ASYMPTOTIC BEHAVIOR OF CONSTRAINED LOCAL MINIMIZERS IN FINITE ELASTICITY

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Abstract. We provide an approximation result for the pure traction problem of linearized elasticity in terms of local minimizers of finite elasticity, under the constraint of vanishing average curl for admissible deformation maps. When suitable rotations are included in the constraint, the limit is shown to be the linear elastic equilibrium associated to rotated loads.

1. Introduction

We consider the pure traction problem of finite elasticity and its relation with the linear elastic problem. If $\Omega \subset \mathbb{R}^3$ is the reference configuration of an elastic body and $\mathbf{y} : \Omega \to \mathbb{R}^3$ is the deformation field, we introduce the global energy

$$\mathcal{G}(\mathbf{y}; \mathcal{L}) := \int_\Omega \mathcal{W}(\mathbf{x}, \nabla \mathbf{y})\, d\mathbf{x} - \mathcal{L}(\mathbf{y} - \mathbf{i}), \tag{1.1}$$

where $\mathbf{i}$ denotes the identity map on Ω and and $\mathcal{W} : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty]$ is the strain energy density. Here, $\mathcal{L}$ is the load functional whose typical form is

$$\mathcal{L}(\mathbf{u}) := \int_\Omega \mathbf{f} \cdot \mathbf{u}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{u}\, dS, \tag{1.2}$$

where $\mathbf{f} \colon \Omega \to \mathbb{R}^3$ is a body force field, $\mathbf{g} \colon \partial\Omega \to \mathbb{R}^3$ is a surface force field, $\mathbf{u} \colon \Omega \to \mathbb{R}^3$ is the displacement field and dS is the surface measure. For every $\mathbf{x} \in \Omega$, the function $\mathcal{W}(\mathbf{x}, \cdot)$ is assumed to be frame indifferent, uniquely minimized at rotations with value 0, C^2-smooth and quadratically growing out of rotations; in addition, $\mathcal{W}$ shall satisfy the natural condition

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = +\infty \quad \text{if} \quad \det \mathbf{F} \le 0.$$

If $h > 0$ is an adimensional parameter, we shall consider the natural rescaling $h^{-2}\mathcal{G}(\cdot; h\mathcal{L})$. We also introduce the associated linear elastic energy

$$\mathcal{F}_0(\mathbf{u}; \mathcal{L}) := \frac{1}{2} \int_\Omega \mathbb{E}(\mathbf{u})\, D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\, \mathbb{E}(\mathbf{u})\, d\mathbf{x} - \mathcal{L}(\mathbf{u}),$$

where $\mathbb{E}(\mathbf{u}) := \mathrm{sym}\nabla \mathbf{u}$ is the infinitesimal strain tensor and $\mathbf{I}$ is the identity matrix. This standard expression is formally obtained by linearization around the identity, writing the deformation field as $\mathbf{i} + h\mathbf{u}$, considering the rescaled energies $h^{-2}\mathcal{G}(\mathbf{i} + h\mathbf{u}; h\mathcal{L})$ and performing a Taylor expansion for small h.

2010 *Mathematics Subject Classification.* 49J45, 74K30, 74K35, 74R10.

Key words and phrases. Calculus of Variations, Linear Elasticity, Finite Elasticity, Traction Problem, Gamma-convergence, Equilibrium Equations, Constrained Minimizers.

Recently, several contributions [11, 15, 16, 19, 20] have analyzed the variational limit of the rescaled functionals $h^{-2}\mathcal{G}(\cdot; h\mathcal{L})$ as $h \to 0$. For this purpose, it is necessary to assume that $\mathcal{L}$ is equilibrated (i.e., with null resultant and null momentum) and that

$$\mathcal{L}(\mathbf{R}\mathbf{x} - \mathbf{x}) \leq 0 \qquad \forall\, \mathbf{R} \in SO(3), \tag{1.3}$$

where $SO(3)$ denotes the special orthogonal group, thus preventing the rescaled energy being driven to $-\infty$ by rigid motions as h becomes small. In contrast to the case of the Dirichlet problem [1, 8, 11, 17, 22], global minimizers (or quasi-minimizers) of $h^{-2}\mathcal{G}(\cdot; h\mathcal{L})$ over $H^1(\Omega, \mathbb{R}^3)$ do not necessarily converge to minimizers of the associated linearized elastic energy $\mathcal{F}_0(\cdot; \mathcal{L})$, as might be expected from classical continuum mechanics literature, so it seems appropriate to shift the attention to the asymptotic behavior of suitably constrained minimizers in order to recover a minimizer of $\mathcal{F}_0(\cdot; \mathcal{L})$ in the limit. Indeed, the results in [11, 19, 20] show that, without further assumptions on the external loads, such as the absence of axes of equilibrium, there holds

$$\inf_{H^1(\Omega,\mathbb{R}^3)} h^{-2}\mathcal{G}(\cdot; h\mathcal{L}) \to \min_{H^1(\Omega,\mathbb{R}^3)} \mathcal{F}_0(\cdot; \mathbf{R}_*\mathcal{L}) \qquad \text{as } h \to 0$$

for some suitable $\mathbf{R}_* \in SO(3)$, and possibly

$$\min_{H^1(\Omega,\mathbb{R}^3)} \mathcal{F}_0(\cdot; \mathbf{R}_*\mathcal{L}) < \min_{H^1(\Omega,\mathbb{R}^3)} \mathcal{F}_0(\cdot; \mathcal{L}).$$

Before introducing constrained approximations, let us mention that an alternative standard way for rigorously obtaining linearized elasticity is to consider the equilibrium problems associated to small loads. Equilibrium configurations of the (rescaled) traction problem for the equilibrated loads $(\mathbf{f}, \mathbf{g})$ are given by the deformations $\mathbf{y} \colon \Omega \to \mathbb{R}^3$ which solve

$$\begin{cases} -\mathrm{div}\,(D\mathcal{W}(\mathbf{x}, \nabla\mathbf{y})) = h\mathbf{f}, & \text{in } \Omega, \\ \quad D\mathcal{W}(\mathbf{x}, \nabla\mathbf{y})\mathbf{n} = h\mathbf{g}, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\mathbf{n}$ denotes the exterior unit normal vector to Ω. Under suitable conditions on $\mathbf{f}$ and $\mathbf{g}$, classical results based on the implicit function theorem, described for instance in [6, 25, 26], allow to prove that there exists a solution $\mathbf{y}_h$ to the above problem in a small neighborhood of the identity and that $\mathbf{u}_h := h^{-1}(\mathbf{y}_h - \mathbf{i}) \to \mathbf{u}_0$ in a suitable Sobolev space over Ω, where $\mathbf{u}_0$ solves the linearized problem

$$\begin{cases} -\mathrm{div}\,(D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\mathbb{E}(\mathbf{u}_0)) = \mathbf{f} & \text{in } \Omega, \\ \quad D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\mathbb{E}(\mathbf{u}_0)\mathbf{n} = \mathbf{g}, & \text{on } \partial\Omega, \end{cases}$$

that is, $\mathbf{u}_0$ is a minimizer of the linearized elastic energy $\mathcal{F}_0(\cdot; \mathcal{L})$. A drawback of this scheme is that it works only if the external loads satisfy the following integrability condition

$$\mathbf{f} \in L^p(\Omega, \mathbb{R}^3), \quad \mathbf{g} \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3) \quad \text{for some } \ p > 3 \tag{1.5}$$

along with further nondegeneracy constraints (the simplest one being again the absence of axes of equilibrium) that we shall discuss in Section 3 along with a precise statement and some literature review.

In this paper we introduce an approximation for linearized elasticity in terms of suitably constrained nonlinear problems. The first step towards this goal will be a suitable definition of local minimizer of functionals (1.1), which takes into account that the expected limit functional $\mathcal{F}_0(\cdot;\mathcal{L})$ can be restricted, without loss of generality, to those displacements $\mathbf{u}$ such that

$$\int_\Omega \operatorname{curl}\mathbf{u} = \mathbf{0},$$

provided that $\mathcal{L}$ is equilibrated. Indeed, if $\mathbf{u} \in H^1(\Omega,\mathbb{R}^3)$ and $\mathbf{w} := |\Omega|^{-1}\int_\Omega \operatorname{curl}\mathbf{u}$, it is immediate to check that $\mathcal{F}_0(\mathbf{u}-\frac{1}{2}\mathbf{w}\wedge\mathbf{x};\mathcal{L}) = \mathcal{F}_0(\mathbf{u};\mathcal{L})$, where $\wedge$ denotes the cross product. In this perspective we shall define constrained local minimizers $\mathbf{y}$ of $\mathcal{G}(\cdot;\mathcal{L})$, the constraint being $\int_\Omega \operatorname{curl}\mathbf{y} = \mathbf{0}$, by requiring that $\mathcal{G}(\mathbf{y}) \le \mathcal{G}(\mathbf{y}+\varepsilon\boldsymbol{\psi})$ for every $\boldsymbol{\psi}$ such that $\int_\Omega \operatorname{curl}\boldsymbol{\psi} = \mathbf{0}$ and for any small enough ε (we refer to Section 2 for the rigorous definition). Accordingly, in our main result (see Theorem 2.5 in Section 2), under additional assumptions on $\mathcal{W}$, we show that if external loads are equilibrated then there exist constrained local minimizers $\mathbf{y}_h \in H^1(\Omega,\mathbb{R}^3)$ for $\mathcal{G}(\cdot;h\mathcal{L})$ such that by letting $\mathbf{u}_h := h^{-1}(\mathbf{y}_h - \mathbf{i})$ there holds

$$\mathbb{E}(\mathbf{u}_h) \rightharpoonup \mathbb{E}(\mathbf{u}_0) \quad \text{weakly in} \quad L^2(\Omega,\mathbb{R}^{3\times 3}),$$

$$h^{-2}\mathcal{G}(\mathbf{y}_h;h\mathcal{L}) \to \mathcal{F}_0(\mathbf{u}_0;\mathcal{L})$$

as $h \to 0$, where $\mathbf{u}_0$ minimizes $\mathcal{F}_0(\cdot;\mathcal{L})$ over $H^1(\Omega,\mathbb{R}^3)$.

Finally, by strengthening the hypotheses on external loads, namely by also assuming (1.3), an almost immediate consequence of the main result is that, given $\mathbf{R} \in SO(3)$ belonging to the rotation kernel

$$\mathcal{S}^0_{\mathcal{L}} := \{\mathbf{R} \in SO(3) \colon \mathcal{L}((\mathbf{R}-\mathbf{I})\mathbf{x}) = 0\}, \tag{1.6}$$

for any small enough h there exists a constrained local minimizer $\mathbf{y}_h$ for $\mathcal{G}(\cdot,h\mathcal{L})$ (this time the constraint for admissible deformations $\mathbf{y}$ being $\int_\Omega \operatorname{curl}\mathbf{R}^T\mathbf{y} = \mathbf{0}$), and

$$\lim_{h\to\infty} h^{-2}\mathcal{G}(\mathbf{y}_h;h\mathcal{L}) = \min_{\in H^1(\Omega,\mathbb{R}^3)} \mathcal{F}_0(\cdot;\mathbf{R}^T\mathcal{L}).$$

This, in a nutshell, is what might be considered the essence of the approximation of minimizers of linear elasticity with constrained local minimizers of finite elasticity: when condition (1.3) is satisfied then close to any $\mathbf{R} \in \mathcal{S}^0_{\mathcal{L}}$ there exists a sequence of constrained local minimizers of functionals (1.1) such that the corresponding energies converge to the energy of the linearly elastic problem where $\mathcal{L}$ is replaced by $\mathbf{R}^T\mathcal{L}$. This further clarifies that global minimizers of (1.1) are not a good choice in order to approximate minimizers of the linear elastic energy $\mathcal{F}_0(\cdot;\mathcal{L})$.

