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Focus Article

Inferring multi-period optimal portfolios via detrending moving average cluster entropy^(a)

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Abstract – Despite half a century of research, there is still no general agreement about the optimal approach to build a robust multi-period portfolio. We address this question by proposing the detrended cluster entropy approach to estimate the weights of a portfolio of high-frequency market indices. The information measure gathered from the markets produces reliable estimates of the weights at varying temporal horizons. The portfolio exhibits a high level of diversity, robustness and stability as not affected by the drawbacks of traditional mean-variance approaches.

focus article

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Introduction. – Markowitz mean-variance approach [1] estimates minimum risk and maximum return for portfolio optimization models in financial decision-making process. Variants of the original mean-variance approach have been later proposed integrating financial concepts such as: capital asset pricing [2], Sharpe ratio [3], excess growth rate [4] Gini-Simpson index [5]. However, contrary to the aim of diversifying, mean-variance approaches yield weights strongly concentrated on some assets, resulting in low diversity of the portfolio.

Traditional approaches have shown even more dramatic limits in multiple horizons and out-of-sample estimates, where non-stationarities and fat tails of the price return distribution come unavoidably into play [6,7]. As high-frequency data become available, microstructure noise increasingly becomes dominant in the returns and volatility series, affecting portfolio performance by reducing the signal-to-noise ratio particularly over multiple periods. Larger sampling intervals could reduce the effect of microstructure noise with the clear disadvantage of not making full use of the available data. Dynamic readjustment of the portfolio, when gathering relevant news, and sequential wealth re-allocation to selected assets in

consecutive trading periods is a key requirement. Regrettably, errors related to the microstructure noise and non-normality of financial series are particularly relevant when portfolio weights should be estimated at different horizons [8,9].

Despite numerous and prominent efforts, the efficacy of quantitative methods of portfolio allocation still remains an open issue, leaving the financial community with the deceiving impression that the *naive equally weighted* $1/\mathcal{N}_A$ portfolio of \mathcal{N}_A assets is not yet significantly outperformed by other approaches to portfolio optimization [10–12].

Over the past decades, a growing wave of interest has been directed towards the analysis of nonlinear interactions arising in complex systems in different contexts. Complex systems exhibit remarkable features related to patterns emerging from the seemingly random structure of time series, due to the interplay of long- and short-range correlated processes [13–20]. Hence, entropy and other information measures have increasingly found applications in complex systems sciences and in particular in economics and finance. As a tool for quantifying *dynamics*, entropy has been adopted for shedding light on fundamental aspects of asset pricing models [21,22]. As a tool for quantifying *diversity*, entropy has been exploited for mitigating drawbacks of traditional portfolio strategies.

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In this specific context, entropy has been considered a convenient instrument of shrinkage and dispersion for the traditional portfolio weights distribution lacking reasonable diversity degree [23–40].

The detrended cluster entropy approach [41–43] has been adopted to investigate several assets over a single period in [44]. An information measure of diversity, the *cluster entropy index* $I(n)$, was put forward by integrating the entropy function over the cluster dimension τ , with the moving average window n as a parameter. It was shown that the *cluster entropy index* $I(n)$ of the volatility is significantly market dependent. The construction of an efficient single-period *static* portfolio based on the *cluster entropy index* $I(n)$ has been proposed and compared to traditional *mean-variance* and *equally weighted* $1/\mathcal{N}_A$ portfolios of \mathcal{N}_A assets.

In [45,46] the cluster entropy approach was extended to multiple temporal horizons \mathcal{M} showing an interesting and significant horizon dependence particularly in long-range correlated markets. Artificially generated price series have been systematically analysed in terms of the cluster entropy dependence on the horizons \mathcal{M} . The cluster entropy for Fractional Brownian Motion (FBM) and Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) time series proved unable to reproduce the real markets dynamics. Conversely, for Autoregressive Fractionally Integrated Moving Average (ARFIMA), the cluster entropy proved able to replicate asset behaviour. Hence, the cluster entropy of real-world series is consistent with the hypothesis of financial processes deviating from *i.i.d.* stochastic processes, proving the ability of the detrended cluster entropy approach of capturing the important statistical features and stylized facts of the real world markets.

