INSTITUT DE FRANCE
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Comptes Rendus
Mathématique

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Volume 358, issue 7 (2020), p. 817-825

Published online: 16 November 2020

https://doi.org/10.5802/crmath.88

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Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

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Theory of functions and function spaces / *Théorie des fonctions et espaces des fonctions*

A short proof of the infinitesimal Hilbertianity of the weighted Euclidean space

Une preuve courte de la Hilbertianité infinitésimale de l'espace euclidien à poids

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Abstract. We provide a quick proof of the following known result: the Sobolev space associated with the Euclidean space, endowed with the Euclidean distance and an arbitrary Radon measure, is Hilbert. Our new approach relies upon the properties of the Alberti–Marchese decomposability bundle. As a consequence of our arguments, we also prove that if the Sobolev norm is closable on compactly-supported smooth functions, then the reference measure is absolutely continuous with respect to the Lebesgue measure.

Résumé. Nous fournissons une preuve courte du résultat connu suivant: l'espace de Sobolev associé à l'espace euclidien muni de sa distance euclidienne et d'une mesure arbitraire de Radon, est un espace d'Hilbert. Notre nouvelle approche repose sur des propriétés du fibré de décomposabilité introduit par Alberti et Marchese. En conséquence de nos arguments, nous prouvons aussi que si la norme de Sobolev est fermable dans les fonctions lisses à support compact, la mesure de référence est absolument continue par rapport à la mesure de Lebesgue.

2020 Mathematics Subject Classification. 53C23, 46E35, 26B05.

Funding. The second and third named authors acknowledge the support by the Academy of Finland, projects 274372, 307333, 312488, and 314789. The first named author is a member of GNAMPA, INdAM.

Manuscript received 6th May 2020, revised 18th June 2020, accepted 19th June 2020.

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Introduction

In recent years, the theory of weakly differentiable functions over an abstract metric measure space $(\mathrm{X},\mathsf{d},\mu)$ has been extensively studied. Starting from the seminal paper [6], several (essentially equivalent) versions of Sobolev space $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ have been proposed in [4, 7, 13]. The definition we shall adopt in this paper is the one via test plans and weak upper gradients, which has been introduced by L. Ambrosio, N. Gigli, and G. Savaré in [4]. In general, $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ is a Banach space, but it might be non-Hilbert: for instance, consider the Euclidean space endowed with the ℓ^∞-norm and the Lebesgue measure (cf. [4, Remark 4.7]).

Those metric measure spaces whose associated Sobolev space is Hilbert (which are said to be *infinitesimally Hilbertian*, cf. [9]) play a very important role. We refer to the introduction of [11] for an account of the main advantages and features of this class of spaces.

The aim of this manuscript is to provide a quick proof of the following result (cf. Theorem 11):

$$(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)\ \text{ is infinitesimally Hilbertian for any Radon measure } \mu\geq 0 \text{ on } \mathbb{R}^d, \qquad (\star)$$

where $\mathsf{d}_{\mathrm{Eucl}}(x,y):=|x-y|$ stands for the Euclidean distance on $\mathbb{R}^d$. This fact has been originally proven in [10], but it can also be alternatively considered as a special case of the main result in [8].

The approach we propose here is more direct and is based upon a differentiability theorem for Lipschitz functions in $\mathbb{R}^d$ with respect to a given Radon measure μ, which was proved by G. Alberti and A. Marchese in [1] and says that it is possible to select the maximal measurable subbundle $V(\mu,\cdot)$ of $T\mathbb{R}^d$ (called the *decomposability bundle* of μ) along which all Lipschitz functions are μ-a.e. differentiable. In Section 2 we explain why the existence of $V(\mu,\cdot)$ yields $(\star)$.

Furthermore, in Section 3 we combine our techniques with a structural result for Radon measures in the Euclidean space by G. De Philippis and F. Rindler [12], to prove the following claim:

$$\text{The Sobolev norm } \|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)} \text{ is closable on } C^\infty_c\text{-functions} \quad\Longrightarrow\quad \mu\ll\mathscr{L}^d.$$

Cf. Definition 14 for the notion of closability we are referring to. This result solves a conjecture that has been posed by M. Fukushima (according to V.I. Bogachev [5, Section 2.6]), which was settled only for $d=1$, see [5, Example 2.6.3 (ii)].

Acknowledgements

The authors would like to thank the anonymous referee for the useful comments on this paper.

1. Preliminaries

1.1. *Sobolev calculus on metric measure spaces*

By *metric measure space* $(\mathrm{X},\mathsf{d},\mu)$ we mean a complete, separable metric space (X,d) together with a non-negative Radon measure $\mu\neq 0$.

We denote by $\mathrm{LIP}(\mathrm{X})$ the space of all real-valued Lipschitz functions on X, whereas $\mathrm{LIP}_c(\mathrm{X})$ stands for the family of all elements of $\mathrm{LIP}(\mathrm{X})$ having compact support. Given any $f\in\mathrm{LIP}(\mathrm{X})$, we shall denote by $\mathrm{lip}(f)\colon\mathrm{X}\to[0,+\infty)$ its *local Lipschitz constant*, which is defined as

$$\mathrm{lip}(f)(x):=\begin{cases}\overline{\lim}_{y\to x}\big|f(x)-f(y)\big|/\mathsf{d}(x,y) & \text{if } x\in\mathrm{X} \text{ is an accumulation point,}\\ 0 & \text{otherwise.}\end{cases}$$

The metric space (X,d) is said to be *proper* provided its bounded, closed subsets are compact.

