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Euler and Pontryagin currents of the Dirac operator

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Abstract
On differential manifolds with spinor structure, it is possible to express the Euler and Pontryagin currents in terms of tensors that also appear as source in the Dirac equation. It is hence possible to tie concepts rooted in geometry and topology to dynamical characters of quantum matter.

Keywords: spin manifolds, characteristic classes, topological currents

1. Introduction

In differential geometry, the curvature of a Riemannian manifold encodes the properties of the space. Among all scalars that are formed from the curvature, some can be expressed as the divergence of a suitable vector: they are called characteristic classes, and they describe topological features of that space. For example, in two-dimensional spaces, the characteristic class is the Euler class χ_2 while, in four-dimensional cases, they are the Euler and Pontryagin classes χ_4 and p_4 respectively. Similar considerations are true also for Riemann–Cartan manifolds, where curvature is accompanied by torsion. In such a case, the Pontryagin class is called Nieh–Yan class [1–3]. For electrodynamics, the Pontryagin class is the $F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$ term. The vector whose divergence is the characteristic class is called topological current, and it can be computed with straightforward manipulations. In electrodynamics, it is $\varepsilon^{\rho\mu\nu\sigma}F_{\mu\nu}A_\sigma$. On torsional manifolds, it is the Hodge dual of torsion [3]. For Riemannian manifolds, it is a combination of spin connections.

In this last case, for our purpose it is not important to give the topological current explicitly in terms of the spin connection. It is enough to state that, for the manner in which it is constructed, it behaves like a true vector only when taken in divergences integrated over the volume of the space. The same is true for topological currents in electrodynamics, where $\varepsilon^{\rho\mu\nu\sigma}F_{\mu\nu}A_{\sigma}$ is not gauge invariant, and it is only when it is taken in a divergence that the spurious term $\nabla_{\rho}(\varepsilon^{\rho\mu\nu\sigma}F_{\mu\nu}\nabla_{\sigma}\varphi)$ is identically equal to zero, ensuring the gauge invariance of the topological current.

Yet, it may be important to have topological currents that are gauge invariant even if they are not appearing in divergences: only in this way, in fact, they can be employed to construct interaction-like terms. For example, take the term $\varepsilon^{\rho\mu\nu\sigma}F_{\mu\nu}A_{\sigma}=K^{\rho}$, as well as the torsion axial-vector W_{ρ}, and write an interaction like $K^{\rho}W_{\rho}$. It is our aim now to check its gauge invariance: the gauge transformation would produce the shift $K^{\rho}W_{\rho}\rightarrow K^{\rho}W_{\rho}-\varepsilon^{\rho\mu\nu\sigma}F_{\mu\nu}\nabla_{\sigma}\varphi W_{\rho}$, with $K^{\rho}W_{\rho}$ being gauge invariant only if $\varepsilon^{\rho\mu\nu\sigma}F_{\mu\nu}\nabla_{\sigma}\varphi W_{\rho}$ is a divergence, and this being the case only if the torsion axial-vector is the gradient of a pseudo-scalar. Because in general this does not happen, the term $K^{\rho}W_{\rho}$ is not gauge invariant. Nevertheless, such terms would still be gauge invariant, and independently on torsion, if K^{ρ} itself were to be gauge invariant. And more in general, if the topological current were to be a true vector.

Because the characteristic classes are given in terms of curvatures, this problem can be reduced to that of asking whether it is possible to write curvatures in terms of true tensors, and not just connections. In this work we shall see that, when the manifold possesses a spin structure, it has enough richness to acquire the possibility to convert spin connection and gauge potential into a real tensor and a gauge invariant vector in terms of which the Riemann curvature and the Maxwell strength can be expressed [4].

This will enables us to express also the topological currents in terms of real vectors. In addition, we will see that the vectors in terms of which the topological currents are expressed also enter as sources into the Dirac field equations, henceforth establishing a link between topological objects and dynamical effects of the Dirac operator.

To help seeing these concepts, we will furnish some examples for different spinorial systems in low-dimensional cases, that is for dimensions $n=2,\ 3,\ 4$ [5, 6].

2. General preliminaries

2.1. Geometrical and topological notions

Let it be given a manifold $\mathscr{M}$ with metric $g_{\sigma\nu}$ and with frame and co-frame e^{a}_{σ} and e^{α}_{s} such that

$$e^{i}_{\sigma}e^{\alpha}_{i}=\delta^{\alpha}_{\sigma}\qquad e^{a}_{\nu}e^{\nu}_{s}=\delta^{a}_{s}\tag{1}$$

verifying

$$e^{\sigma}_{i}e^{\alpha}_{j}g_{\sigma\alpha}=\eta_{ij}\tag{2}$$

with η_{ij} the Minkowskian matrix. This matrix is diagonal with elements that are all unitary up to the sign, and with it we will specify dimension and signature. Greek indices are coordinate indices on the manifold, transforming with diffeomorphisms, while Latin indices are world indices on the tangent space, transforming with elements of the real Lorentz group. With the metric $g_{\sigma\nu}$ and its inverse $g^{\sigma\nu}$ we lower/raise coordinate indices, with the Minkowskian metric and its inverse we lower/raise world indices, while with frame and co-frame we convert coordinate to world indices. This is commonly known in differential geometry although interested readers may find more details in [7].

The spin connection can be introduced by the relation

$$C^{i}{}_{k\mu}=e^{i}_{\sigma}\partial_{\mu}e^{\sigma}_{k}+e^{i}_{\alpha}e^{\sigma}_{k}\Lambda^{\alpha}_{\sigma\mu} \tag{3}$$

where $\Lambda^{\alpha}_{\sigma\mu}$ is the Levi–Civita connection, entirely written in terms of the derivatives of the metric. See again [7].

The Riemann curvature of the Levi–Civita connection can be converted into $R_{ij\mu\nu}=e^{\rho}_{i}e^{\sigma}_{j}R_{\rho\sigma\mu\nu}$ so that

$$R^{i}{}_{j\mu\nu}=\partial_{\mu}C^{i}{}_{j\nu}-\partial_{\nu}C^{i}{}_{j\mu}+C^{i}{}_{k\mu}C^{k}{}_{j\nu}-C^{i}{}_{k\nu}C^{k}{}_{j\mu} \tag{4}$$

as is straightforward to demonstrate [7].

From it we define $R^{i}{}_{j\mu\nu}e^{\mu}_{i}=R_{j\nu}$ called Ricci curvature and hence we can define $R_{a\nu}e^{\nu}_{c}\eta^{ac}=R$ called Ricci scalar, which is a topological invariant in two dimensions. For more scalars we have to consider products of two curvatures as $R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu}$ or $R_{\mu\nu}R^{\mu\nu}$ beside the obvious R^2 amounting to all non-trivial independent contractions, and for which the Gauss–Bonnet term $R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu}-4R_{\mu\nu}R^{\mu\nu}+R^2$ is a topological invariant in four dimensions. In four dimensions it is possible to write the Gauss–Bonnet term like

$$R^{\alpha\nu\mu\rho}R_{\rho\mu\nu\alpha}-4R^{\alpha\mu}R_{\mu\alpha}+R^2=-\frac{1}{4}\varepsilon^{\alpha\sigma\pi\tau}\varepsilon^{\rho\omega\mu\nu}R_{\alpha\sigma\mu\nu}R_{\pi\tau\rho\omega}. \tag{5}$$

It is also possible to have another topological invariant given by $\frac{1}{4}\varepsilon^{\mu\nu\pi\eta}R^{\alpha\sigma}{}_{\mu\nu}R_{\alpha\sigma\pi\eta}$, and which is parity-odd.