In this work, building on the method proposed in [44,45] and the systematic analysis conducted in [46], we discuss how a robust multi-period dynamic portfolio can be constructed via the cluster entropy of *return* and *volatility* of \mathcal{N}_A assets. The cluster entropy index $I(n)$ estimates will be *dynamically* implemented at different temporal horizons \mathcal{M} . Then, the multi-period portfolio will be estimated over $\mathcal{N}_A = 5$ assets and over twelve consecutive monthly periods over one year ($\mathcal{M} = 12$).

Mean-variance approach to portfolio construction. – A portfolio is a vector $w = \{w_1, w_2, w_3, \dots, w_{\mathcal{N}_A}\}$, satisfying $\sum_{i=1}^{\mathcal{N}_A} w_i = 1$, representing the relative allocation of wealth in asset i . Let $r_{t,i}$ be the return of the asset i at time t . The expected return of the portfolio $\mu(r_p)$ can be written in terms of the expected return $\mu(r_i)$ of each asset as

$$\mu(r_p) = w_1\mu(r_1) + \dots + w_{\mathcal{N}_A}\mu(r_{\mathcal{N}_A}). \quad (1)$$

Let σ_i indicate the standard deviation of the return $r_{t,i}$ of the asset i , σ_{ij} the covariance between $r_{t,i}$ and $r_{t,j}$.

The variance of the portfolio return $\sigma^2(r_p)$ can be written as

$$\begin{aligned} \sigma^2(r_p) &= w_1^2\sigma_1^2 + \dots + w_{\mathcal{N}_A}^2\sigma_{\mathcal{N}_A}^2 + \sum_{i=1}^{\mathcal{N}_A} \sum_{\substack{j=1 \\ i \neq j}}^{\mathcal{N}_A} w_i w_j \sigma_i \sigma_j \\ &= \sum_{i=1}^{\mathcal{N}_A} w_i^2 \sigma_i^2 + \sum_{i=1}^{\mathcal{N}_A} \sum_{\substack{j=1 \\ i \neq j}}^{\mathcal{N}_A} \sigma_{ij} w_i w_j. \end{aligned} \quad (2)$$

According to Markowitz portfolio strategy, the weights w_i , for $i = 1, 2, 3, \dots, \mathcal{N}_A$, are chosen to minimize the variance of the return $\sigma^2(r_p)$ (a measure of risk) on portfolio under the constraint of an expected portfolio return $\mu(r_p)$. The performance of the mean-variance portfolio can be maximized in terms of the Sharpe ratio R_S defined as

$$R_S = \frac{\mu(r_p)}{\sqrt{\sigma^2(r_p)}}. \quad (3)$$

The maximization of the Sharpe ratio, with maximum portfolio return, eq. (1), and minimum portfolio variance, eq. (2), is commonly adopted in the standard portfolio theory to nominally yield the optimal weights w_i . However, as eq. (1) and eq. (2) imply normally distributed stationary return series, the approach is flawed at its foundation. Very biased portfolio weights are thus obtained as it could be reasonably expected with asymmetric and heavy tailed return distributions of the real world markets. Several variants of the original theory have been proposed to overcome those limits. The poor accuracy and lack of diversity in the estimation of portfolio weights with the transaction costs involved in the optimization constraints, have been soon recognized as limits of the applicability of mean-variance-based models [10–12].