To introduce the notion of Sobolev space $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ that has been proposed in [4], we first need to recall some terminology. The space $C\big([0,1],\mathrm{X}\big)$ of all continuous curves in X is a complete, separable metric space if endowed with the sup-distance $\mathsf{d}_\infty(\gamma,\sigma) := \max\big\{\mathsf{d}(\gamma_t,\sigma_t)\ \big|\ t \in [0,1]\big\}$. We say that $\gamma \in C\big([0,1],\mathrm{X}\big)$ is *absolutely continuous* provided there exists a function $g \in L^1(0,1)$ such that $\mathsf{d}(\gamma_s,\gamma_t) \le \int_s^t g(r)\,\mathrm{d}r$ holds for all $s,t \in [0,1]$ with $s < t$. The *metric speed* $|\dot\gamma|$ of γ, defined as $|\dot\gamma_t| := \lim_{h\to 0}\mathsf{d}(\gamma_{t+h},\gamma_t)/|h|$ for $\mathscr{L}^1$-a.e. $t \in [0,1]$, is the minimal integrable function (in the $\mathscr{L}^1$-a.e. sense) that can be chosen as g in the previous inequality; cf. [2, Theorem 1.1.2]. A *test plan* over $(\mathrm{X},\mathsf{d},\mu)$ is a Borel probability measure $\boldsymbol{\pi}$ on $C\big([0,1],\mathrm{X}\big)$, concentrated on absolutely continuous curves, such that the following properties are satisfied:

- *Bounded compression.* There exists $\mathrm{Comp}(\boldsymbol{\pi}) > 0$ such that $(\mathrm{e}_t)_*\boldsymbol{\pi} \le \mathrm{Comp}(\boldsymbol{\pi})\,\mu$ holds for all $t \in [0,1]$, where $\mathrm{e}_t\colon C\big([0,1],\mathrm{X}\big) \to \mathrm{X}$ stands for the evaluation map $\gamma \mapsto \mathrm{e}_t(\gamma) := \gamma_t$.
- *Finite kinetic energy.* It holds that $\int\int_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) < +\infty$.

Remark 1. We point out that every test plan $\boldsymbol{\pi}$ on $(\mathrm{X},\mathsf{d},\mu)$ is concentrated on curves that are contained in the support of μ. In order to verify this claim, we argue by contradiction: suppose there exists a Borel set $\Gamma \subseteq C\big([0,1],\mathrm{X}\big)$ such that $\boldsymbol{\pi}(\Gamma) > 0$ and $\gamma\big([0,1]\big) \nsubseteq \mathrm{spt}(\mu)$ for every $\gamma \in \Gamma$. Since the set $\mathrm{spt}(\mu)$ is closed and the curves $\gamma \in \Gamma$ are continuous, we deduce that for every $\gamma \in \Gamma$ there exists $t \in \mathbb{Q}\cap[0,1]$ such that $\gamma_t \notin \mathrm{spt}(\mu)$. This shows that $\Gamma \subseteq \bigcup_{t\in\mathbb{Q}\cap[0,1]} \mathrm{e}_t^{-1}\big(\mathrm{X}\setminus\mathrm{spt}(\mu)\big)$, thus there exists $t \in \mathbb{Q}\cap[0,1]$ such that $\mathrm{e}_t^{-1}\big(\mathrm{X}\setminus\mathrm{spt}(\mu)\big)$ has positive $\boldsymbol{\pi}$-measure. Therefore,

$$0 < (\mathrm{e}_t)_*\boldsymbol{\pi}\big(\mathrm{X}\setminus\mathrm{spt}(\mu)\big) \le \mathrm{Comp}(\boldsymbol{\pi})\,\mu\big(\mathrm{X}\setminus\mathrm{spt}(\mu)\big) = 0,$$

which leads to a contradiction. Consequently, the claim is proven.

Let $f\colon \mathrm{X} \to \mathbb{R}$ be a given Borel function. We say that $G \in L^2(\mu)$ is a *weak upper gradient* of f provided for any test plan $\boldsymbol{\pi}$ on $(\mathrm{X},\mathsf{d},\mu)$ it holds that $f\circ\gamma \in W^{1,1}(0,1)$ for $\boldsymbol{\pi}$-a.e. γ and that

$$\big|(f\circ\gamma)'_t\big| \le G(\gamma_t)\,|\dot\gamma_t| \quad \text{for } (\boldsymbol{\pi}\otimes\mathscr{L}^1)\text{-a.e. } (\gamma,t).$$

The minimal such function G (in the μ-a.e. sense) is called the *minimal weak upper gradient* of f and is denoted by $|Df| \in L^2(\mu)$.

Definition 2 (Sobolev space [4]). *The* Sobolev space $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ *is defined as the family of all those functions* $f \in L^2(\mu)$ *that admit a weak upper gradient* $G \in L^2(\mu)$. *We endow the vector space* $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ *with the Sobolev norm* $\|f\|^2_{W^{1,2}(\mathrm{X},\mathsf{d},\mu)} := \|f\|^2_{L^2(\mu)} + \big\||Df|\big\|^2_{L^2(\mu)}$.

