/abs/1603.04442) [cond-mat.str-el].
- [53] Sagar Vijay, Jeongwan Haah, and Liang Fu. «A New Kind of Topological Quantum Order: A Dimensional Hierarchy of Quasiparticles Built from Stationary Excitations». *Phys. Rev. B* 92.23 (2015), p. 235136. DOI: [10.1103/PhysRevB.92.235136](https://doi.org/10.1103/PhysRevB.92.235136). arXiv: [1505.02576](https://arxiv.org/abs/1505.02576) [cond-mat.str-el].
- [54] Rahul M. Nandkishore and Michael Hermele. «Fractons». *Ann. Rev. Condensed Matter Phys.* 10 (2019), pp. 295–313. DOI: [10.1146/annurev-conmatphys-031218-013604](https://doi.org/10.1146/annurev-conmatphys-031218-013604). arXiv: [1803.11196](https://arxiv.org/abs/1803.11196) [cond-mat.str-el].
- [55] Michael Pretko, Xie Chen, and Yizhi You. «Fracton Phases of Matter». *Int. J. Mod. Phys. A* 35.06 (2020), p. 2030003. DOI: [10.1142/S0217751X20300033](https://doi.org/10.1142/S0217751X20300033). arXiv: [2001.01722](https://arxiv.org/abs/2001.01722) [cond-mat.str-el].

- [56] Nathan Seiberg and Shu-Heng Shao. «Exotic $U(1)$ Symmetries, Duality, and Fractons in 3+1-Dimensional Quantum Field Theory». *SciPost Phys.* 9.4 (2020), p. 046. DOI: [10.21468/SciPostPhys.9.4.046](https://doi.org/10.21468/SciPostPhys.9.4.046). arXiv: [2004.00015](https://arxiv.org/abs/2004.00015) [[cond-mat.str-el](https://arxiv.org/abs/2004.00015)].
- [57] H. B. G. Casimir and D. Polder. «The Influence of retardation on the London-van der Waals forces». *Phys. Rev.* 73 (1948), pp. 360–372. DOI: [10.1103/PhysRev.73.360](https://doi.org/10.1103/PhysRev.73.360).
- [58] H. Saleur. «Lectures on nonperturbative field theory and quantum impurity problems» (Dec. 1998). arXiv: [cond-mat/9812110](https://arxiv.org/abs/cond-mat/9812110).
- [59] H. Saleur. «Lectures on nonperturbative field theory and quantum impurity problems: Part 2» (July 2000). arXiv: [cond-mat/0007309](https://arxiv.org/abs/cond-mat/0007309).
- [60] Eduardo Fradkin and Joel E. Moore. «Entanglement entropy of 2D conformal quantum critical points: hearing the shape of a quantum drum». *Phys. Rev. Lett.* 97 (2006), p. 050404. DOI: [10.1103/PhysRevLett.97.050404](https://doi.org/10.1103/PhysRevLett.97.050404). arXiv: [cond-mat/0605683](https://arxiv.org/abs/cond-mat/0605683).
- [61] M. Mintchev and E. Ragoucy. «Algebraic approach to multiple defects on the line and application to Casimir force». *J. Phys. A* 40 (2007), p. 9515. DOI: [10.1088/1751-8113/40/31/025](https://doi.org/10.1088/1751-8113/40/31/025). arXiv: [0705.1322](https://arxiv.org/abs/0705.1322) [[hep-th](https://arxiv.org/abs/0705.1322)].
- [62] Juan Martin Maldacena. «The Large N limit of superconformal field theories and supergravity». *Adv. Theor. Math. Phys.* 2 (1998), pp. 231–252. DOI: [10.4310/ATMP.1998.v2.n2.a1](https://doi.org/10.4310/ATMP.1998.v2.n2.a1). arXiv: [hep-th/9711200](https://arxiv.org/abs/hep-th/9711200).
- [63] Andrea Amoretti, Alessandro Braggio, Nicola Maggiore, and Nicodemo Magnoli. «Thermo-electric transport in gauge/gravity models». *Adv. Phys. X* 2.2 (2017), pp. 409–427. DOI: [10.1080/23746149.2017.1300509](https://doi.org/10.1080/23746149.2017.1300509).
- [64] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. «Gauge theory correlators from noncritical string theory». *Phys. Lett. B* 428 (1998), pp. 105–114. DOI: [10.1016/S0370-2693\(98\)00377-3](https://doi.org/10.1016/S0370-2693(98)00377-3). arXiv: [hep-th/9802109](https://arxiv.org/abs/hep-th/9802109).
- [65] Andrew Chamblin, Roberto Emparan, Clifford V. Johnson, and Robert C. Myers. «Charged AdS black holes and catastrophic holography». *Phys. Rev. D* 60 (1999), p. 064018. DOI: [10.1103/PhysRevD.60.064018](https://doi.org/10.1103/PhysRevD.60.064018). arXiv: [hep-th/9902170](https://arxiv.org/abs/hep-th/9902170).
- [66] S. W. Hawking and Don N. Page. «Thermodynamics of Black Holes in anti-De Sitter Space». *Commun. Math. Phys.* 87 (1983), p. 577. DOI: [10.1007/BF01208266](https://doi.org/10.1007/BF01208266).
- [67] Subir Sachdev. *Quantum Phase Transitions*. Cambridge University Press, Apr. 2011. DOI: [10.1017/cbo9780511973765](https://doi.org/10.1017/cbo9780511973765).
- [68] J. Wess and J. Bagger. *Supersymmetry and supergravity*. Princeton, NJ, USA: Princeton University Press, 1992. ISBN: 978-0-691-02530-8.
- [69] Danny Birmingham, Mark Rakowski, and George Thompson. «Renormalization of Topological Field Theory». *Nucl. Phys. B* 329 (1990), pp. 83–97. DOI: [10.1016/0550-3213\(90\)90058-L](https://doi.org/10.1016/0550-3213(90)90058-L).
- [70] Stanley Deser, R. Jackiw, and S. Templeton. «Topologically Massive Gauge Theories». *Annals Phys.* 140 (1982). [Erratum: *Annals Phys.* 185, 406 (1988)], pp. 372–411. DOI: [10.1016/0003-4916\(82\)90164-6](https://doi.org/10.1016/0003-4916(82)90164-6).

- [71] L. D. Faddeev and V. N. Popov. «Feynman Diagrams for the Yang-Mills Field». *Phys. Lett. B* 25 (1967). Ed. by Jong-Ping Hsu and D. Fine, pp. 29–30. DOI: [10.1016/0370-2693\(67\)90067-6](https://doi.org/10.1016/0370-2693(67)90067-6).
- [72] A. Bassetto, G. Nardelli, and R. Soldati. *Yang-Mills theories in algebraic noncovariant gauges: Canonical quantization and renormalization*. 1991.
- [73] G. Mack. «Introduction to conformal invariant Quantum Field Theory in two-dimensions and more dimensions». *NATO Advanced Summer Institute on Non-perturbative Quantum Field Theory (Cargese Summer Institute)*. Aug. 1988.
- [74] C. Becchi and O. Piguet. «On the Renormalization of Two-dimensional Chiral Models». *Nucl. Phys. B* 315 (1989), pp. 153–165. DOI: [10.1016/0550-3213\(89\)90452-5](https://doi.org/10.1016/0550-3213(89)90452-5).
- [75] Peter Goddard and David I. Olive. «Kac-Moody and Virasoro Algebras in Relation to Quantum Physics». *Int. J. Mod. Phys. A* 1 (1986), p. 303. DOI: [10.1142/S0217751X86000149](https://doi.org/10.1142/S0217751X86000149).
- [76] A. Blasi and R. Collina. «Chern-Simons model in the Landau gauge and its connection to the Kac-Moody algebra». *Nucl. Phys. B Proc. Suppl.* 18 (1991), pp. 16–21. DOI: [10.1016/0920-5632\(91\)90118-X](https://doi.org/10.1016/0920-5632(91)90118-X).
- [77] A. Blasi, A. Braggio, M. Carrega, D. Ferraro, N. Maggiore, and N. Magnoli. «Non-Abelian BF theory for 2+1 dimensional topological states of matter». *New J. Phys.* 14 (2012), p. 013060. DOI: [10.1088/1367-2630/14/1/013060](https://doi.org/10.1088/1367-2630/14/1/013060). arXiv: [1106.4641](https://arxiv.org/abs/1106.4641) [cond-mat.mes-hall].
- [78] Julian S. Schwinger. «Field theory commutators». *Phys. Rev. Lett.* 3 (1959). Ed. by K. A. Milton, pp. 296–297. DOI: [10.1103/PhysRevLett.3.296](https://doi.org/10.1103/PhysRevLett.3.296).
- [79] Edward Witten. «Topological Quantum Field Theory». *Commun. Math. Phys.* 117 (1988), p. 353. DOI: [10.1007/BF01223371](https://doi.org/10.1007/BF01223371).
- [80] Edward Witten. «Quantum Field Theory and the Jones Polynomial». *Commun. Math. Phys.* 121 (1989). Ed. by Asoke N. Mitra, pp. 351–399. DOI: [10.1007/BF01217730](https://doi.org/10.1007/BF01217730).
- [81] C. Becchi, A. Rouet, and R. Stora. «Renormalization of the Abelian Higgs-Kibble Model». *Commun. Math. Phys.* 42 (1975), pp. 127–162. DOI: [10.1007/BF01614158](https://doi.org/10.1007/BF01614158).
- [82] C. Becchi, A. Rouet, and R. Stora. «Renormalization of Gauge Theories». *Annals Phys.* 98 (1976), pp. 287–321. DOI: [10.1016/0003-4916\(76\)90156-1](https://doi.org/10.1016/0003-4916(76)90156-1).
- [83] Edward Witten. «On the Structure of the Topological Phase of Two-dimensional Gravity». *Nucl. Phys. B* 340 (1990), pp. 281–332. DOI: [10.1016/0550-3213\(90\)90449-N](https://doi.org/10.1016/0550-3213(90)90449-N).
- [84] Edward Witten. «Introduction to cohomological field theories». *Int. J. Mod. Phys. A* 6 (1991), pp. 2775–2792. DOI: [10.1142/S0217751X91001350](https://doi.org/10.1142/S0217751X91001350).
- [85] Edward Witten. «Topological Sigma Models». *Commun. Math. Phys.* 118 (1988), p. 411. DOI: [10.1007/BF01466725](https://doi.org/10.1007/BF01466725).
- [86] Albert S. Schwarz. «The Partition Function of Degenerate Quadratic Functional and Ray-Singer Invariants». *Lett. Math. Phys.* 2 (1978), pp. 247–252. DOI: [10.1007/BF00406412](https://doi.org/10.1007/BF00406412).