To fully appreciate the errors in the weights yielded by the traditional mean-variance approach, the Sharpe-ratio calculation has been implemented on tick-by-tick data of the high-frequency markets described in table 1 by using the code provided by the MATLAB Financial Toolbox. The portfolio weights which maximize the Sharpe ratio are shown in fig. 1 and fig. 2. Raw market data are sampled to yield equally spaced series with equal lengths. Sampling intervals are indicated by Δ . The Sharpe ratio maximization is performed on twelve multiple horizons \mathcal{M} . One can note: i) the unreasonably high variability of the weights of the same assets over consecutive periods and ii) the biased distribution of the portfolio weights oriented towards the riskiest assets rather than a diversified portfolio. The overall scenario results quite scary and disappointing for any investor.

Cluster entropy approach to portfolio construction. – Among several alternatives proposed to the aim of building an effective portfolio, entropy-based tools have been developed supported by the main concept that

Table 1: Asset description. Assets and metadata are downloaded from Bloomberg terminal. Tick duration (time interval between individual transactions) is of the order of second for all the markets. The S&P 500 index includes 500 leading companies, but two types of shares are mentioned for 5 companies thus 505 assets are to be considered for calculations. Analogously, for NASDAQ index the number of members is 2570, for DJIA index is 30, for DAX index is 30 and for FTSEMIB is 40. See Bloomberg website for further details on index composition.

Ticker	Name	Country	Currency	Members	Length
NASDAQ	Nasdaq Composite	US	USD	2570	6982017
S&P500	Standard & Poor 500	US	USD	505	6142443
DJIA	Dow Jones Ind. Avg	US	USD	30	5749145
DAX	Deutscher Aktienindex	DE	Euro	30	7859601
FTSEMIB	Milano Indice di Borsa	UK	Euro	40	11088322

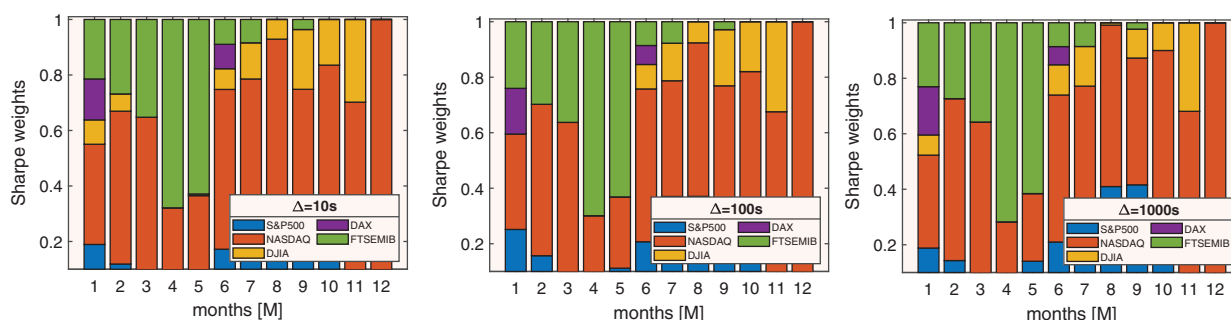


Fig. 1: Portfolio weights w_i vs. investment horizon \mathcal{M} . The weights are obtained by using the standard mean-variance estimates eqs. (1), (2) which maximize the Sharpe ratio R_S , eq. (3). Tick-by-tick high-frequency data for the assets described in table 1 have been used for the estimates. Raw data prices have been downloaded from the Bloomberg terminal. The data are sampled to obtain equally spaced series with equal lengths. Sampling interval is indicated by Δ . The main drawbacks of the traditional portfolio strategy as a results of the non-normality and non-stationarity of the real-world asset price distributions can be easily noted: i) weights are concentrated towards the extremes (*i.e.*, 0 and 1 values are very likely which contradicts the simple principle of high-diversified portfolios); ii) weights exhibit abrupt changes over consecutive horizons (further results of portfolio weights can be found in the figures included in the Supplementary Material `SupplementaryMaterial.pdf` (SM)).