Plan of the paper. In section 2 we introduce all the assumptions of the theory and rigorously state the main results, which are proved in Section 4. In section 3 we give a brief comparison between our approach and the classical results about the asymptotic behavior of equilibrium states of traction problems via implicit function theorem. In Section 5 we revisit some examples of [19] by applying our convergence results of constrained local minimizers.

2. Main results

In this section we introduce the basic notations and assumptions, then we state the main results. In the following, Ω is a bounded open connected Lipschitz subset of $\mathbb{R}^3$, representing the reference configuration of the body. $\mathbb{R}^{3\times 3}$ is the set of real 3×3 matrices. $\mathbb{R}^{3\times 3}_{\rm sym}$ (resp.

$\mathbb{R}^{3\times3}_{\mathrm{skew}}$) is the set of symmetric (resp. skew-symmetric) matrices. $\mathbb{R}^{3\times3}_{+}$ denotes the set of matrices with positive determinant.

Assumptions on the elastic energy density. We let $\mathcal{W} : \Omega \times \mathbb{R}^{3\times3} \to [0,+\infty]$ be a Carathéodory function. We will consider the following assumptions.

$$(\boldsymbol{\mathcal{W}}\mathbf{1}) \qquad \mathcal{W}(\mathbf{x},\mathbf{R}\mathbf{F}) = \mathcal{W}(\mathbf{x},\mathbf{F}) \quad \forall\,\mathbf{R}\in SO(3) \quad \forall\,\mathbf{F}\in\mathbb{R}^{3\times3}, \qquad \text{for a.e. } \mathbf{x}\in\Omega,$$

$$(\boldsymbol{\mathcal{W}}\mathbf{2}) \qquad \min\mathcal{W} = \mathcal{W}(\mathbf{x},\mathbf{I}) = 0 \quad \text{for a.e. } \mathbf{x}\in\Omega.$$

Moreover, we shall consider the following regularity property: there exist an open neighborhood $\mathcal{U}$ of $SO(3)$ in $\mathbb{R}^{3\times3}$, an increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\lim_{t\to0^+}\omega(t) = 0$ and a constant $K > 0$ such that for a.e. $\mathbf{x}\in\Omega$

$$(\boldsymbol{\mathcal{W}}\mathbf{3}) \qquad \begin{aligned} &\mathcal{W}(\mathbf{x},\cdot)\in C^2(\mathcal{U}), \quad |D^2\mathcal{W}(\mathbf{x},\mathbf{I})|\le K \ \text{ and} \\ &|D^2\mathcal{W}(\mathbf{x},\mathbf{F}) - D^2\mathcal{W}(\mathbf{x},\mathbf{G})| \le \omega(|\mathbf{F}-\mathbf{G}|) \quad \forall\,\mathbf{F},\mathbf{G}\in\mathcal{U}. \end{aligned}$$

We introduce a growth conditions from below: there exist $C > 0$ such that for a.e. $\mathbf{x}\in\Omega$

$$(\boldsymbol{\mathcal{W}}\mathbf{4}) \qquad \mathcal{W}(\mathbf{x},\mathbf{F}) \ge C\,d(\mathbf{F},SO(3))^2 \qquad \forall\,\mathbf{F}\in\mathbb{R}^{3\times3},$$

where $\mathrm{dist}(\mathbf{F},SO(3)) := \inf\{|\mathbf{F}-\mathbf{R}| : \mathbf{R}\in SO(3)\}$ and $|\mathbf{F}|^2 := \mathrm{Tr}(\mathbf{F}^T\mathbf{F})$. A second growth condition from below that we shall consider is the following: there exist $C' > 0$, $s \ge 2$, $q \ge \frac{s}{s-1}$ and $r > 1$ such that for a.e. $\mathbf{x}\in\Omega$

$$(\boldsymbol{\mathcal{W}}\mathbf{5}) \qquad \mathcal{W}(\mathbf{x},\mathbf{F}) \ge C'(|\mathbf{F}|^s + |\mathrm{cof}\,\mathbf{F}|^q + (\det\mathbf{F})^r - 1) \qquad \forall\,\mathbf{F}\in\mathbb{R}^{3\times3}$$

Finally, we introduce a polyconvexity condition: for a.e. $\mathbf{x}\in\Omega$

$$(\boldsymbol{\mathcal{W}}\mathbf{6}) \qquad \begin{aligned} &\text{the map } \mathbb{R}^{3\times3}\ni\mathbf{F}\mapsto\mathcal{W}(\mathbf{x},\mathbf{F}) \text{ is polyconvex and} \\ &\mathcal{W}(\mathbf{x},\mathbf{F}) = +\infty \text{ if } \det\mathbf{F}\le0, \qquad \lim_{\det\mathbf{F}\to0^+}\mathcal{W}(\mathbf{x},\mathbf{F}) = +\infty. \end{aligned}$$

Model energy densities. We present here two instances of energy densities $\mathcal{W}$ which satisfy the above assumptions and for which the main result of the present paper (see Theorem 2.5 below) applies. For simplicity, we consider the homogeneous case. Incompressible materials are usually modeled by isochoric-type energies $\mathcal{W}_{\mathrm{iso}}$ defined on the set of matrices with unitary determinant $\{\mathbf{F}\in\mathbb{R}^{3\times3} \colon \det\mathbf{F} = 1\}$ and one can pass to the corresponding compressible model by letting

$$\mathcal{W}(\mathbf{F}) := \begin{cases} \mathcal{W}_{\mathrm{iso}}\left(\dfrac{\mathbf{F}}{(\det\mathbf{F})^{1/3}}\right) + \mathcal{W}_{\mathrm{vol}}(\mathbf{F}) & \text{if } \det\mathbf{F} > 0 \\ +\infty & \text{if } \det\mathbf{F}\le0, \end{cases} \tag{2.1}$$

where $\mathcal{W}_{\mathrm{vol}}(\mathbf{F}) = g(\det\mathbf{F})$ for some convex $g\colon\mathbb{R}_+\to\mathbb{R}$ of class C^2 in a neighborhood of 1 and such that

$$g(t)\ge0 \text{ for all } t>0, \quad g(t)=0 \text{ if and only if } t=1, \quad g''(1)>0, \quad \lim_{t\to0^+} g(t) = +\infty. \tag{2.2}$$

In addition, the function g is required to grow faster than linearly at infinity, i.e.

$$g(t)\ge C''t^r, \quad \text{for } t>0 \text{ sufficiently large and for some } C''>0 \text{ and } r>1. \tag{2.3}$$

In view of the isochoric-volumetric decomposition (2.1), a model energy density is identified by the choice of $\mathcal{W}_{\mathrm{iso}}$, while $\mathcal{W}_{\mathrm{vol}}$ is left in a general form satisfying the above restrictions.

The first example is an energy of Yeoh type, which is defined by choosing

$$\mathcal{W}_{\rm iso}(\mathbf{F}) := \sum_{k=1}^{3} c_k(|\mathbf{F}|^2 - 3)^k \tag{2.4}$$

with coefficients $c_k > 0$. It is easy to check that with this choice the energy density satisfies all the assumptions from ($\boldsymbol{\mathcal{W}1}$) to ($\boldsymbol{\mathcal{W}6}$), provided that $r \geq 2$ in (2.3). Indeed, the validity of ($\boldsymbol{\mathcal{W}1}$), ($\boldsymbol{\mathcal{W}2}$), ($\boldsymbol{\mathcal{W}3}$) is trivial. The validity of ($\boldsymbol{\mathcal{W}6}$) follows from the polyconvexity of the map $\mathbb{R}^{3\times3}_+ \ni \mathbf{F} \mapsto \frac{|\mathbf{F}|^2}{(\det \mathbf{F})^{2/3}}$. The inequality in ($\boldsymbol{\mathcal{W}4}$) is satisfied if $\operatorname{dist}(\mathbf{F}, SO(3))$ is small enough, as shown in [1, Remark 2.8], and therefore ($\boldsymbol{\mathcal{W}4}$) directly follows from the following claim: there are positive constants a_1, a_2 such that $\mathcal{W}(\mathbf{F}) \geq a_1|\mathbf{F}|^3 - a_2$ for every $\mathbf{F} \in \mathbb{R}^{3\times3}$. In order to prove such a claim, it is clear that we can reduce to consider the regime $\operatorname{dist}(\mathbf{F}, SO(3)) \gg 1$ (i.e., $|\mathbf{F}| \gg 1$). By (2.3) there is $t_0 > 1$ such that $g(t) \geq C'' t^r$ for $t \geq t_0$ and the claim is obvious for $\det \mathbf{F} < t_0$ so that we may assume $\det \mathbf{F} \geq t_0$. If $|\mathbf{F}| \leq (\det \mathbf{F})^{2/3}$ we have $g(\det \mathbf{F}) \geq C''(\det \mathbf{F})^r \geq C''|\mathbf{F}|^{3r/2}$ and the claim follows since $r \geq 2$; else if $|\mathbf{F}| \geq (\det \mathbf{F})^{2/3}$ then $|\mathbf{F}|^2/(\det \mathbf{F})^{2/3}$ is large and Young inequality entails the existence of suitable positive constants c_3', c_3'' such that

$$\mathcal{W}(\mathbf{F}) \geq c_3 \left(\frac{|\mathbf{F}|^2}{(\det \mathbf{F})^{2/3}} - 3 \right)^3 + g(\det \mathbf{F}) \geq c_3' \left(\frac{|\mathbf{F}|^6}{(\det \mathbf{F})^2} + (\det \mathbf{F})^r \right) \geq c_3'' |\mathbf{F}|^{\frac{6r}{2+r}}.$$

The claim is proved since $r \geq 2$, and it implies ($\boldsymbol{\mathcal{W}5}$) by means of the elementary inequality $|\operatorname{cof} \mathbf{F}| \leq 2|\mathbf{F}|^2$.