The three above topological invariants characterize the Euler classes χ_2 and χ_4 and the Pontryagin class p_4 respectively. As such we must have that

$$\chi_2:\ \longrightarrow\quad -\frac{1}{2}R=\nabla_{\rho}G^{\rho}_{2} \tag{6}$$

and

$$\chi_4:\ \longrightarrow\quad -\frac{1}{8}\varepsilon^{\alpha\sigma\pi\tau}\varepsilon^{\rho\omega\mu\nu}R_{\alpha\sigma\mu\nu}R_{\pi\tau\rho\omega}=\nabla_{\rho}G^{\rho}_{4} \tag{7}$$

$$p_4:\ \longrightarrow\quad \frac{1}{4}\varepsilon^{\mu\nu\pi\eta}R^{\alpha\sigma}{}_{\mu\nu}R_{\alpha\sigma\pi\eta}=\nabla_{\rho}K^{\rho}_{4} \tag{8}$$

for some bi-dimensional vector G^{ρ}_{2} and four-dimensional vector G^{ρ}_{4} and axial-vector K^{ρ}_{4} (the fractions in front of the curvatures are for later convenience). Further details on characteristic classes can be found in [9].

2.2. *Quantum fields*

Let it be given on $\mathscr{M}$ a spinorial structure determined by the Clifford algebra $\{\boldsymbol{\gamma}^a,\boldsymbol{\gamma}^b\}=2\mathbb{I}\eta^{ab}$ and by a spin-1/2 spinor field. We define $[\boldsymbol{\gamma}_a,\boldsymbol{\gamma}_b]/4=\boldsymbol{\sigma}_{ab}$ as the elements of the complex Lorentz algebra, and their exponentiation are the elements of the complex Lorentz group. When complex Lorentz transformations are accompanied by phase transformations we talk about spinor transformations $\boldsymbol{S}$ [7]. The spin-1/2 spinor fields ψ are columns of four complex objects that are scalars under diffeomorphisms and that transform as $\psi\rightarrow\boldsymbol{S}\psi$ under spinor transformations [8].

We introduce the spinorial connection according to

$$\boldsymbol{C}_\mu = \frac{1}{2} C_{ij\mu} \boldsymbol{\sigma}^{ij} + iqA_\mu \mathbb{I} \tag{9}$$

in terms of the spin connection $C_{ij\mu}$ and an abelian gauge potential A_μ having q as charge. With it

$$\boldsymbol{\nabla}_\mu \psi = \partial_\mu \psi + \boldsymbol{C}_\mu \psi \tag{10}$$

is the covariant derivative of the spinor field (the fact that the full spinor transformation $\boldsymbol{S}$ accounts for complex unitary transformations beside the complex Lorentz transformations is the reason why the connection must contain the generator $\mathbb{I}$ beside the Lorentz generators $\boldsymbol{\sigma}^{ij}$ in order to ensure that the spinorial derivative be covariant under the abelian gauge group beside the Lorentz group).

As usual, we have that

$$[\boldsymbol{\nabla}_\mu, \boldsymbol{\nabla}_\nu]\,\psi = \frac{1}{2} R_{ij\mu\nu} \boldsymbol{\sigma}^{ij} \psi + iqF_{\mu\nu} \psi \tag{11}$$

(again, the fact that the covariant derivative of the spinor contained both abelian gauge and Lorentz group is reflected here in the fact that the commutator encodes both the action of electrodynamics $F_{\mu\nu}$ and that of the curvature of the space-time $R_{ij\mu\nu}$).

The dynamics is assigned by the Dirac equation

$$i\boldsymbol{\gamma}^\mu \boldsymbol{\nabla}_\mu \psi - m\psi = 0 \tag{12}$$

(in which we have assumed no torsion for simplicity).

This construction is valid for any dimension. However, for our purposes, we need to introduce the technique of polar decomposition, and this can be done only after that we specify dimension and signature of the space. In what follows, we will consider spaces of dimension 4 and lower.

3. Polar degrees of freedom and tensorial connections

3.1. Four-dimensions

3.1.1. (1+3)-signature. We begin with the physical space. In it, the Minkowski matrix has four elements, of which one is equal to unity and three are equal to minus unity, while the Levi-Civita completely antisymmetric tensor $\varepsilon_{\alpha\nu\sigma\tau}$ has four indices.

The Clifford matrices, and in particular the sigma matrices, verify $2i\boldsymbol{\sigma}_{ab} = \varepsilon_{abij} \boldsymbol{\sigma}^{ij} \boldsymbol{\pi}$ implicitly defining an additional matrix $\boldsymbol{\pi}$ which is parity-odd[4]. We have

$$\boldsymbol{\gamma}_i \boldsymbol{\gamma}_j \boldsymbol{\gamma}_k = \boldsymbol{\gamma}_i \eta_{jk} - \boldsymbol{\gamma}_j \eta_{ik} + \boldsymbol{\gamma}_k \eta_{ij} - i\varepsilon_{ijkq} \boldsymbol{\gamma}^q \boldsymbol{\pi} \tag{13}$$

[4] This matrix is usually designated as a gamma with an index five, but we will not employ this notation here: the index five simply has no sense, especially in the three- and two-dimensional cases we will consider in the following. In addition, we will indicate it with the boldface Greek letter $\boldsymbol{\pi}$, whose Latin correspondent is p, to mark the fact that it is *parity*-odd (much in the same way in which the generators are denoted with the boldface Greek letter $\boldsymbol{\sigma}$, whose Latin is s, to indicate that they are *spin*-dependent).

showing that products of more than three Clifford matrices can always be reduced to the product of two. Defining $\overline{\psi}=\psi^{\dagger}\boldsymbol{\gamma}^{0}$ we can build the bi-linear spinor quantities

$$S^{a}=\overline{\psi}\boldsymbol{\gamma}^{a}\boldsymbol{\pi}\psi \qquad U^{a}=\overline{\psi}\boldsymbol{\gamma}^{a}\psi \tag{14}$$

$$\Theta=i\overline{\psi}\boldsymbol{\pi}\psi \qquad \Phi=\overline{\psi}\psi \tag{15}$$

which are all real tensors. With them, we have

$$2U_{\mu}S_{\nu}\boldsymbol{\sigma}^{\mu\nu}\boldsymbol{\pi}\psi+U^{2}\psi=0 \tag{16}$$

and

$$U_{a}U^{a}=-S_{a}S^{a}=\Theta^{2}+\Phi^{2} \tag{17}$$

$$U_{a}S^{a}=0 \tag{18}$$

as three examples of Fierz identities (their derivation and general structure is found for instance in [10–13], while in [14–16] they are presented with the structure with which we will employ them in the following of this work). When $\Phi^{2}+\Theta^{2}\neq 0$ it is always possible to write a spinor in polar form, which is given, in chiral representation, as

$$\psi=\phi\, \mathrm{e}^{-\frac{i}{2}\beta\boldsymbol{\pi}}\,\boldsymbol{L}^{-1}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \tag{19}$$

for a pair of functions ϕ and β and for some $\boldsymbol{L}$ with the structure of spinor transformations [17, 18] (we recall here that the spinor transformations encode both abelian gauge and Lorentz transformations). The bi-linear spinor scalars are then given by

$$\Theta=2\phi^{2}\sin\beta \qquad \Phi=2\phi^{2}\cos\beta \tag{20}$$

so that ϕ and β are a scalar and a pseudo-scalar, called module and chiral angle. We can also define

$$S^{a}=2\phi^{2}s^{a} \qquad U^{a}=2\phi^{2}u^{a} \tag{21}$$

being the velocity vector and spin axial-vector. Identities (16)–(18) reduce to

$$2u_{\mu}s_{\nu}\boldsymbol{\sigma}^{\mu\nu}\boldsymbol{\pi}\psi+\psi=0 \tag{22}$$

and

$$u_{a}u^{a}=-s_{a}s^{a}=1 \tag{23}$$

$$u_{a}s^{a}=0 \tag{24}$$

showing that the velocity has only three independent components (the three components of its spatial part) whereas the spin has two independent components (the two angles that, in the rest-frame, its spatial part forms with one given axis, usually chosen as the third). Therefore $\boldsymbol{L}$ is the Lorentz transformation that takes any general spinor into its rest frame with spin aligned along the third axis. As for the sixth parameter of $\boldsymbol{L}$ it can be taken as the angle of the rotation around the third axis (or more in general, around the direction of the spin) or as a global phase (that is, a gauge phase): these two are indistinguishable for spinors in rest-frame and spin-eigenstate. The only two remaining components that cannot be transferred into the frame are the ϕ and β scalars. They are the degrees of freedom.