The Sobolev space $\big(W^{1,2}(\mathrm{X},\mathsf{d},\mu), \|\cdot\|_{W^{1,2}(\mathrm{X},\mathsf{d},\mu)}\big)$ is a Banach space, but in general it is not a Hilbert space. This fact motivates the following definition, which has been proposed by N. Gigli:

Definition 3 (Infinitesimal Hilbertianity [9]). *We say that a metric measure space* $(\mathrm{X},\mathsf{d},\mu)$ *is* infinitesimally Hilbertian *provided its associated Sobolev space* $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ *is a Hilbert space.*

Let us define the *Cheeger energy* functional $\mathsf{E}_{\mathrm{Ch}}\colon L^2(\mu) \to [0,+\infty]$ as

$$\mathsf{E}_{\mathrm{Ch}}(f) := \begin{cases} \frac{1}{2}\int |Df|^2\,\mathrm{d}\mu & \text{if } f \in W^{1,2}(\mathrm{X},\mathsf{d},\mu),\\ +\infty & \text{otherwise.}\end{cases} \tag{1}$$

It holds that the metric measure space $(\mathrm{X},\mathsf{d},\mu)$ is infinitesimally Hilbertian if and only if E_{Ch} satisfies the *parallelogram rule* when restricted to $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$, i.e.,

$$\mathsf{E}_{\mathrm{Ch}}(f+g) + \mathsf{E}_{\mathrm{Ch}}(f-g) = 2\,\mathsf{E}_{\mathrm{Ch}}(f) + 2\,\mathsf{E}_{\mathrm{Ch}}(g) \quad \text{for every } f,g \in W^{1,2}(\mathrm{X},\mathsf{d},\mu). \tag{2}$$

Furthermore, we define the functional $\mathsf{E}_{\mathrm{lip}}\colon L^2(\mu) \to [0,+\infty]$ as

$$\mathsf{E}_{\mathrm{lip}}(f) := \begin{cases} \frac{1}{2}\int \mathrm{lip}^2(f)\,\mathrm{d}\mu & \text{if } f \in \mathrm{LIP}_c(\mathrm{X}),\\ +\infty & \text{otherwise.}\end{cases} \tag{3}$$

Given any $f \in \mathrm{LIP}_c(\mathrm{X})$, it holds that $f \in W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ and $|Df| \le \mathrm{lip}(f)$ in the μ-a.e. sense. This follows from the fact that for any absolutely continuous curve $\gamma\colon [0,1] \to \mathrm{X}$ the function $f\circ\gamma$ is absolutely continuous and that for $\mathscr{L}^1$-a.e. $t \in [0,1]$ one has

$$\big|(f\circ\gamma)'_t\big| = \lim_{h\to 0}\frac{\big|f(\gamma_{t+h})-f(\gamma_t)\big|}{|h|} = \lim_{h\to 0}\frac{\big|f(\gamma_{t+h})-f(\gamma_t)\big|}{\mathsf{d}(\gamma_{t+h},\gamma_t)}\,\frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{|h|} \le \mathrm{lip}(f)(\gamma_t)\,|\dot\gamma_t|,$$

whence $\mathrm{lip}(f)$ is a weak upper gradient of f (notice that $\mathrm{lip}(f)$ is in $L^2(\mu)$, as it is bounded and compactly-supported). Then it holds $\mathsf{E}_{\mathrm{Ch}} \le \mathsf{E}_{\mathrm{lip}}$. Actually, E_{Ch} is the $L^2(\mu)$-relaxation of $\mathsf{E}_{\mathrm{lip}}$:

Theorem 4 (Density in energy [3]). *Let $(\mathrm{X},\mathsf{d},\mu)$ be a metric measure space, with (X,d) proper. Then E_{Ch} is the $L^2(\mu)$-*lower semicontinuous envelope *of $\mathsf{E}_{\mathrm{lip}}$, i.e., it holds that*

$$\mathsf{E}_{\mathrm{Ch}}(f) = \inf \varliminf_{n\to\infty} \mathsf{E}_{\mathrm{lip}}(f_n) \quad \text{for every } f \in L^2(\mu),$$

where the infimum is taken among all sequences $(f_n)_n \subseteq L^2(\mu)$ such that $f_n \to f$ in $L^2(\mu)$.

1.2. *Decomposability bundle*

Let us denote by $\mathrm{Gr}(\mathbb{R}^d)$ the set of all linear subspaces of $\mathbb{R}^d$. Given any $V, W \in \mathrm{Gr}(\mathbb{R}^d)$, we define the distance $\mathsf{d}_{\mathrm{Gr}}(V,W)$ as the Hausdorff distance in $\mathbb{R}^d$ between the closed unit ball of V and that of W. Hence, $\big(\mathrm{Gr}(\mathbb{R}^d),\mathsf{d}_{\mathrm{Gr}}\big)$ is a compact metric space.

Theorem 5 (Decomposability bundle [1]). *Let $\mu \ge 0$ be a given Radon measure on $\mathbb{R}^d$. Then there exists a μ-a.e. unique Borel mapping $V(\mu,\cdot)\colon \mathbb{R}^d \to \mathrm{Gr}(\mathbb{R}^d)$, called the* decomposability bundle *of μ, such that the following properties hold:*

(i) *Any function $f \in \mathrm{LIP}(\mathbb{R}^d)$ is differentiable at μ-a.e. $x \in \mathbb{R}^d$ with respect to $V(\mu,x)$, i.e., there exists a Borel map $\nabla_{\mathrm{AM}} f\colon \mathbb{R}^d \to \mathbb{R}^d$ such that $\nabla_{\mathrm{AM}} f(x) \in V(\mu,x)$ for all $x \in \mathbb{R}^d$ and*

$$\lim_{V(\mu,x)\ni v\to 0}\frac{f(x+v)-f(x)-\nabla_{\mathrm{AM}} f(x)\cdot v}{|v|} = 0 \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{4}$$

(ii) *There exists a function $f_0 \in \mathrm{LIP}(\mathbb{R}^d)$ such that for μ-a.e. point $x \in \mathbb{R}^d$ it holds that f_0 is not differentiable at x with respect to any direction $v \in \mathbb{R}^d \setminus V(\mu,x)$.*