A second example of energy density for hyperelastic materials is given by Ogden type energies, identified by the following choice

$$\mathcal{W}_{\rm iso}(\mathbf{F}) := \sum_{i=1}^{n} c_i \left(\operatorname{Tr}(\mathbf{F}^T\mathbf{F})^{\gamma_i/2} - 3 \right) + \sum_{j=1}^{m} d_k \left(\operatorname{Tr}(\operatorname{cof}\mathbf{F}^T \operatorname{cof}\mathbf{F})^{\delta_j/2} - 3 \right),$$

where n, m are nonnegative integers and $c_i, \gamma_i, d_j, \delta_j \geq 0$. While hypotheses ($\boldsymbol{\mathcal{W}1}$),($\boldsymbol{\mathcal{W}2}$) and ($\boldsymbol{\mathcal{W}3}$) are again easily seen to be fulfilled by this model, some care must be taken concerning the remaining assumptions. In particular, one can prove that ($\boldsymbol{\mathcal{W}5}$) and ($\boldsymbol{\mathcal{W}6}$) hold true, after suitably restricting the values of the exponents γ_i and δ_j. More precisely, in order to obtain polyconvexity of the function $\mathcal{W}$ (i.e. assumption ($\boldsymbol{\mathcal{W}6}$)), one shall assume that

$$\gamma_i \geq \frac{3}{2} \quad \text{for all } i = 1, \dots, n \quad \text{and} \quad \delta_j \geq 3 \quad \text{for all } j = 1, \dots, m.$$

Moreover, if we let $\gamma := \max\{\gamma_i \colon i = 1, \dots, n\}$ and $\delta := \max\{\delta_j \colon j = 1, \dots, m\}$, we have that ($\boldsymbol{\mathcal{W}5}$) holds with $r > 1$ as in (2.3) together with

$$s := \frac{3\gamma r}{3r + \gamma} \quad \text{and} \quad q := \frac{3\delta r}{2\delta + 3r},$$

provided $s \geq 2$ and $q \geq s/(s-1)$. We point out that $\delta \geq 3$ and $r > 1$ imply $q > 1$. For instance, ($\boldsymbol{\mathcal{W}5}$) is satisfied if $\gamma \geq 5/2$ and $r \geq 4$ The proof of ($\boldsymbol{\mathcal{W}5}$) and ($\boldsymbol{\mathcal{W}6}$) can be found in [5, Proposition 6, Proposition 7]. Finally, for the proof of ($\boldsymbol{\mathcal{W}4}$) one can see [1, Remark 2.8] and no further restrictions on γ_i and δ_j are needed.

Energy functionals. Let $\mathcal{W}$ satisfy assumptions ($\boldsymbol{\mathcal{W}}\mathbf{1}$), ($\boldsymbol{\mathcal{W}}\mathbf{2}$), ($\boldsymbol{\mathcal{W}}\mathbf{3}$) and ($\boldsymbol{\mathcal{W}}\mathbf{4}$). We denote by $(H^1(\Omega,\mathbb{R}^3))^*$ the dual of the Sobolev space $H^1(\Omega,\mathbb{R}^3)$. Given $\mathcal{L} \in (H^1(\Omega,\mathbb{R}^3))^*$, we introduce the following energy functionals. We let $\mathcal{G}(\cdot;\mathcal{L}) : H^1(\Omega,\mathbb{R}^3) \to (-\infty,+\infty)$ be defined by

$$\mathcal{G}(\mathbf{y};\mathcal{L}) := \int_\Omega \mathcal{W}(\mathbf{x},\nabla\mathbf{y}(\mathbf{x}))\,d\mathbf{x} - \mathcal{L}(\mathbf{y}-\mathbf{i}), \tag{2.5}$$

and for $\mathbf{u} \in H^1(\Omega,\mathbb{R}^3)$ we let

$$\mathcal{F}(\mathbf{u};\mathcal{L}) := \int_\Omega \mathcal{W}(\mathbf{x},\mathbf{I}+\nabla\mathbf{u}(\mathbf{x}))\,d\mathbf{x} - \mathcal{L}(\mathbf{u}) = \mathcal{G}(\mathbf{i}+\mathbf{u};\mathcal{L}).$$

We define $\mathcal{F}_0 : H^1(\Omega,\mathbb{R}^3) \to (-\infty,+\infty)$ as

$$\mathcal{F}_0(\mathbf{u};\mathcal{L}) := \frac{1}{2}\int_\Omega \mathbb{E}(\mathbf{u})\,D^2\mathcal{W}(x,\mathbf{I})\,\mathbb{E}(\mathbf{u})\,d\mathbf{x} - \mathcal{L}(\mathbf{u}) \tag{2.6}$$

If $h > 0$ we also define the rescaled functionals

$$\mathcal{F}_h(\mathbf{u};\mathcal{L}) := h^{-2}\mathcal{F}(h\mathbf{u};h\mathcal{L}). \tag{2.7}$$

We refer to $\mathcal{L}$ as the *load functional*. Thanks to the Sobolev embedding $H^1(\Omega,\mathbb{R}^3) \hookrightarrow L^6(\Omega,\mathbb{R}^3)$, it can be always written as

$$\mathcal{L}(\mathbf{u}) = \int_\Omega (\mathbf{f}_*\cdot\mathbf{u} + \mathbf{G}_* : \nabla\mathbf{u})\,d\mathbf{x}$$

for suitable $\mathbf{f}_* \in L^{\frac{6}{5}}(\Omega,\mathbb{R}^3)$, $\mathbf{G}_* \in L^2(\Omega,\mathbb{R}^{3\times3})$. We assume that $\mathcal{L}$ is equilibrated, i.e.,

$$\mathcal{L}(\mathbf{c}) = \mathcal{L}(\mathbf{W}\mathbf{x}) = 0 \qquad \text{for every } \mathbf{c} \in \mathbb{R}^3 \text{ and every } \mathbf{W} \in \mathbb{R}^{3\times3}_{\rm skew}. \tag{2.8}$$

We denote with $\|\mathcal{L}\|_*$ its norm in the space $(H^1(\Omega,\mathbb{R}^3))^*$. By (2.8) and by Korn and Poincaré inequalities, there exists a constant $K_\Omega > 0$ such that

$$\mathcal{L}(\mathbf{u}) \le K_\Omega \|\mathcal{L}\|_*\,\|\mathbb{E}\mathbf{u}\|_{L^2(\Omega,\mathbb{R}^{3\times3})}, \tag{2.9}$$

for every $\mathbf{u} \in H^1(\Omega,\mathbb{R}^3)$.

Concerning the elastic energy, the following rigidity inequality is crucial, see [1, 9, 10]: there exists a constant $\tilde{C} = \tilde{C}_\Omega$ such that for every $\mathbf{y} \in H^1(\Omega)$ there is $\mathbf{R} \in SO(3)$ (depending on $\mathbf{y}$) such that

$$\int_\Omega |\nabla\mathbf{y}(\mathbf{x}) - \mathbf{R}|^2\,d\mathbf{x} \le \tilde{C}_\Omega \int_\Omega \mathrm{dist}(\nabla\mathbf{y}(\mathbf{x}),SO(3))^2\,d\mathbf{x}.$$

For every $\mathbf{y} \in H^1(\Omega,\mathbb{R}^3)$, we combine the latter with ($\boldsymbol{\mathcal{W}}\mathbf{4}$) and obtain the existence of $\mathbf{R} \in SO(3)$ (depending on $\mathbf{y}$) such that

$$\int_\Omega |\nabla\mathbf{y}(\mathbf{x}) - \mathbf{R}|^2\,d\mathbf{x} \le C_\Omega \int_\Omega \mathcal{W}(\mathbf{x},\nabla\mathbf{y}(\mathbf{x}))\,d\mathbf{x}, \tag{2.10}$$

where $C_\Omega := \tilde{C}_\Omega/C$ and C is the constant in ($\boldsymbol{\mathcal{W}}\mathbf{4}$).

Main results. We state the main result, showing that there exist suitably constrained local minimizers of the rescaled functionals (2.7) that converge to a global minimizer of the linear elastic energy (4.4) as $h \to 0$. We need the precise notion local minimizer under the constraint of vanishing average curl. In the next definition we make use of the following notation: for $\mathbf{R} \in SO(3)$, we let

$$H^1_{\mathbf{R}}(\Omega, \mathbb{R}^3) := \left\{ \mathbf{u} \in H^1(\Omega, \mathbb{R}^3) : \int_\Omega \operatorname{curl} \mathbf{R}^T \mathbf{u} = \mathbf{0} \right\}.$$

In particular, $H^1_{\mathbf{I}}(\Omega, \mathbb{R}^3)$ is the linear subspace of $H^1(\Omega, \mathbb{R}^3)$ made of vector fields with vanishing average curl.

Definition 2.1 (constrained local minimizer). *Let $\mathcal{W}$ satisfy assumptions ($\boldsymbol{\mathcal{W}}$**1**), ($\boldsymbol{\mathcal{W}}$**2**), ($\boldsymbol{\mathcal{W}}$**3**) and ($\boldsymbol{\mathcal{W}}$**4**) and let $\mathbf{R} \in SO(3)$. We say that $\mathbf{y} \in H^1_{\mathbf{R}}(\Omega, \mathbb{R}^3)$ is a* local minimizer for $\mathcal{G}(\cdot; \mathcal{L})$ over $H^1_{\mathbf{R}}(\Omega, \mathbb{R}^3)$ *if for every $\boldsymbol{\psi} \in H^1_{\mathbf{R}}(\Omega, \mathbb{R}^3)$ there exists $\varepsilon_0 = \varepsilon_0(\mathbf{y}, \boldsymbol{\psi}) > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ there holds $\mathcal{G}(\mathbf{y}; \mathcal{L}) \le \mathcal{G}(\mathbf{y} + \varepsilon \boldsymbol{\psi}; \mathcal{L})$. We say that $\mathbf{u} \in H^1_{\mathbf{R}}(\Omega, \mathbb{R}^3)$ is a* local minimizer for $\mathcal{F}(\cdot; \mathcal{L})$ over $H^1_{\mathbf{R}}(\Omega, \mathbb{R}^3)$ *if $\mathbf{y}(\mathbf{x}) = \mathbf{R}\mathbf{x} + \mathbf{u}$ is a local minimizer for $\mathcal{G}(\cdot; \mathcal{L})$ over $H^1_{\mathbf{R}}(\Omega, \mathbb{R}^3)$.*

Remark 2.2. If $\mathbf{y}(\mathbf{x}) = \mathbf{R}\mathbf{x} + \mathbf{u}$ we have $\operatorname{curl} \mathbf{R}^T \mathbf{y} = \operatorname{curl} \mathbf{R}^T \mathbf{u}$. In the distinguished case $\mathbf{R} = \mathbf{I}$ we have $\operatorname{curl} \mathbf{y} = \operatorname{curl} \mathbf{u}$ and in particular $\mathbf{i} + \mathbf{u} \in H^1_{\mathbf{I}}(\Omega, \mathbb{R}^3)$ if and only if $\mathbf{u} \in H^1_{\mathbf{I}}(\Omega, \mathbb{R}^3)$.

Remark 2.3. Given $h > 0$, $\mathcal{F}_h$ is defined by (2.7). By rescaling ε_0 we notice that $\mathbf{u} \in H^1_{\mathbf{I}}(\Omega, \mathbb{R}^3)$ is a local minimizer for $\mathcal{F}_h(\cdot; \mathcal{L})$ over $H^1_{\mathbf{I}}(\Omega, \mathbb{R}^3)$ if and only if $\mathbf{v} := h\mathbf{u}$ is a local minimizer for $\mathcal{F}(\cdot; h\mathcal{L})$ over $H^1_{\mathbf{I}}(\Omega, \mathbb{R}^3)$.

Remark 2.4. By applying the classical results of [2], if $\mathcal{W}$ satisfies ($\boldsymbol{\mathcal{W}}$**1**), ($\boldsymbol{\mathcal{W}}$**2**), ($\boldsymbol{\mathcal{W}}$**3**), ($\boldsymbol{\mathcal{W}}$**4**), ($\boldsymbol{\mathcal{W}}$**5**), ($\boldsymbol{\mathcal{W}}$**6**) and if $\mathcal{L} \in (H^1(\Omega, \mathbb{R}^3))^*$ satisfies (2.8), then a global minimizer of $\mathcal{F}(\cdot; \mathcal{L})$ over $H^1(\Omega, \mathbb{R}^3)$ does exist. Nevertheless, it has been recently shown in [19] that following a sequence of global minimizers for the rescaled functionals $\mathcal{F}_h(\cdot; \mathcal{L})$, the limit energy as $h \to 0$ is not necessarily equal to $\min_{H^1(\Omega, \mathbb{R}^3)} \mathcal{F}_0(\cdot; \mathcal{L})$. Indeed, in a pure traction problem, this limit can be strictly lower than the minimal value of $\mathcal{F}_0(\cdot; \mathcal{L})$ over $H^1(\Omega, \mathbb{R}^3)$. This motivates the analysis of constrained local minimizers for obtaining the linear elastic energy as limit of rescaled finite elasticity energies.