It is a general result that the logarithmic derivative of an element of a Lie group belong to its Lie algebra, or

$$\boldsymbol{L}^{-1}\partial_\mu\boldsymbol{L} = iq\partial_\mu\zeta\mathbb{I} + \frac{1}{2}\partial_\mu\zeta_{ij}\boldsymbol{\sigma}^{ij} \tag{25}$$

for some ζ_{ij} and ζ [4] (we recall that $\boldsymbol{L}$ is a full spinor transformation, accounting for both gauge and Lorentz groups, and this is the reason why its logarithmic derivative is a linear combination involving both the generator $\mathbb{I}$ and the Lorentz generators $\boldsymbol{\sigma}^{ij}$). With these, we can define

$$R_{ij\mu} = \partial_\mu\zeta_{ij} - C_{ij\mu} \tag{26}$$

$$P_\mu = q\left(\partial_\mu\zeta - A_\mu\right) \tag{27}$$

which are proven to be a real tensor and a gauge invariant vector (for the proof see appendix A or see section III.B of [19]). To identify them separately, we call them space-time and gauge tensorial connections. We can have them both collected together into the single object

$$R^{ij}{}_{\mu} - 2P_\mu u_a s_b \varepsilon^{abij} = -\frac{1}{2}\varepsilon^{ijab}M_{ab\mu} = \Sigma^{ij}{}_{\mu} \tag{28}$$

with $M_{ab\mu}$ and $\Sigma_{ij\mu}$ being the Hodge dual of each other. For this single expression, we will simply talk about tensorial connection. With it, (10) becomes

$$\boldsymbol{\nabla}_\mu\psi = \left(\nabla_\mu \ln\phi\mathbb{I} - \frac{i}{2}\nabla_\mu\beta\boldsymbol{\pi} - \frac{1}{2}\Sigma_{ij\mu}\boldsymbol{\sigma}^{ij}\right)\psi \tag{29}$$

in which identities $2i\boldsymbol{\pi}\boldsymbol{\sigma}_{ab} = \varepsilon_{abcd}\boldsymbol{\sigma}^{cd}$ and (22) have been used. Additionally, we notice that

$$\nabla_\mu s_b = s^a\Sigma_{ab\mu} \qquad \nabla_\mu u_b = u^a\Sigma_{ab\mu} \tag{30}$$

are also valid as general identities (for a general proof, see [20]).

By employing these polar variables, we have

$$R^i{}_{j\mu\nu} = -\left(\nabla_\mu R^i{}_{j\nu} - \nabla_\nu R^i{}_{j\mu} + R^i{}_{k\mu}R^k{}_{j\nu} - R^i{}_{k\nu}R^k{}_{j\mu}\right) \tag{31}$$

$$qF_{\mu\nu} = -\left(\nabla_\mu P_\nu - \nabla_\nu P_\mu\right) \tag{32}$$

showing that the space-time and gauge tensorial connections can be seen as the covariant potentials of Riemann curvature and Maxwell strength (the above relationships are proven in appendix B and a sketch of proof has also been given in section III.B of [19]). They are also known as space-time and gauge curvatures. As before, then, we can have them collected into the single object

$$R^{ij}{}_{\mu\nu} - 2qF_{\mu\nu}u_a s_b \varepsilon^{abij} = -\frac{1}{2}\varepsilon^{ijab}M_{ab\mu\nu} = \Sigma^{ij}{}_{\mu\nu} \tag{33}$$

with $M_{ab\mu\nu}$ and $\Sigma_{ij\mu\nu}$ being a particular Hodge dualization in the two algebra-valued indices. Straightforwardly

$$\Sigma^i{}_{j\mu\nu} = -\left(\nabla_\mu\Sigma^i{}_{j\nu} - \nabla_\nu\Sigma^i{}_{j\mu} + \Sigma^i{}_{k\mu}\Sigma^k{}_{j\nu} - \Sigma^i{}_{k\nu}\Sigma^k{}_{j\mu}\right) \tag{34}$$

which we call simply curvature. With it, (11) becomes

$$[\boldsymbol{\nabla}_{\mu},\boldsymbol{\nabla}_{\nu}]\psi=\frac{1}{2}\Sigma_{ab\mu\nu}\boldsymbol{\sigma}^{ab}\psi. \tag{35}$$

The cyclic permutation of commutators gives

$$\varepsilon^{\kappa\mu\nu\rho}\nabla_{\mu}\Sigma^{ij}{}_{\nu\rho}=0: \tag{36}$$

in absence of electrodynamics, this reduces to the Bianchi identity. On the other hand, with no curvature, it reduces to the Cauchy identity $\varepsilon^{\kappa\mu\nu\rho}\nabla_{\mu}F_{\nu\rho}=0$.

Finally, the Dirac equation is equivalent to the pair

$$\nabla_{\mu}\beta+M_{\mu}+2ms_{\mu}\cos\beta=0 \tag{37}$$

$$\nabla_{\mu}\ln\phi^{2}+\Sigma_{\mu}+2ms_{\mu}\sin\beta=0 \tag{38}$$

with $\Sigma^{\alpha}=\Sigma^{\alpha\nu}{}_{\nu}$ and $M^{\alpha}=M^{\alpha\nu}{}_{\nu}$ as sources [21].

With the tensorial connection (28) one can prove (using the identities of appendix C) that the curvature (33) verifies the following identities

$$-\frac{1}{4}\varepsilon^{\rho\omega\mu\nu}\Sigma_{\alpha\sigma\mu\nu}M^{\alpha\sigma}{}_{\rho\omega}=\nabla_{\mu}G_{4}^{\mu} \tag{39}$$

$$\frac{1}{4}\varepsilon^{\mu\nu\pi\eta}\Sigma^{\alpha\sigma}{}_{\mu\nu}\Sigma_{\alpha\sigma\pi\eta}=\nabla_{\mu}K_{4}^{\mu} \tag{40}$$

for some vector and axial-vector

$$G_{4}^{\mu}=-\frac{1}{4}\varepsilon^{\mu\nu\eta\pi}\varepsilon^{\alpha\sigma\omega\rho}\Sigma_{\omega\rho\nu}\left(\Sigma_{\sigma\alpha\eta\pi}-\frac{2}{3}\Sigma_{\sigma\kappa\eta}\Sigma^{\kappa}{}_{\alpha\pi}\right) \tag{41}$$

$$K_{4}^{\mu}=\frac{1}{2}\varepsilon^{\mu\nu\eta\pi}\Sigma^{\alpha}{}_{\sigma\nu}\left(\Sigma^{\sigma}{}_{\alpha\eta\pi}-\frac{2}{3}\Sigma^{\sigma}{}_{\kappa\eta}\Sigma^{\kappa}{}_{\alpha\pi}\right) \tag{42}$$

written in terms of the tensorial connection alone: when no electrodynamics is present they reduce to

$$G_{4}^{\mu}=-\frac{1}{4}\varepsilon^{\mu\nu\eta\pi}\varepsilon^{\alpha\sigma\omega\rho}R_{\omega\rho\nu}\left(R_{\sigma\alpha\eta\pi}-\frac{2}{3}R_{\sigma\kappa\eta}R^{\kappa}{}_{\alpha\pi}\right) \tag{43}$$

$$K_{4}^{\mu}=\frac{1}{2}\varepsilon^{\mu\nu\eta\pi}R^{\alpha}{}_{\sigma\nu}\left(R^{\sigma}{}_{\alpha\eta\pi}-\frac{2}{3}R^{\sigma}{}_{\kappa\eta}R^{\kappa}{}_{\alpha\pi}\right). \tag{44}$$

When no curvature is present we have

$$G_{4}^{\mu}=0 \tag{45}$$

$$K_{4}^{\mu}=4qF_{\eta\pi}P_{\nu}\varepsilon^{\eta\pi\nu\mu}. \tag{46}$$

These are the Euler and Pontryagin topological currents.

Notice that there is no equivalent of Euler topological current for electrodynamics in 4 dimensions. For the others, we have expressed the Euler and Pontryagin topological currents as space-time and gauge tensorial connections. With the tensorial connection and its Hodge dual one can form vectors that enter as sources in the Dirac field equations.

