ON THE COHOMOLOGY OF SURFACES WITH $p_g = q = 2$ AND MAXIMAL ALBANESE DIMENSION

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ABSTRACT. In this paper we study the cohomology of smooth projective complex surfaces S of general type with invariants $p_g = q = 2$ and surjective Albanese morphism. We show that on a Hodge-theoretic level, the cohomology is described by the cohomology of the Albanese variety and a K3 surface X that we call the *K3 partner* of S. Furthermore, we show that in suitable cases we can geometrically construct the K3 partner X and an algebraic correspondence in $S \times X$ that relates the cohomology of S and X. Finally, we prove the Tate and Mumford–Tate conjectures for those surfaces S that lie in connected components of the Gieseker moduli space that contain a product-quotient or a mixed surface.

1. INTRODUCTION

1.1. Let S be a smooth projective complex surface with invariants $p_g(S) = q(S) = 2$, and assume that the Albanese morphism $\alpha \colon S \to A$ is surjective. The results of this paper are inspired by the following two observations:

(1) The induced map on cohomology $\alpha^* \colon H^*(A, \mathbb{Z}) \to H^*(S, \mathbb{Z})$ is injective. The orthogonal complement $H^2_{\text{new}} = H^*(A, \mathbb{Z})^\perp \subset H^*(S, \mathbb{Z})$ is a Hodge structure of weight 2 with Hodge numbers $(1, n, 1)$, where $n = h^{1,1}(S) - 4$. Such a Hodge structure is said to be of *K3 type*.

(2) Let S' be a smooth projective complex surface with invariant $p_g(S') = 1$. Then Morrison [30] showed that there exists a K3 surface X' together with an isomorphism $\iota' \colon H^2(S', \mathbb{Q})^{\text{tra}} \to H^2(X', \mathbb{Q})^{\text{tra}}$ that preserves the Hodge structure, the integral structure, and the intersection pairing. (Here $(\_)^{\text{tra}}$ denotes the *transcendental* part of the Hodge structure, that is, the orthogonal complement of the Hodge classes.)

These observations lead to the following questions.

1.2. **Question A.** Let S be as before.

Does there exist a K3 surface X together with an isomorphism $\iota \colon (H^2_{\text{new}})^{\text{tra}}_{\mathbb{Q}} \to H^2(X, \mathbb{Q})^{\text{tra}}$ that preserves (H) the Hodge structure or (Z) the integral structure or (P) the intersection pairing?

We give an affirmative answer to this question in Theorem 3.12, showing that there exist an X and an ι that satisfy (H), (Z), and (P). The strategy of the proof is the same as for Morrison's result mentioned above.

1.3. **Question B.** The Hodge conjecture predicts that if ι satisfies (H), then it is algebraic.

> Do there exist X and ι satisfying (H) as above such that ι is algebraic?

In general we are not able to answer this question. However, an interesting class of examples of the surfaces that we consider is formed by so-called *product-quotients*: these are surfaces birational to a surface $(C \times D)/G$, where C and D are curves equipped with a diagonal action by a finite group G (see [3]). In these cases we give an affirmative answer to Question B in Theorem 4.9. The strategy boils down to finding appropriate Prym varieties in the Jacobians of C and D and taking for X the associated Kummer variety.

(Note: Surfaces S for which there is a positive answer to Question B are very much related to the notion of *K3 burgers*, as introduced by Laterveer [24].)

1.4. **Question C.** Since we are not able to settle Question B in general, we may aim for something weaker, sitting in between Question A.(H) and Question B. We use the notion of *motivated cycles* introduced by André [2] (see §5.9 for details).

> Do there exist X and ι satisfying (H) as above such that ι is motivated?

Once again, we are not able to give an answer to this question in general. However, we give sufficient conditions for a positive answer to this question. For example, we show that to decide Question C one may replace S with any other surface in the same connected component of the moduli space of surfaces of general type (Theorem 5.17). In particular, Question C has a positive answer for every surface S that lies in the same connected component as a product-quotient surface or a mixed surface. These last ones are minimal resolutions of quotients $(C \times C)/G$ by a mixed action of a finite group G on $C \times C$, with C a smooth curve (see [14, 15]).

If Question C has a positive answer for the surface S, then we also deduce that the Tate and Mumford–Tate conjectures hold for models of S over finitely generated subfields of $\mathbb{C}$ (Corollary 5.19).

We summarize these results in the following theorem (the conjunction of Theorems 3.12, 4.9, and 5.17 and Corollary 5.19).

1.5. **Theorem.** *Let S be a smooth projective complex surface with invariants $p_g(S) = q(S) = 2$, and assume that the Albanese morphism $\alpha \colon S \to A$ is surjective.*

(1) *Then there exist a K3 surface X and an isomorphism of Hodge structures*

$$\iota \colon (H^2_{\mathrm{new}}(S, \mathbb{Q}))^{\mathrm{tra}} \to H^2(X, \mathbb{Q})^{\mathrm{tra}}.$$

(2) *If S is a product-quotient surface (cf. Definition 2.3) with group G, then there exist X and ι as above and an algebraic cycle on $S \times X$ that induces ι.*

(3) *If S is in the same connected component of the Gieseker moduli space as a product-quotient surface or a mixed surface, then ι is motivated (in the sense of André), and the Tate and Mumford–Tate conjectures hold for S. That is, the Tate and Mumford–Tate conjectures hold for those S that are*

deformation equivalent to a surface that is of general type with $p_g = q = 2$, maximal Albanese dimension, and birational to a quotient of a product of curves.

1.6. **Structure of this text.** In the next section, "On the classification of surfaces with $p_g = q = 2$", we very briefly recall what is known for the surfaces under consideration. It is important to stress that the classification of these surfaces is not yet complete. Hence we present the state of the art up to now. We shall pay particular attention to those surfaces which are product-quotients (here a classification theorem is available) and mixed surfaces (here a classification is still missing, but it is very likely that it will be available in the next few years), recalling definitions, important properties, and the associated group theoretical data. Furthermore, we recall what is known about their moduli space.

In section 3 we discuss the existence of Hodge-theoretical K3 partners for all the surfaces with $p_g = q = 2$, following Morrison's theory. In Theorem 3.12 we prove point (1) of Theorem 1.5.

In section 4 we discuss the problem to find a geometric description where possible of the Hodge-theoretical K3 partners, proving point (2) of Theorem 1.5. Indeed, for those surfaces which are product-quotients we are able to find an algebraic K3 partner.

Finally in the last section we see how the results obtained can be used to prove that the Tate and Mumford–Tate conjectures hold for these surfaces. As already mentioned, here we use the notion of *motivated cycles* introduced by André. Corollary 5.19 proves the last point of Theorem 1.5.

2. On the classification of surfaces with $p_g = q = 2$

2.1. The classification of smooth projective complex surfaces with invariants $p_g = q = 2$ is not complete, although there has been much progress in recent years. We give an overview of the current state of the art. Let S be a minimal surface of general type with $p_g = q = 2$, so that $\chi(S) = 1 - q + p_g = 1$. Recall the following classical general inequalities:

- $K_S^2 \le 9\chi(S)$ (Bogomolov–Miyaoka–Yau).
- $K_S^2 \ge 2p_g$ if $q > 0$ (Debarre).

These yield $4 \le K_S^2 \le 9$ under our assumptions. Except for the case $K_S^2 = 9$, examples of such surfaces have been constructed for every value of K_S^2 in this range. Since we are dealing with irregular surfaces, i.e., with $q > 0$, a useful tool to study them is the Albanese map. The *Albanese variety* of S is the q-dimensional variety $\mathrm{Alb}(S) := H^0(S, \Omega_S^1)^\vee / H_1(S, \mathbb{Z})$. By Hodge theory, $\mathrm{Alb}(S)$ is an abelian variety. For a fixed base point $x_0 \in S$, we define the *Albanese morphism* as

$$\alpha = \alpha_{x_0} \colon S \longrightarrow \mathrm{Alb}(S), \qquad x \mapsto \int_{x_0}^x .$$

The dimension of $\alpha(S)$ ($\le q(S) = 2$) is called the *Albanese dimension* of S and it is denoted by $\mathrm{Albdim}(S)$. If $\mathrm{Albdim}(S) = 2$, we say that S has *maximal Albanese dimension*. For surfaces with $q(S) = 2$, we have two possibilities:

(1) $\mathrm{Albdim}(S) = 1$ and $\alpha(S)$ is a smooth curve of genus 2 or
(2) $\mathrm{Albdim}(S) = 2$ and α is a generically finite cover of an abelian surface.

The first case is completely understood: we have a classification theorem; see [35]. However, these surfaces do not have a surjective Albanese map; therefore they fall outside the scope of this paper and will no longer be considered.

The second case is still open. By [8, section 5] the degree $\deg(\alpha)$ of the Albanese map is a topological invariant. In Table 1 we summarize the state of the art of the classification using K_S^2 and $\deg(\alpha)$ as main invariants.

TABLE 1. State of the art of the classification of minimal complex algebraic surfaces with invariants $p_g = q = 2$. We have indicated, where possible, the number of families (#) and the dimensions of the irreducible component containing them (dim). Moreover, we point out if some members of the family are product-quotient surfaces (PQ) or mixed surfaces (MS); see Definitions 2.3 and 2.4. In the last column, we give references to more detailed descriptions of the class. Finally, as a consequence of the present paper, we put a checkmark in the column MTC if we prove the Tate and Mumford–Tate conjectures for a class.

№	K_S^2	$\deg(\alpha)$	#	dim	Name	MTC	PQ/MS	Reference
1	8	2	2	0^2			No	[41]
2	8	≤ 6	4	$3^3, 4$	SIP	✓	Yes	[35]
3	7	3	1	3		✓	Yes	[6, 40]
4	7	2	?	≥ 2			?	[42]
5	6	4	1	4		✓	Yes	[38]
6	6	2	1	3			?	[37]
7	6	2	2	4^2		✓	Yes	[37] [39]
8	5	3	1	4	Chen–Hacon	✓	Yes	[36]
9	4	2	1	4	Catanese	✓	Yes	[9, 35]

2.2. In the rest of this section we will describe the examples № $= 2, 3, 5, 7, 8, 9$ of Table 1 in more detail.

The examples № $= 1, 4, 6$ will not be treated in this paper for the following reasons. For the surfaces in № $= 4$, at the time of writing, very few things are known about their moduli spaces. Despite the abundance of information about their moduli, for the surfaces in № $= 1$ and 6 none of the methods we develop here seem to work; principally this is due to the small dimensions of their moduli.

2.3. **Definition.** Let G be a finite group acting on two compact Riemann surfaces C_1, C_2 of respective genera $g_1, g_2 \geq 2$. Consider the diagonal action of G on $C_1 \times C_2$. In this situation we say for short: the action of G on $C_1 \times C_2$ is *unmixed*. By [7] we may assume w.l.o.g. that G acts faithfully on both factors.

The minimal resolution S of the singularities of $T = (C_1 \times C_2)/G$ is called a *product-quotient surface*. If the action of G on the product $C_1 \times C_2$ is free we will speak of *surfaces isogenous to a product of unmixed type* (SIP). In this case T is already smooth.

2.4. **Definition.** Let C be a smooth projective curve, and let G be a finite subgroup of $\mathrm{Aut}(C)^2 \rtimes \mathbb{Z}/2$. Assume that there are elements in G exchanging the two factors of $C \times C$. In this case we say that the action is *mixed*.

The minimal resolution of singularities $S \longrightarrow T = (C \times C)/G$ is called a *mixed surface*. A *surface isogenous to a product of mixed type* (also SIP) is a mixed surface where G acts freely on $C \times C$.