We are in a position to state our main result

Theorem 2.5. *Suppose that ($\boldsymbol{\mathcal{W}}$**1**), ($\boldsymbol{\mathcal{W}}$**2**), ($\boldsymbol{\mathcal{W}}$**3**), ($\boldsymbol{\mathcal{W}}$**4**), ($\boldsymbol{\mathcal{W}}$**5**), ($\boldsymbol{\mathcal{W}}$**6**) hold true. Let $\mathcal{L} \in (H^1(\Omega, \mathbb{R}^3))^*$ satisfy (2.8). Then there exist a vanishing sequence $(h_j)_{j \in \mathbb{N}} \subset (0, 1)$ and a sequence $(\mathbf{u}_j)_{j \in \mathbb{N}} \subset H^1_{\mathbf{I}}(\Omega, \mathbb{R}^3)$ such that $\mathbf{u}_j$ is a local minimizer for $\mathcal{F}_{h_j}(\cdot; \mathcal{L})$ over $H^1_{\mathbf{I}}(\Omega, \mathbb{R}^3)$ for any $j \in \mathbb{N}$ according to* Definition 2.1*, and moreover*

$$\mathbb{E}(\mathbf{u}_j) \rightharpoonup \mathbb{E}(\mathbf{u}_*) \quad \textit{weakly in} \quad L^2(\Omega, \mathbb{R}^{3\times 3}) \ \textit{as}\ j \to \infty,$$

$$\lim_{j \to \infty} \mathcal{F}_{h_j}(\mathbf{u}_j; \mathcal{L}) = \mathcal{F}_0(\mathbf{u}_*; \mathcal{L}),$$

where $\mathbf{u}_*$ is a minimizer for $\mathcal{F}_0(\cdot; \mathcal{L})$ over $H^1(\Omega, \mathbb{R}^3)$.

The above Theorem 2.5 expresses the fact that the minimizer of the linear elastic energy $\mathcal{F}_0(\cdot; \mathcal{L})$ (which is unique up to infinitesimal rigid displacements) gets approximated by a sequence of constrained local minimizers of rescaled finite elasticity functionals. We stress that the vanishing average curl constraint disappears in the limit problem, since the stored linear elastic energy and the load functional are invariant by infinitesimal rigid displacements, due to (2.8). Since only local minimizers are involved, the statements requires (2.8) and no further assumptions on the external loads. In particular, assumption (1.3), appearing in [19] for the analysis of the asymptotic behavior of global minimizers (or quasi-minimizers) of rescaled finite elasticity functionals, is not required. Similarly, assumptions about axes of equilibrium of external loads (see Section 3) are not required as well.

However, if besides (2.8) we additionally assume (1.3), we can draw some interesting consequences of Theorem 2.5. Under assumption 1.3, it is not difficult to see that the *rotation kernel* of $\mathcal{L}$, defined by (1.6), is a subgroup of $SO(3)$, see [17, Remark 2.2]. $\mathcal{S}^0_{\mathcal{L}}$ contains the identity matrix and it is possibly reduced to it. Moreover, if $\mathbf{R} \in \mathcal{S}^0_{\mathcal{L}}$, then for every $\mathbf{S} \in SO(3)$ we get

$$\mathbf{R}^T\mathcal{L}((\mathbf{S}-\mathbf{I})\mathbf{x}) = \mathcal{L}((\mathbf{RS}-\mathbf{I})\mathbf{x}) - \mathcal{L}((\mathbf{R}-\mathbf{I})\mathbf{x}) = \mathcal{L}((\mathbf{RS}-\mathbf{I})\mathbf{x}) \le 0,$$

that is, $\mathbf{R}^T\mathcal{L}$ still satisfies (1.3) (hence it also satisfies(2.8), see [17, Remark 2.1]) for every $\mathbf{R} \in \mathcal{S}^0_{\mathcal{L}}$. Therefore the map $\beta : \mathcal{S}^0_{\mathcal{L}} \to \mathbb{R}$

$$\beta(\mathbf{R}) := \min\{\mathcal{F}_0(\mathbf{u}; \mathbf{R}^T\mathcal{L}) : \mathbf{u} \in H^1(\Omega, \mathbb{R}^3)\} \tag{2.11}$$

is well defined and continuous. We notice that the energy identity satisfied by solutions to the minimization problem (2.11) implies that $\beta(\mathbf{R}) = -\frac{1}{2}\mathcal{B}(\mathbf{R}^T\mathcal{L}, \mathbf{R}^T\mathcal{L})$, where $\mathcal{B}(\cdot,\cdot)$ is the Betti form associated to couples of equilibrated loads, defined by

$$\mathcal{B}(\mathcal{L}_1, \mathcal{L}_2) := \int_\Omega \mathbb{E}\mathbf{u}_1\, D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\, \mathbb{E}\mathbf{u}_2\, d\mathbf{x}.$$

Here, $\mathbf{u}_i$ are solutions the the linear elastic problem with external loads $\mathcal{L}_i$, $i = 1, 2$. The degeneracy properties of the Betti form are relevant in the analysis of the nonlinear elastic equations for pure traction problems, see [4]. A nondegeneracy condition on β also appears in the following results which are straightforward consequences of Theorem 2.5.

Corollary 2.6. *Assume that* ($\boldsymbol{\mathcal{W}}$**1**)*,* ($\boldsymbol{\mathcal{W}}$**2**)*,* ($\boldsymbol{\mathcal{W}}$**3**)*,* ($\boldsymbol{\mathcal{W}}$**4**)*,* ($\boldsymbol{\mathcal{W}}$**5**)*,* ($\boldsymbol{\mathcal{W}}$**6**) *hold true and that* $\mathcal{L} \in (H^1(\Omega, \mathbb{R}^3))^*$ *satisfies* (2.8) *and* (1.3) *Let* $\mathbf{R} \in \mathcal{S}^0_{\mathcal{L}}$*. Then there exist a vanishing sequence* $(h_j)_{j\in\mathbb{N}} \subset (0, 1)$ *and a sequence* $(\mathbf{y}_j)_{j\in\mathbb{N}} \subset H^1_{\mathbf{R}}(\Omega, \mathbb{R}^3)$ *such that* $\mathbf{y}_j$ *is a local minimizer for* $\mathcal{G}(\cdot, h_j\mathcal{L})$ *over* $H^1_{\mathbf{R}}(\Omega : \mathbb{R}^3)$ *for every* $j \in \mathbb{N}$ *according to* Definition 2.1 *and*

$$\lim_{j\to\infty} h_j^{-2}\mathcal{G}(\mathbf{y}_j, h_j\mathcal{L}) = \beta(\mathbf{R}).$$

Corollary 2.7. *In the same assumptions of* Corollary 2.6*, suppose that* $\mathcal{S}^0_{\mathcal{L}} \setminus \{\mathbf{I}\} \neq \emptyset$ *and that* β *from* (2.11) *is not a constant map. Then for every* $n \in \mathbb{N}$ *there exist* $\{\mathbf{R}_k : k = 1, 2...n\} \subset \mathcal{S}^0_{\mathcal{L}}$ *and* $h_0 = h_0(n) > 0$ *such that for every* $h \in (0, h_0)$ *the functional* $\mathcal{G}(\cdot\,; h\mathcal{L})$ *has a local minimizer* $\mathbf{y}_k$ *over* $H^1_{\mathbf{R}_k}(\Omega, \mathbb{R}^3)$ *for every* $k = 1, 2...n$ *(according to* Definition 2.1*) and* $\mathcal{G}(\mathbf{y}_k; h\mathcal{L}) \neq \mathcal{G}(\mathbf{y}_{k'}; h\mathcal{L})$ *if* $k \neq k'$*.*

The last corollary shows that the nonlinear elastic energy might have several constrained local minimizers (each corresponding to a different constraint) at different energy levels. The condition requiring that β is not a constant map is verified for instance in the example from [19, Theorem 2.7], which we will further discuss in Section 5.

3. The classical approach via implicit function theorem

In this section we briefly review some known results concerning existence and asymptotic behavior of equilibrium configurations of hyperelastic bodies, that make use of a perturbative method which relies on the implicit function theorem. This is done for the sake of a comparison with the assumptions of our result in Theorem 2.5. Indeed, the classical results that we are going to describe hereafter require stronger summability and nondegeneracy assumptions on external loads, as well as more regularity of $\mathcal{W}$ (in turn, they do not need polyconvexity).

Throughout this section, we assume $\Omega \subset \mathbb{R}^3$ to be of class C^2, and besides ($\boldsymbol{\mathcal{W}}\mathbf{1}$) and ($\boldsymbol{\mathcal{W}}\mathbf{2}$) we shall consider the following additional regularity hypothesis on the strain energy density

($\boldsymbol{\mathcal{W}}\mathbf{7}$) $\quad \mathcal{W} \in C^3(\overline{\Omega} \times \mathcal{U})$ for some $\mathcal{U} \subseteq \mathbb{R}^{3\times 3}$ open neighbourhood of $SO(3)$,

along with the coercivity condition

($\boldsymbol{\mathcal{W}}\mathbf{8}$) there exists $C_* > 0$ s.t. $\mathbf{F}^T D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\mathbf{F} \geq C_*|\mathbf{F}|^2$ for all $\mathbf{F} \in \mathbb{R}^{3\times 3}$ and all $\mathbf{x} \in \Omega$.

Given a couple of external (dead) loads $\mathbf{f}\colon \Omega \to \mathbb{R}^3$ and $\mathbf{g}\colon \partial\Omega \to \mathbb{R}^3$ acting on the elastic body Ω as forces of body and surface type, respectively, equilibrium configurations are given by the deformations $\mathbf{y}\colon \Omega \to \mathbb{R}^3$ which solve

$$\begin{cases} -\mathrm{div}\,(D\mathcal{W}(\mathbf{x}, \nabla\mathbf{y})) = \mathbf{f}, & \text{in } \Omega, \\ D\mathcal{W}(\mathbf{x}, \nabla\mathbf{y})\mathbf{n} = \mathbf{g}, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

In view of the above assumptions on $\mathcal{W}$, it is trivial to observe that, if there are no external forces, i.e. $\mathbf{f} = \mathbf{g} = \mathbf{0}$, then the identity map $\mathbf{i}$ solves (3.1), as well as the map $\mathbf{y}(\mathbf{x}) = \mathbf{R}\mathbf{x} + \mathbf{c}$, for every $\mathbf{R} \in SO(3)$ and every $\mathbf{c} \in \mathbb{R}^3$. Therefore one may expect that problem (3.1) admits a solution when the couple $(\mathbf{f}, \mathbf{g})$ consists of a small perturbation of $(\mathbf{0}, \mathbf{0})$ and that the gradient of such solution is close, in some suitable sense, to the identity matrix (or at least to a rotation). This approach is based on a careful use of the implicit function theorem and it has been pursued for the first time in the works by Stoppelli [23, 24], where loads are of the form $(h\mathbf{f}, h\mathbf{g})$, for some fixed $(\mathbf{f}, \mathbf{g})$ and small parameter $h > 0$. This approach for traction problems of the type (3.1) has been further developed by several authors, see e.g. [3, 4, 12, 13]. We also mention the comprehensive book [26] for a thorough description and [6, Section 6.7] for an exhaustive exposition of the literature concerning this topic (see also [7, 25]).