- [87] Nicola Maggiore. «From Chern–Simons to Tomonaga–Luttinger». *Int. J. Mod. Phys. A* 33.02 (2018), p. 1850013. DOI: [10.1142/S0217751X18500136](https://doi.org/10.1142/S0217751X18500136). arXiv: [1712.08744](https://arxiv.org/abs/1712.08744) [hep-th].
- [88] Andrea Amoretti, Alessandro Braggio, Giacomo Caruso, Nicola Maggiore, and Nicodemo Magnoli. «Introduction of a boundary in topological field theories». *Phys. Rev. D* 90.12 (2014), p. 125006. DOI: [10.1103/PhysRevD.90.125006](https://doi.org/10.1103/PhysRevD.90.125006). arXiv: [1410.2728](https://arxiv.org/abs/1410.2728) [hep-th].
- [89] X. G. Wen. «Chiral Luttinger Liquid and the Edge Excitations in the Fractional Quantum Hall States». *Phys. Rev. B* 41 (1990), pp. 12838–12844. DOI: [10.1103/PhysRevB.41.12838](https://doi.org/10.1103/PhysRevB.41.12838).
- [90] Gerald V. Dunne. «Aspects of Chern-Simons theory». *Les Houches Summer School in Theoretical Physics, Session 69: Topological Aspects of Low-dimensional Systems*. July 1998. arXiv: [hep-th/9902115](https://arxiv.org/abs/hep-th/9902115).
- [91] David Tong. «Lectures on the Quantum Hall Effect». June 2016. arXiv: [1606.06687](https://arxiv.org/abs/1606.06687) [hep-th].
- [92] H. Aratyn. «fermions from bosons in (2+1)-dimensions». *Phys. Rev. D* 28 (1983), pp. 2016–2018. DOI: [10.1103/PhysRevD.28.2016](https://doi.org/10.1103/PhysRevD.28.2016).
- [93] H. Aratyn. «A Bose representation for the massless Dirac field in four-dimensions». *Nucl. Phys. B* 227 (1983), pp. 172–188. DOI: [10.1016/0550-3213\(83\)90148-7](https://doi.org/10.1016/0550-3213(83)90148-7).
- [94] Charles L. Kane. «Lectures on Bosonization». URL: <https://api.semanticscholar.org/CorpusID:41254493>.
- [95] Liang Fu, C. Kane, and E. Mele. «Topological Insulators in Three Dimensions». *Phys. Rev. Lett.* 98.10 (2007), p. 106803. DOI: [10.1103/PhysRevLett.98.106803](https://doi.org/10.1103/PhysRevLett.98.106803). arXiv: [cond-mat/0607699](https://arxiv.org/abs/cond-mat/0607699).
- [96] X. G. Wen. «Gapless Boundary Excitations in the Quantum Hall States and in the Chiral Spin States». *Phys. Rev. B* 43 (1991), pp. 11025–11036. DOI: [10.1103/PhysRevB.43.11025](https://doi.org/10.1103/PhysRevB.43.11025).
- [97] X. G. Wen. «Electrodynamical Properties of Gapless Edge Excitations in the Fractional Quantum Hall States». *Phys. Rev. Lett.* 64 (1990), p. 2206. DOI: [10.1103/PhysRevLett.64.2206](https://doi.org/10.1103/PhysRevLett.64.2206).
- [98] C. L. Kane and Matthew P. A. Fisher. «Impurity scattering and transport of fractional quantum Hall edge states». *Physical Review B* 51.19 (1995), pp. 13449–13466. DOI: [10.1103/physrevb.51.13449](https://doi.org/10.1103/physrevb.51.13449). URL: <https://doi.org/10.1103/PhysRevB.51.13449>.
- [99] Masayuki Hashisaka and Toshimasa Fujisawa. «Tomonaga–Luttinger-liquid nature of edge excitations in integer quantum Hall edge channels». *Reviews in Physics* 3 (2018), pp. 32–43. ISSN: 2405-4283. DOI: <https://doi.org/10.1016/j.revip.2018.07.001>. URL: <https://www.sciencedirect.com/science/article/pii/S2405428318300078>.
- [100] Xiao-Gang Wen. «Edge transport properties of the fractional quantum Hall states and weak-impurity scattering of a one-dimensional charge-density wave». *Phys. Rev. B* 44.11 (1991), p. 5708. DOI: [10.1103/PhysRevB.44.5708](https://doi.org/10.1103/PhysRevB.44.5708).

- [101] R. Floreanini and R. Jackiw. «Selfdual Fields as Charge Density Solitons». *Phys. Rev. Lett.* 59 (1987), p. 1873. DOI: [10.1103/PhysRevLett.59.1873](https://doi.org/10.1103/PhysRevLett.59.1873).
- [102] B. E. Kane, D. C. Tsui, and G. Weimann. «Evidence for edge currents in the integral quantum Hall effect». *Phys. Rev. Lett.* 59 (12 1987), pp. 1353–1356. DOI: [10.1103/PhysRevLett.59.1353](https://doi.org/10.1103/PhysRevLett.59.1353). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.59.1353>.
- [103] Robert M. Wald. *General Relativity*. Chicago, USA: Chicago Univ. Pr., 1984. DOI: [10.7208/chicago/9780226870373.001.0001](https://doi.org/10.7208/chicago/9780226870373.001.0001).
- [104] Steven Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. New York: John Wiley and Sons, 1972. ISBN: 978-0-471-92567-5, 978-0-471-92567-5.
- [105] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, July 2019. ISBN: 978-0-8053-8732-2, 978-1-108-48839-6, 978-1-108-77555-7.
- [106] R. d’Inverno. *Introducing Einstein’s relativity*. 1992. ISBN: 978-0-19-859686-8.
- [107] Noboru Nakanishi. «Covariant Quantization of the Electromagnetic Field in the Landau Gauge». *Prog. Theor. Phys.* 35 (1966), pp. 1111–1116. DOI: [10.1143/PTP.35.1111](https://doi.org/10.1143/PTP.35.1111).
- [108] B. Lautrup. «Canonical quantum electrodynamics in covariant gauges» (1967).
- [109] Shmuel Elitzur, Gregory W. Moore, Adam Schwimmer, and Nathan Seiberg. «Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory». *Nucl. Phys. B* 326 (1989), pp. 108–134. DOI: [10.1016/0550-3213\(89\)90436-7](https://doi.org/10.1016/0550-3213(89)90436-7).
- [110] Marc Geiller. «Edge modes and corner ambiguities in 3d Chern–Simons theory and gravity». *Nucl. Phys. B* 924 (2017), pp. 312–365. DOI: [10.1016/j.nuclphysb.2017.09.010](https://doi.org/10.1016/j.nuclphysb.2017.09.010). arXiv: [1703.04748 \[gr-qc\]](https://arxiv.org/abs/1703.04748).
- [111] C. Nash and S. Sen. *Topology and geometry for physicists*. 1983.
- [112] W. Warner F. *Foundations of Differentiable Manifolds and Lie Groups*. Springer Verlag, 1983.
- [113] M. Basler. «Functional methods for arbitrary densities in curved space-time». *Fortsch. Phys.* 41 (1993), pp. 1–43.
- [114] J. Frohlich and T. Kerler. «Universality in quantum Hall systems». *Nucl. Phys. B* 354 (1991), pp. 369–417. DOI: [10.1016/0550-3213\(91\)90360-A](https://doi.org/10.1016/0550-3213(91)90360-A).
- [115] F. D. M. Haldane. «Luttinger liquid theory of one-dimensional quantum fluids. I. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas». *J. Phys. C* 14 (1981), pp. 2585–2609. DOI: [10.1088/0022-3719/14/19/010](https://doi.org/10.1088/0022-3719/14/19/010).
- [116] I. Safi. «Propriétés d’un fil quantique connecté à des fils de mesure». *Ann. Phys. Fr.* 22.5 (1997). DOI: [10.1051/anphys:199705001](https://doi.org/10.1051/anphys:199705001).
- [117] E. Perfetto, G. Stefanucci, H. Kamata, and T. Fujisawa. «Time-resolved charge fractionalization in inhomogeneous Luttinger liquids». *Physical Review B* 89.20 (2014). DOI: [10.1103/physrevb.89.201413](https://doi.org/10.1103/physrevb.89.201413). URL: <https://doi.org/10.1103/PhysRevB.89.201413>.