We denote by $G_0 \lhd G$ the index two subgroup, i.e., the subgroup of elements that do not exchange the factors. In general the singularities of T are rather complicated. Assuming that G_0 acts freely, i.e., $(C \times C)/G_0$ is a surface isogenous to a product, T is smooth and we call it a *semi-isogenous mixed surface*.

2.5. The families of surfaces № $= 2$ in Table 1 are all the surfaces isogenous to a product of unmixed type (see [35]), and a brief description of them can be read from Table 2. There one can find the genus of C_1 and C_2, the group G, and the number of branch points (with multiplicities) of the covering $C_i \longrightarrow C_i/G$ for $i = 1, 2$. Notice first that the curves C_i/G are elliptic curves for $i = 1, 2$; second that for a complete description one needs a system of generators for the group G. The families of surfaces № $= 5, 8$, and 9 contain subloci of product-quotient surfaces. To briefly describe these particular members we add information about the singularities of $(C_1 \times C_2)/G$ to Table 2.

TABLE 2. Classification of product-quotient surfaces of unmixed type with $p_g = q = 2$. In the column labeled η we give the number of irreducible components of the exceptional divisor of the minimal resolution of $(C_1 \times C_2)/G$. The last column (№) mentions the component of the Gieseker moduli space described in Table 1 that contains the family of product-quotient surfaces as a subfamily.

K_S^2	$g(C_1)$	$g(C_2)$	G	branch	sing.	η	dim	№
8	3	3	V_4	$(2^2), (2^2)$	–	0	4	3
8	3	4	S_3	$(3), (2^2)$	–	0	3	3
8	3	5	D_4	$(2), (2^2)$	–	0	3	3
6	4	4	A_4	$(2), (2)$	$2 \times \frac{1}{2}(1,1)$	2	2	5
5	3	3	S_3	$(3), (3)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$	3	2	8
4	3	3	Q_8	$(2), (2)$	$4 \times \frac{1}{2}(1,1)$	4	2	9
4	3	3	D_4	$(2), (2)$	$4 \times \frac{1}{2}(1,1)$	4	2	9
4	2	2	C_2	$(2^2), (2^2)$	$4 \times \frac{1}{2}(1,1)$	4	4	9

On the other hand, the families of surfaces № $= 3$ and 7 contain subloci of mixed surfaces as shown in [39, 40]. Table 3 describes families of mixed surfaces that are of interest to us. The surfaces described by the first row are surfaces isogenous to a product of mixed type. The semi-isogenous mixed surfaces $T = (C \times C)/G$ with $p_g(T) = q(T) = 2$ and $K_T^2 > 0$ form 9 families [6]. We will not be able to say anything about the existence of algebraic K3 partners for these surfaces (section 4),

but we will study them in section 5. Since the questions studied in section 5 will turn out to be invariant under deformation we only need to study one subfamily per component of the Gieseker moduli space. Of the 9 families mentioned above, 4 are subfamilies of components of the Gieseker moduli space that also contain families of product-quotient surfaces that we already described (see [6] and [39]). The only families for which this is not the case are listed in Table 3; they are families in № $= 2, 3, 7$ described in Table 1.

TABLE 3. The families of mixed surfaces with $p_g = q = 2$ that are not subfamilies of components of the Gieseker moduli space that also contain families of product-quotient surfaces that we described before.

K_T^2	$g(C)$	G	G^0	branch	dim	№
8	2	C_4	C_2	–	3	2
7	4	C_6	C_3	$(2, -4)$	3	3
6	5	D_4	$(C_2)^2$	$(3, -8)$	4	7
6	5	D_4	$(C_2)^2$	$(3, -8)$	4	7
6	5	$C_2 \times C_4$	C_4	$(2, -4)^2$	4	7

2.6. As already remarked, the surfaces in component № $= 2$ are all isogenous to a product. Hence the weak rigidity theorem of Catanese [7] tells us that for each family their moduli space consists of one connected irreducible component in the subspace $\mathcal{M}_{8,2,2}$ ($\mathcal{M}_{K^2, p_g, q}$) of the Gieseker moduli space of surfaces of general type $\mathcal{M}_{8,1}$ ($\mathcal{M}_{K^2, \chi}$). Moreover each member of the family is isogenous to a product.

The property of preserving an isotrivial fibration is no longer true for the families № $= 3, 5, 7, 8, 9$. Indeed, their moduli space is bigger in some sense. To be precise let us first analyze the families of surfaces in Table 2 with $K_S^2 < 8$. These families form an irreducible sublocus of $\mathcal{M}_{K^2, 2, 2}$, but they sit inside a bigger connected component.

The connected component with $K_S^2 = 4$ was studied in [9]. We have that all three families in Table 2 with $K_S^2 = 4$ belong to the same connected component of dimension 4. The surfaces with $K_S^2 = 5$ form an irreducible component of dimension 2 sitting inside the connected component of Chen–Hacon surfaces described in [36], which has dimension 4. Finally the family of surfaces with $K_S^2 = 6$ is a 2-dimensional irreducible component inside an irreducible component of $\mathcal{M}_{6,2,2}$ of dimension 4; this component is studied in [38].

The first entry of Table 3 is again isogenous to a product of mixed type. Hence its moduli space is 3-dimensional and is described by the weak rigidity theorem of Catanese [7].

The moduli space of the surfaces relative to the second entry of Table 3 is described by Pignatelli and Polizzi in [40]. In this case the moduli space is a generically smooth, irreducible, open, and normal subset of the Gieseker moduli space $\mathcal{M}_{7,2,2}$. For the general member of the family the Albanese surface is simple, but some

specific surfaces admit an irrational fibration over an elliptic curve. Finally, the last three entries of Table 3 are described by Pignatelli in [39], more precisely, in Proposition 6.5, Corollary 6.6, and Proposition 6.7. There it is shown that these families are subfamilies contained in the irreducible components I_a and I_b of surfaces of general type with $p_g = q = 2$, $K^2 = 6$, and Albanese map of degree 2 studied in [36]. As a last remark, notice that in [36] a third family of these surfaces is studied, labeled II. This family corresponds to № = 6 in Table 1, and up to now it is unknown if it contains a subfamily of mixed surfaces.

3. HODGE-THEORETIC K3 PARTNERS

3.1. Let S be a smooth projective complex surface with invariants $p_g(S) = q(S) = 2$, and assume that the Albanese morphism $\alpha\colon S \to A$ is surjective. Recall Question A from the introduction:

> Does there exist a K3 surface X together with an isomorphism $\iota\colon (H^2_{\mathrm{new}})^{\mathrm{tra}}_{\mathbb{Q}} \to H^2(X,\mathbb{Q})^{\mathrm{tra}}$ that preserves (H) the Hodge structure or (Z) the integral structure or (P) the intersection pairing?

In Theorem 3.12 we give an affirmative answer to this question.

3.2. A *Hodge lattice* V is a free $\mathbb{Z}$-module of finite rank, endowed with a polarized Hodge structure such that the polarization on V makes V into a lattice—a symmetric bilinear form on a free $\mathbb{Z}$-module of finite rank. In particular, the weight (as Hodge structure) of V is always even.

3.3. *Notation.* There is some risk of confusing notation: if V is a Hodge lattice, then $V(k)$ may denote either the k-th Tate twist of the Hodge structure on V or it may denote the k-th twist of the lattice structure on V. In this paper we use the notation $V(k)$ only for Tate twists of the Hodge structure.

3.4. **Definition.** Let V be a Hodge lattice of K3 type (that is, the Hodge structure on V is of K3 type). A *K3 partner* of V is a complex K3 surface X together with an isomorphism of Hodge structures $\iota\colon V^{\mathrm{tra}}_{\mathbb{Q}} \to H^2(X,\mathbb{Q})^{\mathrm{tra}}$. Following the terminology of Morrison (p. 181 of [30]) we say that a K3 partner (X,ι) is *strict* if ι maps the intersection form on $V^{\mathrm{tra}}_{\mathbb{Q}}$ to the intersection form on $H^2(X,\mathbb{Q})^{\mathrm{tra}}$. The K3 partner (X,ι) is *integral* if ι is compatible with an isomorphism of integral Hodge lattices $V^{\mathrm{tra}} \to H^2(X,\mathbb{Z})^{\mathrm{tra}}$.

3.5. Let Λ denote the even unimodular lattice $E_8\langle -1\rangle^2 \oplus U^3$. The lattice Λ goes by the name *K3 lattice*, since there is an isometry $H^2(X,\mathbb{Z}) \cong \Lambda$ for every complex K3 surface X. For an integer d, let Λ_d denote the lattice $E_8\langle -1\rangle^2 \oplus U^2 \oplus \langle -2d\rangle$. Observe that $\Lambda_d \hookrightarrow \Lambda$. The signature of Λ is $(3,19)$, whereas the signature of Λ_d is $(2,19)$.

Recall that if M is a lattice, then the pairing on M defines a natural map $M \hookrightarrow M^\vee$, and the cokernel $M^\vee/M$ is called the *discriminant group* A_M. The minimal number of generators of A_M is denoted by $\ell(A_M)$.

3.6. **Theorem.** *Let L and M be even lattices with signatures (s_+, s_-) and (t_+, t_-), respectively. Assume that L is unimodular. Then there exists a unique primitive embedding $M \hookrightarrow L$ if the following conditions hold:*
  (1) $t_+ < s_+$ *and* $t_- < s_-$, *and*
  (2) $\mathrm{rk}(L) - \mathrm{rk}(M) \geq \ell(A_M) + 2$.

Proof. This is a slightly weaker form of Theorem 1.14.4 of [32].   □

The following corollary is part of an observation by Morrison; see Corollary 2.10 of [29].

3.7. Corollary. *Let M be an even lattice with signature $(2, n)$ for some integer $0 \leq n \leq 8$. Then there exists a unique primitive embedding $M \hookrightarrow \Lambda$ into the K3 lattice introduced above.*

3.8. Let L be a lattice with signature (s_+, s_-), and assume that $2 \leq s_+ \leq 3$ and $s_- \leq 19$. Define $\Omega(L) = \{x \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0, (x, \bar{x}) > 0\}$. Note that $\Omega(L)$ is an analytic open subset of the quadric in $\mathbb{P}(L \otimes \mathbb{C})$ defined by the equation $(x, x) = 0$. If $M \hookrightarrow L$ is an embedding of two such lattices, then there is a natural holomorphic map $\Omega(M) \to \Omega(L)$.

Observe that there is a natural bijection

$$
\left\{
\begin{array}{l}
\text{Hodge structures of K3 type on } L \text{ such} \\
\text{that for every non-zero } x \in L^{2,0} \text{ one has} \\
(x, x) = 0, \ (x, \bar{x}) > 0, \text{ and } x \perp L^{1,1}
\end{array}
\right\} \longrightarrow \Omega(L),
$$

obtained by mapping a Hodge structure on L to the point $L^{2,0}$ in $\Omega(L)$.

3.9. We denote with $\mathcal{F}^{\text{full}}_{2d,\mathbb{K},\mathbb{C}}$ the moduli space of degree $2d$ primitively polarized K3 surfaces with full level $\mathbb{K}$-structure for an admissible subgroup $\mathbb{K} \subset \mathrm{SO}(\Lambda_d)(\hat{\mathbb{Z}})$; see [44] for details.