We refer to $\nabla_{\mathrm{AM}} f$ as the *Alberti–Marchese gradient* of f. It readily follows from (4) that $\nabla_{\mathrm{AM}} f$ is uniquely determined (up to μ-a.e. equality) and that for every $f, g \in \mathrm{LIP}(\mathbb{R}^d)$ it holds that

$$\nabla_{\mathrm{AM}}(f \pm g)(x) = \nabla_{\mathrm{AM}} f(x) \pm \nabla_{\mathrm{AM}} g(x) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{5}$$

Remark 6. Theorem 5 was actually proven under the additional assumption of μ being a finite measure. However, the statement depends only on the null sets of μ, not on the measure μ itself. Therefore, in order to obtain Theorem 5 as a consequence of the original result in [1], it is sufficient to replace μ with the following Borel probability measure on $\mathbb{R}^d$:

$$\widetilde{\mu} := \sum_{j=1}^{\infty}\frac{\mu|_{B_j(\bar x)}}{2^j\,\mu\big(B_j(\bar x)\big)}, \quad \text{for some } \bar x \in \mathrm{spt}(\mu).$$

Observe, indeed, that the measure $\widetilde{\mu}$ satisfies $\mu \ll \widetilde{\mu} \ll \mu$.

Remark 7. Given any function $f \in \mathrm{LIP}(\mathbb{R}^d)$, it holds that

$$\big|\nabla_{\mathrm{AM}} f(x)\big| \le \mathrm{lip}(f)(x) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{6}$$

Indeed, fix any point $x \in \mathbb{R}^d$ such that f is differentiable at x with respect to $V(\mu,x)$. Then for all $v \in V(\mu,x)\setminus\{0\}$ it holds that $\nabla_{\mathrm{AM}} f(x)\cdot v = |v| \lim_{h\searrow 0}\big(f(x+hv)-f(x)\big)/|hv| \le |v|\,\mathrm{lip}(f)(x)$ by (4), thus accordingly $\big|\nabla_{\mathrm{AM}} f(x)\big| = \sup\big\{\nabla_{\mathrm{AM}} f(x)\cdot v \,\big|\, v \in V(\mu,x),\, |v| \le 1\big\} \le \mathrm{lip}(f)(x)$.

2. Universal infinitesimal Hilbertianity of the Euclidean space

The objective of this section is to show that the Euclidean space is *universally infinitesimally Hilbertian*, meaning that it is infinitesimally Hilbertian when equipped with any Radon measure. Let us briefly outline the strategy of our argument, which is based upon the structure of the decomposability bundle described in Subsection 1.2. Given any Radon measure $\mu \geq 0$ on $\mathbb{R}^d$, consider the associated decomposability bundle $V(\mu,\cdot)$. Since the Alberti–Marchese gradient ∇_{AM} is a linear operator, its induced Dirichlet energy functional E_{AM} on $L^2(\mu)$ is a quadratic form. Hence, the proof of the infinitesimal Hilbertianity of $(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ follows along these lines:

(a) The maximality of $V(\mu,\cdot)$ ensures that the curves selected by a test plan $\boldsymbol{\pi}$ on $(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ are "tangent" to $V(\mu,\cdot)$, namely, $\dot\gamma_t \in V(\mu,\gamma_t)$ for $(\boldsymbol{\pi}\otimes\mathscr{L}^1)$-a.e. (γ,t). See Lemma 8.

(b) Given a compactly-supported Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$, we can deduce from item a) that the modulus of the gradient $\nabla_{\mathrm{AM}} f$ is a weak upper gradient of f; cf. Proposition 10.

(c) Since Lipschitz functions with compact support are dense in energy in $W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ (recall Theorem 4) we conclude from item b) that the Cheeger energy E_{Ch} is the lower semicontinuous envelope of E_{AM}. This grants that E_{Ch} is a quadratic form, thus accordingly the space $(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ is infinitesimally Hilbertian. See Theorem 11.

First of all, we prove that any given test plan over the weighted Euclidean space is "tangent" (in a suitable sense) to the Alberti–Marchese decomposability bundle:

Lemma 8. *Let $\mu \geq 0$ be a given Radon measure on $\mathbb{R}^d$. Let $\boldsymbol{\pi}$ be a test plan on $(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$. Then for $\boldsymbol{\pi}$-a.e. γ it holds that*

$$\dot\gamma_t \in V(\mu,\gamma_t) \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in[0,1].$$

Proof. Let f_0 be an L-Lipschitz function as in Theorem 5 (ii). Set $B \subseteq C\big([0,1],\mathbb{R}^d\big)\times[0,1]$ as

$$B := \big\{(\gamma,t)\;\big|\;\gamma \text{ and } f_0\circ\gamma \text{ are differentiable at } t, \text{ and } \dot\gamma_t \notin V(\mu,\gamma_t)\big\}.$$

It holds that the set B is Borel measurable; we postpone the verification of this fact to Remark 9. In order to prove the statement, it suffices to show that $(\boldsymbol{\pi}\otimes\mathscr{L}^1)(B)=0$. Indeed, by definition test plans are concentrated on absolutely continuous curves, thus in particular both γ and $f_0\circ\gamma$ are absolutely continuous (and accordingly $\mathscr{L}^1$-almost everywhere differentiable) for $\boldsymbol{\pi}$-a.e. γ.