- [118] H. Kamata, N. Kumada, M. Hashisaka, K. Muraki, and T. Fujisawa. «Fractionalized wave packets from an artificial Tomonaga–Luttinger liquid». *Nature Nanotechnology* 9.3 (2014), pp. 177–181. DOI: [10.1038/nnano.2013.312](https://doi.org/10.1038/nnano.2013.312). URL: <https://doi.org/10.1038/2Fnnano.2013.312>.
- [119] Paul Brasseur, Ngoc Han Tu, Yoshiaki Sekine, Koji Muraki, Masayuki Hashisaka, Toshimasa Fujisawa, and Norio Kumada. «Charge fractionalization in artificial Tomonaga-Luttinger liquids with controlled interaction strength». *Physical Review B* 96.8 (2017). DOI: [10.1103/physrevb.96.081101](https://doi.org/10.1103/physrevb.96.081101). URL: <https://doi.org/10.1103/2Fphysrevb.96.081101>.
- [120] Chaojing Lin, Masayuki Hashisaka, Takafumi Akiho, Koji Muraki, and Toshimasa Fujisawa. «Time-resolved investigation of plasmon mode along interface channels in integer and fractional quantum Hall regimes». *Physical Review B* 104.12 (2021). DOI: [10.1103/physrevb.104.125304](https://doi.org/10.1103/physrevb.104.125304). URL: <https://doi.org/10.1103/2Fphysrevb.104.125304>.
- [121] Manohar Kumar, Antti Laitinen, and Pertti Hakonen. «Unconventional fractional quantum Hall states and Wigner crystallization in suspended Corbino graphene». *Nature Communications* 9.1 (2018). DOI: [10.1038/s41467-018-05094-8](https://doi.org/10.1038/s41467-018-05094-8). URL: <https://doi.org/10.1038/2Fs41467-018-05094-8>.
- [122] Glenn Wagner, Fernando de Juan, and Dung X. Nguyen. «Landau levels in curved space realized in strained graphene». *SciPost Phys. Core* 5 (2022), p. 029. DOI: [10.21468/SciPostPhysCore.5.2.029](https://doi.org/10.21468/SciPostPhysCore.5.2.029). arXiv: [1911.02028](https://arxiv.org/abs/1911.02028) [cond-mat.str-el].
- [123] T. Can, M. Laskin, and P. Wiegmann. «Fractional Quantum Hall Effect in a Curved Space: Gravitational Anomaly and Electromagnetic Response». *Phys. Rev. Lett.* 113 (2014), p. 046803. DOI: [10.1103/PhysRevLett.113.046803](https://doi.org/10.1103/PhysRevLett.113.046803). arXiv: [1402.1531](https://arxiv.org/abs/1402.1531) [cond-mat.str-el].
- [124] Xiao-Gang Wen. «Topological orders and edge excitations in FQH states». *Adv. Phys.* 44.5 (1995), pp. 405–473. DOI: [10.1080/00018739500101566](https://doi.org/10.1080/00018739500101566). arXiv: [cond-mat/9506066](https://arxiv.org/abs/cond-mat/9506066).
- [125] Congjun Wu, B. Andrei Bernevig, and Shou-Cheng Zhang. «Helical Liquid and the Edge of Quantum Spin Hall Systems». *Phys. Rev. Lett.* 96 (10 2006), p. 106401. DOI: [10.1103/PhysRevLett.96.106401](https://doi.org/10.1103/PhysRevLett.96.106401). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.96.106401>.
- [126] Chao-Xing Liu, Xiao-Liang Qi, Xi Dai, Zhong Fang, and Shou-Cheng Zhang. «Quantum Anomalous Hall Effect in Hg(1-y)Mn(y)Te Quantum Wells». *Phys. Rev. Lett.* 101 (2008), p. 146802. DOI: [10.1103/PhysRevLett.101.146802](https://doi.org/10.1103/PhysRevLett.101.146802). arXiv: [0802.2711](https://arxiv.org/abs/0802.2711) [cond-mat.mes-hall].
- [127] Rui Yu, Wei Zhang, Hai-Jun Zhang, Shou-Cheng Zhang, Xi Dai, and Zhong Fang. «Quantized Anomalous Hall Effect in Magnetic Topological Insulators». *Science* 329.5987 (2010), pp. 61–64. DOI: [10.1126/science.1187485](https://doi.org/10.1126/science.1187485). arXiv: [1002.0946](https://arxiv.org/abs/1002.0946) [cond-mat.mes-hall].
- [128] X. G. Wen. *Quantum field theory of many-body systems: From the origin of sound to an origin of light and electrons*. 2004.

- [129] Alessio Calzona, Matteo Carrega, Giacomo Dolcetto, and Maura Sassetti. «Time-resolved pure spin fractionalization and spin-charge separation in helical Luttinger liquid based devices». *Physical Review B* 92.19 (2015). DOI: [10.1103/physrevb.92.195414](https://doi.org/10.1103/physrevb.92.195414). URL: <https://doi.org/10.1103%2Fphysrevb.92.195414>.
- [130] Tobias Meng and Daniel Loss. «Strongly anisotropic spin response as a signature of the helical regime in Rashba nanowires». *Physical Review B* 88.3 (2013). DOI: [10.1103/physrevb.88.035437](https://doi.org/10.1103/physrevb.88.035437). URL: <https://doi.org/10.1103%2Fphysrevb.88.035437>.
- [131] P. Štředa and P. Šeba. «Antisymmetric Spin Filtering in One-Dimensional Electron Systems with Uniform Spin-Orbit Coupling». *Phys. Rev. Lett.* 90 (25 2003), p. 256601. DOI: [10.1103/PhysRevLett.90.256601](https://link.aps.org/doi/10.1103/PhysRevLett.90.256601). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.90.256601>.
- [132] S. Heedt, N. Traverso Ziani, F. Crépin, W. Prost, St. Trellenkamp, J. Schubert, D. Grützmacher, B. Trauzettel, and Th. Schäpers. «Signatures of interaction-induced helical gaps in nanowire quantum point contacts». *Nature Physics* 13.6 (2017), pp. 563–567. DOI: [10.1038/nphys4070](https://doi.org/10.1038/nphys4070). URL: <https://doi.org/10.1038%2Fnphys4070>.
- [133] Alexey A. Soluyanov, Dominik Gresch, Zhijun Wang, QuanSheng Wu, Matthias Troyer, Xi Dai, and B. Andrei Bernevig. «Type-II Weyl semimetals». *Nature* 527.7579 (2015), pp. 495–498. DOI: [10.1038/nature15768](https://doi.org/10.1038/nature15768). URL: <https://doi.org/10.1038%2Fnature15768>.
- [134] Serguei Tchoumakov, Marcello Civelli, and Mark O. Goerbig. «Magnetic-Field-Induced Relativistic Properties in Type-I and Type-II Weyl Semimetals». *Phys. Rev. Lett.* 117 (8 2016), p. 086402. DOI: [10.1103/PhysRevLett.117.086402](https://link.aps.org/doi/10.1103/PhysRevLett.117.086402). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.117.086402>.
- [135] M. O. Goerbig, J.-N. Fuchs, G. Montambaux, and F. Piéchon. *Physical Review B* 78.4 (2008). DOI: [10.1103/physrevb.78.045415](https://doi.org/10.1103/physrevb.78.045415). URL: <https://doi.org/10.1103%2Fphysrevb.78.045415>.
- [136] Giacomo Rebola, Dario Ferraro, Ramiro H. Rodriguez, François D. Parmentier, Patrice Roche, and Maura Sassetti. «Electronic Wave-Packets in Integer Quantum Hall Edge Channels: Relaxation and Dissipative Effects». *Entropy* 23.2 (2021), p. 138. DOI: [10.3390/e23020138](https://doi.org/10.3390/e23020138). URL: <https://doi.org/10.3390%2Fe23020138>.
- [137] John L. Cardy. «Boundary conformal field theory» (Nov. 2004). arXiv: [hep-th/0411189](https://arxiv.org/abs/hep-th/0411189).
- [138] A. Blasi and R. Collina. «The Chern-Simons model with boundary: A Cohomological approach». *Int. J. Mod. Phys. A* 7 (1992), pp. 3083–3104. DOI: [10.1142/S0217751X92001381](https://doi.org/10.1142/S0217751X92001381).
- [139] Andrea Cappelli and Lorenzo Maffi. «Bulk-Boundary Correspondence in the Quantum Hall Effect». *J. Phys. A* 51.36 (2018), p. 365401. DOI: [10.1088/1751-8121/aad0ab](https://doi.org/10.1088/1751-8121/aad0ab). arXiv: [1801.03759](https://arxiv.org/abs/1801.03759) [hep-th].