3.10. Proposition. *Let B be a connected algebraic variety over $\mathbb{C}$, and let $\mathcal{V}/B$ be a polarized variation of $\mathbb{Z}$-Hodge structures of weight 2 of K3 type. Let b be a point of B and assume that $\mathcal{V}_b$ is a lattice with signature $(2, n)$ that admits a primitive embedding $\mathcal{V}_b \hookrightarrow \Lambda$. Then there is an étale open $B^\circ \to B$ such that b lies in the image of B°, a K3 space $f: \mathcal{X} \to B^\circ$, and a morphism of variations of Hodge structures $\mathcal{V}|_{B^\circ} \to \mathrm{R}^2 f_* \mathbb{Z}$ that is fiberwise a primitive embedding and a Hodge isometry on the transcendental lattices.*

Proof. Let $\tilde{B} \to B$ be a universal cover of B, and let $\tilde{b}$ be a point of $\tilde{B}$ lying above b. Let L denote the lattice underlying the fibre of $\mathcal{V}_{\tilde{b}}$. Fix a primitive embedding $L \hookrightarrow \Lambda$. Under the induced map $\Omega(L) \to \Omega(\Lambda)$ the Hodge structure on $\mathcal{V}_{\tilde{b}}$ maps to a point $x \in \Omega(\Lambda)$. By the surjectivity of the period map for K3 surfaces there exists a complex K3 surface X, with $H^2(X, \mathbb{Z}) \cong \Lambda$, such that the Hodge structure on $H^2(X, \mathbb{Z})$ corresponds to x.

Let $d \in \mathbb{Z}$ and $\mathbb{K} \subset \mathrm{SO}(\Lambda_d)(\hat{\mathbb{Z}})$ be such that X admits a primitive polarization of degree $2d$ with full level $\mathbb{K}$-structure. This means that $[X] \in \mathcal{F}^{\text{full}}_{2d,\mathbb{K},\mathbb{C}}$. Since L has signature $(2, n)$ we get a primitive embedding $L \hookrightarrow \Lambda_d$. This yields a diagram

$$
\begin{array}{ccc}
\tilde{B} \longrightarrow \Omega(L) \longrightarrow \Omega(\Lambda_d) \\
\downarrow \qquad\qquad\qquad\qquad \downarrow \\
B \qquad\qquad\qquad \mathbb{K}\backslash\Omega(\Lambda_d)
\end{array}
$$

The composite map $\tilde{B} \to \mathbb{K}\backslash\Omega(\Lambda_d)$ factors via a finite cover $B_{\mathbb{K}}$ of B:

$$\begin{array}{ccccc}
\tilde{B} & \longrightarrow & \Omega(L) & \longrightarrow & \Omega(\Lambda_d) \\
\downarrow & \searrow & & & \downarrow \\
B & \xleftarrow[\text{fin ét}]{} & B_{\mathbb{K}} & \longrightarrow & \mathbb{K}\backslash\Omega(\Lambda_d)
\end{array}$$

By a theorem of Borel (see Theorem 6.4.1 of [20]) the map $B_{\mathbb{K}} \to \mathbb{K}\backslash\Omega(\Lambda_d)$ is algebraic. By Proposition 3.2.11 of [43], there is an open immersion $\mathcal{F}^{\text{full}}_{2d,\mathbb{K},\mathbb{C}} \to \text{Sh}_{\mathbb{K}}(\text{SO}(\Lambda_d), \Omega(\Lambda_d))$, where the latter is the Shimura variety parameterizing polarized Hodge structures on $\Lambda_d \otimes \mathbb{Q}$ with a level $\mathbb{K}$-structure.

We now have a diagram

$$\begin{array}{ccccccc}
\tilde{B} & \longrightarrow & \Omega(L) & \longrightarrow & \Omega(\Lambda_d) & & \\
\downarrow & \searrow & & & \downarrow & & \\
B & \xleftarrow[\text{fin ét}]{} & B_{\mathbb{K}} & \longrightarrow & \mathbb{K}\backslash\Omega(\Lambda_d) & \longrightarrow & \text{Sh}_{\mathbb{K}}(\text{SO}(\Lambda_d), \Omega(\Lambda_d)) \\
& & \uparrow & & & & \uparrow \\
& & B^\circ & \xrightarrow{\hspace{3.5cm}} & & & \mathcal{F}^{\text{full}}_{2d,\mathbb{K},\mathbb{C}}
\end{array}$$

where the bottom right rectangle is cartesian. By our choice of d and $\mathbb{K}$, we know that $[X] \in \mathcal{F}^{\text{full}}_{2d,\mathbb{K},\mathbb{C}}$ and $b \in B$ are in the image of B° under the respective maps. In particular B° is non-empty. Pulling back the universal family of K3 surfaces from $\mathcal{F}^{\text{full}}_{2d,\mathbb{K},\mathbb{C}}$ to B° we end up with a K3 space $f\colon \mathcal{X} \to B^\circ$ and a morphism of variations of Hodge structures $\mathcal{V}|_{B^\circ} \to \mathrm{R}^2 f_* \mathbb{Z}$. By construction it is fiberwise a primitive embedding and a Hodge isometry on the transcendental lattices. $\qquad\square$

3.11. Lemma. *Let S be a smooth projective complex surface with invariants $p_g = q = 2$. Let $\alpha\colon S \to A$ be the Albanese morphism, and assume that α is surjective. Define $H^2_{\text{new}} = (H^*(A, \mathbb{Z})/\text{tors})^\perp \subset H^*(S, \mathbb{Z})/\text{tors}$. Then H^2_{new} has rank $12 - K_S^2$. In particular $(H^2_{\text{new}})^{\text{tra}}$ has signature $(2, n)$ with $n \leq 10 - K_S^2$.*

Proof. By our assumptions we have $\chi_S = p_g - q + 1 = 1$. Noether's formula gives $e + K_S^2 = 12 \cdot \chi_S = 12$. Observe that $e = \sum_{i=0}^{4} (-1)^i b_i$, where $b_0 = b_4 = 1$ and $b_1 = b_3 = 2q = 4$. Therefore

$$b_2 = 12 - K_S^2 + 2 \cdot 4 - 2 \cdot 1 = 18 - K_S^2.$$

Finally, b_2 is the sum of the ranks of H^2_{new} and $H^2(A, \mathbb{Z})$. The latter has rank 6, and we conclude that H^2_{new} has rank $12 - K_S^2$. Since $(H^2_{\text{new}})^{\text{tra}}$ is a transcendental Hodge structure of K3 type, it must have signature $(2, n)$, with $n \leq \text{rk}(H^2_{\text{new}}) - 2$. $\qquad\square$

3.12. Theorem. *Let S be a smooth projective complex surface with invariants $p_g = q = 2$, and let $\alpha\colon S \to A$ and H^2_{new} be as in the preceding Lemma 3.11. Then there exist a complex K3 surface X and a morphism $\iota\colon (H^2_{\text{new}})^{\text{tra}}_{\mathbb{Q}} \to H^2(X, \mathbb{Q})^{\text{tra}}$ that preserves (H) the Hodge structure, (Z) the integral structure, and (P) the intersection pairing. In other words, we have a positive answer to Question A of the introduction.*

Proof. Since $4 \leq K_S^2 \leq 9$, Lemma 3.11 shows that the lattice $(H^2_{\text{new}})^{\text{tra}}$ has signature $(2, n)$ with $n \leq 8$. Note that $(H^2_{\text{new}})^{\text{tra}}$ is even by the Wu formula: for every $v \in H^2(S, \mathbb{Z})$ we have $(v, v) \equiv (v, c_1(S))$ (mod 2), and we have $(H^2_{\text{new}})^{\text{tra}} \subset c_1(S)^\perp$

since $(H_{\mathrm{new}}^2)^{\mathrm{tra}}$ is by definition perpendicular to all Hodge classes. The result follows from Corollary 3.7 and Proposition 3.10. $\qquad\square$

3.13. *Remark.* Let B be a connected component of the Gieseker moduli space of minimal smooth projective surfaces with invariants $p_g = q = 2$. Assume that $\dim(B) > 0$. It is natural to ask if there are two classes $[S_1], [S_2] \in B$ such that the Hodge structures $H_{\mathrm{new}}^2(S_1, \mathbb{Q})$ and $H_{\mathrm{new}}^2(S_2, \mathbb{Q})$ are not isomorphic. In other words: does the "new part" of the Hodge structure have a non-trivial period domain?

If this question has a positive answer, then one may apply the techniques of Moonen [27] to the universal family over B, which would be particularly relevant for our results in section 5. For example, if the answer is positive, one may use a monodromy argument to show that $H^2(S)$ is an abelian motive in the sense of André for all $[S] \in B$.

If B contains a sublocus of product-quotient surfaces, then we can give a positive answer to this question. (However, in this case we have direct arguments for our results in section 5 and do not need the techniques of [27].) We have spent ample time on this question for the three components with $K^2 = 6$ in № = 7 of Table 1, but we have not been able to answer it in these cases.

4. Algebraic K3 partners

Let S be a smooth projective complex surface with invariants $p_g(S) = q(S) = 2$, and assume that the Albanese morphism $\alpha \colon S \to A$ is surjective. In this section we attempt to answer Question B of the introduction:

> Does there exist a K3 surface X together with an isomorphism $\iota \colon (H_{\mathrm{new}}^2)_{\mathbb{Q}}^{\mathrm{tra}} \to H^2(X, \mathbb{Q})^{\mathrm{tra}}$ that is algebraic?

Since Theorem 3.12 provides an affirmative answer to Question A, the Hodge conjecture predicts a positive answer to Question B as well. In Theorem 4.9 we show that this is indeed the case for certain surfaces that are product-quotients (cf. Definition 2.3). If the essence of the proof must be captured in one sentence, it would be the following: the algebraic correspondence inducing ι is built from the Kummer K3 surface associated with a suitable 2-dimensional isogeny factor of the product of the Jacobians of the curves that are used in the construction of the product-quotient surface S.

4.1. **Outline of this section.** This section is the most technical part of this paper. It is organized as follows. First we recall some facts about Chow motives of surfaces. Proposition 4.5 describes a natural decomposition of $h^2(S, \mathbb{Q})$ for product-quotient surfaces S of unmixed type. In Theorem 4.9 we give the general proof for the existence of an algebraic correspondence inducing ι. This proof relies on a case-by-case computation, for which we refer to a `MAGMA`-script with which we tabulate the output. The final part of this section illustrates this proof by discussing one of the cases in detail, as an example.

4.2. **Chow motives of surfaces.** For an introduction to the theory of Chow motives we refer to the excellent paper [45] of Scholl. Let $\mathcal{M}_{\mathrm{rat}}$ denote the category of Chow motives over $\mathbb{C}$. We recall that $\mathcal{M}_{\mathrm{rat}}$ is an additive, $\mathbb{Q}$-linear, pseudoabelian category (Theorem 1.6 of [45]). There exists a functor $h \colon \mathrm{SmPr}_{/\mathbb{C}}^{\mathrm{op}} \to \mathcal{M}_{\mathrm{rat}}$ from the category of smooth projective varieties over $\mathbb{C}$ to the category of Chow motives.

We denote with $\mathrm{CH}^i(X)$ (resp., $\mathrm{CH}^i(M)$) the i-th Chow group of a smooth projective variety X (resp., a motive $M \in \mathcal{M}_{\mathrm{rat}}$). In general, it is not known whether the Künneth projectors π_i are algebraic, so it does not (yet) make sense to speak of the summand $h^i(X) \subset h(X)$ for an arbitrary smooth projective variety $X/\mathbb{C}$. However, a so-called Chow–Künneth decomposition does exist for curves [25], for surfaces [31], and for abelian varieties [13]. For algebraic surfaces there is in fact the following theorem, which strengthens the decomposition of the Chow motive. Statement and proof are copied from Theorem 2.2 of [24].