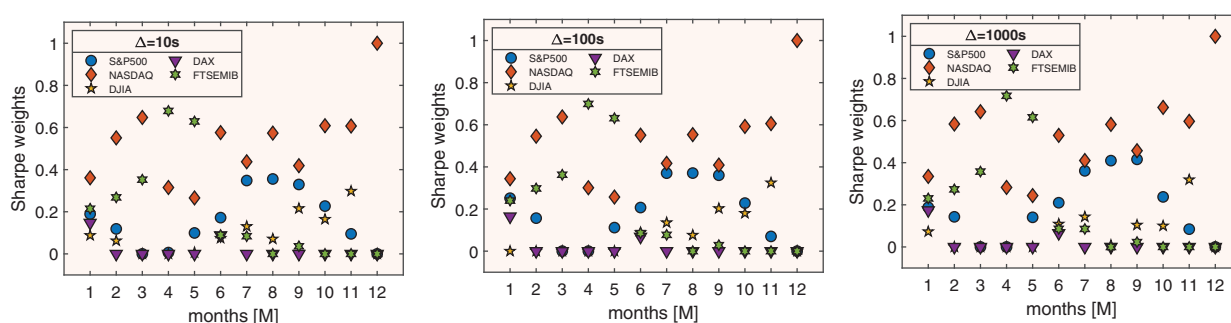


Fig. 2: The same as in fig. 1 but with scattered plots.

entropy itself is a measure of diversity. Entropy-based portfolio inference has been based on Shannon entropy, defined as

$$S(P_i) = - \sum_i P_i \ln P_i, \quad (4)$$

P_i being the probability associated to a given stochastic variable relevant to the asset i . First attempts have been applied to introduce the portfolio weights obtained by Markowitz-based approaches into eq. (4). Portfolio weights $w = (w_1, w_2, \dots, w_{\mathcal{N}_A})'$ among \mathcal{N}_A risky assets, with $w_i \geq 0, i = 1, 2, \dots, \mathcal{N}_A$ and $\sum_{i=1}^{\mathcal{N}_A} w_i = 1$, have the

structure of a probability distribution, thus Shannon entropy can be written as

$$S(w_i) = - \sum_{i=1}^{\mathcal{N}_A} w_i \ln w_i. \quad (5)$$

One can immediately note that with *equally distributed naive weights* $u_i = 1/\mathcal{N}_A$ for all i , $S(w_i)$ reach its maximum value $\ln \mathcal{N}_A$. Conversely, when $w_i = 1$ for the asset i and $w_i = 0$ for the others, then $S(w_i) = 0$. In [25] portfolio optimization is based on the Kullback-Leibler minimum cross-entropy principle:

$$S(w_i, u_i) = - \sum_{i=1}^{\mathcal{N}_A} w_i \ln \frac{w_i}{u_i}, \quad (6)$$

$S(w_i, u_i)$ is minimized with respect to the reference distribution u_i . If the probability of the weights $u_i = 1/\mathcal{N}_A$ is taken as reference the approach provide a shrinkage of the poorly diversified traditional weights towards the uniform distribution.

When entropy is used, the portfolio is generally shrinked toward an equally weighted portfolio, which corresponds to the maximally diversified portfolio yield by the *equally distributed naive weights* $1/\mathcal{N}_A$ rule. However the portfolios yielded either by eq. (5) or by eq. (6) are still affected by the native limitation of operating with the weights w_i estimated by the traditional approach and thus requiring normally and stationary distributed data. Very critical performances are obtained when multiple horizons and out-of-sample estimates are considered.

The *Detrending Moving Average (DMA) cluster entropy method* goes beyond this limit as the portfolio weights are obtained by using the market data probability distribution function to maximize diversity and minimize risk in portfolio optimization. The *DMA cluster entropy* approach to portfolio optimization relies on the general Shannon functional eq. (4) with the probability distribution function of each asset i defined as $P_i = P_i(\tau, n)$. The probability $P_i(\tau, n)$ is associated to the duration of the *clusters* obtained by the intersection with moving average $\tilde{y}_n(t)$, with n the moving average window [41,42]. The simplest type of moving average is defined at each t as the average of the n past observation from t to $t - n + 1$, $\tilde{y}_n(t) = 1/n \sum_{k=0}^{n-1} y(t - k)$. The detrending moving average clusters can be generated with higher-order polynomials as shown in [47,48].