- [140] Andrea Cappelli, Enrico Randellini, and Jacopo Sisti. «Three-dimensional Topological Insulators and Bosonization». *JHEP* 05 (2017), p. 135. DOI: [10.1007/JHEP05\(2017\)135](https://doi.org/10.1007/JHEP05(2017)135). arXiv: [1612.05212](https://arxiv.org/abs/1612.05212) [cond-mat.str-el].
- [141] Andreas Schnyder, Shinsei Ryu, Akira Furusaki, and Andreas Ludwig. «Classification of topological insulators and superconductors in three spatial dimensions». *Phys. Rev. B* 78.19 (2008), p. 195125. DOI: [10.1103/PhysRevB.78.195125](https://doi.org/10.1103/PhysRevB.78.195125). arXiv: [0803.2786](https://arxiv.org/abs/0803.2786) [cond-mat.mes-hall].
- [142] Andrea Amoretti, Alberto Blasi, Nicola Maggiore, and Nicodemo Magnoli. «Three-dimensional dynamics of four-dimensional topological BF theory with boundary». *New J. Phys.* 14 (2012), p. 113014. DOI: [10.1088/1367-2630/14/11/113014](https://doi.org/10.1088/1367-2630/14/11/113014). arXiv: [1205.6156](https://arxiv.org/abs/1205.6156) [hep-th].
- [143] A. Blasi, N. Maggiore, N. Magnoli, and S. Storage. «Maxwell-Chern-Simons Theory With Boundary». *Class. Quant. Grav.* 27 (2010), p. 165018. DOI: [10.1088/0264-9381/27/16/165018](https://doi.org/10.1088/0264-9381/27/16/165018). arXiv: [1002.3227](https://arxiv.org/abs/1002.3227) [hep-th].
- [144] Nicola Maggiore. «Holographic reduction of Maxwell-Chern-Simons theory». *Eur. Phys. J. Plus* 133.7 (2018), p. 281. DOI: [10.1140/epjp/i2018-12130-y](https://doi.org/10.1140/epjp/i2018-12130-y). arXiv: [1807.09960](https://arxiv.org/abs/1807.09960) [hep-th].
- [145] Alberto Blasi and Nicola Maggiore. «Topologically protected duality on the boundary of Maxwell-BF theory». *Symmetry* 11 (2019), p. 921. DOI: [10.3390/sym11070921](https://doi.org/10.3390/sym11070921). arXiv: [1907.08764](https://arxiv.org/abs/1907.08764) [hep-th].
- [146] Jingbo Wang. «Black hole as topological insulator (II): the boundary modes» (June 2017). arXiv: [1706.01630](https://arxiv.org/abs/1706.01630) [gr-qc].
- [147] Dimitra Karabali and V. P. Nair. «Boundary Conditions as Dynamical Fields». *Phys. Rev. D* 92.12 (2015), p. 125003. DOI: [10.1103/PhysRevD.92.125003](https://doi.org/10.1103/PhysRevD.92.125003). arXiv: [1507.03880](https://arxiv.org/abs/1507.03880) [hep-th].
- [148] Michael Pretko and Leo Radzihovsky. «Fracton-Elasticity Duality». *Phys. Rev. Lett.* 120.19 (2018), p. 195301. DOI: [10.1103/PhysRevLett.120.195301](https://doi.org/10.1103/PhysRevLett.120.195301). arXiv: [1711.11044](https://arxiv.org/abs/1711.11044) [cond-mat.str-el].
- [149] Andrey Gromov and Piotr Surówka. «On duality between Cosserat elasticity and fractons». *SciPost Phys.* 8.4 (2020), p. 065. DOI: [10.21468/SciPostPhys.8.4.065](https://doi.org/10.21468/SciPostPhys.8.4.065). arXiv: [1908.06984](https://arxiv.org/abs/1908.06984) [cond-mat.str-el].
- [150] Alessio Caddeo, Carlos Hoyos, and Daniele Musso. «Emergent dipole gauge fields and fractons». *Phys. Rev. D* 106.11 (2022), p. L111903. DOI: [10.1103/PhysRevD.106.L111903](https://doi.org/10.1103/PhysRevD.106.L111903). arXiv: [2206.12877](https://arxiv.org/abs/2206.12877) [cond-mat.str-el].
- [151] Dung Xuan Nguyen, Andrey Gromov, and Sergej Moroz. «Fracton-elasticity duality of two-dimensional superfluid vortex crystals: defect interactions and quantum melting». *SciPost Phys.* 9 (2020), p. 076. DOI: [10.21468/SciPostPhys.9.5.076](https://doi.org/10.21468/SciPostPhys.9.5.076). arXiv: [2005.12317](https://arxiv.org/abs/2005.12317) [cond-mat.quant-gas].
- [152] Andrey Gromov, Andrew Lucas, and Rahul M. Nandkishore. «Fracton hydrodynamics». *Phys. Rev. Res.* 2.3 (2020), p. 033124. DOI: [10.1103/PhysRevResearch.2.033124](https://doi.org/10.1103/PhysRevResearch.2.033124). arXiv: [2003.09429](https://arxiv.org/abs/2003.09429) [cond-mat.str-el].
- [153] Jian-Keng Yuan, Shuai A. Chen, and Peng Ye. «Fractonic Superfluids». *Phys. Rev. Res.* 2.2 (2020), p. 023267. DOI: [10.1103/PhysRevResearch.2.023267](https://doi.org/10.1103/PhysRevResearch.2.023267). arXiv: [1911.02876](https://arxiv.org/abs/1911.02876) [cond-mat.str-el].

- [154] Juven Wang and Shing-Tung Yau. «Non-Abelian gauged fracton matter field theory: Sigma models, superfluids, and vortices». *Phys. Rev. Res.* 2.4 (2020), p. 043219. DOI: [10 . 1103 / PhysRevResearch . 2 . 043219](https://doi.org/10.1103/PhysRevResearch.2.043219). arXiv: [1912 . 13485](https://arxiv.org/abs/1912.13485) [[cond-mat.str-el](https://arxiv.org/abs/1912.13485)].
- [155] Kevin T. Grosvenor, Carlos Hoyos, Francisco Peña Benítez, and Piotr Surówka. «Hydrodynamics of ideal fracton fluids». *Phys. Rev. Res.* 3.4 (2021), p. 043186. DOI: [10 . 1103 / PhysRevResearch . 3 . 043186](https://doi.org/10.1103/PhysRevResearch.3.043186). arXiv: [2105 . 01084](https://arxiv.org/abs/2105.01084) [[cond-mat.str-el](https://arxiv.org/abs/2105.01084)].
- [156] Aleksander Głódkowski, Francisco Peña Benítez, and Piotr Surówka. «Hydrodynamics of dipole-conserving fluids». *Phys. Rev. E* 107.3 (2023), p. 034142. DOI: [10.1103/PhysRevE.107.034142](https://doi.org/10.1103/PhysRevE.107.034142). arXiv: [2212.06848](https://arxiv.org/abs/2212.06848) [[cond-mat.str-el](https://arxiv.org/abs/2212.06848)].
- [157] Shriya Pai and Michael Pretko. «Dynamical Scar States in Driven Fracton Systems». *Phys. Rev. Lett.* 123.13 (2019), p. 136401. DOI: [10.1103/PhysRevLett.123.136401](https://doi.org/10.1103/PhysRevLett.123.136401). arXiv: [1903.06173](https://arxiv.org/abs/1903.06173) [[cond-mat.stat-mech](https://arxiv.org/abs/1903.06173)].
- [158] Vedika Khemani, Michael Hermele, and Rahul Nandkishore. «Localization from Hilbert space shattering: From theory to physical realizations». *Phys. Rev. B* 101.17 (2020), p. 174204. DOI: [10 . 1103 / PhysRevB . 101 . 174204](https://doi.org/10.1103/PhysRevB.101.174204). arXiv: [1904 . 04815](https://arxiv.org/abs/1904.04815) [[cond-mat.stat-mech](https://arxiv.org/abs/1904.04815)].
- [159] Pablo Sala, Tibor Rakovszky, Ruben Verresen, Michael Knap, and Frank Pollmann. «Ergodicity-breaking arising from Hilbert space fragmentation in dipole-conserving Hamiltonians». *Phys. Rev. X* 10.1 (2020), p. 011047. DOI: [10 . 1103 / PhysRevX . 10 . 011047](https://doi.org/10.1103/PhysRevX.10.011047). arXiv: [1904.04266](https://arxiv.org/abs/1904.04266) [[cond-mat.str-el](https://arxiv.org/abs/1904.04266)].
- [160] Michael Pretko. «Generalized Electromagnetism of Subdimensional Particles: A Spin Liquid Story». *Phys. Rev. B* 96.3 (2017), p. 035119. DOI: [10 . 1103 / PhysRevB . 96 . 035119](https://doi.org/10.1103/PhysRevB.96.035119). arXiv: [1606.08857](https://arxiv.org/abs/1606.08857) [[cond-mat.str-el](https://arxiv.org/abs/1606.08857)].
- [161] Michael Pretko. «Subdimensional Particle Structure of Higher Rank $U(1)$ Spin Liquids». *Phys. Rev. B* 95.11 (2017), p. 115139. DOI: [10 . 1103 / PhysRevB . 95 . 115139](https://doi.org/10.1103/PhysRevB.95.115139). arXiv: [1604.05329](https://arxiv.org/abs/1604.05329) [[cond-mat.str-el](https://arxiv.org/abs/1604.05329)].
- [162] Daniel Bulmash and Maissam Barkeshli. «The Higgs Mechanism in Higher-Rank Symmetric $U(1)$ Gauge Theories». *Phys. Rev. B* 97.23 (2018), p. 235112. DOI: [10.1103/PhysRevB.97.235112](https://doi.org/10.1103/PhysRevB.97.235112). arXiv: [1802.10099](https://arxiv.org/abs/1802.10099) [[cond-mat.str-el](https://arxiv.org/abs/1802.10099)].
- [163] Wilbur Shirley, Kevin Slagle, Zhenghan Wang, and Xie Chen. «Fracton Models on General Three-Dimensional Manifolds». *Phys. Rev. X* 8.3 (2018), p. 031051. DOI: [10.1103/PhysRevX.8.031051](https://doi.org/10.1103/PhysRevX.8.031051). arXiv: [1712.05892](https://arxiv.org/abs/1712.05892) [[cond-mat.str-el](https://arxiv.org/abs/1712.05892)].
- [164] Han Ma, Ethan Lake, Xie Chen, and Michael Hermele. «Fracton topological order via coupled layers». *Phys. Rev. B* 95.24 (2017), p. 245126. DOI: [10 . 1103 / PhysRevB . 95 . 245126](https://doi.org/10.1103/PhysRevB.95.245126). arXiv: [1701.00747](https://arxiv.org/abs/1701.00747) [[cond-mat.str-el](https://arxiv.org/abs/1701.00747)].
- [165] Han Ma, Michael Hermele, and Xie Chen. «Fracton topological order from the Higgs and partial-confinement mechanisms of rank-two gauge theory». *Phys. Rev. B* 98.3 (2018), p. 035111. DOI: [10 . 1103 / PhysRevB . 98 . 035111](https://doi.org/10.1103/PhysRevB.98.035111). arXiv: [1802 . 10108](https://arxiv.org/abs/1802.10108) [[cond-mat.str-el](https://arxiv.org/abs/1802.10108)].
- [166] Dominic J. Williamson, Zhen Bi, and Meng Cheng. «Fractonic Matter in Symmetry-Enriched $U(1)$ Gauge Theory». *Phys. Rev. B* 100.12 (2019), p. 125150. DOI: [10.1103/PhysRevB.100.125150](https://doi.org/10.1103/PhysRevB.100.125150). arXiv: [1809.10275](https://arxiv.org/abs/1809.10275) [[cond-mat.str-el](https://arxiv.org/abs/1809.10275)].