Set $B_t := \big\{\gamma\;\big|\;(\gamma,t)\in B\big\}$ for every $t\in[0,1]$. Let G be the set of all $x\in\mathbb{R}^d$ such that f_0 is not differentiable at x with respect to any direction $v\in\mathbb{R}^d\setminus V(\mu,x)$. Thus, $\mu(\mathbb{R}^d\setminus G)=0$ by Theorem 5. We claim that the inclusion $\mathrm{e}_t(B_t)\subseteq\mathbb{R}^d\setminus G$ holds for every $t\in[0,1]$. Indeed, for every $\gamma\in B_t$ one has that

$$\begin{aligned}\left|\frac{f_0(\gamma_t+h\dot\gamma_t)-f_0(\gamma_t)}{h}-(f_0\circ\gamma)'_t\right| &\leq \left|\frac{f_0(\gamma_t+h\dot\gamma_t)-f_0(\gamma_{t+h})}{h}\right|+\left|\frac{f_0(\gamma_{t+h})-f_0(\gamma_t)}{h}-(f_0\circ\gamma)'_t\right|\\ &\leq L\left|\frac{\gamma_{t+h}-\gamma_t}{h}-\dot\gamma_t\right|+\left|\frac{f_0(\gamma_{t+h})-f_0(\gamma_t)}{h}-(f_0\circ\gamma)'_t\right|,\end{aligned}$$

so by letting $h\to 0$ we conclude that f_0 is differentiable at γ_t in the direction $\dot\gamma_t$, i.e., $\gamma_t\notin G$. Therefore, we conclude that $\boldsymbol{\pi}(B_t)\leq\boldsymbol{\pi}\big(\mathrm{e}_t^{-1}(\mathbb{R}^d\setminus G)\big)\leq\mathrm{Comp}(\boldsymbol{\pi})\,\mu(\mathbb{R}^d\setminus G)=0$ for all $t\in[0,1]$. This grants that $(\boldsymbol{\pi}\otimes\mathscr{L}^1)(B)=0$ by Fubini theorem, whence the statement follows. □

Remark 9 (Some measurability issues). Let us verify that the set B in the proof of Lemma 8 is Borel measurable. To do so, we first fix some notation: we denote by $\mathrm{e}\colon C\big([0,1],\mathbb{R}^d\big)\times[0,1]\to\mathbb{R}^d$ the evaluation map $\mathrm{e}(\gamma,t):=\gamma_t$. Observe that e, as well as $f_0\circ\mathrm{e}$ are continuous functions. Denote by D the set where both e and $f_0\circ\mathrm{e}$ are differentiable in the time direction, which is classical to see that is Borel. Next consider the following functions

$$g_h(\gamma,t) := \mathrm{dist}\bigg(V(\mu,\gamma_t),\frac{\gamma_{t+h}-\gamma_t}{h}\bigg);$$

every g_h is a Borel function since $V(\mu,\cdot)\circ \mathrm{e}$ is. But then also $g := \liminf_{h\to 0} g_h$ is a Borel function; notice now that $B = D\cap\{(\gamma,t) : g(\gamma,t)=0\}$, which is clearly a Borel set, as claimed.

As a consequence of Lemma 8, we can readily prove that the modulus of the Alberti–Marchese gradient of a given Lipschitz function is a weak upper gradient of the function itself:

Proposition 10. *Let $\mu\geq 0$ be a Radon measure on $\mathbb{R}^d$. Let $f\in \mathrm{LIP}_c(\mathbb{R}^d)$ be given. Then the function $|\nabla_{\mathrm{AM}} f|\in L^2(\mu)$ is a weak upper gradient of f.*

Proof. Let $\boldsymbol{\pi}$ be any test plan over $(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$. We claim that for $\boldsymbol{\pi}$-a.e. γ it holds

$$(f\circ\gamma)'_t = \nabla_{\mathrm{AM}} f(\gamma_t)\cdot\dot\gamma_t \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in[0,1]. \tag{7}$$

Indeed, for $(\boldsymbol{\pi}\otimes\mathscr{L}^1)$-a.e. (γ,t) we have that f is differentiable at γ_t with respect to $V(\mu,\gamma_t)$ and that $\dot\gamma_t\in V(\mu,\gamma_t)$; this stems from item i) of Theorem 5 and Lemma 8. Hence, (4) yields

$$\begin{aligned}\left|\nabla_{\mathrm{AM}} f(\gamma_t)\cdot\dot\gamma_t-(f\circ\gamma)'_t\right| &= \lim_{h\searrow 0}\left|\frac{f(\gamma_t+h\dot\gamma_t)-f(\gamma_t)}{h}-(f\circ\gamma)'_t\right|\\ &\leq \varlimsup_{h\searrow 0}\left|\frac{f(\gamma_t+h\dot\gamma_t)-f(\gamma_{t+h})}{h}\right|+\varlimsup_{h\searrow 0}\left|\frac{f(\gamma_{t+h})-f(\gamma_t)}{h}-(f\circ\gamma)'_t\right|\\ &\leq L\varlimsup_{h\searrow 0}\left|\frac{\gamma_{t+h}-\gamma_t}{h}-\dot\gamma_t\right|+\varlimsup_{h\searrow 0}\left|\frac{f(\gamma_{t+h})-f(\gamma_t)}{h}-(f\circ\gamma)'_t\right|=0,\end{aligned}$$

where we denoted by $L\geq 0$ the Lipschitz constant of f. This proves the claim (7). In particular, for $\boldsymbol{\pi}$-a.e. curve γ it holds

$$\left|(f\circ\gamma)'_t\right|\leq\left|\nabla_{\mathrm{AM}} f(\gamma_t)\right||\dot\gamma_t| \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in[0,1].$$

Given that $|\nabla_{\mathrm{AM}} f|\in L^2(\mu)$ by (6), we conclude that $|Df|\leq|\nabla_{\mathrm{AM}} f|$ holds in the μ-a.e. sense. □

We are now in a position to prove the universal infinitesimal Hilbertianity of the Euclidean space, as an immediate consequence of Proposition 10 and of the linearity of ∇_{AM}:

Theorem 11 (Infinitesimal Hilbertianity of weighted $\mathbb{R}^d$). *Let $\mu\geq 0$ be a Radon measure on $\mathbb{R}^d$. Then the metric measure space $(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ is infinitesimally Hilbertian.*