Consecutive intersections of the time series and of the moving average series yield several sets of *clusters*, defined as the portion of the time series between two consecutive intersections of $y(t)$ with $\tilde{y}_n(t)$. The cluster duration is equal to $\tau_j \equiv |t_j - t_{j-1}|$ where t_{j-1} and t_j refers to consecutive intersections of $y(t)$ and $\tilde{y}_n(t)$. For each moving average window n , the probability distribution function $P_i(\tau_j, n)$, *i.e.*, the frequency of the cluster lengths τ_j , can be obtained by counting the number of clusters $\mathcal{N}_C(\tau_j, n)$

with length τ_j , $j \in \{1, N - n - 1\}$. The probability distribution function $P_i(\tau_j, n)$ results in

$$P_i(\tau_j, n) \sim \tau_j^{-D} \mathcal{F}(\tau_j, n), \quad (7)$$

with $D = 2 - H$ the fractal dimension with H the Hurst exponent of the series ($0 < H < 1$). These relationships involve the widely accepted framework of power-law scaling of temporal correlation (see, *e.g.*, [45,46,49,50]). The term $\mathcal{F}(\tau_j, n)$ in eq. (7) takes the form

$$\mathcal{F}(\tau_j, n) \equiv e^{-\tau_j/n}, \quad (8)$$

to account for the drop-off of the power-law behavior for and the onset of the exponential decay when $\tau \geq n$ due to the finiteness of n . When $n \rightarrow 1$ the lengths τ of clusters tend to be centered around a single value. When $n \rightarrow N$, that is when n tends to the length of the whole sequence, only one cluster with $\tau = N$ is generated. Intermediate values of n produces the broad distribution of cluster durations.

When the probability distribution eq. (7) is fed into the Shannon functional eq. (4), the entropy $S_i(\tau_j, n)$ of the cluster lifetime τ_j distribution of the asset i results in

$$S_i(\tau_j, n) = S_0 + \log \tau_j^D + \frac{\tau_j}{n}, \quad (9)$$

where S_0 is a constant, $\log \tau_j^D$ and τ_j/n respectively arises from power-law and exponentially correlated cluster duration. The subscript j refers to each single cluster duration variable and will be suppressed in the forthcoming discussion for simplicity.

The cluster entropy index $I_i(n)$ of a relevant quantity, *e.g.*, the return, for a given asset i can be described in discrete form as

$$I_i(n) = \sum_{\tau=1}^m S_i(\tau, n) + \sum_{\tau=m}^N S_i(\tau, n). \quad (10)$$

The first sum is referred to the power law regime of the cluster duration probability distribution. The second sum is referred to the linear regime of the cluster duration probability distribution (*i.e.*, the excess entropy term with respect to the logarithmic one). The index m represents the threshold value of the cluster lifetime between the regimes.

Results. – As recalled in the Introduction, the cluster entropy and related portfolio's weights have been estimated over a single period, *i.e.*, a single temporal horizon of about six years [44]. In this work, the entropy ability to quantify dynamics and heterogeneity will be exploited to estimate the weights of a *multi-period portfolio*. Such a construction is possible as the cluster entropy estimates involve horizon dependence. The values of the cluster entropy weights can be directly compared to the results obtained by using the traditional mean-variance approach and Sharpe ratio maximization on the same data shown in fig. 1 and fig. 2.