- [167] Kevin Slagle. «Foliated Quantum Field Theory of Fracton Order». *Phys. Rev. Lett.* 126.10 (2021), p. 101603. DOI: [10.1103/PhysRevLett.126.101603](https://doi.org/10.1103/PhysRevLett.126.101603). arXiv: [2008.03852](https://arxiv.org/abs/2008.03852) [hep-th].
- [168] Kevin Slagle and Yong Baek Kim. «Quantum Field Theory of X-Cube Fracton Topological Order and Robust Degeneracy from Geometry». *Phys. Rev. B* 96.19 (2017), p. 195139. DOI: [10.1103/PhysRevB.96.195139](https://doi.org/10.1103/PhysRevB.96.195139). arXiv: [1708.04619](https://arxiv.org/abs/1708.04619) [cond-mat.str-el].
- [169] Kevin Slagle, David Aasen, and Dominic Williamson. «Foliated Field Theory and String-Membrane-Net Condensation Picture of Fracton Order». *SciPost Phys.* 6.4 (2019), p. 043. DOI: [10.21468/SciPostPhys.6.4.043](https://doi.org/10.21468/SciPostPhys.6.4.043). arXiv: [1812.01613](https://arxiv.org/abs/1812.01613) [cond-mat.str-el].
- [170] Kevin Slagle, Abhinav Prem, and Michael Pretko. «Symmetric Tensor Gauge Theories on Curved Spaces». *Annals Phys.* 410 (2019), p. 167910. DOI: [10.1016/j.aop.2019.167910](https://doi.org/10.1016/j.aop.2019.167910). arXiv: [1807.00827](https://arxiv.org/abs/1807.00827) [cond-mat.str-el].
- [171] Leo Bidussi, Jelle Hartong, Emil Have, Jørgen Musaeus, and Stefan Prohazka. «Fractons, dipole symmetries and curved spacetime». *SciPost Phys.* 12.6 (2022), p. 205. DOI: [10.21468/SciPostPhys.12.6.205](https://doi.org/10.21468/SciPostPhys.12.6.205). arXiv: [2111.03668](https://arxiv.org/abs/2111.03668) [hep-th].
- [172] Akash Jain and Kristan Jensen. «Fractons in curved space». *SciPost Phys.* 12.4 (2022), p. 142. DOI: [10.21468/SciPostPhys.12.4.142](https://doi.org/10.21468/SciPostPhys.12.4.142). arXiv: [2111.03973](https://arxiv.org/abs/2111.03973) [hep-th].
- [173] Lazaros Tsaloukidis, José J. Fernández-Melgarejo, Javier Molina-Vilaplana, and Piotr Surówka. «Fracton-elasticity duality on curved manifolds» (Apr. 2023). arXiv: [2304.12242](https://arxiv.org/abs/2304.12242) [hep-th].
- [174] Han Yan. «Hyperbolic fracton model, subsystem symmetry, and holography». *Phys. Rev. B* 99.15 (2019), p. 155126. DOI: [10.1103/PhysRevB.99.155126](https://doi.org/10.1103/PhysRevB.99.155126). arXiv: [1807.05942](https://arxiv.org/abs/1807.05942) [hep-th].
- [175] Michael Pretko. «Higher-Spin Witten Effect and Two-Dimensional Fracton Phases». *Phys. Rev. B* 96.12 (2017), p. 125151. DOI: [10.1103/PhysRevB.96.125151](https://doi.org/10.1103/PhysRevB.96.125151). arXiv: [1707.03838](https://arxiv.org/abs/1707.03838) [cond-mat.str-el].
- [176] Tomas Brauner, Sean A. Hartnoll, Pavel Kovtun, Hong Liu, Márk Mezei, Alberto Nicolis, Riccardo Penco, Shu-Heng Shao, and Dam Thanh Son. «Snowmass White Paper: Effective Field Theories for Condensed Matter Systems». *Snowmass 2021*. Mar. 2022. arXiv: [2203.10110](https://arxiv.org/abs/2203.10110) [hep-th].
- [177] Michael Pretko. «Emergent gravity of fractons: Mach's principle revisited». *Phys. Rev. D* 96.2 (2017), p. 024051. DOI: [10.1103/PhysRevD.96.024051](https://doi.org/10.1103/PhysRevD.96.024051). arXiv: [1702.07613](https://arxiv.org/abs/1702.07613) [cond-mat.str-el].
- [178] Alberto Blasi and Nicola Maggiore. «The theory of symmetric tensor field: From fractons to gravitons and back». *Phys. Lett. B* 833 (2022), p. 137304. DOI: [10.1016/j.physletb.2022.137304](https://doi.org/10.1016/j.physletb.2022.137304). arXiv: [2207.05956](https://arxiv.org/abs/2207.05956) [hep-th].
- [179] D. Tong. *Lectures on Gauge Theory*. URL: <https://www.damtp.cam.ac.uk/user/tong/gaugetheory.html>.
- [180] Edward Witten. «Dyons of Charge $e\theta/2\pi$ ». *Phys. Lett. B* 86 (1979), pp. 283–287. DOI: [10.1016/0370-2693\(79\)90838-4](https://doi.org/10.1016/0370-2693(79)90838-4).