4.3. Theorem. *Let S be a smooth projective surface over $\mathbb{C}$. There exists a self-dual Chow–Künneth decomposition $\{\pi_i\}$ of S, with the further property that there is a splitting*

$$\pi_2 = \pi_2^{\mathrm{alg}} + \pi_2^{\mathrm{tra}} \in \mathrm{CH}^2(S \times S)$$

in orthogonal idempotents, defining a splitting $h^2(S) = h^2(S)^{\mathrm{alg}} \oplus h^2(S)^{\mathrm{tra}}$ with Chow groups

$$\mathrm{CH}^i(h^2(S)^{\mathrm{alg}}) = \begin{cases} \mathrm{NS}(S) & \text{if } i = 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{CH}^i(h^2(S)^{\mathrm{tra}}) = \begin{cases} \mathrm{CH}^2_{\mathrm{AJ}}(S) & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathrm{CH}^2_{\mathrm{AJ}}(S)$ denotes the kernel of the Abel–Jacobi map.

Proof. The Chow–Künneth decomposition is given in Proposition 7.2.1 of [21]. The further splitting into an algebraic and transcendental component is Proposition 7.2.3 of [21]. □

4.4. Motives of quotient varieties and blow-ups. If X is a smooth projective variety and G is a finite group that acts on X, then we define $h(X/G) = h(X)^G$. This leads to a satisfactory theory of motives of quotient varieties, as is explained in [12]; see also Example 16.1.13 in [18]. Since the intersection theory of quotient varieties (and more generally Alexander schemes [22, 49]) behaves analogously to that of smooth varieties, the blow-up formula generalizes to this setting: Let Y be a variety that is the quotient of a smooth variety by the action of a finite group. Let $X \subset Y$ be a smooth subvariety of codimension $r > 1$. Consider the blow-up square

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Then there is a split exact sequence of motives [25]:

$$0 \to h(X) \otimes \mathbb{L}^{\oplus r} \to (h(\tilde{X}) \otimes \mathbb{L}) \oplus h(Y) \to h(\tilde{Y}) \to 0.$$

(Here $\mathbb{L}$ denotes the *Lefschetz motive*, which is the formal dual of $h^2(\mathbb{P}^1)$.) We also refer to [1] for more information about the motive of the blow-up of a quotient variety.

4.5. Proposition. *Let S be a product-quotient surface of unmixed type with curves C_1 and C_2 and with group G. Then the second Künneth component of S is given by $h^2(S) \cong U \oplus Z \oplus E$, where*

$$U := \left(h^2(C_1) \otimes h^0(C_2)\right) \oplus \left(h^0(C_1) \otimes h^2(C_2)\right),$$

$$Z := \left(h^1(C_1) \otimes h^1(C_2)\right)^G,$$

$$E := \mathbb{Q}(-1)^{\oplus \eta},$$

and where η is the number of exceptional divisors introduced in the minimal desingularization of the quotient surface.

Proof. Let η denote the number of exceptional divisors introduced in the minimal desingularization of the quotient surface. Recall that for the surfaces that we are interested in we gave the value of η in Table 2. Observe that $h^2(S) \cong h^2((C_1 \times C_2)/G) \oplus E$. Notice that if the action of G on $C_1 \times C_2$ is free, then $E = 0$. By the Künneth formula we obtain

$$h^2(C_1 \times C_2) = \left(h^2(C_1) \otimes h^0(C_2)\right) \oplus \left(h^0(C_1) \otimes h^2(C_2)\right) \oplus \left(h^1(C_1) \otimes h^1(C_2)\right).$$

By definition we have $h^2((C_1 \times C_2)/G) = h^2(C_1 \times C_2)^G$, and since G acts trivially on $h^0(C_i)$ and $h^2(C_i)$, we get the result. $\qquad\square$

4.6. *Remark.* Note that we can further decompose the Chow motive Z of Proposition 4.5 as $Z \cong Z_1 \oplus Z_2$ where

$$Z_1 := h^1(C_1)^G \otimes h^1(C_2)^G, \quad Z_2 := \bigoplus_{W \in \hat{G} - \chi_1} \left(h^1(C_1)^{(W)} \otimes h^1(C_2)^{(W^\vee)}\right)^G,$$

where $\hat{G}$ is the set of isomorphism classes of irreducible representations of G over $\mathbb{Q}$, χ_1 is the trivial representation, and $(\_)^{(W)}$ denotes the W-isotypical component.

4.7. *Remark.* Observe that if W is a finite-dimensional $\mathbb{Q}$-vector space, then we may view it as a Chow motive, as follows: It is determined by the identity

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(M, W) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(M, \mathbb{Q}) \otimes W,$$

and it is non-canonically isomorphic to $\mathbb{Q}^{\oplus \dim(W)}$.

If G is a finite group and W is equipped with a finite-dimensional representation of G, then we may view the representation $G \to \mathrm{Aut}(W)$ as an action of G on the Chow motive W.

4.8. We now state the main theorem of this section, which gives a partial answer to Question B of the introduction. The rest of this section is dedicated to its proof. We refer to §4.17 for an explicit computation that illustrates the general argument of this proof.

4.9. Theorem. *Let S be a minimal surface of general type with $p_g = q = 2$, of maximal Albanese dimension isogenous to a product of unmixed type. Then there exist a K3 surface X and a correspondence in $S \times X$ that induces an isomorphism between $H^2(X, \mathbb{Q})^{\mathrm{tra}}$ and $H^2(S, \mathbb{Q})^{\mathrm{tra}}_{\mathrm{new}}$.*

4.10. The proof is done in several steps and will be completed in §4.16. Following Proposition 4.5 and Remark 4.6 we decompose $h^2(S) = U \oplus Z_1 \oplus Z_2 \oplus E$. First of all we see that for $i = 1, 2$ the quotients $C_i/G =: E_i$ are elliptic curves. The product $E_1 \times E_2$ is the Albanese variety of S, and we have $h^2(E_1 \times E_2) = U \oplus Z_1$. Secondly, let η be the number of irreducible components of the exceptional divisor of the minimal resolution of the quotient surface (see Tables 2 and 3; if there is no branch locus, then $\eta = 0$). This gives $E = \mathbb{Q}(-1)^\eta$. For the purpose of this theorem we are interested in the remaining term Z_2. To find an algebraic K3 partner X for S, we will find an abelian surface A as isogeny factor of $J(C_1) \times J(C_2)$ such that $Z_2 = h^2(A)^{\mathrm{tra}} \oplus \mathbb{Q}(-1)^k$ for some $k \in \mathbb{Z}_{\geq 0}$. We may then take the minimal resolution of singularities of the Kummer surface $\mathrm{Km}(A) = A/\langle -1 \rangle$ for the K3 surface X. To find the isogeny factor A, we proceed by decomposing (up to isogeny) the Jacobians $J(C_i)$ of C_i for $i = 1, 2$ as products of simple abelian varieties following [34].

4.11. Now let A be an abelian variety of dimension g with a faithful action of a finite group G. There is an induced homomorphism of $\mathbb{Q}$-algebras

$$\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(A).$$

Any element $\alpha \in \mathbb{Q}[G]$ defines an abelian subvariety

$$A^\alpha := \mathrm{Im}(m\rho(\alpha)) \subset A,$$

where m is some positive integer such that $m\rho(\alpha) \in \mathrm{End}(A)$. This definition does not depend on the chosen integer m.

We will now describe the so-called *isotypical decomposition* of the abelian variety A with group action by G. Begin with the decomposition of $\mathbb{Q}[G]$ as a product of simple $\mathbb{Q}$-algebras $Q_1 \times \cdots \times Q_r$. The factors Q_i correspond canonically to the rational irreducible representations W_i of the group G, because each one is generated by a unit element $e_i \in Q_i$ which may be considered as a central idempotent of $\mathbb{Q}[G]$.

The corresponding decomposition of $1 \in \mathbb{Q}[G]$,

$$1 = e_1 + \cdots + e_r,$$

induces an isogeny, via ρ above,

(4.11.1) $$A^{e_1} \times \cdots \times A^{e_r} \to A,$$

which is given by addition. Note that the components A^{e_i} are G-stable complex subtori of A with $\mathrm{Hom}_G(A^{e_i}, A^{e_j}) = 0$ for $i \neq j$. The decomposition (4.11.1) is the isotypical decomposition mentioned above.

4.12. The isotypical components A^{e_i} can be decomposed further, using the decomposition of Q_i into a product of minimal left ideals. Fix an $i \in \{1, \ldots, r\}$, and let W_i be the irreducible rational representation of G corresponding to the idempotent e_i. We will now recall some facts from representation theory; see §12.2 of [46] for details.

Write D_i for the simple algebra $\mathrm{End}_G(W_i)$, and observe that $Q_i = \mathrm{Mat}_{n_i}(D_i^\circ)$ is a matrix algebra of degree n_i over the opposite algebra for some $n_i \in \mathbb{Z}_{\geq 1}$. Recall that the Schur index of W_i is the degree m_i of D_i over its centre. If χ_i is the character of one of the irreducible summands of $W_i \otimes_{\mathbb{Q}} \mathbb{C}$, then $\deg \chi_i = m_i \cdot n_i$. There is a set of primitive idempotents $\{\pi_{i,1}, \ldots, \pi_{i,n_i}\}$ in $\mathrm{Mat}_{n_i}(D_i^\circ) = Q_i \subset \mathbb{Q}[G]$ such that

$$e_i = \pi_{i,1} + \cdots + \pi_{i,n_i}.$$

(We warn the reader that the $\pi_{i,j}$ are not G-equivariant, and hence the abelian subvarieties $A^{\pi_{i,j}}$ are not G-stable.) The abelian subvarieties $A^{\pi_{i,j}}$ are mutually isogenous for $j = 1, \ldots, n_i$. Let B_i be any one of these isogenous factors; we call it a *reduced factor* of A^{e_i} and n_i the *multiplicity* of the reduced factor: $B_i^{n_i} \to A^{e_i}$ is an isogeny. Replacing the factors in (4.11.1) for every $i = 1, \ldots, r$, we get an isogeny, called the *group algebra decomposition* of the G-abelian variety A,

$$(4.12.1) \qquad\qquad B_1^{n_1} \times \cdots \times B_r^{n_r} \to A.$$

Note that whereas (4.11.1) is uniquely determined, (4.12.1) is not. It depends on the choice of the π_{ij} as well as the choice of the B_i. However, the dimension and the isogeny class of the abelian varieties B_i are independent of choices.

4.13. *Remark.* If $D_i = \mathbb{Q}$, then we get a G-equivariant isomorphism $h^1(A^{e_i}) \cong h^1(B_i) \otimes W_i$ of Chow motives, where G acts trivially on $h^1(B_i)$.

4.14. While the factors in (4.12.1) are not necessarily easy to determine, we may compute their dimension in the case of a Jacobian variety. Let C be a compact Riemann surface equipped with an action of a finite group G and consider the induced action of G on $J(C)$. Define V to be the representation of G on $H_1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. We use the same notation as at the beginning of this section, so the quotient C/G has genus g_0 and the cover $\pi \colon C \to C/G$ has r branch points $\{q_1, \ldots, q_r\}$ where each q_i has corresponding monodromy g_i. The tuple $(g_1, \ldots, g_r)$ is called the *generating vector* for the action [5].