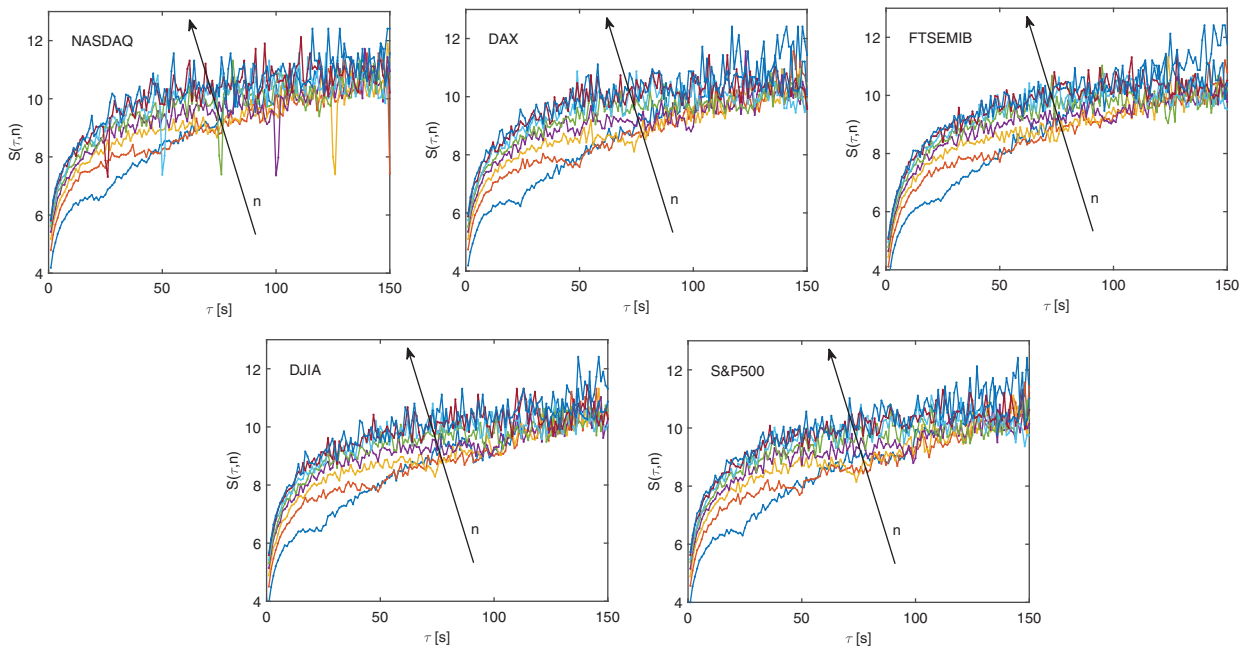


Fig. 3: Cluster entropy $S(\tau, n)$ calculated according to eq. (4) for the probability distribution function eq. (7) of the volatility series of the linear return of tick-by-tick data of the S&P500, NASDAQ, DJIA, DAX and FTSEMIB assets (further details in table 1). Time series have same length $N = 492023$. The volatility is calculated according to eq. (11) with window $T = 180$ s for all the five graphs. The plots refer to the horizon $\mathcal{M} = 12$, *i.e.*, twelve monthly periods sampled out of the year 2018. The different plots refer to different values of the moving average window n (here n ranges from 25 s to 200 s with step 25 s). Further results of the cluster entropy for a broad set of relevant parameters (volatility window T and temporal horizon \mathcal{M}) can be found in the figures included in the SM.