We now outline the results, starting by fixing the functional framework. We consider the Nemitsky operators associated with $-\mathrm{div}\,(D\mathcal{W}(\mathbf{x}, \nabla\mathbf{y}(\mathbf{x})))$ and $D\mathcal{W}(\mathbf{x}, \nabla\mathbf{y}(\mathbf{x}))\mathbf{n}$, that is

$$\begin{gathered} W^{2,p}(\Omega, \mathbb{R}^3) \to L^p(\Omega, \mathbb{R}^3) \times W^{1-1/p,p}(\Omega, \mathbb{R}^3), \\ \mathbf{y} \mapsto \begin{pmatrix} -\mathrm{div}\,(D\mathcal{W}(\cdot, \nabla\mathbf{y}(\cdot))) \\ D\mathcal{W}(\mathbf{x}, \nabla\mathbf{y}(\mathbf{x}))\mathbf{n} \end{pmatrix}. \end{gathered} \tag{3.2}$$

We remark that it is hereafter fundamental to have $p > 3$ in order to obtain that $W^{1,p}(\Omega, \mathbb{R}^3)$ is a Banach algebra along with the continuous embedding

$$W^{1,p}(\Omega, \mathbb{R}^3) \hookrightarrow L^\infty(\Omega, \mathbb{R}^3). \tag{3.3}$$

In particular, this ensures that the Nemitsky operator (3.2) is of class C^1, see [26, Lemma 2.1, Chapter V]. This motivates assumptions (1.5) for external loads. Given a small parameter $h > 0$, we are then interested in existence and asymptotic behavior of solutions to the rescaled nonlinear problem (1.4), which formally coincide with critical points of the functional $\mathcal{G}(\cdot; h\mathcal{L})$ defined by (2.5), with the load functional given by (1.2). We easily notice that, if (1.4) possesses a solution, the forces must necessarily satisfy the following compatibility condition

$$\int_\Omega \mathbf{f}\,d\mathbf{x} + \int_{\partial\Omega} \mathbf{g}\,dS = 0. \tag{3.4}$$

In addition, after simple computations, one can observe that, if $\mathbf{y}_h \in W^{2,p}(\Omega, \mathbb{R}^3)$ solves (1.4), then it must necessarily satisfy

$$\int_\Omega \mathbf{y}_h \wedge \mathbf{f}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{y}_h \wedge \mathbf{g}\, dS = \mathbf{0}. \tag{3.5}$$

When (3.5) holds, we say that the loads $\mathbf{f}$ and $\mathbf{g}$ are equilibrated with respect to the deformation $\mathbf{y}_h$. (3.4) can be imposed as an a priori condition on the loads while (3.5) may be only regarded as an a posteriori condition. Nevertheless, we assume the loads $\mathbf{f}$ and $\mathbf{g}$ to satisfy

$$\int_\Omega \mathbf{x} \wedge \mathbf{f}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{x} \wedge \mathbf{g}\, dS = \mathbf{0}. \tag{3.6}$$

Loosely speaking, a justification of this assumption lies in the fact that a solution to (1.4) (thus verifying (3.5)) is expected to be a small perturbation of the identity map $\mathbf{i}$. We just say that loads are equilibrated if (3.4) and (3.6) hold. Moreover, heuristically speaking, another natural condition which, together with (3.6), goes in the direction of ruling out the chance that $\nabla \mathbf{y}_h$ converges to a rotation $\mathbf{R} \neq \mathbf{I}$ is the absence of axes of equilibrium for the couple $(\mathbf{f}, \mathbf{g})$. We recall that $\mathbf{a} \in \mathbb{R}^3$ is said to be an *axis of equilibrium* for $(\mathbf{f}, \mathbf{g})$ if, for any rotation $\mathbf{R} \in SO(3)$ around $\mathbf{a}$, there holds

$$\int_\Omega \mathbf{x} \wedge \mathbf{R}^T \mathbf{f}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{x} \wedge \mathbf{R}^T \mathbf{g}\, dS = 0,$$

which is equivalent to

$$\int_\Omega \mathbf{R}\mathbf{x} \wedge \mathbf{f}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{R}\mathbf{x} \wedge \mathbf{g}\, dS = 0.$$

The presence of axes of equilibrium for a couple $(\mathbf{f}, \mathbf{g})$ is linked with the *astatic load matrix*, defined as

$$K_{\mathbf{f},\mathbf{g}} := \int_\Omega \mathbf{x} \otimes \mathbf{f}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{x} \otimes \mathbf{g}\, dS.$$

In particular, if we call $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ the eigenvalues of the matrix $K_{\mathbf{f},\mathbf{g}}$ (which is symmetric in view of (3.6)), it is known that the absence of axes of equilibrium is equivalent to the following assumption:

$$\lambda_i + \lambda_j \neq 0, \quad \text{for all } i, j = 1, \dots, 3,\ i \neq j. \tag{3.7}$$

We refer to [3, §3] for a more detailed characterization of this property. We also observe that the couple of assumptions (3.4) and (3.6) coincides with (2.8). Furthermore, we notice that if loads are equilibrated and satisfy (1.3), assumption (3.7) (i.e., absence of axes of equilibrium) can be rephrased by prescribing that the rotation kernel (1.6) of $\mathcal{L}$ only contains the identity matrix (this can be seen for instance by combining [20, Proposition 6.2] and [3, Proposition 3.8]).

The following result is contained in [26, Corollary 6.9, Chapter V]. We point out that for such a result the assumptions on the lack of axes of equilibrium for the loads $(\mathbf{f}, \mathbf{g})$ could be relaxed, see for instance Theorem 6.8 and Corollary 6.10 in Chapter V of [26] and the thorough analysis in [3, 4]. However, to the best of our knowledge, even the results with weaker assumptions do not allow the full generality for the equilibrated couple $(\mathbf{f}, \mathbf{g})$: for instance, they do not cover the case in which every axis in $\mathbb{R}^3$ is an axis of equilibrium for $(\mathbf{f}, \mathbf{g})$ (which is equivalent to saying that the rotation kernel of $\mathcal{L}$ is the whole of $SO(3)$), a case which we consider in our example in Section 5.

Theorem 3.1 ([26, Corollary 6.9, Chapter V])**.** *Suppose that Ω is of class C^2 and that ($\boldsymbol{\mathcal{W}}$**1**), ($\boldsymbol{\mathcal{W}}$**2**), ($\boldsymbol{\mathcal{W}}$**7**) and ($\boldsymbol{\mathcal{W}}$**8**) are satisfied. Let $p > 3$. Let $\mathbf{f} \in L^p(\Omega, \mathbb{R}^3)$ and $\mathbf{g} \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ satisfy (3.4) and (3.6) and let us assume that the eigenvalues $(\lambda_i)_{i=1,\dots,3}$ of the astatic load $K_{\mathbf{f},\mathbf{g}}$ satisfy (3.7). Then there exists $h_0 > 0$ such that, for any $h \in (0, h_0)$ there exists a deformation $\mathbf{y}_h \in W^{2,p}(\Omega, \mathbb{R}^3)$ which solves (1.4). Moreover the map*

$$\begin{aligned} \mathbf{Y}\colon [0, h_0) &\to W^{2,p}(\Omega, \mathbb{R}^3), \\ h &\mapsto \mathbf{Y}(h) = \mathbf{y}_h \end{aligned} \tag{3.8}$$

extended to the identity $\mathbf{i}$ *for* $h = 0$*, belongs to* $C^1([0, h_0); W^{2,p}(\Omega, \mathbb{R}^3))$.

It is now immediate to investigate the asymptotic behavior of the deformation $\mathbf{y}_h$ as $h \to 0$. A direct consequence of Theorem 3.1 is that

$$\mathbf{y}_h \to \mathbf{i} \quad \text{in } W^{2,p}(\Omega, \mathbb{R}^3) \text{ as } h \to 0$$

which, in view of (3.3), in turn implies that

$$\nabla \mathbf{y}_h \to \mathbf{I}, \quad \in L^\infty(\Omega, \mathbb{R}^3) \text{ as } h \to 0.$$

Being the map $\mathbf{Y}$ from (3.8) of class C^1, we can perform a first order Taylor expansion at $h = 0$, which guarantees the existence of a function $\mathbf{u}_0 \in W^{2,p}(\Omega, \mathbb{R}^3)$ such that

$$\mathbf{u}_h := \frac{\mathbf{y}_h - \mathbf{i}}{h} \to \mathbf{u}_0 \quad \text{in } W^{2,p}(\Omega, \mathbb{R}^3) \text{ as } h \to 0.$$

Since

$$h \nabla \mathbf{u}_h \to 0 \quad \text{in } L^\infty(\Omega, \mathbb{R}^3) \text{ as } h \to 0,$$

we can carry out a Taylor expansion of $D\mathcal{W}(\mathbf{x}, \cdot)$ near the identity, that is

$$D\mathcal{W}(\mathbf{x}, \mathbf{I} + h\nabla \mathbf{u}_h(\mathbf{x})) = h\, D^2\mathcal{W}(\mathbf{x}, \boldsymbol{\xi}_h(\mathbf{x}))\nabla \mathbf{u}_h(\mathbf{x}) \quad \text{for every } \mathbf{x} \in \Omega, \tag{3.9}$$

for some $\boldsymbol{\xi}_h(\mathbf{x})$ belonging to the segment $[\mathbf{I}, \mathbf{I} + h\nabla \mathbf{u}_h(\mathbf{x})]$, and thus $\boldsymbol{\xi}_h$ is uniformly converging to $\mathbf{I}$ as $h \to 0^+$. Now, in view of the equation (1.4) satisfied by $\mathbf{y}_h = \mathbf{i} + h\mathbf{u}_h$ and of (3.9) we deduce that

$$\int_\Omega D^2\mathcal{W}(\mathbf{x}, \boldsymbol{\xi}_h(\mathbf{x}))\nabla \mathbf{u}_h(\mathbf{x}) : \nabla \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}\, dS$$

for all $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3)$. Then, thanks to the regularity of $\mathcal{W}$ we can pass to the limit as $h \to 0$ and obtain

$$\int_\Omega D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\nabla \mathbf{u}_0(\mathbf{x}) : \nabla \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}\, dS$$

for all $\mathbf{v} \in H^1(\Omega, \mathbb{R}^3)$, i.e. $\mathbf{u}_0$ weakly solves

$$\begin{cases} -\mathrm{div}\, (D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\nabla \mathbf{u}_0) = \mathbf{f} & \text{in } \Omega, \\ \quad (D^2\mathcal{W}(\mathbf{x}, \mathbf{I})\nabla \mathbf{u}_0)\mathbf{n} = \mathbf{g} & \text{on } \partial\Omega. \end{cases}$$

This means that $\mathbf{u}_0$ is an equilibrium configuration for the linearized elastic problem with loads $(\mathbf{f}, \mathbf{g})$ and as such it minimizes functional $\mathcal{F}_0(\cdot; \mathcal{L})$ over $H^1(\Omega, \mathbb{R}^3)$.