We now copy equation (2.14) from [5] and explain the notation afterwards: the Hurwitz character χ_V associated to V is

$$(4.14.1) \qquad\qquad \chi_V = 2\chi_1 + 2\,(g_0 - 1)\,\rho_{1*} + \sum_{i=1}^{r} \left(\rho_{1*} - \rho_{\langle g_i \rangle} \right).$$

Here χ_1 is the trivial character on G, and ρ_{1*} is the character of the regular representation. The character ρ_H is the induced character on G of the trivial character of the subgroup H. (When $H = \langle g_i \rangle$, this subgroup is the stabilizer, or isotropy group, of a point in the fiber of the branch point q_i.)

With this definition of χ_V in place, we have the equality

$$(4.14.2) \qquad\qquad \dim B_i = \frac{1}{2} \dim_{\mathbb{Q}} \pi_{i,j} V = \frac{1}{2} \langle \psi_i, \chi_V \rangle,$$

where ψ_i denotes the character of the $\mathbb{Q}$-irreducible representation of G corresponding to W_i. See [23, 33] for details.

4.15. We will now complete the proof of Theorem 4.9. We may calculate the dimension of the B_i's for each class of surfaces in the statement, either by hand or using the MAGMA script as in [34]. The result of this computation is given in Table 4.

It is a coincidence that in the table all the characters that appear are actually self-dual and defined over $\mathbb{Q}$. One can check that for each row in the table, there is only one character χ that appears in column char(C_1) such that the dual character $\chi^\vee$ appears in column char(C_2).

Let W be the irreducible rational representation of G that corresponds to χ. We will complete the proof by a case distinction. First assume that $G \neq Q_8$. In this case, one may check that the Schur index of W is 1, and in fact $D = \mathrm{End}_G(W) = \mathbb{Q}$. In other words, we are in the situation of Remark 4.13. Let L_1 be a reduced factor

TABLE 4. The group algebra decomposition of the Jacobian varieties $J(C_i)$. The columns $\mathrm{char}(C_i) - \chi_1$ have to be read in the following way: each list $[d, n, k]$ represents a non-trivial isotypical factor of $J(C_i)$ corresponding to a simple rational representation W of G. Here d is the dimension of a reduced factor, n is the multiplicity of the factor, and k is the identifying number according to the MAGMA character table of the group G of an irreducible character appearing in $W \otimes_{\mathbb{Q}} \mathbb{C}$.

G	$\mathrm{char}(C_1) - \chi_1$	$J(C_1)/E_1$	$\mathrm{char}(C_2) - \chi_1$	$J(C_2)/E_2$
V_4	$[1,1,2],\ [1,1,4]$	$E \times L_1$	$[1,1,3],\ [1,1,4]$	$E \times L_2$
S_3	$[1,2,3]$	L_1^2	$[1,1,2],\ [1,2,3]$	$E \times L_2^2$
D_4	$[1,2,5]$	L_1^2	$[1,1,3],\ [1,1,4],\ [1,2,5]$	$E \times E' \times L_2^2$
A_4	$[1,3,4]$	L_1^3	$[1,3,4]$	L_2^3
S_3	$[1,2,3]$	L_1^2	$[1,2,3]$	L_2^2
Q_8	$[2,1,5]$	A	$[2,1,5]$	A'
D_4	$[1,2,5]$	L_1^2	$[1,2,5]$	L_2^2
C_2	$[1,1,2]$	L_1	$[1,1,2]$	L_2

of $J(C_1)$ corresponding with χ, and denote with L_2 a reduced factor of $J(C_2)$ that corresponds to $\chi^{\vee}$. Thus we have

$$h^1(J(C_1))^{(W)} \cong h^1(L_1) \otimes W, \qquad h^1(J(C_2))^{(W^{\vee})} \cong h^1(L_2) \otimes W^{\vee},$$

as motives with an action of G. Consequently, we find that

$$\left(h^1(J(C_1))^{(W)} \otimes h^1(J(C_2))^{(W^{\vee})} \right)^G \cong h^1(L_1) \otimes h^1(L_2) \otimes (W \otimes W^{\vee})^G$$
$$\cong h^1(L_1) \otimes h^1(L_2).$$

We conclude that $Z_2 \cong h^1(L_1) \otimes h^1(L_2) \cong h^2(L_1 \times L_2)^{\mathrm{tra}} \oplus \mathbb{Q}(-1)^{\oplus k}$ for some $k \in \mathbb{Z}_{\geq 0}$, and thus $L_1 \times L_2$ is the 2-dimensional isogeny factor A of $J(C_1) \times J(C_2)$ that we are looking for.

4.16. **The case $G = Q_8$.** In the case where $G = Q_8$, we find that $D = \mathrm{End}_G(W) = \mathbb{H}$, which has Schur index 2. In this case we cannot use the methods employed so far to prove that the Jacobian is isogenous to a product of elliptic curves as in all the other cases.

Nevertheless, in [17] it is proven that in this case the curves C_1 and C_2, which are of genus $g = 3$, admit a bigger automorphism group. Indeed, their automorphism group is isomorphic to $(C_4 \times C_2) \rtimes C_2$, which readily contains Q_8. More precisely, in [17] it is shown that the curves C of genus $g = 3$ and with automorphism Q_8 and $(C_4 \times C_2) \rtimes C_2$ give rise to the same subvariety in the moduli space of curves which is the family (34) of Table 2 in [16]. Therefore, we can try to decompose the Jacobian of C using this larger group.

Performing the calculation relative to this larger group we have (in the notation of Table 4)

$$(C_4 \times C_2) \rtimes C_2 \qquad [1,2,9] \qquad L_1^2 \qquad [1,2,9] \qquad L_2^2.$$

To conclude, notice that the ninth character of $(C_4 \times C_2) \rtimes C_2$ is not self-dual, but we have to restrict it to Q_8. Recalling that $Q_8 \triangleleft (C_4 \times C_2) \rtimes C_2$, one sees that all the conditions of Problem 5.2 on page 65 of [19] are fulfilled. Hence the restriction of this character to Q_8 is the only 2-dimensional irreducible representation which is self-dual. We remark that the L_i must be CM elliptic curves, since $\mathbb{H}$ injects into $M_2(\mathrm{End}(L_i)_{\mathbb{Q}})$.

Now recall that $\mathbb{H} \otimes \mathbb{H} \cong \mathrm{End}(\chi_1 \oplus \chi_i \oplus \chi_j \oplus \chi_k) \cong M_4(\mathbb{Q})$. We conclude that $h^1(L_1^2) \otimes h^1(L_2^2) \cong \left(h^1(L_1) \otimes h^1(L_2)\right) \otimes (\chi_1 \oplus \chi_i \oplus \chi_j \oplus \chi_k)$ as Chow motives with an action of G. In particular we have $\left(h^1(L_1^2) \otimes h^1(L_2^2)\right)^G \cong h^1(L_1) \otimes h^1(L_2)$. This concludes the proof of Theorem 4.9.

4.17. **The case $K^2 = 8$ and $G = V_4$.** As an illustration of the proof of Theorem 4.9 and in particular the computations performed by the MAGMA script, we will now study one example in detail. This example will occupy us for the next few pages.

Let E_1 and E_2 be elliptic curves, marked with two points $\{p_{i,1}, p_{i,2}\} \subset E_i$. As a group we take $G = V_4$. Let $\{\alpha_i, \beta_i\}$ be generators of $\pi_1(E_i, 0)$ and let $\gamma_{i,j}$ be a loop in $\pi_1(E_i, 0)$ around $p_{i,j}$. Then we define two G-covers $f_i \colon C_i \to E_i$, using the Riemann existence theorem (see, e.g., [26, section III]), by the following epimorphisms of groups:

(4.17.1)

$$
\begin{array}{ll}
\pi_1(E_1 - \{p_{1,j}\}, 0) \longrightarrow G & \qquad \pi_1(E_2 - \{p_{2,j}\}, 0) \longrightarrow G \\
\alpha_1 \longmapsto (0,1) =: a_1 & \qquad \alpha_2 \longmapsto (1,0) =: a_2 \\
\beta_1 \longmapsto (0,0) =: b_1 & \qquad \beta_2 \longmapsto (0,0) =: b_2 \\
\gamma_{1,j} \longmapsto (1,0) =: c_{1,j} & \qquad \gamma_{2,j} \longmapsto (0,1) =: c_{2,j}
\end{array}
$$

By construction $\{p_{i,1}, p_{i,2}\} \subset E_i$ is the branch locus, and above each $p_{i,j}$, there are two ramification points with branching orders 2. Therefore, by the Riemann–Hurwitz formula the C_i are curves of genus 3.

For a non-trivial element $g \in G$, let χ_g denote the non-trivial character of G that annihilates g, let $\rho_{\langle g \rangle}$ denote the character of G induced from the trivial character of the subgroup generated by g, and let ρ_{1*} be the regular character. We get

$$\mathbb{Q}[G] = \mathbb{Q}^{\chi_{(0,0)}} \times \mathbb{Q}^{\chi_{(0,1)}} \times \mathbb{Q}^{\chi_{(1,0)}} \times \mathbb{Q}^{\chi_{(1,1)}}.$$

We proceed by calculating the Hurwitz character (4.14.1) relative to the first quotient ($i = 1$). Starting from the ramification data of the curve C_1, we get

$$\chi_1 = (1,1,1,1), \quad \rho_{1*} = (4,0,0,0), \quad \rho_{\langle 1,0 \rangle} = (2,2,0,0),$$

where the induced trivial representation is calculated using the formula of Exercise 3.19(b) in [19]. Therefore, the Hurwitz character is $\chi_V = (6, -2, 2, 2)$. Now we use equation (4.14.2) to compute that the dimensions of the reduced factors B_χ of $J(C_1)$ are respectively $1, 1, 0, 1$ for $\chi = \chi_{(0,0)}, \chi_{(0,1)}, \chi_{(1,0)}, \chi_{(1,1)}$. Analogously, we compute that the dimension of the reduced factors B_χ of $J(C_2)$ are respectively $1, 0, 1, 1$ for $\chi = \chi_{(0,0)}, \chi_{(0,1)}, \chi_{(1,0)}, \chi_{(1,1)}$.

These results relate to row 1 of Table 4 in the following way: $[1,1,2]$ means that there is a 1-dimensional reduced factor with multiplicity 1 corresponding to the

second character in the MAGMA character table of V_4; this is the character $\chi_{(0,1)}$. This is exactly the isogeny factor $B_{\chi_{(0,1)}}$ of $J(C_1)$. Similarly $[1, 1, 3]$ corresponds to the isogeny factor $B_{\chi_{(1,0)}}$ of $J(C_2)$. Both $J(C_1)$ and $J(C_2)$ have an isogeny factor $B_{\chi_{(1,1)}}$, which corresponds to $[1, 1, 4]$ in Table 4 and thus to the elliptic curves L_1 and L_2.

4.18. Now we will explain how to construct geometrically the algebraic K3 partner X of $S = (C_1 \times C_2)/G$ and a correspondence that induces the isomorphism $H^2_{\text{new}}(S, \mathbb{Q})^{\text{tra}} \to H^2(X, \mathbb{Q})^{\text{tra}}$. The K3 partner X will turn out to be the minimal resolution of Kummer surface associated with $L_1 \times L_2$.