- [181] G. Rosenberg and M. Franz. «Witten effect in a crystalline topological insulator». *Phys. Rev. B* 82 (2010), p. 035105. DOI: [10.1103/PhysRevB.82.035105](https://doi.org/10.1103/PhysRevB.82.035105). arXiv: [1001.3179](https://arxiv.org/abs/1001.3179) [[cond-mat.mes-hall](#)].
- [182] Fiona J. Burnell, Trithep Devakul, Pranay Gorantla, Ho Tat Lam, and Shu-Heng Shao. «Anomaly inflow for subsystem symmetries». *Phys. Rev. B* 106.8 (2022), p. 085113. DOI: [10.1103/PhysRevB.106.085113](https://doi.org/10.1103/PhysRevB.106.085113). arXiv: [2110.09529](https://arxiv.org/abs/2110.09529) [[cond-mat.str-el](#)].
- [183] Yizhi You, Trithep Devakul, S. L. Sondhi, and F. J. Burnell. «Fractonic Chern-Simons and BF theories». *Phys. Rev. Res.* 2.2 (2020), p. 023249. DOI: [10.1103/PhysRevResearch.2.023249](https://doi.org/10.1103/PhysRevResearch.2.023249). arXiv: [1904.11530](https://arxiv.org/abs/1904.11530) [[cond-mat.str-el](#)].
- [184] Yizhi You, F. J. Burnell, and Taylor L. Hughes. «Multipolar Topological Field Theories: Bridging Higher Order Topological Insulators and Fractons». *Phys. Rev. B* 103.24 (2021), p. 245128. DOI: [10.1103/PhysRevB.103.245128](https://doi.org/10.1103/PhysRevB.103.245128). arXiv: [1909.05868](https://arxiv.org/abs/1909.05868) [[cond-mat.str-el](#)].
- [185] Andrea Cappelli and Enrico Randellini. «Multipole Expansion in the Quantum Hall Effect». *JHEP* 03 (2016), p. 105. DOI: [10.1007/JHEP03\(2016\)105](https://doi.org/10.1007/JHEP03(2016)105). arXiv: [1512.02147](https://arxiv.org/abs/1512.02147) [[cond-mat.str-el](#)].
- [186] Alex Rasmussen, Yi-Zhuang You, and Cenke Xu. *Stable Gapless Bose Liquid Phases without any Symmetry*. 2016. arXiv: [1601.08235](https://arxiv.org/abs/1601.08235) [[cond-mat.str-el](#)].
- [187] Michael Pretko. «The Fracton Gauge Principle». *Phys. Rev. B* 98.11 (2018), p. 115134. DOI: [10.1103/PhysRevB.98.115134](https://doi.org/10.1103/PhysRevB.98.115134). arXiv: [1807.11479](https://arxiv.org/abs/1807.11479) [[cond-mat.str-el](#)].
- [188] Beni Yoshida. «Exotic topological order in fractal spin liquids». *Phys. Rev. B* 88.12 (2013), p. 125122. DOI: [10.1103/PhysRevB.88.125122](https://doi.org/10.1103/PhysRevB.88.125122). arXiv: [1302.6248](https://arxiv.org/abs/1302.6248) [[cond-mat.str-el](#)].
- [189] Abhinav Prem, Michael Pretko, and Rahul Nandkishore. «Emergent Phases of Fractonic Matter». *Phys. Rev. B* 97.8 (2018), p. 085116. DOI: [10.1103/PhysRevB.97.085116](https://doi.org/10.1103/PhysRevB.97.085116). arXiv: [1709.09673](https://arxiv.org/abs/1709.09673) [[cond-mat.str-el](#)].
- [190] Abhinav Prem, Jeongwan Haah, and Rahul Nandkishore. «Glassy quantum dynamics in translation invariant fracton models». *Phys. Rev. B* 95.15 (2017), p. 155133. DOI: [10.1103/PhysRevB.95.155133](https://doi.org/10.1103/PhysRevB.95.155133). arXiv: [1702.02952](https://arxiv.org/abs/1702.02952) [[cond-mat.stat-mech](#)].
- [191] Sergey Bravyi and Jeongwan Haah. «Quantum Self-Correction in the 3D Cubic Code Model». *Phys. Rev. Lett.* 111 (20 2013), p. 200501. DOI: [10.1103/PhysRevLett.111.200501](https://doi.org/10.1103/PhysRevLett.111.200501). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.111.200501>.
- [192] Bowen Shi and Yuan-Ming Lu. «Deciphering the nonlocal entanglement entropy of fracton topological orders». *Phys. Rev. B* 97.14 (2018), p. 144106. DOI: [10.1103/PhysRevB.97.144106](https://doi.org/10.1103/PhysRevB.97.144106). arXiv: [1705.09300](https://arxiv.org/abs/1705.09300) [[cond-mat.str-el](#)].
- [193] Riccardo Argurio, Carlos Hoyos, Daniele Musso, and Daniel Naegels. «Fractons in effective field theories for spontaneously broken translations». *Phys. Rev. D* 104.10 (2021), p. 105001. DOI: [10.1103/PhysRevD.104.105001](https://doi.org/10.1103/PhysRevD.104.105001). arXiv: [2107.03073](https://arxiv.org/abs/2107.03073) [[hep-th](#)].

- [194] Michael Pretko. «Finite-temperature screening of $U(1)$ fractons». *Phys. Rev. B* 96 (11 2017), p. 115102. DOI: [10.1103/PhysRevB.96.115102](https://doi.org/10.1103/PhysRevB.96.115102). URL: <https://link.aps.org/doi/10.1103/PhysRevB.96.115102>.
- [195] Andrey Gromov. «Towards classification of Fracton phases: the multipole algebra». *Phys. Rev. X* 9.3 (2019), p. 031035. DOI: [10.1103/PhysRevX.9.031035](https://doi.org/10.1103/PhysRevX.9.031035). arXiv: [1812.05104](https://arxiv.org/abs/1812.05104) [cond-mat.str-el].
- [196] Pranay Gorantla, Ho Tat Lam, Nathan Seiberg, and Shu-Heng Shao. «Global dipole symmetry, compact Lifshitz theory, tensor gauge theory, and fractons». *Phys. Rev. B* 106.4 (2022), p. 045112. DOI: [10.1103/PhysRevB.106.045112](https://doi.org/10.1103/PhysRevB.106.045112). arXiv: [2201.10589](https://arxiv.org/abs/2201.10589) [cond-mat.str-el].
- [197] Michael Pretko, Zhengzheng Zhai, and Leo Radzihovsky. «Crystal-to-Fracton Tensor Gauge Theory Dualities». *Phys. Rev. B* 100.13 (2019), p. 134113. DOI: [10.1103/PhysRevB.100.134113](https://doi.org/10.1103/PhysRevB.100.134113). arXiv: [1907.12577](https://arxiv.org/abs/1907.12577) [cond-mat.str-el].
- [198] Cenke Xu. «Gapless bosonic excitation without symmetry breaking: An algebraic spin liquid with soft gravitons». *Phys. Rev. B* 74 (22 2006), p. 224433. DOI: [10.1103/PhysRevB.74.224433](https://doi.org/10.1103/PhysRevB.74.224433). URL: <https://link.aps.org/doi/10.1103/PhysRevB.74.224433>.
- [199] Zheng-Cheng Gu and Xiao-Gang Wen. «Emergence of helicity ± 2 modes (gravitons) from qbit models». *Nucl. Phys. B* 863 (2012), pp. 90–129. DOI: [10.1016/j.nuclphysb.2012.05.010](https://doi.org/10.1016/j.nuclphysb.2012.05.010). arXiv: [0907.1203](https://arxiv.org/abs/0907.1203) [gr-qc].
- [200] Cenke Xu and Petr Horava. «Emergent Gravity at a Lifshitz Point from a Bose Liquid on the Lattice». *Phys. Rev. D* 81 (2010), p. 104033. DOI: [10.1103/PhysRevD.81.104033](https://doi.org/10.1103/PhysRevD.81.104033). arXiv: [1003.0009](https://arxiv.org/abs/1003.0009) [hep-th].
- [201] Barbara M. Terhal. «Quantum error correction for quantum memories». *Rev. Mod. Phys.* 87 (2 2015), pp. 307–346. DOI: [10.1103/RevModPhys.87.307](https://doi.org/10.1103/RevModPhys.87.307). URL: <https://link.aps.org/doi/10.1103/RevModPhys.87.307>.
- [202] Han Ma, A. T. Schmitz, S. A. Parameswaran, Michael Hermele, and Rahul M. Nandkishore. «Topological Entanglement Entropy of Fracton Stabilizer Codes». *Phys. Rev. B* 97.12 (2018), p. 125101. DOI: [10.1103/PhysRevB.97.125101](https://doi.org/10.1103/PhysRevB.97.125101). arXiv: [1710.01744](https://arxiv.org/abs/1710.01744) [cond-mat.str-el].
- [203] Zheng-Cheng Gu and Xiao-Gang Wen. «A Lattice bosonic model as a quantum theory of gravity» (June 2006). arXiv: [gr-qc/0606100](https://arxiv.org/abs/gr-qc/0606100).
- [204] Bahram Mashhoon. *Gravitoelectromagnetism: A Brief Review*. 2008. arXiv: [gr-qc/0311030](https://arxiv.org/abs/gr-qc/0311030) [gr-qc].
- [205] Athanasios Chatzistavrakidis, Georgios Karagiannis, and Peter Schupp. «Torsion-induced gravitational θ term and gravitoelectromagnetism». *Eur. Phys. J. C* 80.11 (2020), p. 1034. DOI: [10.1140/epjc/s10052-020-08600-9](https://doi.org/10.1140/epjc/s10052-020-08600-9). arXiv: [2007.06632](https://arxiv.org/abs/2007.06632) [gr-qc].
- [206] R. D. Peccei and Helen R. Quinn. «CP Conservation in the Presence of Instantons». *Phys. Rev. Lett.* 38 (1977), pp. 1440–1443. DOI: [10.1103/PhysRevLett.38.1440](https://doi.org/10.1103/PhysRevLett.38.1440).
- [207] R. D. Peccei and Helen R. Quinn. «Constraints Imposed by CP Conservation in the Presence of Instantons». *Phys. Rev. D* 16 (1977), pp. 1791–1797. DOI: [10.1103/PhysRevD.16.1791](https://doi.org/10.1103/PhysRevD.16.1791).