4. Proof of the main results

Let us state a preliminary result about Korn inequalities. We state it for $W^{1,p}(\Omega,\mathbb{R}^3)$ vector fields with generic $p\in(1,+\infty)$. It is based on the standard Korn inequality, see [21], which yields the existence of a constant $D_{p,\Omega}$ such that for every $\mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^{3\times 3})$ there holds

$$\min_{\mathbf{W}\in\mathbb{R}^{3\times 3}_{\mathrm{skew}}}\|\nabla\mathbf{u}-\mathbf{W}\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\le D_{p,\Omega}\,\|\mathbb{E}\mathbf{u}\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}. \tag{4.1}$$

Lemma 4.1. *Let $p\in(1,+\infty)$. There are constants $Z_{p,\Omega}$ and $Q_{p,\Omega}$ (only depending on p,Ω) such that*

$$\int_\Omega|\nabla\mathbf{u}|^p\le Z_{p,\Omega}\int_\Omega|\mathbb{E}\mathbf{u}|^p\quad \textit{for every } \mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^3)\ \textit{s.t.}\ \int_\Omega\operatorname{curl}\mathbf{u}=0 \tag{4.2}$$

and

$$\int_\Omega|\nabla\mathbf{u}|^p\le Q_{p,\Omega}\int_\Omega|\mathbb{E}\mathbf{u}|^p\quad \textit{for every } \mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^3)\ \textit{s.t.}\ \int_\Omega|\nabla\mathbf{u}|^{p-2}\operatorname{curl}\mathbf{u}=0. \tag{4.3}$$

Proof. Suppose first that $\int_\Omega|\nabla\mathbf{u}|^{p-2}\operatorname{curl}\mathbf{u}=0$. Then for every $\mathbf{W}\in\mathbb{R}^{3\times3}_{\mathrm{skew}}$

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_\Omega|\nabla\mathbf{u}+\varepsilon\mathbf{W}|^p=p\int_\Omega|\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}:\mathbf{W}=p\int_\Omega|\nabla\mathbf{u}|^{p-2}\operatorname{skew}(\nabla\mathbf{u}):\mathbf{W}=0,$$

implying that the unique projection of $\nabla\mathbf{u}$ on the vector space of constant skew-symmetric tensor fields over Ω is $\mathbf{0}$. Therefore, (4.1) gives (4.3).

In order to prove that (4.2) holds, assume by contradiction that there exists a sequence $(\mathbf{u}_j)_{j\in\mathbb{N}}\subset W^{1,p}(\Omega,\mathbb{R}^3)$ such that $\int_\Omega\operatorname{curl}\mathbf{u}_j=\mathbf{0}$ and $\int_\Omega|\nabla\mathbf{u}_j|^p=1$ for every $j\in\mathbb{N}$, and $\int_\Omega|\mathbb{E}\mathbf{u}_j|^p\to0$ as $j\to\infty$. By (4.1) and since $\int_\Omega\operatorname{skew}(\nabla\mathbf{u}_j)=\mathbf{0}$, there exists a sequence $(\mathbf{W}_j)_{j\in\mathbb{N}}\subset\mathbb{R}^{3\times3}_{\mathrm{skew}}$ such that

$$\begin{aligned}|\Omega||\mathbf{W}_j|=\left|\int_\Omega(\operatorname{skew}(\nabla\mathbf{u}_j)-\mathbf{W}_j)\right|&\le|\Omega|^{\frac{p-1}{p}}\|\nabla\mathbf{u}_j-\mathbf{W}_j\|_{L^p(\Omega,\mathbb{R}^{3\times3})}\\&\le|\Omega|^{\frac{p-1}{p}}D_{p,\Omega}\,\|\mathbf{E}\mathbf{u}_j\|_{L^p(\Omega,\mathbb{R}^{3\times3})},\end{aligned}$$

and since $\int_\Omega|\mathbb{E}\mathbf{u}_j|^p\to0$ we conclude that $\mathbf{W}_j\to\mathbf{0}$ and that $\int_\Omega|\nabla\mathbf{u}_j-\mathbf{W}_j|^p\to0$ as $j\to\infty$. But then we have

$$\int_\Omega|\nabla\mathbf{u}_j|^p\le2^{p-1}\int_\Omega|\nabla\mathbf{u}_j-\mathbf{W}_j|^p+2^{p-1}|\Omega||\mathbf{W}_j|^p$$

and the right hand side goes to 0 as $j\to\infty$, contradicting the fact that $\int_\Omega|\nabla\mathbf{u}_j|^p=1$ for every $j\in\mathbb{N}$ and thus proving (4.2). ■

Let us now state a simple result about the minimization of the linear elastic energy.

Lemma 4.2. *Let $M>0$. Suppose that $(\boldsymbol{\mathcal{W}}\mathbf{1})$, $(\boldsymbol{\mathcal{W}}\mathbf{2})$, $(\boldsymbol{\mathcal{W}}\mathbf{3})$, $(\boldsymbol{\mathcal{W}}\mathbf{4})$, hold and that $\mathcal{L}$ satisfies* (2.8) *and $\|\mathcal{L}\|_*\le CM/K_\Omega$, where K_Ω is the constant in* (2.9) *and C is the constant in* $(\boldsymbol{\mathcal{W}}\mathbf{4})$. *If $\mathbf{u}_*\in H^1(\Omega,\mathbb{R}^3)$ minimizes $\mathcal{F}_0(\cdot;\mathcal{L})$ over $H^1(\Omega,\mathbb{R}^3)$, then $\int_\Omega|\mathbb{E}\mathbf{u}_*|^2\,d\mathbf{x}\le M^2$.*

Proof. Let us consider a solution $\mathbf{u}_* \in H^1(\Omega,\mathbb{R}^3)$ to the linear elastic problem

$$\min\left\{\mathcal{F}_0(\cdot;\mathcal{L}) : \mathbf{u}\in H^1(\Omega,\mathbb{R}^3)\right\}, \tag{4.4}$$

whose existence is ensured by the compatibility conditions (2.8) by standard arguments. Being a minimizer, $\mathbf{u}_*$ satisfies the energy identity

$$\int_\Omega \mathbb{E}\mathbf{u}_*\, D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathbb{E}\mathbf{u}_* = \mathcal{L}(\mathbf{u}_*). \tag{4.5}$$

From (4.5), by the basic ellipticity estimate $\mathrm{sym}\mathbf{F}\, D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathrm{sym}\mathbf{F} \ge C|\mathrm{sym}\mathbf{F}|^2$, where C is the constant in ($\boldsymbol{\mathcal{W}}\mathbf{4}$), by taking into account (2.9) we get

$$C\int_\Omega |\mathbb{E}\mathbf{u}_*|^2 \le \int_\Omega \mathbb{E}\mathbf{u}_*\, D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathbb{E}\mathbf{u}_* = \mathcal{L}(\mathbf{u}_*) \le K_\Omega\|\mathcal{L}\|_*\,\|\mathbb{E}\mathbf{u}_*\|_{L^2(\Omega,\mathbb{R}^{3\times3})},$$

that is,

$$\|\mathbb{E}\mathbf{u}_*\|^2_{L^2(\Omega,\mathbb{R}^{3\times3})} \le C^{-2}K_\Omega^2\,\|\mathcal{L}\|_*^2.$$

Thanks to the the assumption $\|\mathcal{L}\|_* \le CM/K_\Omega$, we conclude. ∎

For every $R>0$ and for every $\mathcal{L}\in(H^1(\Omega,\mathbb{R}^3))^*$, in the sequel we shall consider the following constrained minimization problems of finite elasticity

$$\min\left\{\mathcal{F}(\mathbf{u};\mathcal{L}) : \mathbf{u}\in H^1(\Omega,\mathbb{R}^3),\ \int_\Omega |\mathbb{E}\mathbf{u}|^2 \le R\right\} \tag{4.6}$$

$$\min\left\{\mathcal{F}(\mathbf{u};\mathcal{L}) : \mathbf{u}\in H^1(\Omega,\mathbb{R}^3),\ \int_\Omega |\mathbb{E}\mathbf{u}|^2 \le R,\ \int_\Omega \mathrm{curl}\,\mathbf{u} = \mathbf{0}\right\} \tag{4.7}$$

Lemma 4.3. *Suppose that* ($\boldsymbol{\mathcal{W}}\mathbf{1}$), ($\boldsymbol{\mathcal{W}}\mathbf{2}$), ($\boldsymbol{\mathcal{W}}\mathbf{3}$), ($\boldsymbol{\mathcal{W}}\mathbf{4}$), ($\boldsymbol{\mathcal{W}}\mathbf{5}$), ($\boldsymbol{\mathcal{W}}\mathbf{6}$) *hold and that* $\mathcal{L}\in(H^1(\Omega,\mathbb{R}^3))^*$ *satisfies* (2.8)*. Let* $R>0$*. Then there exists a solution to problem* (4.6) *and there exists a solution to problem* (4.7)*.*

Proof. Let $(\mathbf{u}_n)_{n\in\mathbb{N}}\subset H^1(\Omega,\mathbb{R}^3)$ denote a minimizing sequence for problem (4.6). Since $\mathcal{F}(\mathbf{0};\mathcal{L})=0$, it is not restrictive to assume that $\mathcal{F}(\mathbf{u}_n;\mathcal{L})\le 1$ for every $n\in\mathbb{N}$. Moreover, by 2.10 there exists a sequence $(\mathbf{R}_n)_{n\in\mathbb{N}}\subset SO(3)$ such that

$$\begin{aligned}\int_\Omega |\mathbf{I}+\nabla\mathbf{u}_n-\mathbf{R}_n|^2\,d\mathbf{x} &\le C_\Omega\int_\Omega \mathcal{W}(\mathbf{x},\mathbf{I}+\nabla\mathbf{u}_n)\,d\mathbf{x}\\ &\le C_\Omega\mathcal{F}(\mathbf{u}_n)+C_\Omega\mathcal{L}(\mathbf{u}_n)\le C_\Omega+C_\Omega\mathcal{L}(\mathbf{u}_n).\end{aligned} \tag{4.8}$$

From (2.9) and (4.8), since $\int_\Omega|\mathbb{E}\mathbf{u}_n|^2\le R$, we deduce that the sequence $(\nabla\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega,\mathbb{R}^{3\times3})$. Thanks to Poincaré inequality and to (2.8), we deduce that up to subsequences $\nabla\mathbf{u}_n$ weakly converge to $\nabla\mathbf{u}$ for some $\mathbf{u}\in H^1(\Omega,\mathbb{R}^3)$, that $\mathcal{L}(\mathbf{u}_n)\to\mathcal{L}(\mathbf{u})$, and that $\mathbb{E}\mathbf{u}_n$ weakly converge to $\mathbb{E}\mathbf{u}$ in $L^2(\Omega,\mathbb{R}^3)$. The weak $L^2(\Omega,\mathbb{R}^{3\times3})$ lower semicontinuity of the map $\mathbf{F}\mapsto\int_\Omega|\mathbf{F}|^2$ yields $\int_\Omega|\mathbb{E}\mathbf{u}|^2\le R$. On the other hand, with the notation $\mathbf{y}_n(\mathbf{x}):=\mathbf{x}+h\mathbf{u}_n(\mathbf{x})$, the assumption ($\boldsymbol{\mathcal{W}}\mathbf{5}$) implies, thanks to the classical results by Ball [2, §6], that $\mathrm{cof}\nabla\mathbf{y}_n$ (resp. $\det\nabla\mathbf{y}_n$) weakly converge in $L^q(\Omega,\mathbb{R}^{3\times3})$ to $\mathrm{cof}\nabla\mathbf{y}$ (resp. weakly converge in $L^r(\Omega)$ to $\det\nabla\mathbf{y}$). By the polyconvexity assumption $\boldsymbol{\mathcal{W}}\mathbf{6}$ on the Carathéodory function $\mathcal{W}$, and since $\mathcal{L}(\mathbf{u}_n)\to\mathcal{L}(\mathbf{u})$ as $n\to\infty$, we deduce that

$$\mathcal{F}(\mathbf{u};\mathcal{L})\le\liminf_{n\to+\infty}\mathcal{F}(\mathbf{u}_n;\mathcal{L}).$$

Since $(\mathbf{u}_n)_n$ is by assumption a minimizing sequence, we conclude that $\mathbf{u} \in H^1(\Omega, \mathbb{R}^3)$ is a solution to problem (4.6).