Observe that f_i factors as in the following diagram:

(4.18.1)

$$
\begin{array}{ccccc}
& & C_i & & \\
& \swarrow^{\phi_{i,(0,1)}} & \downarrow^{\phi_{i,(1,0)}} & \searrow^{\phi_{i,(1,1)}} & \\
C_i/\langle(0,1)\rangle & & C_i/\langle(1,0)\rangle & & C_i/\langle(1,1)\rangle \\
& \searrow_{\psi_{i,(0,1)}} & \downarrow^{\psi_{i,(1,0)}} & \swarrow_{\psi_{i,(1,1)}} & \\
& & E_i & &
\end{array}
$$

Using the Riemann–Hurwitz formula we compute the following genera for the quotient curves:

$$g(C_1/\langle(0,1)\rangle) = 2, \quad g(C_1/\langle(1,0)\rangle) = 1, \quad g(C_1/\langle(1,1)\rangle) = 2;$$

$$g(C_2/\langle(0,1)\rangle) = 1, \quad g(C_2/\langle(1,0)\rangle) = 2, \quad g(C_2/\langle(1,1)\rangle) = 2.$$

Pushing the preceding diagram through the Jacobian functor, we obtain the diagram

$$
\begin{array}{ccccc}
& & J(C_i) & & \\
& \swarrow & \downarrow & \searrow & \\
J(C_i/\langle(0,1)\rangle) & & J(C_i/\langle(1,0)\rangle) & & J(C_i/\langle(1,1)\rangle) \\
& \searrow & \downarrow & \swarrow & \\
& & E_i & &
\end{array}
$$

This leads to the following isogenies of abelian varieties:

(4.18.2)
$$
\begin{array}{ll}
J(C_1/\langle(0,1)\rangle) \sim E_1 \times P(\psi_{1,(0,1)}), & J(C_2/\langle(0,1)\rangle) \sim E_2, \\
J(C_1/\langle(1,0)\rangle) \sim E_1, & J(C_2/\langle(1,0)\rangle) \sim E_2 \times P(\psi_{2,(1,0)}), \\
J(C_1/\langle(1,1)\rangle) \sim E_1 \times P(\psi_{1,(1,1)}), & J(C_2/\langle(1,1)\rangle) \sim E_2 \times P(\psi_{2,(1,1)}),
\end{array}
$$

where $P(\psi)$ denotes the *Prym–Tyurin variety* associated to the cover ψ: it is the kernel of the induced map $J(\psi)$ between the Jacobians (see also [4], paragraph 12.2). Observe that L_i is isogenous to $P(\psi_{i,(1,1)})$. Finally by (4.18.2) we have

(4.18.3)
$$
\begin{aligned}
J(C_1) &\sim J(C_1/\langle(0,1)\rangle) \times P(\phi_{1,(0,1)}) \\
&\sim E_1 \times P(\psi_{1,(0,1)}) \times P(\phi_{1,(0,1)}),
\end{aligned}
$$

(4.18.4)
$$
\begin{aligned}
J(C_2) &\sim J(C_2/\langle(1,0)\rangle) \times P(\phi_{2,(1,0)}) \\
&\sim E_2 \times P(\psi_{2,(1,0)}) \times P(\phi_{2,(1,0)}).
\end{aligned}
$$

We now go back to the surface $S = (C_1 \times C_2)/V_4$. Since $\eta = 0$, we get

$$h^2(S) \cong h^2(E_1 \times E_2) \oplus h^1(P(\psi_{1,(1,1)})) \otimes h^1(P(\psi_{2,(1,1)})).$$

Let us go further and build an algebraic K3 partner of S. To do that we consider the abelian surface $A = P(\psi_{1,(1,1)}) \times P(\psi_{2,(1,1)})$ and divide modulo the natural involution ι. In this way we get a singular Kummer surface.

According to the proposition above and Theorem 3 of [47] the minimal resolution $\mathrm{Km}(A)$ of the singularities of A/ι is a K3 surface, and there exists a double cover X of $\mathrm{Km}(A)$ whose transcendental part of $H^2(X)$ is isomorphic to the transcendental part of $H^2(A)$. Now consider the following diagram:

$$
\begin{array}{ccccccccc}
C_1 \times C_2 & \longrightarrow & J(C_1) \times J(C_2) & \longleftarrow & A & & \mathrm{Km}(A) & \longleftarrow & X \\
\downarrow & & & & \downarrow & & & & \\
S & & & & A/\iota & & & &
\end{array}
$$

All the morphisms in this diagram induce correspondences, and by composing these correspondences we obtain an isomorphism $h^2(S)^{\mathrm{tra}}_{\mathrm{new}} \to h^2(X)^{\mathrm{tra}}$ that induces an isomorphism of Hodge structures $H^2_{\mathrm{new}}(S, \mathbb{Q})^{\mathrm{tra}} \to H^2(X, \mathbb{Q})^{\mathrm{tra}}$.

5. Motivated K3 partners

Let S be a smooth projective complex surface with invariants $p_g(S) = q(S) = 2$, and assume that the Albanese morphism $\alpha\colon S \to A$ is surjective. In this section we attempt to answer Question C of the introduction:

> Does there exist a K3 surface X together with an isomorphism $\iota\colon (H^2_{\mathrm{new}})^{\mathrm{tra}}_{\mathbb{Q}} \to H^2(X, \mathbb{Q})^{\mathrm{tra}}$ that is motivated in the sense of Yves André?

5.1. In this section we prove the Tate and Mumford–Tate conjectures for surfaces that fall into type № $= 3, 4, 6, 8$, and 9 of Table 1. We will use the language of motives and specifically motivated cycles as introduced by André [2].

This section is organized as follows: First we introduce notation and recall the definition of the Mumford–Tate group and the ℓ-adic monodromy groups. Then we will recall three conjectures that are connected in the following sense: if two of the conjectures hold, then so does the third. These conjectures are (i) the Hodge conjecture, (ii) its ℓ-adic analogue known as the Tate conjecture, and (iii) the Mumford–Tate conjecture.

Starting from §5.9 we recall the definition of motivated cycles in the sense of André [2], and we quote the main theorems that describe the resulting category of motives. Once we have all the machinery in place we turn our attention to the proof of the Tate and Mumford–Tate conjectures for the surfaces mentioned above.

5.2. *Notation.* Let $K \subset \mathbb{C}$ be a field, and let X be a smooth projective variety over K. We denote with $H^i(X)$ the singular cohomology group $H^i_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$. It is naturally endowed with a pure Hodge structure of weight i. Let ℓ be a prime number, and let $\bar{K} \subset \mathbb{C}$ be the algebraic closure of $K \in \mathbb{C}$. We denote with $H^i_\ell(X)$ the ℓ-adic étale cohomology group $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$. It is naturally endowed with a Galois representation $\mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(H^i_\ell(X))$.

Artin's comparison theorem between étale cohomology and singular cohomology gives an isomorphism of vector spaces

$$H_\ell^i(X) \cong H_\ell^i(X_\mathbb{C}) \xrightarrow{\sim} H^i(X) \otimes \mathbb{Q}_\ell$$

that is functorial in X.

Recall from §4.2 that we denote with $\mathrm{CH}^*(X)$ the Chow ring of X with $\mathbb{Q}$-coefficients. Recall the cycle class map $\mathrm{cl}\colon \mathrm{CH}^i(X_\mathbb{C}) \to H^{2i}(X)(i)$ for singular cohomology. There is also a cycle class map $\mathrm{cl}_\ell\colon \mathrm{CH}^i(X) \to H_\ell^{2i}(X)(i)$ for étale cohomology. These are compatible with the comparison isomorphism $H_\ell^i(X) \cong H^i(X) \otimes \mathbb{Q}_\ell$; we get the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{CH}^i(X) & \xrightarrow{\quad\quad\quad\mathrm{cl}_\ell\quad\quad\quad} & H_\ell^{2i}(X)(i) \\
\downarrow & & \downarrow{\scriptstyle\sim} \\
\mathrm{CH}^i(X_\mathbb{C}) & \xrightarrow{\mathrm{cl}} H^{2i}(X)(i) \xrightarrow{-\otimes 1} & H^{2i}(X)(i) \otimes \mathbb{Q}_\ell
\end{array}
$$

5.3. Mumford–Tate groups.

Let V be a $\mathbb{Q}$-Hodge structure. There is a representation of $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_\mathrm{m}$ on $V \otimes \mathbb{R}$: on complex points $(z, \bar{z})$ acts on $v \in V^{pq}$ by $v \mapsto z^{-p}\bar{z}^{-q}v$. (The minus signs are a historical convention.) Write $h = h_V$ for this representation $\mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$.

The *Mumford–Tate group* of V is the smallest algebraic subgroup $G \subset \mathrm{GL}(V)$ over $\mathbb{Q}$ such that $G_\mathbb{R}$ contains the image of h_V. We denote the Mumford–Tate group of V by $G_\mathrm{MT}(V)$. Alternatively, $G_\mathrm{MT}(V)$ may be defined using the Tannakian formalism. It is the algebraic group over $\mathbb{Q}$ associated with the Tannakian subcategory of $\mathbb{Q}$-HS generated by V. If V is polarizable, then this subcategory generated by V is semisimple, and hence $G_\mathrm{MT}(V)$ is a reductive algebraic group.

Two more remarks are in place: First, observe that $G_\mathrm{MT}(V)$ is a connected algebraic group, since $\mathbb{S}$ is connected. Second, note that the subspace $V \cap V_\mathbb{C}^{0,0}$ of *Hodge classes* in V is exactly the space of invariants $V^{G_\mathrm{MT}(V)}$.

5.4. ℓ-adic monodromy groups.

Let $K \subset \mathbb{C}$ be a field of finite transcendence degree over $\mathbb{Q}$. Let ℓ be a prime number, let V be a finite-dimensional $\mathbb{Q}_\ell$-vector space, and let $\rho\colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(V)$ be a representation that is continuous for the ℓ-adic topology on V.

The *ℓ-adic monodromy group* of V is the smallest algebraic subgroup $G \subset \mathrm{GL}(V)$ over $\mathbb{Q}_\ell$ such that $G(\mathbb{Q}_\ell)$ contains the image of ρ. We denote the ℓ-adic monodromy group of V with $G_\ell(V)$. In general, the algebraic group $G_\ell(V)$ is not connected; the identity component is denoted $G_\ell^\circ(V)$.

An element of V is called a *Tate class* if it is invariant under an open subgroup of $\mathrm{Gal}(\bar{K}/K)$. In particular, the subspace of Tate classes in V is exactly the space of invariants $V^{G_\ell^\circ(V)}$.

In general, the algebraic group $G_\ell^\circ(V)$ is not reductive.

5.5. Conjecture (Hodge).

Let X be a smooth projective variety over $\mathbb{C}$. Then the image of $\mathrm{cl}\colon \mathrm{CH}^i(X) \to H^{2i}(X)(i)$ *is the subspace of Hodge classes* $H^{2i}(X)(i) \cap H^{2i}(X)(i)_\mathbb{C}^{0,0}$.

5.6. Conjecture (Tate).

Let X be a smooth projective variety over a number field K. Then the image of $\mathrm{cl}_\ell\colon \mathrm{CH}^i(X_{\bar{K}}) \to H_\ell^{2i}(X)(i)$ *spans the space of Tate classes* $H_\ell^{2i}(X)(i)^{G_\ell^\circ(X)}$.

5.7. **Conjecture** (Mumford–Tate). *Let X be a smooth projective variety over a number field K. The comparison isomorphism $H_\ell^i(X) \to H^i(X) \otimes \mathbb{Q}_\ell$ induces an isomorphism $G_\ell^\circ(H_\ell^i(X)) \to G_{\mathrm{MT}}(H^i(X)) \otimes \mathbb{Q}_\ell$.*

5.8. *Remark.* To illustrate how these conjectures fit together, we make the following claims.
  (1) If the Mumford–Tate conjecture is true for X, then the Hodge conjecture for X is equivalent to the Tate conjecture for X.
  (2) If the Tate conjecture is true for all smooth projective varieties X over K, then the ℓ-adic monodromy groups are reductive. This follows from [28].
  (3) If the Hodge and Tate conjectures are true for all X, then the Mumford–Tate conjecture is true for all X.