- [208] P. Sikivie. «Experimental Tests of the Invisible Axion». *Phys. Rev. Lett.* 51 (1983). Ed. by M. A. Srednicki. [Erratum: *Phys.Rev.Lett.* 52, 695 (1984)], pp. 1415–1417. DOI: [10.1103/PhysRevLett.51.1415](https://doi.org/10.1103/PhysRevLett.51.1415).
- [209] Frank Wilczek. «Two Applications of Axion Electrodynamics». *Phys. Rev. Lett.* 58 (1987), p. 1799. DOI: [10.1103/PhysRevLett.58.1799](https://doi.org/10.1103/PhysRevLett.58.1799).
- [210] A. Zee. «GRAVITOMAGNETIC POLE AND MASS QUANTIZATION». *Phys. Rev. Lett.* 55 (1985). [Erratum: *Phys.Rev.Lett.* 56, 1101 (1986)], pp. 2379–2381. DOI: [10.1103/PhysRevLett.55.2379](https://doi.org/10.1103/PhysRevLett.55.2379).
- [211] Kurt Hinterbichler. «Theoretical Aspects of Massive Gravity». *Rev. Mod. Phys.* 84 (2012), pp. 671–710. DOI: [10.1103/RevModPhys.84.671](https://doi.org/10.1103/RevModPhys.84.671). arXiv: [1105.3735 \[hep-th\]](https://arxiv.org/abs/1105.3735).
- [212] Y. S. Wu and A. Zee. «Membranes, Higher Hopf Maps, and Phase Interactions». *Phys. Lett. B* 207 (1988), pp. 39–43. DOI: [10.1016/0370-2693\(88\)90882-9](https://doi.org/10.1016/0370-2693(88)90882-9).
- [213] Yi-Hsien Du, Umang Mehta, Dung Xuan Nguyen, and Dam Thanh Son. «Volume-preserving diffeomorphism as nonabelian higher-rank gauge symmetry». *SciPost Phys.* 12.2 (2022), p. 050. DOI: [10.21468/SciPostPhys.12.2.050](https://doi.org/10.21468/SciPostPhys.12.2.050). arXiv: [2103.09826 \[cond-mat.str-el\]](https://arxiv.org/abs/2103.09826).
- [214] Alberto Blasi and Nicola Maggiore. «Massive gravity and Fierz-Pauli theory». *Eur. Phys. J. C* 77.9 (2017), p. 614. DOI: [10.1140/epjc/s10052-017-5205-y](https://doi.org/10.1140/epjc/s10052-017-5205-y). arXiv: [1706.08140 \[hep-th\]](https://arxiv.org/abs/1706.08140).
- [215] Alberto Blasi and Nicola Maggiore. «Massive deformations of rank-2 symmetric tensor theory (a.k.a. BRS characterization of Fierz–Pauli massive gravity)». *Class. Quant. Grav.* 34.1 (2017), p. 015005. DOI: [10.1088/1361-6382/34/1/015005](https://doi.org/10.1088/1361-6382/34/1/015005). arXiv: [1512.01025 \[hep-th\]](https://arxiv.org/abs/1512.01025).
- [216] L. D. Landau and E. M. Lifschits. *The Classical Theory of Fields*. Vol. Volume 2. Course of Theoretical Physics. Oxford: Pergamon Press, 1975. ISBN: 978-0-08-018176-9.
- [217] R. D. Peccei. «The Strong CP problem and axions». *Lect. Notes Phys.* 741 (2008). Ed. by Markus Kuster, Georg Raffelt, and Berta Beltran, pp. 3–17. DOI: [10.1007/978-3-540-73518-2_1](https://doi.org/10.1007/978-3-540-73518-2_1). arXiv: [hep-ph/0607268](https://arxiv.org/abs/hep-ph/0607268).
- [218] Giulio Gambuti and Nicola Maggiore. «A note on harmonic gauge(s) in massive gravity». *Phys. Lett. B* 807 (2020), p. 135530. DOI: [10.1016/j.physletb.2020.135530](https://doi.org/10.1016/j.physletb.2020.135530). arXiv: [2006.04360 \[gr-qc\]](https://arxiv.org/abs/2006.04360).
- [219] Giulio Gambuti and Nicola Maggiore. «Fierz–Pauli theory reloaded: from a theory of a symmetric tensor field to linearized massive gravity». *Eur. Phys. J. C* 81.2 (2021), p. 171. DOI: [10.1140/epjc/s10052-021-08962-8](https://doi.org/10.1140/epjc/s10052-021-08962-8). arXiv: [2102.10813 \[gr-qc\]](https://arxiv.org/abs/2102.10813).
- [220] N. Cabibbo and G. Parisi. «Exponential Hadronic Spectrum and Quark Liberation». *Phys. Lett. B* 59 (1975), pp. 67–69. DOI: [10.1016/0370-2693\(75\)90158-6](https://doi.org/10.1016/0370-2693(75)90158-6).
- [221] Adam Miklos Halasz, A. D. Jackson, R. E. Shrock, Misha A. Stephanov, and J. J. M. Verbaarschot. «On the phase diagram of QCD». *Phys. Rev. D* 58 (1998), p. 096007. DOI: [10.1103/PhysRevD.58.096007](https://doi.org/10.1103/PhysRevD.58.096007). arXiv: [hep-ph/9804290](https://arxiv.org/abs/hep-ph/9804290).

- [222] G. Baym and G. Grinstein. «Phase Transition in the Sigma Model at Finite Temperature». *Phys. Rev. D* 15 (1977), pp. 2897–2912. DOI: [10.1103/PhysRevD.15.2897](https://doi.org/10.1103/PhysRevD.15.2897).
- [223] M. Grater and C. Wetterich. «Kosterlitz-Thouless phase transition in the two-dimensional linear sigma model». *Phys. Rev. Lett.* 75 (1995), pp. 378–381. DOI: [10.1103/PhysRevLett.75.378](https://doi.org/10.1103/PhysRevLett.75.378). arXiv: [hep-ph/9409459](https://arxiv.org/abs/hep-ph/9409459).
- [224] Marc Henneaux and Claudio Teitelboim. «Duality in linearized gravity». *Phys. Rev. D* 71 (2005), p. 024018. DOI: [10.1103/PhysRevD.71.024018](https://doi.org/10.1103/PhysRevD.71.024018). arXiv: [gr-qc/0408101](https://arxiv.org/abs/gr-qc/0408101).
- [225] Claudio Bunster, Marc Henneaux, and Sergio Hortner. «Gravitational Electric-Magnetic Duality, Gauge Invariance and Twisted Self-Duality». *J. Phys. A* 46 (2013). [Erratum: *J.Phys.A* 46, 269501 (2013)], p. 214016. DOI: [10.1088/1751-8113/46/21/214016](https://doi.org/10.1088/1751-8113/46/21/214016). arXiv: [1207.1840 \[hep-th\]](https://arxiv.org/abs/1207.1840).
- [226] Diego M. Hofman and Nabil Iqbal. «Goldstone modes and photonization for higher form symmetries». *SciPost Phys.* 6.1 (2019), p. 006. DOI: [10.21468/SciPostPhys.6.1.006](https://doi.org/10.21468/SciPostPhys.6.1.006). arXiv: [1802.09512 \[hep-th\]](https://arxiv.org/abs/1802.09512).
- [227] Kurt Hinterbichler, Diego M. Hofman, Austin Joyce, and Grégoire Mathys. «Gravity as a gapless phase and biform symmetries». *JHEP* 02 (2023), p. 151. DOI: [10.1007/JHEP02\(2023\)151](https://doi.org/10.1007/JHEP02(2023)151). arXiv: [2205.12272 \[hep-th\]](https://arxiv.org/abs/2205.12272).
- [228] Paul A. M. Dirac. «Generalized Hamiltonian dynamics». *Can. J. Math.* 2 (1950), pp. 129–148. DOI: [10.4153/CJM-1950-012-1](https://doi.org/10.4153/CJM-1950-012-1).
- [229] D. Dalmazi and R. R. Lino dos Santos. «The dimensional reduction of linearized spin-2 theories invariant under transverse diffeomorphisms». *Eur. Phys. J. C* 81.6 (2021), p. 547. DOI: [10.1140/epjc/s10052-021-09297-0](https://doi.org/10.1140/epjc/s10052-021-09297-0). arXiv: [2010.12051 \[hep-th\]](https://arxiv.org/abs/2010.12051).
- [230] C. Aragone and A. Khoudeir. «Selfdual massive gravity». *Phys. Lett. B* 173 (1986), pp. 141–144. DOI: [10.1016/0370-2693\(86\)90234-0](https://doi.org/10.1016/0370-2693(86)90234-0).
- [231] D. Dalmazi and Elias L. Mendonca. «Dual descriptions of spin two massive particles in D=2+1 via master actions». *Phys. Rev. D* 79 (2009), p. 045025. DOI: [10.1103/PhysRevD.79.045025](https://doi.org/10.1103/PhysRevD.79.045025). arXiv: [0812.0161 \[hep-th\]](https://arxiv.org/abs/0812.0161).
- [232] Stanley Deser and James G. McCarthy. «Selfdual formulations of D=3 gravity theories». *Phys. Lett. B* 246 (1990). [Addendum: *Phys.Lett.B* 248, 473 (1990)], pp. 441–444. DOI: [10.1016/0370-2693\(90\)90627-I](https://doi.org/10.1016/0370-2693(90)90627-I).
- [233] Andrey Gromov. «Chiral Topological Elasticity and Fracton Order». *Phys. Rev. Lett.* 122.7 (2019), p. 076403. DOI: [10.1103/PhysRevLett.122.076403](https://doi.org/10.1103/PhysRevLett.122.076403). arXiv: [1712.06600 \[cond-mat.str-el\]](https://arxiv.org/abs/1712.06600).
- [234] D. Boito, L. N. S. de Andrade, G. de Sousa, R. Gama, and C. Y. M. London. «On Maxwell’s electrodynamics in two spatial dimensions». *Rev. Bras. Ens. Fis.* 42 (2020), e20190323. DOI: [10.1590/1806-9126-RBEF-2019-0323](https://doi.org/10.1590/1806-9126-RBEF-2019-0323). arXiv: [1809.07368 \[physics.class-ph\]](https://arxiv.org/abs/1809.07368).
- [235] Clay Córdova, Po-Shen Hsin, and Nathan Seiberg. «Time-Reversal Symmetry, Anomalies, and Dualities in (2+1)d». *SciPost Phys.* 5.1 (2018), p. 006. DOI: [10.21468/SciPostPhys.5.1.006](https://doi.org/10.21468/SciPostPhys.5.1.006). arXiv: [1712.08639 \[cond-mat.str-el\]](https://arxiv.org/abs/1712.08639).