3.2. Two-dimensions

3.2.1. (1+1)-signature. The $(1+1)$-dimensional case has the Minkowski matrix with two elements of opposite sign, while the Levi–Civita completely antisymmetric tensor has only two indices.

The Clifford matrices, and in particular the sigma matrices, verify $2\boldsymbol{\sigma}_{ab}=\varepsilon_{ab}\boldsymbol{\pi}$ defining the $\boldsymbol{\pi}$ matrix. We have

$$\boldsymbol{\gamma}_i\boldsymbol{\gamma}_j\boldsymbol{\gamma}_k=\boldsymbol{\gamma}_i\eta_{jk}-\boldsymbol{\gamma}_j\eta_{ik}+\boldsymbol{\gamma}_k\eta_{ij} \tag{47}$$

as a general identity. Defining $\overline{\psi}=\psi^\dagger\boldsymbol{\gamma}^0$ we can build the bi-linear spinor quantities as

$$U^a=\overline{\psi}\boldsymbol{\gamma}^a\psi \tag{48}$$

$$\Theta=i\overline{\psi}\boldsymbol{\pi}\psi \qquad \Phi=\overline{\psi}\psi \tag{49}$$

which are all real tensors. They verify

$$U_aU^a=\Phi^2+\Theta^2 \tag{50}$$

as Fierz identities. When $\Phi^2+\Theta^2\neq 0$ we can always write the spinor in polar form, which is given, in a representation for which $\boldsymbol{\gamma}^0=\boldsymbol{\sigma}^1$ and $\boldsymbol{\gamma}^1=-i\boldsymbol{\sigma}^2$ (where $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$ are two Pauli matrices), as

$$\psi=\phi\,\mathrm{e}^{-\frac{i}{2}\beta\boldsymbol{\pi}}\,\boldsymbol{L}^{-1}\begin{pmatrix}1\\1\end{pmatrix} \tag{51}$$

for a pair of functions ϕ and β and for some $\boldsymbol{L}$ with the structure of a spinor transformation. In this form

$$\Theta=2\phi^2\sin\beta \qquad \Phi=2\phi^2\cos\beta \tag{52}$$

so that ϕ and β are a scalar and a pseudo-scalar. Then

$$U^a=2\phi^2u^a \tag{53}$$

is the velocity vector. It verifies

$$u_au^a=1 \tag{54}$$

as normalization condition. Therefore ϕ and β are the two variables that represent the two true degrees of freedom.

The four-dimensional construction of both tensorial connections can be re-done exactly also in the two-dimensional case. Yet, now the space-time tensorial connection is

$$R_{\alpha\beta\mu}=R_\alpha g_{\beta\mu}-R_\beta g_{\alpha\mu} \tag{55}$$

where R_α is a real vector. With it and P_μ we have

$$\boldsymbol{\nabla}_\mu\psi=\left(\nabla_\mu\ln\phi\mathbb{I}-\frac{i}{2}\nabla_\mu\beta\boldsymbol{\pi}-\frac{1}{2}R^\alpha\varepsilon_{\alpha\mu}\boldsymbol{\pi}-iP_\mu\mathbb{I}\right)\psi. \tag{56}$$

Notice that in two dimensions we cannot collect both tensorial connections together.

In fact, the Dirac field equation re-written as

$$\nabla_{\mu}\beta - 2P^{\alpha}\varepsilon_{\alpha\mu} + 2mu^{\alpha}\varepsilon_{\alpha\mu}\cos\beta = 0 \tag{57}$$

$$\nabla_{\mu}\ln\phi^{2} + R_{\mu} + 2mu^{\alpha}\varepsilon_{\alpha\mu}\sin\beta = 0 \tag{58}$$

shows that the gauge tensorial connection appears only in the field equation for the chiral angle while the space-time tensorial connection appears only in the field equation for the module. In this sense, they remain decoupled.

In two dimensions, the Ricci curvatures are

$$R_{ab} = -\nabla_{i}R^{i}g_{ab} \tag{59}$$

$$R = -2\nabla_{i}R^{i} \tag{60}$$

meaning that the Einstein tensor is zero, as expected. In the last relation we see that

$$-\frac{1}{2}R = \nabla_{\rho}G_{2}^{\rho} \tag{61}$$

for some vector

$$G_{2}^{\rho} = R^{\rho} \tag{62}$$

which is the Euler topological current.

Additionally, one can see that

$$\frac{1}{2}qF_{\alpha\nu}\varepsilon^{\alpha\nu} = \nabla_{\mu}K_{2}^{\mu} \tag{63}$$

for some axial-vector

$$K_{2}^{\mu} = P_{\nu}\varepsilon^{\nu\mu} \tag{64}$$

which is the Pontryagin topological current.

As before, there is no Euler current for electrodynamics. And as known from differential geometry, there is no Pontryagin current for curvature, in 2 dimensions.

3.2.2. (0+2)-signature. The $(0+2)$-dimensional case has Minkowski matrix in which the two elements have the same sign.

The Clifford and sigma matrices verify $\boldsymbol{\gamma}^{a}\boldsymbol{\pi} = i\varepsilon^{ab}\boldsymbol{\gamma}_{b}$ as well as $2i\boldsymbol{\sigma}_{ab} = \varepsilon_{ab}\boldsymbol{\pi}$ defining the $\boldsymbol{\pi}$ matrix. We have that equation (47) is still valid. However, the adjoint is now $\overline{\psi} = \psi^{\dagger}$ and hence the bi-linear spinor quantities are

$$S^{a} = \overline{\psi}\boldsymbol{\gamma}^{a}\psi \tag{65}$$

$$\Theta = \overline{\psi}\boldsymbol{\pi}\psi \qquad \Phi = \overline{\psi}\psi \tag{66}$$

all being real tensors. They verify

$$S_{a}S^{a} = \Phi^{2} - \Theta^{2} \tag{67}$$

where $\Phi^2 \geqslant \Theta^2$ since in this signature the norm of vectors is always positive. It is always possible to write the spinor in polar form, which is given, in the representation where $\boldsymbol{\gamma}^i = \boldsymbol{\sigma}^i$ (with $\boldsymbol{\sigma}^i$ two Pauli matrices), according to

$$\psi = \phi\, \mathrm{e}^{-\frac{1}{2}\eta\boldsymbol{\pi}} \boldsymbol{L}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{68}$$

for some ϕ and η which can be proven to be real. Then

$$\Theta = -2\phi^2 \sinh\eta \qquad \Phi = 2\phi^2 \cosh\eta \tag{69}$$

so that ϕ and η are a scalar and a pseudo-scalar. Also

$$S^a = 2\phi^2 s^a \tag{70}$$

is the spin axial-vector. It verifies

$$s_a s^a = 1 \tag{71}$$

as normalization. So ϕ and η are the degrees of freedom of the system. Notice, however, that the change of signature entailed a corresponding change from circular to hyperbolic functions.

The space tensorial connection given in (55) is the same as it does not depend on the signature. However now

$$\boldsymbol{\nabla}_\mu \psi = \left(\nabla_\mu \ln\phi \mathbb{I} - \frac{1}{2} \nabla_\mu \eta \boldsymbol{\pi} - R^\alpha \boldsymbol{\sigma}_{\alpha\mu} - i P_\mu \mathbb{I} \right) \psi. \tag{72}$$

Again, regardless of the signature, in two dimensions the space and gauge tensorial connections cannot be collected within a single tensorial connection.

The Dirac field equations are

$$\nabla_k \eta - 2P^b \varepsilon_{bk} - 2m \cosh\eta \varepsilon_{kb} s^b = 0 \tag{73}$$

$$\nabla_k \ln\phi^2 + R_k + 2m \sinh\eta \varepsilon_{kb} s^b = 0 \tag{74}$$

still with the gauge and space tensorial connections that appear as two separate external sources.

The structure of the Ricci scalar is also unchanged, so that $G_2^\rho = R^\rho$ is still the Euler topological current.