Proof. First of all, let us define the *Alberti–Marchese energy* functional $\mathsf{E}_{\mathrm{AM}}: L^2(\mu)\to[0,+\infty]$ as

$$\mathsf{E}_{\mathrm{AM}}(f) := \begin{cases}\frac{1}{2}\int|\nabla_{\mathrm{AM}} f|^2\,\mathrm{d}\mu & \text{if } f\in\mathrm{LIP}_c(\mathbb{R}^d),\\ +\infty & \text{otherwise.}\end{cases}$$

Since $|Df|\leq|\nabla_{\mathrm{AM}} f|\leq \mathrm{lip}(f)$ holds μ-a.e. for any $f\in\mathrm{LIP}_c(\mathbb{R}^d)$ by Proposition 10 and (6), we have that $\mathsf{E}_{\mathrm{Ch}}\leq\mathsf{E}_{\mathrm{AM}}\leq\mathsf{E}_{\mathrm{lip}}$, where E_{Ch} and $\mathsf{E}_{\mathrm{lip}}$ are defined as in (1) and (3), respectively. In view of Theorem 4, we deduce that E_{Ch} is the $L^2(\mu)$-lower semicontinuous envelope of E_{AM}. Thanks to the identities in (5), we also know that E_{AM} satisfies the parallelogram rule when restricted to $\mathrm{LIP}_c(\mathbb{R}^d)$, which means that

$$\mathsf{E}_{\mathrm{AM}}(f+g)+\mathsf{E}_{\mathrm{AM}}(f-g)=2\,\mathsf{E}_{\mathrm{AM}}(f)+2\,\mathsf{E}_{\mathrm{AM}}(g) \quad \text{for every } f,g\in\mathrm{LIP}_c(\mathbb{R}^d). \tag{8}$$

Fix $f,g\in W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$. Let us choose any two sequences $(f_n)_n,(g_n)_n\subseteq\mathrm{LIP}_c(\mathbb{R}^d)$ such that

- $f_n\to f$ and $g_n\to g$ in $L^2(\mu)$,
- $\mathsf{E}_{\mathrm{AM}}(f_n)\to\mathsf{E}_{\mathrm{Ch}}(f)$ and $\mathsf{E}_{\mathrm{AM}}(g_n)\to\mathsf{E}_{\mathrm{Ch}}(g)$.

In particular, observe that $f_n+g_n\to f+g$ and $f_n-g_n\to f-g$ in $L^2(\mu)$. Therefore, it holds that

$$\begin{aligned}\mathsf{E}_{\mathrm{Ch}}(f+g)+\mathsf{E}_{\mathrm{Ch}}(f-g)&\leq\varliminf_{n\to\infty}\big(\mathsf{E}_{\mathrm{AM}}(f_n+g_n)+\mathsf{E}_{\mathrm{AM}}(f_n-g_n)\big)\overset{(8)}{=}2\lim_{n\to\infty}\big(\mathsf{E}_{\mathrm{AM}}(f_n)+\mathsf{E}_{\mathrm{AM}}(g_n)\big)\\ &=2\,\mathsf{E}_{\mathrm{Ch}}(f)+2\,\mathsf{E}_{\mathrm{Ch}}(g).\end{aligned}$$

By replacing f and g with $f+g$ and $f-g$, respectively, we conclude that the converse inequality is verified as well. Consequently, the Cheeger energy E_{Ch} satisfies the parallelogram rule (2), thus $W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ is a Hilbert space. This completes the proof of the statement. □

Remark 12. As a byproduct of the proof of Theorem 11, we see that for all $f \in W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ there exists a sequence $(f_n)_n \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$ such that $f_n \to f$ and $|\nabla_{\mathrm{AM}} f_n| \to |Df|$ in $L^2(\mu)$.

Example 13. Given an arbitrary Radon measure μ on $\mathbb{R}^d$, it might happen that

$$|Df| \neq |\nabla_{\mathrm{AM}} f| \quad \text{for some } f \in \mathrm{LIP}_c(\mathbb{R}^d).$$

For instance, consider the measure $\mu := \mathscr{L}^1|_C$ on $\mathbb{R}$, where $C \subseteq \mathbb{R}$ is any Cantor set of positive Lebesgue measure. Since the support of μ is totally disconnected, we know from Remark 1 that all test plans on $(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ are concentrated on constant curves, thus every $f \in L^2(\mu)$ admits the null function as a weak upper gradient. This means that $W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu) = L^2(\mu)$ and $|Df| = 0$ μ-a.e. for all $f \in L^2(\mu)$. However, it holds $V(\mu,x) = \mathbb{R}$ for $\mathscr{L}^1$-a.e. $x \in C$ by Rademacher theorem, whence for any $f \in \mathrm{LIP}(\mathbb{R})$ we have $\nabla_{\mathrm{AM}} f(x) = f'(x)$ for $\mathscr{L}^1$-a.e. $x \in C$.

3. Closability of the Sobolev norm on smooth functions

The aim of this conclusive section is to address a problem that has been raised by M. Fukushima (as reported in [5, Section 2.6]). Namely, we provide a (negative) answer to the following question: *Does there exist a singular Radon measure μ on $\mathbb{R}^2$ for which the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^2,\mathsf{d}_{\mathrm{Eucl}},\mu)}$ is closable on compactly-supported smooth functions (in the sense of Definition 14 below)?*

Actually, we are going to prove a stronger result: *Given any Radon measure μ on $\mathbb{R}^d$ that is not absolutely continuous with respect to $\mathscr{L}^d$, it holds that $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)}$ is not closable on compactly-supported smooth functions.* Cf. Theorem 19 below.