- [236] Davide Gaiotto, Anton Kapustin, Zohar Komargodski, and Nathan Seiberg. «Theta, Time Reversal, and Temperature». *JHEP* 05 (2017), p. 091. doi: [10.1007/JHEP05\(2017\)091](https://doi.org/10.1007/JHEP05(2017)091). arXiv: [1703.00501](https://arxiv.org/abs/1703.00501) [hep-th].
- [237] Francesco Benini, Po-Shen Hsin, and Nathan Seiberg. «Comments on global symmetries, anomalies, and duality in $(2 + 1)d$ ». *JHEP* 04 (2017), p. 135. doi: [10.1007/JHEP04\(2017\)135](https://doi.org/10.1007/JHEP04(2017)135). arXiv: [1702.07035](https://arxiv.org/abs/1702.07035) [cond-mat.str-el].
- [238] Diego Delmastro, Davide Gaiotto, and Jaume Gomis. «Global anomalies on the Hilbert space». *JHEP* 11 (2021), p. 142. doi: [10.1007/JHEP11\(2021\)142](https://doi.org/10.1007/JHEP11(2021)142). arXiv: [2101.02218](https://arxiv.org/abs/2101.02218) [hep-th].
- [239] Anton Kapustin, Ryan Thorngren, Alex Turzillo, and Zitao Wang. «Fermionic Symmetry Protected Topological Phases and Cobordisms». *JHEP* 12 (2015), p. 052. doi: [10.1007/JHEP12\(2015\)052](https://doi.org/10.1007/JHEP12(2015)052). arXiv: [1406.7329](https://arxiv.org/abs/1406.7329) [cond-mat.str-el].
- [240] Maissam Barkeshli, Parsa Bonderson, Chao-Ming Jian, Meng Cheng, and Kevin Walker. «Reflection and time reversal symmetry enriched topological phases of matter: path integrals, non-orientable manifolds, and anomalies». *Commun. Math. Phys.* 374.2 (2019), pp. 1021–1124. doi: [10.1007/s00220-019-03475-8](https://doi.org/10.1007/s00220-019-03475-8). arXiv: [1612.07792](https://arxiv.org/abs/1612.07792) [cond-mat.str-el].
- [241] Clay Córdova, Kantaro Ohmori, Shu-Heng Shao, and Fei Yan. «Decorated \mathbb{Z}_2 symmetry defects and their time-reversal anomalies». *Phys. Rev. D* 102.4 (2020), p. 045019. doi: [10.1103/PhysRevD.102.045019](https://doi.org/10.1103/PhysRevD.102.045019). arXiv: [1910.14046](https://arxiv.org/abs/1910.14046) [hep-th].
- [242] Mauro Cirio, Giandomenico Palumbo, and Jiannis K. Pachos. « $(3+1)$ - dimensional topological quantum field theory from a tight-binding model of interacting spinless fermions». *Phys. Rev. B* 90.8 (2014), p. 085114. doi: [10.1103/PhysRevB.90.085114](https://doi.org/10.1103/PhysRevB.90.085114). arXiv: [1309.2380](https://arxiv.org/abs/1309.2380) [cond-mat.str-el].
- [243] Ali H. Chamseddine and M. Reuter. «Induced Two-dimensional Quantum Gravity and $Sl(2,r)$ Kac-Moody Current Algebra». *Nucl. Phys. B* 317 (1989), pp. 757–771. doi: [10.1016/0550-3213\(89\)90542-7](https://doi.org/10.1016/0550-3213(89)90542-7).
- [244] Laurent Houart. «Kac-Moody algebras in gravity and M-theories». *AIP Conf. Proc.* 841.1 (2006). Ed. by Lysiane Mornas and Joaquin Diaz Alonso, pp. 298–305. doi: [10.1063/1.2218185](https://doi.org/10.1063/1.2218185). arXiv: [hep-th/0511009](https://arxiv.org/abs/hep-th/0511009).
- [245] S. Tomonaga. «Remarks on Bloch’s Method of Sound Waves applied to Many-Fermion Problems». *Prog. Theor. Phys.* 5 (1950), pp. 544–569. doi: [10.1143/PTP.5.544](https://doi.org/10.1143/PTP.5.544).
- [246] J. M. Luttinger. «An Exactly Soluble Model of a Many-Fermion System». *J. Math. Phys.* 4 (1963), pp. 1154–1162. doi: [10.1063/1.1704046](https://doi.org/10.1063/1.1704046).
- [247] Steven Weinberg. «Infrared photons and gravitons». *Phys. Rev.* 140 (1965), B516–B524. doi: [10.1103/PhysRev.140.B516](https://doi.org/10.1103/PhysRev.140.B516).
- [248] Temple He, Vyacheslav Lysov, Prahar Mitra, and Andrew Strominger. «BMS supertranslations and Weinberg’s soft graviton theorem». *JHEP* 05 (2015), p. 151. doi: [10.1007/JHEP05\(2015\)151](https://doi.org/10.1007/JHEP05(2015)151). arXiv: [1401.7026](https://arxiv.org/abs/1401.7026) [hep-th].
- [249] Daniel Kapec, Vyacheslav Lysov, Sabrina Pasterski, and Andrew Strominger. «Higher-dimensional supertranslations and Weinberg’s soft graviton theorem». *Ann. Math. Sci. Appl.* 02 (2017), pp. 69–94. doi: [10.4310/AMSA.2017.v2.n1.a2](https://doi.org/10.4310/AMSA.2017.v2.n1.a2). arXiv: [1502.07644](https://arxiv.org/abs/1502.07644) [gr-qc].

- [250] Charles W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. San Francisco: W. H. Freeman, 1973. ISBN: 978-0-7167-0344-0, 978-0-691-17779-3.
- [251] Steven Weinberg and Edward Witten. «Limits on Massless Particles». *Phys. Lett. B* 96 (1980), pp. 59–62. DOI: [10.1016/0370-2693\(80\)90212-9](https://doi.org/10.1016/0370-2693(80)90212-9).
- [252] Eric Poisson, Adam Pound, and Ian Vega. «The Motion of point particles in curved spacetime». *Living Rev. Rel.* 14 (2011), p. 7. DOI: [10.12942/lrr-2011-7](https://doi.org/10.12942/lrr-2011-7). arXiv: [1102.0529 \[gr-qc\]](https://arxiv.org/abs/1102.0529).
- [253] Dmitri V. Vassilevich. «Spectral problems from quantum field theory». *Contemp. Math.* 366 (2005), pp. 3–22. arXiv: [math-ph/0403052](https://arxiv.org/abs/math-ph/0403052).
- [254] M Blau. «Lecture Notes on General Relativity». 2021. URL: <http://www.blau.itp.unibe.ch/newlecturesGR.pdf>.
- [255] Taichiro Kugo and Nobuyoshi Ohta. «Covariant Approach to the No-ghost Theorem in Massive Gravity». *PTEP* 2014 (2014), 043B04. DOI: [10.1093/ptep/ptu046](https://doi.org/10.1093/ptep/ptu046). arXiv: [1401.3873 \[hep-th\]](https://arxiv.org/abs/1401.3873).