5.9. **Motivated cycles.** Let K be a subfield of $\mathbb{C}$, and let X be a smooth projective variety over K. A class γ in $H^{2i}(X)$ is called a *motivated* cycle of degree i if there exists an auxiliary smooth projective variety Y over K such that γ is of the form $\pi_*(\alpha \cup \star\beta)$, where $\pi\colon X \times Y \to X$ is the projection, α and β are algebraic cycle classes in $H^*(X \times Y)$, and $\star\beta$ is the image of β under the Hodge star operation. (Alternatively, one may use the Lefschetz star operation; see section 1 of [2].)

Every algebraic cycle is motivated, and under the Lefschetz standard conjecture the converse holds as well. The set of motivated cycles naturally forms a graded $\mathbb{Q}$-algebra. The category of motives over K, denoted Mot_K, consists of objects (X, p, m), where X is a smooth projective variety over K, p is an idempotent motivated cycle on $X \times X$, and m is an integer. A morphism $(X, p, m) \to (Y, q, n)$ is a motivated cycle γ of degree $n - m$ on $Y \times X$ such that $q\gamma p = \gamma$. We denote with $\mathrm{H}_{\mathrm{mot}}(X)$ the object $(X, \Delta, 0)$, where Δ is the class of the diagonal in $X \times X$. The Künneth projectors π_i are motivated cycles, and we denote by $\mathrm{H}_{\mathrm{mot}}^i(X)$ the object $(X, \pi_i, 0)$. Observe that $\mathrm{H}_{\mathrm{mot}}(X) = \bigoplus_i \mathrm{H}_{\mathrm{mot}}^i(X)$. This gives contravariant functors $\mathrm{H}_{\mathrm{mot}}(\_)$ and $\mathrm{H}_{\mathrm{mot}}^i(\_)$ from the category of smooth projective varieties over K to Mot_K.

5.10. **Theorem.** *The category Mot_K is Tannakian over $\mathbb{Q}$, semisimple, graded, and polarized. Every classical cohomology theory of smooth projective varieties over K factors via Mot_K.*

Proof. See théorème 0.4 of [2]. □

5.11. **Definition.** Let K be a subfield of $\mathbb{C}$. An *abelian motive* over K is an object of the Tannakian subcategory of Mot_K generated by objects of the form $\mathrm{H}_{\mathrm{mot}}(X)$, where X is an abelian variety, or $X = \mathrm{Spec}(L)$ for some finite extension L/K, with $L \subset \mathbb{C}$.

We denote the category of abelian motives over K by AbMot_K.

5.12. **Example.** If C/K is a curve, then $\mathrm{H}_{\mathrm{mot}}^1(C)$ is an abelian motive: it is isomorphic to $\mathrm{H}_{\mathrm{mot}}^1(J(C))$, where $J(C)$ denotes the Jacobian of C.

If X/K is a K3 surface, then $\mathrm{H}_{\mathrm{mot}}^2(X)$ is an abelian motive, by théorème 7.1 of [2].

The Lefschetz motive $\mathbb{Q}(-1)$ is abelian, because any class of a hyperplane section in an abelian variety A will give a splitting $\mathrm{H}_{\mathrm{mot}}^2(A) \cong M \oplus \mathbb{Q}(-1)$.

5.13. **Theorem.** *The Hodge realization functor $H\colon \mathrm{AbMot}_\mathbb{C} \to \mathbb{Q}\text{-HS}$ is a full functor.*

Proof. See théorème 0.6.2 of [2]. □

5.14. Theorem. *Let B be a reduced connected scheme of finite type over $\mathbb{C}$. Let $f\colon X \to B$ be a smooth projective morphism, and let ξ be a global section of the sheaf $\mathrm{R}^{2i} f_* \mathbb{Q}(i)$. If there is a point $0 \in B(\mathbb{C})$ such that ξ_0 is motivated, then ξ_b is motivated for all $b \in B(\mathbb{C})$.*

Proof. See théorème 0.5 of [2]. □

5.15. By Theorem 5.10, the singular cohomology and ℓ-adic cohomology functors factor via Mot_K. This means that if M is a motive, then we can attach to it a Hodge structure $H(M)$ and an ℓ-adic Galois representation $H_\ell(M)$. The comparison isomorphism between singular cohomology and ℓ-adic cohomology extends to an isomorphism of vector spaces $H_\ell(M) \cong H(M) \otimes \mathbb{Q}_\ell$ that is natural in the motive M.

We shall write $G_{\mathrm{MT}}(M)$ for $G_{\mathrm{MT}}(H(M))$. Similarly, we write $G_\ell(M)$ (resp., $G_\ell^\circ(M)$) for $G_\ell(H_\ell(M))$ (resp., $(G_\ell^\circ(H_\ell(M))))$. The Mumford–Tate conjecture extends to motives: for the motive M it asserts that the comparison isomorphism $H_\ell(M) \cong H(M) \otimes \mathbb{Q}_\ell$ induces an isomorphism $G_\ell^\circ(M) \cong G_{\mathrm{MT}}(M) \otimes \mathbb{Q}_\ell$.

5.16. We now have the notation and theory in place to answer Question C (see §1.4) about surfaces of general type with $p_g = q = 2$. We give a partial answer to this question in the following results.

5.17. Theorem. *Let $\alpha\colon S \to B$ be a smooth projective family of surfaces of general type with invariants $p_g = q = 2$ and dominant Albanese morphism. Assume that B is connected, and assume that there is one point $0 \in B$ such that the motive $\mathrm{H}_{\mathrm{mot}}^2(S_0)$ of the fibre S_0 is an abelian motive. Then for every point $b \in B$, there exist a K3 surface X_b and an isomorphism of motives $\mathrm{H}_{\mathrm{mot}}^2(S_b)_{\mathrm{new}}^{\mathrm{tra}} \cong \mathrm{H}_{\mathrm{mot}}^2(X_b)^{\mathrm{tra}}$. In particular, the motive $\mathrm{H}_{\mathrm{mot}}^2(S_b)$ is abelian.*

Proof. The main idea of the proof is as follows: Using Proposition 3.10 we construct a family $X \to B$ of Hodge-theoretic K3 partners. We then use Theorem 5.13 to prove that $\mathrm{H}_{\mathrm{mot}}^2(S_0)_{\mathrm{new}}^{\mathrm{tra}}$ is isomorphic to $\mathrm{H}_{\mathrm{mot}}^2(X_0)_{\mathrm{new}}^{\mathrm{tra}}$. Finally, this isomorphism spreads out to the other fibers via Theorem 5.14. We now make this sketch precise.

By replacing $\alpha\colon S \to B$ with the pullback along $\tilde{B} \to B$, we may and do assume that B is simply connected. Let $\mathcal{V}$ denote the subvariation of Hodge structures of $\mathrm{R}^2 \alpha_* \mathbb{Q}$ whose fiber $\mathcal{V}_b$ at a point $b \in B$ is $H^2(S_b, \mathbb{Q})_{\mathrm{new}}$. Fix a point $b \in B$. By Proposition 3.10 we find that there are an open $B_b^\circ \subset B$ such that $b \in B_b^\circ$, a K3 space $f\colon \mathcal{X} \to B_b^\circ$, and a morphism of variations of Hodge structures $\iota\colon \mathcal{V}|_{B_b^\circ} \to \mathrm{R}^2 f_* \mathbb{Q}$ that is fiberwise a primitive embedding and a Hodge isometry on the transcendental lattices. We may view ι as a global section of the sheaf $\mathcal{V}^\vee|_{B_b^\circ} \otimes \mathrm{R}^2 f_* \mathbb{Q}$ which is a subsheaf of $\mathrm{R}^4 (\alpha, f)_* \mathbb{Q}(2)$. Note that we may and do assume that $b \in B_0^\circ$; indeed, if $b \notin B_0^\circ$, then we first prove the statement for all points in B_0° and then rerun the proof with a point $0' \in B_b^\circ \cap B_0^\circ$.

Recall from Example 5.12 that $\mathrm{H}_{\mathrm{mot}}^2(X_0)$ is an abelian motive. Also note that $\mathrm{H}_{\mathrm{mot}}^2(S_0)$ is abelian by assumption. Hence ι_0 is motivated, by Theorem 5.13. By Theorem 5.14, we see that ι_b is motivated as well. This means that we obtain an isomorphism $\mathrm{H}_{\mathrm{mot}}^2(S_b)_{\mathrm{new}}^{\mathrm{tra}} \to \mathrm{H}_{\mathrm{mot}}^2(X_b)^{\mathrm{tra}}$. In particular, the motive $\mathrm{H}_{\mathrm{mot}}^2(S_b)_{\mathrm{new}}^{\mathrm{tra}}$ is

abelian. To conclude that $\mathrm{H}^2_{\mathrm{mot}}(S)$ is abelian, observe that $\mathrm{H}^2_{\mathrm{mot}}(S) \cong \mathrm{H}^2_{\mathrm{mot}}(S)^{\mathrm{tra}}_{\mathrm{new}} \oplus \mathrm{H}^2_{\mathrm{mot}}(S)^{\mathrm{tra}}_{\mathrm{old}} \oplus \mathbb{Q}(-1)^r$. The term $\mathrm{H}^2_{\mathrm{mot}}(S)^{\mathrm{tra}}_{\mathrm{old}}$ is abelian, because it is the part coming from the Albanese surface, whose motive is abelian by definition. $\qquad\square$

5.18. Lemma. *Let S' be a smooth surface, and let G be a finite group acting faithfully on S'. Let S be the resolution of singularities of S'/G. If $\mathrm{H}^2_{\mathrm{mot}}(S')$ is an abelian motive, then $\mathrm{H}^2_{\mathrm{mot}}(S)$ is an abelian motive. In particular, this is the case if S is one of the surfaces mentioned in Tables 2 and 3.*

Proof. (Note that for $i \neq 2$, the motives $\mathrm{H}^i_{\mathrm{mot}}(S')$ and $\mathrm{H}^i_{\mathrm{mot}}(S)$ are always abelian motives.)

We have $\mathrm{H}^2_{\mathrm{mot}}(S'/G) = \mathrm{H}^2_{\mathrm{mot}}(S')^G$, which is an abelian motive, because it is a submotive of the abelian motive $\mathrm{H}^2_{\mathrm{mot}}(S')$.

By the blow-up formula for motives of quotient varieties (see §4.4) we see that $\mathrm{H}^2_{\mathrm{mot}}(S)$ is the sum of $\mathrm{H}^2_{\mathrm{mot}}(S/G)$ and (Tate twists of) the motives of centers of blow-ups. Since the center of blow-up in this case is zero-dimensional, we conclude that $\mathrm{H}^2_{\mathrm{mot}}(S)$ is a sum of $\mathrm{H}^2_{\mathrm{mot}}(S/G)$ and a Tate motive. Hence it is an abelian motive. $\qquad\square$

5.19. Corollary. *Let K be a finitely generated subfield of $\mathbb{C}$. Let S be a surface of general type over K with invariants $p_g = q = 2$ and dominant Albanese morphism. Assume that S lies in one of the connected components of the Gieseker moduli space of surfaces of general type that contain a surface that is either a product-quotient or a mixed surface (that is, one of the types № 2, 3, 5,7, 8, or 9 in Table 1). Then the Tate and Mumford–Tate conjectures are true for S.*

Proof. We first prove the Mumford–Tate conjecture for S. Let A be the Albanese variety of S. By Lemma 5.18 we may apply Theorem 5.17 to see that there exists a K3 surface X such that $\mathrm{H}^2_{\mathrm{mot}}(S)^{\mathrm{tra}}_{\mathrm{new}} \cong \mathrm{H}^2_{\mathrm{mot}}(X)^{\mathrm{tra}}$. Possibly after replacing K by a finitely generated extension we may assume that X is defined over K. Hence the motive $\mathrm{H}_{\mathrm{mot}}(S)$ is an object in the Tannakian subcategory of Mot_K generated by $\mathrm{H}_{\mathrm{mot}}(A)$ and $\mathrm{H}_{\mathrm{mot}}(X)$. Therefore it suffices to prove the Mumford–Tate conjecture for $\mathrm{H}_{\mathrm{mot}}(A) \oplus \mathrm{H}_{\mathrm{mot}}(X)$. This follows from the main result of [10]. (See also [48] and [11] for more general results on the Mumford–Tate conjecture for direct sums of abelian motives.)