3.3. Three-dimensions

3.3.1. Any-signature. The three-dimensional case that we are going to consider is peculiar in many senses, one of which being that, regardless the signature, the polar decomposition always yields the same outcome. Then, it is with no loss of generality that it is possible to take into account only one signature, which will be chosen as that of the pure space. The Minkowski matrix is the identity in three dimensions, and the Levi-Civita fully antisymmetric tensor has three indices.

Clifford and sigma matrices verify $2\boldsymbol{\sigma}^{ab} = i\varepsilon^{abc}\boldsymbol{\gamma}_c$ with no $\boldsymbol{\pi}$ matrix defined. We have now

$$\boldsymbol{\gamma}_i \boldsymbol{\gamma}_j \boldsymbol{\gamma}_k = \boldsymbol{\gamma}_i \eta_{jk} - \boldsymbol{\gamma}_j \eta_{ik} + \boldsymbol{\gamma}_k \eta_{ij} + i\varepsilon_{ijk}\mathbb{I}. \tag{75}$$

With $\overline{\psi}=\psi^{\dagger}$ the bi-linear spinors are

$$S^a=\overline{\psi}\boldsymbol{\gamma}^a\psi \tag{76}$$
$$\Phi=\overline{\psi}\psi \tag{77}$$

all being real tensors. They verify

$$S_a\boldsymbol{\gamma}^a\psi-\Phi\psi=0 \tag{78}$$

and

$$S_aS^a=\Phi^2 \tag{79}$$

as Fierz identities. It is always possible to write

$$\psi=\phi\boldsymbol{L}^{-1}\begin{pmatrix}1\\0\end{pmatrix} \tag{80}$$

with ϕ a real function. In polar form

$$\Phi=\phi^2 \tag{81}$$

showing that ϕ is a scalar. As usual

$$S^a=\phi^2s^a \tag{82}$$

is the spin vector. It verifies

$$s_a\boldsymbol{\gamma}^a\psi-\psi=0 \tag{83}$$

and

$$s_as^a=1 \tag{84}$$

as normalization. As usual, ϕ is the unique degree of freedom.

As in 4 and 2, also in three dimensions $R_{ab\nu}$ and P_ν have the same definition. They can be collected together into

$$R^{\alpha\nu}{}_{\mu}+2P_\mu s_\rho\varepsilon^{\rho\alpha\nu}=\varepsilon^{\alpha\nu\rho}M_{\rho\mu}=\Sigma^{\alpha\nu}{}_{\mu} \tag{85}$$

where $M_{\rho\mu}$ and $\Sigma^{\alpha\nu}{}_{\mu}$ are the hodge dual of each other. Then we can write

$$\boldsymbol{\nabla}_\mu\psi=\left(\nabla_\mu\ln\phi\mathbb{I}-\frac{1}{2}\Sigma_{\alpha\nu\mu}\boldsymbol{\sigma}^{\alpha\nu}\right)\psi \tag{86}$$

in which $2\boldsymbol{\sigma}^{ab}=i\varepsilon^{abc}\boldsymbol{\gamma}_c$ and (83) have been used.

With polar variables we have

$$R^{ij}{}_{\mu\nu}+2qF_{\mu\nu}s_k\varepsilon^{kij}=\Sigma^{ij}{}_{\mu\nu} \tag{87}$$

as the compact form of the curvature.

The Dirac equation is equivalent to

$$M^{\rho}{}_{\rho} - 2m = 0 \tag{88}$$

$$\Sigma_{\mu\nu}{}^{\nu} + \nabla_{\mu} \ln \phi^2 = 0 \tag{89}$$

in which we see a peculiar occurrence. Because the spinor has a single degree of freedom, in three dimensions there are three differential equations determining the three derivatives of that degree of freedom. The Dirac equations are two complex, or four real, conditions. Of these four conditions, three preserve the status of differential equations. However, one must become a constraint. This is what happened to equation (88).

There is, in three dimensions, no characteristic class, which means that now there is a priori no way to have currents whose divergence is zero in flat spaces. Nevertheless, one can still define currents that are divergenceless, whether in flat spaces or not, according to

$$G_3^{\mu} = \varepsilon^{\mu\nu\sigma} \varepsilon^{\omega\alpha\eta} s_{\omega} \left(\Sigma_{\alpha\eta\nu\sigma} + 2 s^i s^j \Sigma_{i\alpha\nu} \Sigma_{j\eta\sigma} \right) \tag{90}$$

which is a type of Euler-like topological current: without any curvature of the manifold, the above (90) reduces to

$$G_3^{\mu} = 4qF_{\nu\sigma} \varepsilon^{\nu\sigma\mu} \tag{91}$$

as expected for the magnetic components. On the other hand, without electrodynamics (90) becomes

$$G_3^{\mu} = \varepsilon^{\mu\nu\sigma} \varepsilon^{\omega\alpha\eta} s_{\omega} \left(R_{\alpha\eta\nu\sigma} + 2 s^i s^j R_{i\alpha\nu} R_{j\eta\sigma} \right) \tag{92}$$

recovering the results of [6]. Notice however that by taking the trace $R_{\alpha\nu\sigma} g^{\nu\sigma} = R_{\alpha}$ one can prove that

$$G_3^{\mu} = \varepsilon^{\mu\nu\alpha} \nabla_{\nu} R_{\alpha} \tag{93}$$

is divergenceless in curved spaces and as such it is another well-defined type of Euler-like topological current.

4. Higher-order equations

4.1. Two-dimensions

If we take the Dirac equation and square it by applying onto the spinor the differential operator twice we obtain what in literature is known as Lichnerowicz equation [22].

Consider now two dimensions, for which in any signature

$$\begin{aligned} \gamma_a \gamma_b \gamma_c \gamma_d = 2 \left(g_{ab} \boldsymbol{\sigma}_{cd} - g_{ac} \boldsymbol{\sigma}_{bd} + g_{ad} \boldsymbol{\sigma}_{bc} + g_{cd} \boldsymbol{\sigma}_{ab} - g_{bd} \boldsymbol{\sigma}_{ac} + g_{bc} \boldsymbol{\sigma}_{ad} \right) \\ + \left(g_{ab} g_{cd} - g_{ac} g_{bd} + g_{ad} g_{bc} \right) \mathbb{I}. \end{aligned} \tag{94}$$

With it, the Dirac equation squares to give

$$\boldsymbol{\nabla}^{\alpha} \boldsymbol{\nabla}_{\alpha} \psi + iqF_{ab} \boldsymbol{\sigma}^{ab} \psi - \frac{1}{4} R \psi + m^2 \psi = 0 \tag{95}$$

which is the Lichnerowicz equation.

If we specialize, for instance, to the case of alternating signature, we would have $2\boldsymbol{\sigma}_{ab}=\varepsilon_{ab}\boldsymbol{\pi}$ and therefore

$$\boldsymbol{\nabla}^{\alpha}\boldsymbol{\nabla}_{\alpha}\psi+i\nabla_{\alpha}K_2^{\alpha}\boldsymbol{\pi}\psi+\frac{1}{2}\nabla_{\alpha}G_2^{\alpha}\psi+m^2\psi=0 \tag{96}$$

(the case of homogeneous signature would have been identical up to the presence of the imaginary unit). This is the Klein–Gordon equation modulo two contributions that are simply the characteristic classes.

That the Lichnerowicz equation contains all and only the characteristic classes is a feature of two dimensions. In four-dimensional spaces the characteristic classes would still appear, but only when going to fourth-order differential.

4.2. four-dimensions

As a dimensional analysis would show, in second-order differential equations there can be no 4-dimensional characteristic class. To make them appear, one must consider the square of the Lichnerowicz equation itself.