Let $f \in C_c^\infty(\mathbb{R}^d)$ be given. Then we denote by $\nabla f\colon \mathbb{R}^d \to \mathbb{R}^d$ its classical gradient. Note that the identity $|\nabla f| = \mathrm{lip}(f)$ holds. Given a Radon measure μ on $\mathbb{R}^d$, it is immediate to check that

$$\nabla_{\mathrm{AM}} f(x) = \pi_x\big(\nabla f(x)\big) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d, \tag{9}$$

where $\pi_x\colon \mathbb{R}^d \to V(\mu,x)$ stands for the orthogonal projection map. We denote by $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ the space of all (equivalence classes, up to μ-a.e. equality, of) Borel maps $v\colon \mathbb{R}^d \to \mathbb{R}^d$ with $|v| \in L^2(\mu)$. It holds that $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ is a Hilbert space if endowed with the norm $v \mapsto \big(\int |v|^2\,\mathrm{d}\mu\big)^{1/2}$.

Definition 14 (Closability of the Sobolev norm on smooth functions). *Let μ be a Radon measure on $\mathbb{R}^d$. Then the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)}$ is* closable on compactly-supported smooth functions *provided the following property is verified: if a sequence $(f_n)_n \subseteq C_c^\infty(\mathbb{R}^d)$ satisfies $f_n \to 0$ in $L^2(\mu)$ and $\nabla f_n \to v$ in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ for some element $v \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, then it holds that $v = 0$.*

Remark 15. We point out that in Definition 14 the requirement $\nabla f_n \to v$ can be replaced by the weak convergence $\nabla f_n \rightharpoonup v$. Indeed, suppose to have a sequence $(f_n)_n \subseteq C_c^\infty(\mathbb{R}^d)$ and an element $v \in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ such that $f_n \to 0$ in $L^2(\mu)$ and $\nabla f_n \rightharpoonup v$ weakly in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$. By virtue of Banach–Saks theorem, there is a subsequence $(n_k)_k$ for which the functions $\widetilde{f}_k := \frac{1}{k}\sum_{i=1}^k f_{n_i}$ satisfy $\nabla \widetilde{f}_k \to v$ strongly in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$. Since we also have that $\widetilde{f}_k \to 0$ in $L^2(\mu)$, we deduce from Definition 14 that $v = 0$.

In order to provide some alternative characterisations of the above-defined closability property, we need to recall the following improvement of Theorem 4 in the weighted Euclidean space case:

Theorem 16 (Density in energy of smooth functions (Lemma 2.9 in [10])). *Let μ be a Radon measure on $\mathbb{R}^d$. Then $\mathsf{E}_{\rm Ch}$ is the $L^2(\mu)$-lower semicontinuous envelope of $\mathsf{E}_{C_c^\infty}\colon L^2(\mu)\to[0,+\infty]$,*

$$\mathsf{E}_{C_c^\infty}(f):=\begin{cases}\frac{1}{2}\int|\nabla f|^2\,\mathrm{d}\mu & \text{if } f\in C_c^\infty(\mathbb{R}^d),\\ +\infty & \text{otherwise,}\end{cases}$$

i.e., it holds that

$$\mathsf{E}_{\rm Ch}(f)=\inf\varliminf_{n\to\infty}\mathsf{E}_{C_c^\infty}(f_n)\quad \text{for every } f\in L^2(\mu),$$

where the infimum is taken among all sequences $(f_n)_n\subseteq L^2(\mu)$ such that $f_n\to f$ in $L^2(\mu)$.

Lemma 17. *Let μ be a Radon measure on $\mathbb{R}^d$. Then the following conditions are equivalent:*

(i) *The Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\rm Eucl},\mu)}$ is closable on compactly-supported smooth functions.*
(ii) *The functional $\mathsf{E}_{\rm lip}$ (see (3)) is $L^2(\mu)$-lower semicontinuous when restricted to $C_c^\infty(\mathbb{R}^d)$.*
(iii) *The identity $|Df|=|\nabla f|$ holds μ-a.e. on $\mathbb{R}^d$, for every function $f\in C_c^\infty(\mathbb{R}^d)$.*

Proof. (i) $\Longrightarrow$ (ii)**.** Fix any $f\in C_c^\infty(\mathbb{R}^d)$ and $(f_n)_n\subseteq C_c^\infty(\mathbb{R}^d)$ such that $f_n\to f$ in $L^2(\mu)$. We claim that

$$\int|\nabla f|^2\,\mathrm{d}\mu\leq\varliminf_{n\to\infty}\int|\nabla f_n|^2\,\mathrm{d}\mu. \tag{10}$$

Without loss of generality, we may assume the right-hand side in (10) is finite. Therefore, we can find a subsequence $(f_{n_k})_k$ of $(f_n)_n$ and an element $v\in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ such that $\lim_k\int|\nabla f_{n_k}|^2\,\mathrm{d}\mu=\varliminf_n\int|\nabla f_n|^2\,\mathrm{d}\mu$ and $\nabla f_{n_k}\rightharpoonup v$ in the weak topology of $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$. Since $f_{n_k}-f\to 0$ in $L^2(\mu)$ and $\nabla(f_{n_k}-f)\rightharpoonup v-\nabla f$ weakly in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, we deduce from item (i) that $v=\nabla f$ (here Remark 15 plays a role). Consequently, we have that $\nabla f_n\rightharpoonup\nabla f$ in the weak topology of $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, thus proving (10) by semicontinuity of the norm. In other words, it holds that $\mathsf{E}_{\rm lip}(f)\leq\varliminf_n\mathsf{E}_{\rm lip}(f_n)$, which yields the validity of item (ii).