Recall that the Hodge conjecture is true for S, by the Lefschetz-$(1,1)$ theorem. Therefore the Tate conjecture for S is true, since it follows from the conjunction of the Hodge conjecture and the Mumford–Tate conjecture. Indeed, if $\gamma \subset H^2_\ell(S)(1)$ is a Tate class, then this means that it is fixed by $G^\circ_\ell(S)$. We have just proven the Mumford–Tate conjecture for S, so we know that $G^\circ_\ell(S) \cong G_{\mathrm{MT}}(S) \otimes \mathbb{Q}_\ell$. This means that $\gamma \in H^2_\ell(S)(1) \cong H^2(S)(1) \otimes \mathbb{Q}_\ell$ is a $\mathbb{Q}_\ell$-linear combination of $G_{\mathrm{MT}}(S)$-invariant classes in $H^2(S)(1)$. Those $G_{\mathrm{MT}}(S)$-invariant classes are precisely Hodge classes, and by the Lefschetz-$(1,1)$ theorem we know that they are in the image of the cycle class map. We conclude that γ is in the $\mathbb{Q}_\ell$-span of the image of the ℓ-adic cycle class map. $\qquad\square$

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References

[1] Reza Akhtar and Roy Joshua, *On murre's conjectures for blow-ups: a constructive approach*, arXiv:1309.6403, 2013.

[2] Yves André, *Pour une théorie inconditionnelle des motifs* (French), Inst. Hautes Études Sci. Publ. Math. **83** (1996), 5–49. MR1423019

[3] I. Bauer and R. Pignatelli, *The classification of minimal product-quotient surfaces with $p_g = 0$*, Math. Comp. **81** (2012), no. 280, 2389–2418, DOI 10.1090/S0025-5718-2012-02604-4. MR2945163

[4] Christina Birkenhake and Herbert Lange, *Complex abelian varieties*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 2004. MR2062673

[5] S. Allen Broughton, *Classifying finite group actions on surfaces of low genus*, J. Pure Appl. Algebra **69** (1991), no. 3, 233–270, DOI 10.1016/0022-4049(91)90021-S. MR1090743

[6] Nicola Cancian and Davide Frapporti, *On semi-isogenous mixed surfaces*, Math. Nachr. **291** (2018), no. 2-3, 264–283, DOI 10.1002/mana.201600436. MR3767139

[7] Fabrizio Catanese, *Fibred surfaces, varieties isogenous to a product and related moduli spaces*, Amer. J. Math. **122** (2000), no. 1, 1–44. MR1737256

[8] F. Catanese, *A superficial working guide to deformations and moduli*, Handbook of moduli. Vol. I, Adv. Lect. Math. (ALM), vol. 24, Int. Press, Somerville, MA, 2013, pp. 161–215. MR3184164

[9] Ciro Ciliberto and Margarida Mendes Lopes, *On surfaces with $p_g = q = 2$ and non-birational bicanonical maps*, Adv. Geom. **2** (2002), no. 3, 281–300, DOI 10.1515/advg.2002.014. MR1924760

[10] Johan Commelin, *The Mumford-Tate conjecture for the product of an abelian surface and a K3 surface*, Doc. Math. **21** (2016), 1691–1713. MR3603933

[11] Johan Commelin, *The mumford–tate conjecture for products of abelian varieties*, arXiv:1804.06840, 2018.

[12] Sebastian del Baño Rollin and Vicente Navarro Aznar, *On the motive of a quotient variety*, Collect. Math. **49** (1998), no. 2-3, 203–226. MR1677089

[13] Christopher Deninger and Jacob Murre, *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. **422** (1991), 201–219. MR1133323

[14] Davide Frapporti and Roberto Pignatelli, *Mixed quasi-étale quotients with arbitrary singularities*, Glasg. Math. J. **57** (2015), no. 1, 143–165, DOI 10.1017/S0017089514000184. MR3292683

[15] Davide Frapporti, *Mixed quasi-étale surfaces, new surfaces of general type with $p_g = 0$ and their fundamental group*, Collect. Math. **64** (2013), no. 3, 293–311, DOI 10.1007/s13348-013-0084-5. MR3084399

[16] Paola Frediani, Alessandro Ghigi, and Matteo Penegini, *Shimura varieties in the Torelli locus via Galois coverings*, Int. Math. Res. Not. IMRN **20** (2015), 10595–10623, DOI 10.1093/imrn/rnu272. MR3455876

[17] Paola Frediani, Matteo Penegini, and Paola Porru, *Shimura varieties in the Torelli locus via Galois coverings of elliptic curves*, Geom. Dedicata **181** (2016), 177–192, DOI 10.1007/s10711-015-0118-0. MR3475744

[18] William Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR1644323

[19] William Fulton and Joe Harris, *Representation theory: A first course*, Graduate Texts in Mathematics, vol. 129, Readings in Mathematics, Springer-Verlag, New York, 1991. MR1153249

[20] Daniel Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016. MR3586372

[21] Bruno Kahn, Jacob P. Murre, and Claudio Pedrini, *On the transcendental part of the motive of a surface*, Algebraic cycles and motives. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 344, Cambridge Univ. Press, Cambridge, 2007, pp. 143–202. MR2187153

[22] Shun-Ichi Kimura, *A cohomological characterization of Alexander schemes*, Invent. Math. **137** (1999), no. 3, 575–611, DOI 10.1007/s002220050336. MR1709866

[23] Herbert Lange and Anita M. Rojas, *Polarizations of isotypical components of Jacobians with group action*, Arch. Math. (Basel) **98** (2012), no. 6, 513–526, DOI 10.1007/s00013-012-0400-4. MR2935657

[24] Robert Laterveer, *Algebraic cycles and triple K3 burgers*, Ark. Mat. **57** (2019), no. 1, 157–189, DOI 10.4310/ARKIV.2019.v57.n1.a9. MR3951279

[25] Ju. I. Manin, *Correspondences, motifs and monoidal transformations* (Russian), Mat. Sb. (N.S.) **77 (119)** (1968), 475–507. MR0258836

[26] Rick Miranda, *Algebraic curves and Riemann surfaces*, Graduate Studies in Mathematics, vol. 5, American Mathematical Society, Providence, RI, 1995. MR1326604

[27] Ben Moonen, *On the Tate and Mumford-Tate conjectures in codimension 1 for varieties with $h^{2,0} = 1$*, Duke Math. J. **166** (2017), no. 4, 739–799, DOI 10.1215/00127094-3774386. MR3619305

[28] Ben Moonen, *A remark on the Tate conjecture*, J. Algebraic Geom. **28** (2019), no. 3, 599–603. MR3959072

[29] D. R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. **75** (1984), no. 1, 105–121, DOI 10.1007/BF01403093. MR728142

[30] David R. Morrison, *Isogenies between algebraic surfaces with geometric genus one*, Tokyo J. Math. **10** (1987), no. 1, 179–187, DOI 10.3836/tjm/1270141802. MR899482

[31] J. P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204, DOI 10.1515/crll.1990.409.190. MR1061525

[32] V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238. MR525944

[33] Jennifer Paulhus, *Decomposing Jacobians of curves with extra automorphisms*, Acta Arith. **132** (2008), no. 3, 231–244, DOI 10.4064/aa132-3-3. MR2403651

[34] Jennifer Paulhus and Anita M. Rojas, *Completely decomposable Jacobian varieties in new genera*, Exp. Math. **26** (2017), no. 4, 430–445, DOI 10.1080/10586458.2016.1199981. MR3684576

[35] Matteo Penegini, *The classification of isotrivially fibred surfaces with $p_g = q = 2$*, with an appendix by Sönke Rollenske, Collect. Math. **62** (2011), no. 3, 239–274, DOI 10.1007/s13348-011-0043-y. MR2825713

[36] Matteo Penegini and Francesco Polizzi, *On surfaces with $p_g = q = 2$, $K^2 = 5$ and Albanese map of degree 3*, Osaka Journal of Mathematics **50** (2013), no. 3, 643–686. MR3128997

[37] Matteo Penegini and Francesco Polizzi, *Surfaces with $p_g = q = 2$, $K^2 = 6$, and Albanese map of degree 2*, Canad. J. Math. **65** (2013), no. 1, 195–221, DOI 10.4153/CJM-2012-007-0. MR3004463

[38] Matteo Penegini and Francesco Polizzi, *A new family of surfaces with $p_g = q = 2$ and $K^2 = 6$ whose Albanese map has degree 4*, J. Lond. Math. Soc. (2) **90** (2014), no. 3, 741–762, DOI 10.1112/jlms/jdu048. MR3291798

[39] Roberto Pignatelli, *Quotients of the square of a curve by a mixed action, further quotients and albanese morphisms*, arXiv:1708.01750, 2017.

[40] Roberto Pignatelli and Francesco Polizzi, *A family of surfaces with $p_g = q = 2$, $K^2 = 7$ and Albanese map of degree 3*, Math. Nachr. **290** (2017), no. 16, 2684–2695, DOI 10.1002/mana.201600202. MR3722504

[41] Francesco Polizzi, Carlos Rito, and Xavier Roulleau, *A pair of rigid surfaces with $p_g = q = 2$ and $k^2 = 8$ whose universal cover is not the bidisk*, International Mathematics Research Notices, posted on 2018, DOI 10.1093/imrn/rny107.

[42] Carlos Rito, *New surfaces with $K^2 = 7$ and $p_g = q \leq 2$*, Asian J. Math. **22** (2018), no. 6, 1117–1126, DOI 10.4310/AJM.2018.v22.n6.a7. MR3919554

[43] Jordan Rizov, *Moduli of k3 surfaces and abelian varieties*, PhD thesis, University of Utrecht, 2005.

[44] Jordan Rizov, *Moduli stacks of polarized K3 surfaces in mixed characteristic*, Serdica Math. J. **32** (2006), no. 2-3, 131–178. MR2263236

[45] A. J. Scholl, *Classical motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 163–187, DOI 10.1090/pspum/055.1/1265529. MR1265529

[46] Jean-Pierre Serre, *Linear representations of finite groups*, translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42, Springer-Verlag, New York-Heidelberg, 1977. MR0450380

[47] T. Shioda and H. Inose, *On singular K3 surfaces*, Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 119–136. MR0441982

[48] Adrian Vasiu, *Some cases of the Mumford-Tate conjecture and Shimura varieties*, Indiana Univ. Math. J. **57** (2008), no. 1, 1–75, DOI 10.1512/iumj.2008.57.3513. MR2400251

[49] Angelo Vistoli, *Alexander duality in intersection theory*, Compositio Math. **70** (1989), no. 3, 199–225. MR1002043

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