So let us be on a four-dimensional manifold. Here, let us consider the four-fold application of the differential operator $i\boldsymbol{\gamma}^k\boldsymbol{\nabla}_k\psi=\boldsymbol{\nabla}\psi$ given by $\boldsymbol{\nabla}^4\psi\equiv\boldsymbol{\gamma}^a\boldsymbol{\gamma}^b\boldsymbol{\gamma}^c\boldsymbol{\gamma}^d\boldsymbol{\nabla}_a\boldsymbol{\nabla}_b\boldsymbol{\nabla}_c\boldsymbol{\nabla}_d\psi$ and which ends up being determined by the product of 4 gamma matrices. This product is computed to be

$$\begin{aligned}\boldsymbol{\gamma}_a\boldsymbol{\gamma}_b\boldsymbol{\gamma}_c\boldsymbol{\gamma}_d=&2\left(g_{ab}\boldsymbol{\sigma}_{cd}-g_{ac}\boldsymbol{\sigma}_{bd}+g_{ad}\boldsymbol{\sigma}_{bc}+g_{cd}\boldsymbol{\sigma}_{ab}-g_{bd}\boldsymbol{\sigma}_{ac}+g_{bc}\boldsymbol{\sigma}_{ad}\right)\\&+\left(g_{ab}g_{cd}-g_{ac}g_{bd}+g_{ad}g_{bc}\right)\mathbb{I}+i\varepsilon_{abcd}\boldsymbol{\pi}\end{aligned} \tag{97}$$

in which each term is proportional to either the identity or the generators or the $i\varepsilon_{ijab}\boldsymbol{\pi}$ matrix. Of all these terms, those that might contain a characteristic class could only be those that have no differential operator acting on the spinor field and therefore, because of (35), those that are proportional to the $i\varepsilon_{ijab}\boldsymbol{\pi}$ matrix. In fact, we have that

$$\begin{aligned}&i\varepsilon^{abcd}\boldsymbol{\pi}\boldsymbol{\nabla}_a\boldsymbol{\nabla}_b\boldsymbol{\nabla}_c\boldsymbol{\nabla}_d\psi=\frac{i}{16}\varepsilon^{abcd}\Sigma_{ijcd}\Sigma_{pqab}\boldsymbol{\sigma}^{ij}\boldsymbol{\sigma}^{pq}\boldsymbol{\pi}\psi\\&=-\frac{1}{32}\varepsilon^{ijpq}\Sigma_{abij}\left(i\Sigma^{ab}{}_{pq}\boldsymbol{\pi}+M^{ab}{}_{pq}\right)\psi=-\frac{i}{8}\nabla_{\alpha}K_4^{\alpha}\boldsymbol{\pi}\psi+\frac{1}{8}\nabla_{\alpha}G_4^{\alpha}\psi\end{aligned} \tag{98}$$

in which (36), (39)and (40) were used. As easy to see, this contribution provides all and only characteristic classes.

5. Conclusion

In this work, we have shown, through various examples in different dimensions, how the Euler and the Pontryagin topological currents can be re-expressed in terms of real tensors, which were also shown to be found as sources in the Dirac differential field equations. We have deepened the tie between spinor dynamics and topological currents by showing that in any equation resulting from applying the Dirac operator as many times as the space dimension we find exactly the characteristic classes of that space.

The possibility to link topological features of manifolds to dynamical characters of spinors defined on those manifolds is the idea at the center of fundamental results like the Atiyah-Singer index theorem [23]. We regard results like the one presented here as another step, albeit small, toward clarifying the connections between spinors and topology.

Data availability statement

The data that support the findings of this study are openly available at the following URL/DOI: https://inspirehep.net/literature/2759232.

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Conflict of interest

There is no conflict of interest.

Appendix A. Tensorial connections

In this paper, section 3 contains some results that are central points, which therefore should be proved in detail. In the following appendices, we will give the proofs for the results of sub-section 3.1 only, as the others would be demonstrated in analogous ways.

In this first appendix, we prove that the tensorial connections are in fact tensors. To this purpose, we start from recalling that a spinor is defined to transform, under diffeomorphisms, in such a way that each of its components is a scalar. Its covariant derivative is then a covariant vector. Because there is nothing new in this, we will not consider diffeomorphisms for now. Instead, under Lorentz transformations, and more precisely the complex representation of Lorentz transformations, or spinorial transformation, $\boldsymbol{S}$, the spinor transforms as $\psi \rightarrow \boldsymbol{S}\psi$. In order for the spinorial covariant derivative (10) to transform in the same way $\boldsymbol{\nabla}_\mu\psi \rightarrow \boldsymbol{S}\boldsymbol{\nabla}_\mu\psi$ we must require the spinorial connection to transform as

$$\boldsymbol{C}_\mu \rightarrow \boldsymbol{S}\left(\boldsymbol{C}_\mu - \boldsymbol{S}^{-1}\partial_\mu \boldsymbol{S}\right)\boldsymbol{S}^{-1}. \tag{A1}$$

Now, in the decomposition (19), the action of the spinorial transformation $\psi \rightarrow \boldsymbol{S}\psi$ gives

$$\phi\, \mathrm{e}^{-\frac{i}{2}\beta\boldsymbol{\pi}}\, \boldsymbol{L}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \boldsymbol{S}\phi\, \mathrm{e}^{-\frac{i}{2}\beta\boldsymbol{\pi}}\, \boldsymbol{L}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{A2}$$

and because ϕ and β are scalars, we have $\phi \rightarrow \phi$ and $\beta \rightarrow \beta$ and so

$$\boldsymbol{L}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \boldsymbol{S}\boldsymbol{L}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{A3}$$

where $[\boldsymbol{S},\boldsymbol{\pi}]=0$ was also taken into account. Therefore

$$\boldsymbol{L}^{-1}\to\boldsymbol{S}\boldsymbol{L}^{-1} \tag{A4}$$

which is just the composition of spinorial transformations. By using (A1) and (A4) we can see that

$$\begin{aligned}&\partial_\mu\boldsymbol{L}^{-1}\boldsymbol{L}+\boldsymbol{C}_\mu\to\partial_\mu\left(\boldsymbol{S}\boldsymbol{L}^{-1}\right)\left(\boldsymbol{L}\boldsymbol{S}^{-1}\right)+\boldsymbol{S}(\boldsymbol{C}_\mu-\boldsymbol{S}^{-1}\partial_\mu\boldsymbol{S})\boldsymbol{S}^{-1}\\&=\partial_\mu\boldsymbol{S}\boldsymbol{L}^{-1}\boldsymbol{L}\boldsymbol{S}^{-1}+\boldsymbol{S}\partial_\mu\boldsymbol{L}^{-1}\boldsymbol{L}\boldsymbol{S}^{-1}+\boldsymbol{S}\boldsymbol{C}_\mu\boldsymbol{S}^{-1}-\boldsymbol{S}\boldsymbol{S}^{-1}\partial_\mu\boldsymbol{S}\boldsymbol{S}^{-1}\\&=\boldsymbol{S}\partial_\mu\boldsymbol{L}^{-1}\boldsymbol{L}\boldsymbol{S}^{-1}+\boldsymbol{S}\boldsymbol{C}_\mu\boldsymbol{S}^{-1}=\boldsymbol{S}\left(\partial_\mu\boldsymbol{L}^{-1}\boldsymbol{L}+\boldsymbol{C}_\mu\right)\boldsymbol{S}^{-1}\end{aligned} \tag{A5}$$

showing that $\partial_\mu\boldsymbol{L}^{-1}\boldsymbol{L}+\boldsymbol{C}_\mu$ transforms as a spinorial matrix.

It is a general result of the Lie theory that the logarithmic derivative of an element of a Lie group belong to its Lie algebra, so that

$$\boldsymbol{L}^{-1}\partial_\mu\boldsymbol{L}=iq\partial_\mu\zeta\mathbb{I}+\frac{1}{2}\partial_\mu\zeta_{ij}\boldsymbol{\sigma}^{ij} \tag{A6}$$

for some ζ_{ij} and ζ. Combining this with (9) allows us to see that the object $\partial_\mu\boldsymbol{L}^{-1}\boldsymbol{L}+\boldsymbol{C}_\mu$ can be written as

$$\begin{aligned}&\partial_\mu\boldsymbol{L}^{-1}\boldsymbol{L}+\boldsymbol{C}_\mu=-iq\partial_\mu\zeta\mathbb{I}-\frac{1}{2}\partial_\mu\zeta_{ij}\boldsymbol{\sigma}^{ij}+\frac{1}{2}C_{ij\mu}\boldsymbol{\sigma}^{ij}+iqA_\mu\mathbb{I}\\&=-iq\left(\partial_\mu\zeta-A_\mu\right)\mathbb{I}-\frac{1}{2}\left(\partial_\mu\zeta_{ij}-C_{ij\mu}\right)\boldsymbol{\sigma}^{ij}=-iP_\mu\mathbb{I}-\frac{1}{2}R_{ij\mu}\boldsymbol{\sigma}^{ij}\end{aligned} \tag{A7}$$

where (26) and (27) were used.