The same argument shows that there exists a solution to problem (4.7), since the weak $L^2(\Omega, \mathbb{R}^{3\times 3})$ convergence of $\nabla \mathbf{u}_n$ to $\nabla \mathbf{u}$ implies that $\int_\Omega \operatorname{curl} \mathbf{u}_n \to \int_\Omega \operatorname{curl} \mathbf{u}$. ■

Remark 4.4. It is worth noticing that a solution $\mathbf{u}$ to problem (4.7) is not a priori guaranteed to be a constrained local minimizer in the sense of Definition 2.1. Indeed it may happen that $\mathbf{u}$ satisfies $\int_\Omega |\mathbb{E}\mathbf{u}|^2 = R$, so that for instance by taking $\boldsymbol{\psi} = \mathbf{u}$, for every $\varepsilon > 0$ the function $\mathbf{u} + \varepsilon \boldsymbol{\psi}$ is not admissible for problem (4.7).

Remark 4.5. Let $h > 0$ and $M > 0$. Then by Lemma 4.3 there exists a solution $\mathbf{v}_h$ to the problem

$$\min \left\{ \mathcal{F}(\mathbf{v}; h\mathcal{L}) : \mathbf{v} \in H^1(\Omega, \mathbb{R}^3),\ \int_\Omega |\mathbb{E}\mathbf{v}|^2 \le M^2 h^2,\ \int_\Omega \operatorname{curl} \mathbf{v} = 0 \right\},$$

and thus $\mathbf{u}_h =: h^{-1}\mathbf{v}_h$ is a solution to

$$\min \left\{ \mathcal{F}_h(\mathbf{u}; \mathcal{L}) : \mathbf{u} \in H^1(\Omega, \mathbb{R}^3),\ \int_\Omega |\mathbb{E}\mathbf{u}|^2 \le M^2,\ \int_\Omega \operatorname{curl} \mathbf{u} = \mathbf{0} \right\}. \tag{4.9}$$

The next lemma provides convergence to constrained linear elasticity and generalizes results from [16, 18].

Lemma 4.6. *Suppose that* $(\boldsymbol{\mathcal{W}}\mathbf{1})$, $(\boldsymbol{\mathcal{W}}\mathbf{2})$, $(\boldsymbol{\mathcal{W}}\mathbf{3})$, $(\boldsymbol{\mathcal{W}}\mathbf{4})$, $(\boldsymbol{\mathcal{W}}\mathbf{5})$, $(\boldsymbol{\mathcal{W}}\mathbf{6})$ *hold and that* $\mathcal{L} \in (H^1(\Omega, \mathbb{R}^3))^*$ *satisfies* (2.8)*. Let* $M > 0$ *and let* $(h_j)_{j\in\mathbb{N}} \subset (0,1)$ *be a vanishing sequence. For any* $j \in \mathbb{N}$*, let* $\mathbf{u}_j := \mathbf{u}_{h_j}$*, where* $\mathbf{u}_{h_j}$ *is a solution to problem* (4.9) *(with* h_j *in place of* h*, see* Remark 4.5*). Then there exists* $\mathbf{u}_* \in H^1(\Omega, \mathbb{R}^3)$ *such that* $\int_\Omega |\mathbb{E}\mathbf{u}_*|^2 \le M^2$ *and such that, up to subsequences,* $\mathbb{E}\mathbf{u}_j \rightharpoonup \mathbb{E}\mathbf{u}_*$ *weakly in* $L^2(\Omega, \mathbb{R}^{3\times 3})$ *as* $j \to \infty$ *and*

$$\lim_{j\to+\infty} \mathcal{F}_{h_j}(\mathbf{u}_j; \mathcal{L}) = \mathcal{F}_0(\mathbf{u}_*\,; \mathcal{L}) = \min_{H^1(\Omega,\mathbb{R}^3)} \left\{ \mathcal{F}_0(\mathbf{u}\,; \mathcal{L}) : \int_\Omega |\mathbb{E}\mathbf{u}|^2 \le M^2 \right\}. \tag{4.10}$$

Proof. **Step 1 (lower bound).** Suppose that $\sup_{j\in\mathbb{N}} \mathcal{F}_{h_j}(\tilde{\mathbf{u}}_j; \mathcal{L}) < +\infty$ and that $\mathbb{E}\tilde{\mathbf{u}}_j \rightharpoonup \mathbb{E}\tilde{\mathbf{u}}$ weakly in $L^2(\Omega, \mathbb{R}^{3\times 3})$ for some sequence $(\tilde{\mathbf{u}}_j)_{j\in\mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)$ such that $\int_\Omega \operatorname{curl} \tilde{\mathbf{u}}_j = \mathbf{0}$ for every $j \in \mathbb{N}$ and some $\tilde{\mathbf{u}} \in H^1(\Omega, \mathbb{R}^3)$. By the weak $L^2(\Omega, \mathbb{R}^{3\times 3})$ lower semicontinuity of the map $\mathbf{F} \mapsto \int_\Omega |\mathbf{F}|^2$, we get

$$\liminf_{j\to+\infty} \int_\Omega |\mathbb{E}\tilde{\mathbf{u}}_j|^2 \ge \int_\Omega |\mathbb{E}\tilde{\mathbf{u}}|^2. \tag{4.11}$$

By applying [18, Lemma 5.1] we obtain the existence of $\mathbf{W} \in \mathbb{R}^{3\times 3}_{\text{skew}}$ such that $\sqrt{h_j}\nabla\tilde{\mathbf{u}}_j \to \mathbf{W}$ in $L^2(\Omega, \mathbb{R}^{3\times 3})$ as $j \to \infty$. However, since $\int_\Omega \operatorname{curl} \tilde{\mathbf{u}}_j = \mathbf{0}$, by Lemma 4.1 we deduce that the sequence $(\nabla\tilde{\mathbf{u}}_j)_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$, thus forcing $\mathbf{W} \equiv \mathbf{0}$. From [18, Lemma 5.2] we also deduce that $\mathbf{1}_{B_j}\mathbb{E}(\tilde{\mathbf{u}}_j) \rightharpoonup \mathbb{E}(\tilde{\mathbf{u}})$ weakly in $L^2(\Omega, \mathbb{R}^{3\times 3})$ as $j \to \infty$, where $B_j := \{\mathbf{x} \in \Omega : |\sqrt{h_j}\nabla\tilde{\mathbf{u}}_j(\mathbf{x})| \le 1\}$. We stress that [18, Lemma 5.1, Lemma 5.2] are stated under the additional incompressibility constraint requiring $\mathcal{W}(\mathbf{F}) = +\infty$ if $\det \mathbf{F} \ne 1$, but they are true (with the very same proof) even without such a constraint.

By the frame indifference assumption $(\boldsymbol{\mathcal{W}}\mathbf{1})$, there exists a function $\mathcal{V}$ such that for a.e. $\mathbf{x} \in \Omega$

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = \mathcal{V}(\mathbf{x}, \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})) \qquad \forall\, \mathbf{F} \in \mathbb{R}^{3\times 3}.$$

By ($\boldsymbol{\mathcal{W}}\mathbf{3}$) we deduce that for a.e. $\mathbf{x} \in \Omega$ the function $\mathcal{V}(\mathbf{x},\cdot)$ is C^2 smooth in a suitable neighborhood $\tilde{\mathcal{U}}$ of the origin in $\mathbb{R}^{3\times 3}$, with a $\mathbf{x}$-independent modulus of continuity $\eta : \mathbb{R}_+ \to \mathbb{R}$, which is increasing and vanishing at 0^+. By following the proof of [18, Lemma 5.3], we define

$$\mathbb{D}\tilde{\mathbf{u}}_j := \mathbb{E}\tilde{\mathbf{u}}_j + \frac{1}{2}h_j \nabla\tilde{\mathbf{u}}_j^T \nabla\tilde{\mathbf{u}}_j.$$

By ($\boldsymbol{\mathcal{W}}\mathbf{2}$), ($\boldsymbol{\mathcal{W}}\mathbf{3}$) and by a Taylor expansion, we get the existence of $j_0 \in \mathbb{N}$ such that for every $j > j_0$ the estimate

$$\left| \mathcal{V}(\mathbf{x}, h_j\mathbb{D}\tilde{\mathbf{u}}_j) - \frac{h_j^2}{2}\,\mathbb{D}\tilde{\mathbf{u}}_j^T\, D^2\mathcal{V}(\mathbf{x},\mathbf{0})\;\mathbb{D}\tilde{\mathbf{u}}_j \right| \le h_j^2 \eta(h_j|\mathbb{D}\tilde{\mathbf{u}}_j|)|\mathbb{D}\tilde{\mathbf{u}}_j|^2$$

holds for a.e. $\mathbf{x} \in B_j$, since $h_j|\mathbb{D}\tilde{\mathbf{u}}_j(\mathbf{x})| \le 2\sqrt{h_j}$ and thus $h_j\mathbb{D}\tilde{\mathbf{u}}_j(\mathbf{x}) \in \tilde{\mathcal{U}}$ for a.e. $\mathbf{x} \in B_j$ if j is large enough. We deduce

$$\begin{aligned}
\mathcal{F}_{h_j}(\tilde{\mathbf{u}}_j) &\ge \frac{1}{h_j^2}\int_{B_j} \mathcal{W}(\mathbf{x}, \mathbf{I} + h_j\nabla\tilde{\mathbf{u}}_j)\,d\mathbf{x} - \mathcal{L}(\tilde{\mathbf{u}}_j) = \frac{1}{h_j^2}\int_{B_j} \mathcal{V}(\mathbf{x}, h_j\mathbb{D}\tilde{\mathbf{u}}_j)\,d\mathbf{x} - \mathcal{L}(\tilde{\mathbf{u}}_j) \\
&\ge \int_{B_j} \frac{1}{2}\mathbb{D}\tilde{\mathbf{u}}_j^T\, D^2\mathcal{V}(\mathbf{x},\mathbf{0})\,\mathbb{D}\tilde{\mathbf{u}}_j\,d\mathbf{x} - \int_{B_j} \eta(h_j|\mathbb{D}\tilde{\mathbf{u}}_j|)|\mathbb{D}\tilde{\mathbf{u}}_j|^2\,d\mathbf{x} - \mathcal{L}(\tilde{\mathbf{u}}_j) \\
&\ge \frac{1}{2}\int_\Omega (\mathbf{1}_{B_j}\mathbb{D}\tilde{\mathbf{u}}_j)\, D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,(\mathbf{1}_{B_j}\mathbb{D}\tilde{\mathbf{u}}_j)\,d\mathbf{x} - \eta(2\sqrt{h_j})\int_\Omega |\mathbf{1}_{B_j}\mathbb{D}\tilde{\mathbf{u}}_j|^2\,d\mathbf{x} - \mathcal{L}(\tilde{\mathbf{u}}_j)
\end{aligned}$$