The construction of the *multi-period portfolio* is carried on by implementing the procedure as follows. The portfolio strategy is applied on five market time series. The datasets include tick-by-tick prices y_t from Jan. 1 to Dec. 31, 2018 downloaded from the Bloomberg terminal (further details are provided in table 1). Given the price series y_t and the related time series of the returns r_t , the volatility is defined as

$$\sigma_{t,T} = \sqrt{\frac{\sum_{t=k}^{k+T} (r_t - \mu_{t,T})^2}{T-1}}, \quad (11)$$

with the volatility window ranging from k and $k+T$ and $\mu_{t,T}$ the expected returns over the window T defined as

$$\mu_{t,T} = \frac{1}{T} \sum_{t=k}^{k+T} r_t. \quad (12)$$

The cluster entropy $S_i(\tau_j, n)$ of the return and volatility series is estimated by using eq. (7) into eq. (4) for different horizons \mathcal{M} . In particular, twelve monthly horizons out of one year have been considered. Results obtained on volatility series (assets described in table 1) are plotted in fig. 3. The behaviour is consistent with the expectations provided by eq. (9): a logarithmic trend is observed at small cluster lengths τ , whereas a linear increase appears at larger τ values.

The current approach takes a different perspective compared to the traditional portfolio strategy. The cluster entropy is straightaway estimated from the financial market data without using the hypothesis of Gaussian returns. To obtain the portfolio weights, the cluster entropy index $I_i(n)$ of the return and volatility of each single market i is estimated by using eq. (10). Then the average index I_i is calculated over the set of moving average values n :

$$I_i = \sum_n I_i(n). \quad (13)$$

The quantity I_i is a cumulative figure of diversity, a number suitable to quantify information content and compare different markets. For each market i and volatility windows T the index I_i calculated according to eq. (13) is normalized as follows:

$$w_{i,C} = \frac{I_i}{\sum_{i=1}^{\mathcal{N}_A} I_i}, \quad (14)$$

to satisfy the condition $\sum_{i=1}^{\mathcal{N}_A} w_{i,C} = 1$.

The quantities $w_{i,C}$ defined as in eq. (14) provide the portfolio probability of the riskiest assets for the case of high-risk propensity of the investor. Alternative estimates based on the cluster entropy index I might be easily carried on for low-risk profiles.

The weights $w_{i,C}$ are plotted in fig. 4 and fig. 5 for the five assets. At short horizons \mathcal{M} and small volatility windows T , the weights take values close to 0.2 as it would

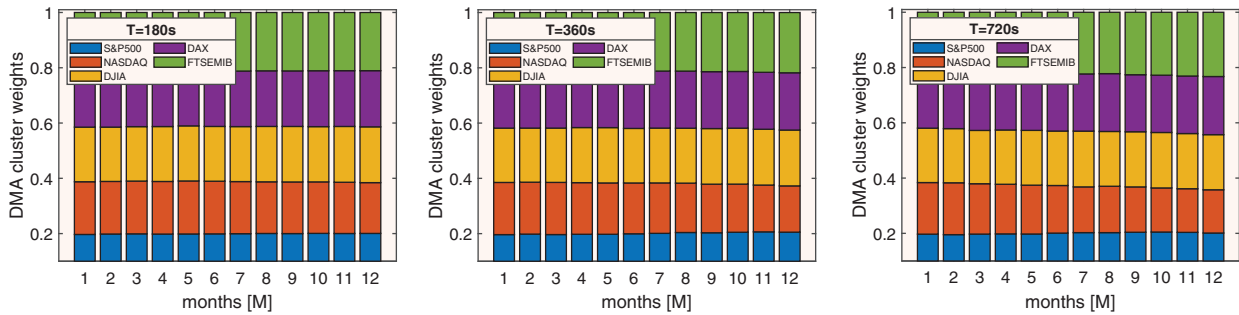


Fig. 4: Portfolio weights $w_{i,c}$ vs. investment horizon \mathcal{M} . The weights have been calculated according to the detrended cluster entropy approach eqs. (10), (14) by considering the case of high-risk propensity, *i.e.*, by maximizing variance/volatility. The cluster entropy refers to the volatility series estimated according to eq. (11) respectively with window $T = 180$ s, $T = 360$ s and $T = 720$ s. At small horizons \mathcal{M} and small volatility windows T the weights take values close to those expected according to the $1/\mathcal{N}_{\mathcal{A}}$ uniform distribution. This behaviour is related to the high unpredictability of the price and the associated risk (volatility) at short \mathcal{M} . As horizon \mathcal{M} and volatility window T increase, a less diversified set of values is obtained. The decreased level of diversity is consistent with the increased amount of information gathered along the series as time advances. (Further data of the portfolio weights can be found in the SM.)