(ii) $\Longrightarrow$ (iii)**.** Let $f\in C_c^\infty(\mathbb{R}^d)$ be given. Theorem 16 yields existence of a sequence $(f_n)_n\subseteq C_c^\infty(\mathbb{R}^d)$ such that $f_n\to f$ and $|\nabla f_n|\to|Df|$ in $L^2(\mu)$. Therefore, item (ii) ensures that

$$\frac{1}{2}\int|\nabla f|^2\,\mathrm{d}\mu=\mathsf{E}_{\rm lip}(f)\leq\varliminf_{n\to\infty}\mathsf{E}_{\rm lip}(f_n)=\lim_{n\to\infty}\frac{1}{2}\int|\nabla f_n|^2\,\mathrm{d}\mu=\frac{1}{2}\int|Df|^2\,\mathrm{d}\mu.$$

Since $|Df|\leq|\nabla f|$ holds μ-a.e. on $\mathbb{R}^d$, we conclude that $|Df|=|\nabla f|$ μ-a.e., thus proving item (iii).

(iii) $\Longrightarrow$ (i)**.** We argue by contradiction: suppose that there exists a sequence $(f_n)_n\subseteq C_c^\infty(\mathbb{R}^d)$ such that $f_n\to 0$ in $L^2(\mu)$ and $\nabla f_n\to v$ in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ for some $v\in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)\setminus\{0\}$. Fix any $k\in\mathbb{N}$ such that $\|\nabla f_k-v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}\leq\frac{1}{3}\|v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}$. In particular, $\|\nabla f_k\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}\geq\frac{2}{3}\|v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}$. Let us define $g_n:=f_k-f_n\in C_c^\infty(\mathbb{R}^d)$ for every $n\in\mathbb{N}$. Since $g_n\to f_k$ in $L^2(\mu)$ and $\nabla g_n\to\nabla f_k-v$ in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ as $n\to\infty$, we conclude that

$$\|\nabla f_k\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}\geq\frac{2}{3}\|v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}>\frac{1}{3}\|v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}\geq\|\nabla f_k-v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}=\lim_{n\to\infty}\|\nabla g_n\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)},$$

whence $\mathsf{E}_{\rm lip}(f_k)>\lim_n\mathsf{E}_{\rm lip}(g_n)$. This contradicts the lower semicontinuity of $\mathsf{E}_{\rm lip}$ on $C_c^\infty(\mathbb{R}^d)$. Consequently, item (i) is proven. □

The last ingredient we need is the following result proven by G. De Philippis and F. Rindler:

Theorem 18 (Weak converse of Rademacher theorem [12]). *Let μ be a Radon measure on $\mathbb{R}^d$. Suppose all Lipschitz functions $f\colon\mathbb{R}^d\to\mathbb{R}$ are μ-a.e. differentiable. Then it holds that $\mu\ll\mathscr{L}^d$.*

We are finally in a position to prove the following statement concerning closability:

Theorem 19 (Failure of closability for singular measures). *Let $\mu\geq 0$ be a given Radon measure on $\mathbb{R}^d$. Suppose that μ is not absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^d$. Then the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\rm Eucl},\mu)}$ is not closable on compactly-supported smooth functions.*

Proof. First of all, Theorem 18 grants the existence of a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and a Borel set $P \subseteq \mathbb{R}^d$ such that $\mu(P) > 0$ and f is not differentiable at any point of P. Recalling Theorem 5, we see that $V(\mu,x) \neq \mathbb{R}^d$ for μ-a.e. $x \in P$, so that for μ-a.e. $x \in P$ there exists a vector $v \in \mathbb{Q}^d$ such that $v \notin V(\mu,x)$. Defining $P_v := \{x \in P : v \notin V(\mu,x)\}$ for every $v \in \mathbb{Q}^d$, we thus have that P is contained (up to μ-negligible sets) in the countable union $\bigcup_{v\in\mathbb{Q}^d} P_v$, thus necessarily there exists $v \in \mathbb{Q}^d$ such that $\mu(P_v) > 0$. By inner regularity of μ, we can find a compact set $K \subseteq P_v$ such that $\mu(K) > 0$. Observe that $v \notin V(\mu,x)$ for every $x \in K$. Now pick any $g \in C_c^\infty(\mathbb{R}^d)$ such that $\nabla g(x) = v$ holds for all $x \in K$. Then Proposition 10 and (9) yield

$$|Dg|(x) \leq |\nabla_{\mathrm{AM}}\, g|(x) = \left|\pi_x\big(\nabla g(x)\big)\right| = \left|\pi_x(v)\right| < |v| = |\nabla g|(x) \quad \text{for } \mu\text{-a.e. } x \in K,$$

thus accordingly $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)}$ is not closable on compactly-supported smooth functions by Lemma 17. Hence, the statement is achieved. □

Remark 20. The converse of Theorem 19 might fail. In fact in the real line [5, Theorem 2.6.4] gives a complete characterization for closability of $\|\cdot\|_{W^{1,2}(\mathbb{R},\mathsf{d}_{\mathrm{Eucl}},\mu)}$, and it requires also other conditions on μ apart from being absolutely continuous with respect to the Lebesgue measure.

For instance, the measure μ described in Example 13 is absolutely continuous with respect to $\mathscr{L}^1$, but the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R},\mathsf{d}_{\mathrm{Eucl}},\mu)}$ is not closable on compactly-supported smooth functions as a consequence of Lemma 17.

Remark 21. All the results in Section 3, in particular Theorem 19 can be easily generalized to the Sobolev norms $\|\cdot\|_{W^{1,p}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)}$ for any $1 < p < \infty$, but we stick with the case $p = 2$ for ease of notation.

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