When this decomposition is subject to the transformation law (A5) we see that

$$-iP_\mu\mathbb{I}-\frac{1}{2}R_{ij\mu}\boldsymbol{\sigma}^{ij}\to\boldsymbol{S}\left(-iP_\mu\mathbb{I}-\frac{1}{2}R_{ij\mu}\boldsymbol{\sigma}^{ij}\right)\boldsymbol{S}^{-1} \tag{A8}$$

and as $\mathbb{I}$ and $\boldsymbol{\sigma}^{ij}$ are linearly independent the above splits into

$$R_{ij\mu}\boldsymbol{\sigma}^{ij}\to R_{ij\mu}\boldsymbol{S}\boldsymbol{\sigma}^{ij}\boldsymbol{S}^{-1} \tag{A9}$$

$$P_\mu\to P_\mu. \tag{A10}$$

Recalling that the complex and real Lorentz transformations are linked by $\boldsymbol{S}\boldsymbol{\sigma}^{ab}\boldsymbol{S}^{-1}(\Lambda)^i_a(\Lambda)^j_b=\boldsymbol{\sigma}^{ij}$ we have

$$R_{ij\mu}\to R_{ab\mu}\left(\Lambda^{-1}\right)^a_i\left(\Lambda^{-1}\right)^b_j. \tag{A11}$$

Thus, $R_{ij\mu}$ and P_μ are gauge invariant and transform as a scalar and an antisymmetric tensor under real Lorentz transformations.

If we were now to use the tetrads to convert tangent space indices into manifold indices as discussed in section 2, and re-introduce the diffeomorphisms left out at the beginning of this appendix, we would have

$$R_{\alpha'\nu'\mu'}\to R_{\alpha\nu\mu}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^\mu}{\partial x^{\mu'}} \tag{A12}$$

$$P_{\mu'}\to P_\mu\frac{\partial x^\mu}{\partial x^{\mu'}} \tag{A13}$$

as the complete transformation law of real tensors.

Appendix B. Curvature tensors

In the first appendix we have demonstrated that the tensorial connections are real tensors (either under diffeomorphisms or under real Lorentz transformations, according to whether they are in the Greek or Latin index configuration). In this second appendix we prove that they are also the covariant potentials of their respective curvatures. To this purpose, consider the tensor-valued spinorial matrix $\partial_\mu \boldsymbol{L}^{-1}\boldsymbol{L}+\boldsymbol{C}_\mu$ and set

$$\partial_\mu \boldsymbol{L}^{-1}\boldsymbol{L}+\boldsymbol{C}_\mu=-\boldsymbol{B}_\mu \tag{B1}$$

for compactness. Due to its transformation properties, its covariant derivative is given by

$$\boldsymbol{\nabla}_\mu \boldsymbol{B}_\nu=\partial_\mu \boldsymbol{B}_\nu+[\boldsymbol{C}_\mu,\boldsymbol{B}_\nu]-\boldsymbol{B}_\rho \Lambda^\rho_{\nu\mu} \tag{B2}$$

where we recall that $\Lambda^\sigma_{\nu\mu}$ the Levi–Civita connection. To see that this is the case, just impose the transformation law

$$\boldsymbol{B}_{\mu'} \to \boldsymbol{S}\boldsymbol{B}_\mu \boldsymbol{S}^{-1}\frac{\partial x^\mu}{\partial x^{\mu'}} \tag{B3}$$

and check that, with (A1) and the transformation law of the Levi–Civita connection, one has indeed

$$\boldsymbol{\nabla}_{\alpha'}\boldsymbol{B}_{\mu'} \to \boldsymbol{S}\boldsymbol{\nabla}_\alpha \boldsymbol{B}_\mu \boldsymbol{S}^{-1}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\alpha}{\partial x^{\alpha'}}. \tag{B4}$$

Then

$$\boldsymbol{\nabla}_\mu \boldsymbol{B}_\nu-\boldsymbol{\nabla}_\nu \boldsymbol{B}_\mu+[\boldsymbol{B}_\mu,\boldsymbol{B}_\nu]=\partial_\mu \boldsymbol{B}_\nu-\partial_\nu \boldsymbol{B}_\mu+[\boldsymbol{C}_\mu,\boldsymbol{B}_\nu]-[\boldsymbol{C}_\nu,\boldsymbol{B}_\mu]+[\boldsymbol{B}_\mu,\boldsymbol{B}_\nu] \tag{B5}$$

$$=-(\partial_\mu \boldsymbol{C}_\nu-\partial_\nu \boldsymbol{C}_\mu+[\boldsymbol{C}_\mu,\boldsymbol{C}_\nu])-\partial_\mu\left(\partial_\nu \boldsymbol{L}^{-1}\boldsymbol{L}\right)+\partial_\nu\left(\partial_\mu \boldsymbol{L}^{-1}\boldsymbol{L}\right)+\left[\partial_\mu \boldsymbol{L}^{-1}\boldsymbol{L},\partial_\nu \boldsymbol{L}^{-1}\boldsymbol{L}\right]$$

in which the last line can be worked out as

$$\begin{aligned}&\partial_\mu\left(\boldsymbol{L}^{-1}\partial_\nu \boldsymbol{L}\right)-\partial_\nu\left(\boldsymbol{L}^{-1}\partial_\mu \boldsymbol{L}\right)+\left(\boldsymbol{L}^{-1}\partial_\mu \boldsymbol{L}\right)\left(\boldsymbol{L}^{-1}\partial_\nu \boldsymbol{L}\right)-\left(\boldsymbol{L}^{-1}\partial_\nu \boldsymbol{L}\right)\left(\boldsymbol{L}^{-1}\partial_\mu \boldsymbol{L}\right)\\ &=\partial_\mu \boldsymbol{L}^{-1}\partial_\nu \boldsymbol{L}+\boldsymbol{L}^{-1}\partial_\mu\partial_\nu \boldsymbol{L}-\partial_\nu \boldsymbol{L}^{-1}\partial_\mu \boldsymbol{L}-\boldsymbol{L}^{-1}\partial_\nu\partial_\mu \boldsymbol{L}\\ &\quad-\boldsymbol{L}^{-1}\boldsymbol{L}\partial_\mu \boldsymbol{L}^{-1}\partial_\nu \boldsymbol{L}+\boldsymbol{L}^{-1}\boldsymbol{L}\partial_\nu \boldsymbol{L}^{-1}\partial_\mu \boldsymbol{L}=\boldsymbol{L}^{-1}\partial_\mu\partial_\nu \boldsymbol{L}-\boldsymbol{L}^{-1}\partial_\nu\partial_\mu \boldsymbol{L}=0:\end{aligned} \tag{B6}$$

therefore

$$\boldsymbol{\nabla}_\mu \boldsymbol{B}_\nu-\boldsymbol{\nabla}_\nu \boldsymbol{B}_\mu+[\boldsymbol{B}_\mu,\boldsymbol{B}_\nu]=-(\partial_\mu \boldsymbol{C}_\nu-\partial_\nu \boldsymbol{C}_\mu+[\boldsymbol{C}_\mu,\boldsymbol{C}_\nu]). \tag{B7}$$

Expression (11) can be taken as an implicit definition of Riemann curvature and Maxwell strength, in terms of which

$$\partial_\mu \boldsymbol{C}_\nu-\partial_\nu \boldsymbol{C}_\mu+[\boldsymbol{C}_\mu,\boldsymbol{C}_\nu]=\frac{1}{2}R_{ij\mu\nu}\boldsymbol{\sigma}^{ij}+iqF_{\mu\nu}\mathbb{I}. \tag{B8}$$