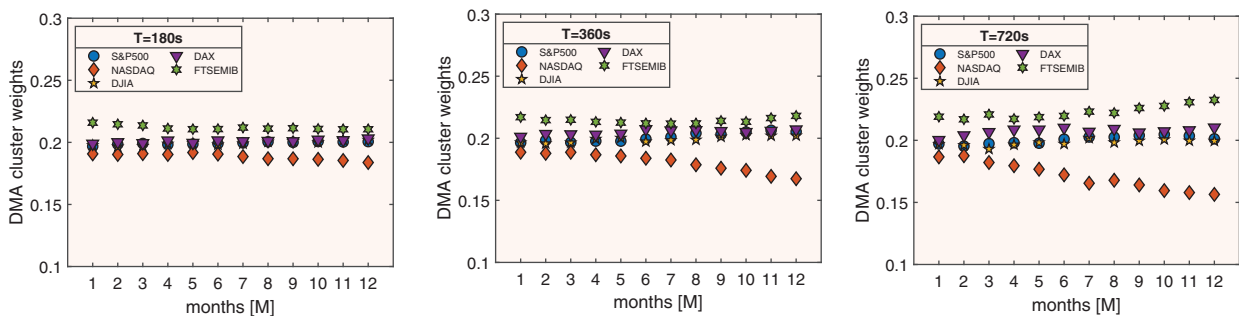


Fig. 5: The same as fig. 4 but with scattered plots.

be expected by a uniform wealth allocation. The weights distribution, with values close to $1/\mathcal{N}_{\mathcal{A}}$, is related to the low predictability degree of price series and associated risk (volatility) with the limited amount of data at short periods and volatility windows. As \mathcal{M} and T increase, a less diversified weights distribution emerges consistently with the increased amount of information gathered along the increased temporal horizon. Further results of the portfolio weights according to cluster entropy model can be found in the SM.

Conclusion. – A multi-period portfolio strategy is proposed based on the detrending moving average cluster entropy approach. An innovative, dynamic and robust perspective of the investment strategy is offered by the structure of the portfolio that importantly does not rely on the flawed assumptions of normal and stationary distribution of market data.

The strategy has been implemented on return and volatility of the high-frequency market data series (items in table 1) over multiple consecutive horizons \mathcal{M} . The high degree of stability, diversity and reliability of the portfolio weights can be indeed appreciated by the results shown in fig. 4 and in fig. 5. A continuous set of values of portfolio weights with a smooth and sound dependence

on the horizon can be observed. The entropy-based estimate of the portfolio is straightaway obtained from the stationary detrended distribution of the financial series rather than using the unrealistic mean-variance hypothesis of Gaussian returns. At short volatility windows (*e.g.*, $T = 180$ s in fig. 5), weights take values close to the equally distributed $1/\mathcal{N}_{\mathcal{A}}$ portfolio. These results are consistent with expected investment strategies where volatility (risk) does not give a prominent contribution. As T increases, the volatility plays a relevant role in the weights estimate which deviates from the uniform distribution.

The proposed approach uses a stationary set of variables (*i.e.*, the detrended cluster durations τ_j of the return and volatility series rather than non-stationary and non-normal variables as asset returns and volatility) [41–46]. The basic drawbacks of the traditional mean-variance approach are thus removed at their roots. Clustering methods have demonstrated the ability to obtain sound analysis of data with applications in various fields including portfolio strategies [51–57]. Our approach is based on the joint adoption of clustering and information measure. Several developments can be envisaged, as for example cluster entropy portfolio optimization based on the cross-correlation cluster distance measures and Kullback-Leibler entropy.

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