On the other hand, because of (A7) we know that

$$\boldsymbol{B}_\mu=iP_\mu \mathbb{I}+\frac{1}{2}R_{ij\mu}\boldsymbol{\sigma}^{ij}. \tag{B9}$$

Substituting these last two decompositions into (B7) we get

$$\frac{1}{2}\left(R_{ij\mu\nu}+\nabla_{\mu}R_{ij\nu}-\nabla_{\nu}R_{ij\mu}+R_{ik\mu}R^{k}{}_{j\nu}-R_{ik\nu}R^{k}{}_{j\mu}\right)\boldsymbol{\sigma}^{ij}+i\left(qF_{\mu\nu}+\nabla_{[\mu}P_{\nu]}\right)\mathbb{I}=0 \tag{B10}$$

in which $\boldsymbol{\nabla}_{\mu}\boldsymbol{\sigma}_{ij}=0$ and $[\boldsymbol{\sigma}_{ab},\boldsymbol{\sigma}_{cd}]=\eta_{ad}\boldsymbol{\sigma}_{bc}-\eta_{ac}\boldsymbol{\sigma}_{bd}+\eta_{bc}\boldsymbol{\sigma}_{ad}-\eta_{bd}\boldsymbol{\sigma}_{ac}$ have also been used. Linear independence gives now

$$R_{ij\mu\nu}+\nabla_{\mu}R_{ij\nu}-\nabla_{\nu}R_{ij\mu}+R_{ik\mu}R^{k}{}_{j\nu}-R_{ik\nu}R^{k}{}_{j\mu}=0 \tag{B11}$$

$$qF_{\mu\nu}+\nabla_{[\mu}P_{\nu]}=0 \tag{B12}$$

as the Riemann curvature and Maxwell strength in (31) and (32).

With (31) and (32), using (28) and (33) allows one to check that (34) holds. The proof requires straightforward algebra.

Appendix C. Characteristic classes

In the second appendix we have proved (31) and (32), from which (34) is derived after some algebraic manipulation. In this last appendix we want to finally show that with (34), one can obtain (39) and (40) for some (41) and (42). In this perspective, we start by considering that for any rank-3 tensor antisymmetric in two indices, and in particular the tensorial connection, we have the following identities

$$\frac{1}{2}\varepsilon^{ijab}\varepsilon^{\mu\nu\rho\sigma}\Sigma_{ij\nu}\Sigma_{ak\rho}\Sigma^{k}{}_{b\sigma}\equiv\varepsilon^{\mu\nu\rho\sigma}M^{b}{}_{a\nu}M^{a}{}_{k\rho}M^{k}{}_{b\sigma} \tag{C1}$$

$$\varepsilon^{\eta\pi\tau\omega}\varepsilon^{\mu\nu\rho\sigma}\Sigma^{k}{}_{\eta\mu}\Sigma_{k\pi\nu}\Sigma^{j}{}_{\tau\rho}\Sigma_{j\omega\sigma}\equiv-\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\eta\pi\tau\omega}M_{\eta\pi\mu}M_{\tau\alpha\nu}M^{\alpha}{}_{\kappa\sigma}M^{\kappa}{}_{\omega\rho}\equiv0 \tag{C2}$$

which can be proven very easily by considering the fact that $M_{\tau\alpha\nu}$ and $\Sigma_{\tau\alpha\nu}$ are the Hodge dual of one another, and the general identities involving products of the completely antisymmetric Levi-Civita pseudo-tensor. Similarly, one can demonstrate that

$$\varepsilon^{\mu\nu\rho\sigma}\nabla_{\mu}\Sigma_{ij\nu}\Sigma^{i}{}_{k\rho}\Sigma^{jk}{}_{\sigma}\equiv\frac{1}{3}\nabla_{\mu}\left(\varepsilon^{\mu\nu\eta\pi}\Sigma^{\alpha}{}_{\sigma\nu}\Sigma^{\sigma}{}_{\kappa\eta}\Sigma^{\kappa}{}_{\alpha\pi}\right) \tag{C3}$$

$$\varepsilon^{ijab}\varepsilon^{\mu\nu\rho\sigma}\nabla_{\mu}\Sigma_{ij\nu}\Sigma_{ak\rho}\Sigma^{k}{}_{b\sigma}\equiv\frac{2}{3}\nabla_{\mu}\left(\varepsilon^{\mu\nu\rho\sigma}M^{b}{}_{a\nu}M^{a}{}_{k\rho}M^{k}{}_{b\sigma}\right) \tag{C4}$$

are analogous identities at the differential level.

With the above identities, and definition (34), it is now straightforward to see that $\varepsilon^{\mu\nu\pi\eta}\Sigma^{\alpha\sigma}{}_{\mu\nu}\Sigma_{\alpha\sigma\pi\eta}$ is given by

$$\begin{aligned}
\frac{1}{2}\varepsilon^{\mu\nu\pi\eta}\Sigma^{\alpha\sigma}{}_{\mu\nu}\Sigma_{\alpha\sigma\pi\eta}&=\varepsilon^{\mu\nu\pi\eta}\Sigma_{\alpha\sigma\pi\eta}\left(\nabla_{\nu}\Sigma^{\alpha\sigma}{}_{\mu}+\Sigma^{\alpha}{}_{\rho\nu}\Sigma^{\rho\sigma}{}_{\mu}\right)\\
&=\varepsilon^{\mu\nu\pi\eta}\Sigma_{\alpha\sigma\pi\eta}\nabla_{\nu}\Sigma^{\alpha\sigma}{}_{\mu}+\varepsilon^{\mu\nu\pi\eta}\Sigma_{\alpha\sigma\pi\eta}\Sigma^{\alpha}{}_{\rho\nu}\Sigma^{\rho\sigma}{}_{\mu}\\
&=\varepsilon^{\mu\nu\pi\eta}\Sigma_{\alpha\sigma\pi\eta}\nabla_{\nu}\Sigma^{\alpha\sigma}{}_{\mu}+2\varepsilon^{\mu\nu\pi\eta}\left(\nabla_{\eta}\Sigma_{\alpha\sigma\pi}+\Sigma_{\alpha\kappa\eta}\Sigma^{\kappa}{}_{\sigma\pi}\right)\Sigma^{\alpha}{}_{\rho\nu}\Sigma^{\rho\sigma}{}_{\mu}\\
&=\varepsilon^{\mu\nu\pi\eta}\Sigma_{\alpha\sigma\pi\eta}\nabla_{\nu}\Sigma^{\alpha\sigma}{}_{\mu}+2\varepsilon^{\mu\nu\pi\eta}\nabla_{\eta}\Sigma_{\alpha\sigma\pi}\Sigma^{\alpha}{}_{\rho\nu}\Sigma^{\rho\sigma}{}_{\mu}\\
&=\nabla_{\nu}\left(\varepsilon^{\mu\nu\pi\eta}\Sigma_{\alpha\sigma\pi\eta}\Sigma^{\alpha\sigma}{}_{\mu}\right)-\frac{2}{3}\nabla_{\mu}\left(\varepsilon^{\mu\nu\eta\pi}\Sigma^{\alpha}{}_{\sigma\nu}\Sigma^{\sigma}{}_{\kappa\eta}\Sigma^{\kappa}{}_{\alpha\pi}\right)
\end{aligned} \tag{C5}$$

where (36) was also used. The last line can be collected as

$$\frac{1}{2}\varepsilon^{\mu\nu\pi\eta}\Sigma^{\alpha\sigma}{}_{\mu\nu}\Sigma_{\alpha\sigma\pi\eta}=\nabla_{\mu}\left[\varepsilon^{\mu\nu\eta\pi}\Sigma^{\alpha\sigma}{}_{\nu}\left(\Sigma_{\sigma\alpha\eta\pi}-\frac{2}{3}\Sigma_{\sigma\kappa\eta}\Sigma^{\kappa}{}_{\alpha\pi}\right)\right] \tag{C6}$$

which, up to a normalizing factor, is the Pontryagin class (40)–(42).

The Euler class (39)–(41) would be found with an analogous, albeit slightly longer, manipulation.

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