for every $j > j_0$, where the last inequality is due $D^2\mathcal{W}(\mathbf{x},\mathbf{I}) = D^2\mathcal{V}(\mathbf{x},\mathbf{0})$ and to the monotonicity of the modulus of continuity η, and we also recall that $\eta(2\sqrt{h_j}) \to 0$ as $j \to \infty$. By recalling that $\sqrt{h_j}\nabla\tilde{\mathbf{u}}_j$ strongly converge to zero in $L^2(\Omega,\mathbb{R}^{3\times 3})$, hence a.e. in Ω up to subsequences, and since $(\mathbf{1}_{B_j}h_j\nabla\tilde{\mathbf{u}}_j^T\nabla\tilde{\mathbf{u}}_j)_{j\in\mathbb{N}}$ is a bounded sequence in $L^2(\Omega,\mathbb{R}^{3\times 3})$, we deduce that up to subsequences $\mathbf{1}_{B_j}h_j\nabla\tilde{\mathbf{u}}_j^T\nabla\tilde{\mathbf{u}}_j$ weakly converge to zero in $L^2(\Omega, R^{3\times 3})$ so that $\mathbf{1}_{B_j}\mathbb{D}\tilde{\mathbf{u}}_j$ weakly converge in $L^2(\Omega,\mathbb{R}^{3\times 3})$ to $\mathbb{E}\tilde{\mathbf{u}}$ as $j \to \infty$. Since the map $\mathbf{F} \mapsto \int_\Omega \mathbf{F}^T\, D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathbf{F}\,d\mathbf{x}$ is weakly $L^2(\Omega,\mathbb{R}^{3\times 3})$ lower semicontinuous, and since $\mathcal{L} \in (H^1(\Omega,\mathbb{R}^3))^*$ and (2.8) imply $\mathcal{L}(\tilde{\mathbf{u}}_j) \to \mathcal{L}(\tilde{\mathbf{u}})$ thanks to Korn and Poincaré inequalities, we conclude that

$$\liminf_{j\to+\infty} \mathcal{F}_{h_j}(\tilde{\mathbf{u}}_j;\mathcal{L}) \ge \frac{1}{2}\int_\Omega \mathbb{E}\tilde{\mathbf{u}}\, D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathbb{E}\tilde{\mathbf{u}}\,d\mathbf{x} - \mathcal{L}(\tilde{\mathbf{u}}).$$

Step 2 (upper bound). Let now $\bar{\mathbf{u}} \in H^1(\Omega,\mathbb{R}^3)$ be such that $\int_\Omega |\mathbb{E}\bar{\mathbf{u}}|^2 \le M^2$. Let $\delta_j := h_j^{1/5}$ and $\bar{\mathbf{u}}_j := \bar{\mathbf{u}} * \rho_j$, where $\rho_j(\mathbf{x}) := \delta_j^{-3}\rho(\mathbf{x}/\delta_j)$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ is the standard unit symmetric mollifier. We notice that $\bar{\mathbf{u}}_j \to \bar{\mathbf{u}}$ in $H^1(\Omega,\mathbb{R}^3)$ as $j \to +\infty$ and that the elementary estimate $\|\rho_j\|_{W^{1,\infty}(\mathbb{R}^3)} \le 2\delta_j^{-4}\|\rho\|_{W^{1,\infty}(\mathbb{R}^3)}$ holds. Therefore, by Young inequality we obtain

$$\|\bar{\mathbf{u}}_j\|_{W^{1,\infty}(\Omega,\mathbb{R}^3)} \le \|\bar{\mathbf{u}}\|_{L^1(\Omega',\mathbb{R}^3)}\|\rho_j\|_{W^{1,\infty}(\mathbb{R}^3)} \le 2\delta_j^{-4}\|\rho\|_{W^{1,\infty}(\mathbb{R}^3)}\|\bar{\mathbf{u}}\|_{L^1(\Omega',\mathbb{R}^3)}$$

where Ω' is a suitable open neighbor of Ω (and $\bar{\mathbf{u}}$ is understood as a not relabeled $H^1(\Omega',\mathbb{R}^3)$ extension). We deduce $h\|\bar{\mathbf{u}}_j\|_{W^{1,\infty}(\Omega,\mathbb{R}^3)} \to 0$ as $j \to +\infty$. Therefore $\mathbf{I} + h_j\nabla\bar{\mathbf{u}}_j \in \mathcal{U}$ for a.e. $\mathbf{x}$ in Ω if j is large enough, where $\mathcal{U}$ is the neighborhood of $SO(3)$ that appears in assumption

($\boldsymbol{\mathcal{W}}$**3**), which then implies, by Taylor expansion,

$$\begin{aligned}\limsup_{j\to+\infty}|\mathcal{F}_{h_j}(\bar{\mathbf{u}}_j;\mathcal{L})-\mathcal{F}_0(\bar{\mathbf{u}}_j;\mathcal{L})| &\le \limsup_{j\to+\infty}\int_\Omega\left|\frac{1}{h_j^2}\mathcal{W}(\mathbf{x},\mathbf{I}+h_j\nabla\bar{\mathbf{u}}_j)-\frac12\,\nabla\bar{\mathbf{u}}_j^T D^2\mathcal{W}(\mathbf{x},\mathbf{I})\nabla\bar{\mathbf{u}}_j\right| \\ &\le \limsup_{j\to+\infty}\int_\Omega \omega(h_j|\nabla\bar{\mathbf{u}}_j|)\,|\nabla\bar{\mathbf{u}}_j|^2 \\ &\le \limsup_{j\to+\infty}\,\|\omega(h_j|\nabla\bar{\mathbf{u}}_j|)\|_{L^\infty(\Omega)}\int_\Omega|\nabla\bar{\mathbf{u}}_j|^2=0.\end{aligned}$$

The latter limit is zero since $h_j\nabla\bar{\mathbf{u}}_j\to0$ in $L^\infty(\Omega,\mathbb{R}^{3\times3})$, as ω is increasing with $\lim_{t\to0^+}\omega(t)=0$, and since $\bar{\mathbf{u}}_j\to\bar{\mathbf{u}}$ in $H^1(\Omega,\mathbb{R}^3)$ as $j\to+\infty$, which also implies $\mathcal{F}_0(\bar{\mathbf{u}}_j;\mathcal{L})\to\mathcal{F}_0(\bar{\mathbf{u}};\mathcal{L})$, so that by writing $|\mathcal{F}_{h_j}(\bar{\mathbf{u}}_j;\mathcal{L})-\mathcal{F}_0(\bar{\mathbf{u}};\mathcal{L})|\le|\mathcal{F}_{h_j}(\bar{\mathbf{u}}_j;\mathcal{L})-\mathcal{F}_0(\bar{\mathbf{u}}_j;\mathcal{L})|+|\mathcal{F}_0(\bar{\mathbf{u}}_j;\mathcal{L})-\mathcal{F}_0(\bar{\mathbf{u}};\mathcal{L})|$ we deduce

$$\lim_{j\to+\infty}\mathcal{F}_{h_j}(\bar{\mathbf{u}}_j;\mathcal{L})=\mathcal{F}_0(\bar{\mathbf{u}};\mathcal{L}).$$

Moreover, by using Young inequality again we deduce that for any $j\in\mathbb{N}$

$$\int_\Omega|\mathbb{E}\bar{\mathbf{u}}_j|^2\le\int_\Omega|\mathbb{E}\bar{\mathbf{u}}|^2\le M^2. \tag{4.12}$$

Step 3 (convergence). Since we are assuming

$$\mathbf{u}_j\in\operatorname{argmin}_{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3)}\left\{\mathcal{F}_{h_j}(\mathbf{u};\mathcal{L}):\int_\Omega|\mathbb{E}\mathbf{u}|^2\le M^2,\ \int_\Omega\operatorname{curl}\mathbf{u}=\mathbf{0}\right\} \tag{4.13}$$

for every $j\in\mathbb{N}$, it is readily seen that that $\sup_{j\in\mathbb{N}}\mathcal{F}_{h_j}(\mathbf{u}_j;\mathcal{L})<+\infty$. Moreover, since $\int_\Omega|\mathbb{E}\mathbf{u}_j|^2\le M^2$ for every $j\in\mathbb{N}$, up to subsequences there exists $\mathbf{u}_*\in H^1(\Omega,\mathbb{R}^3)$ such that $\mathbb{E}(\mathbf{u}_j)\rightharpoonup\mathbb{E}(\mathbf{u}_*)$ weakly in $L^2(\Omega,\mathbb{R}^{3\times3})$, see [16, Lemma 3.2]. By Step 1 we get $\int_\Omega|\mathbb{E}\mathbf{u}_*|^2\le M^2$, see (4.11), and

$$\frac12\int_\Omega\mathbb{E}(\mathbf{u}_*)\,D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathbb{E}\mathbf{u}_*\,d\mathbf{x}-\mathcal{L}(\mathbf{u}_*)\le\liminf_{j\to+\infty}\mathcal{F}_{h_j}(\mathbf{u}_j;\mathcal{L}). \tag{4.14}$$

To every $\bar{\mathbf{u}}\in H^1(\Omega,\mathbb{R}^3)$ such that $\int_\Omega|\mathbb{E}\bar{\mathbf{u}}|^2\le M^2$, we associate the sequence $(\bar{\mathbf{u}}_j)_{j\in\mathbb{N}}\subset H^1(\Omega,\mathbb{R}^3)$ constructed as in Step 2. Therefore, such a sequence converges to $\bar{\mathbf{u}}$ strongly in $H^1(\Omega,\mathbb{R}^3)$ with $h_j\|\nabla\bar{\mathbf{u}}_j\|_{L^\infty(\Omega,\mathbb{R}^{3\times3})}\to0$ as $j\to+\infty$, and it satisfies $\int_\Omega|\mathbb{E}\bar{\mathbf{u}}_j|^2\le M^2$ for every $j\in\mathbb{N}$, see (4.12). For every $j\in\mathbb{N}$ we let $\mathbf{w}_j:=|\Omega|^{-1}\int_\Omega\operatorname{curl}\bar{\mathbf{u}}_j$ so that

$$\int_\Omega\operatorname{curl}(\bar{\mathbf{u}}_j-\tfrac12\mathbf{w}_j\wedge\mathbf{x})=0,$$

since $\operatorname{curl}(\mathbf{a}\wedge\mathbf{x})=2\mathbf{a}$ for every $\mathbf{a}\in\mathbb{R}^3$. Thus $\mathbf{w}_j\to\mathbf{w}:=|\Omega|^{-1}\int_\Omega\operatorname{curl}\bar{\mathbf{u}}$ as $j\to\infty$. We deduce that $\bar{\mathbf{u}}_j-\frac12\mathbf{w}_j\wedge\mathbf{x}$ strongly converge in $H^1(\Omega,\mathbb{R}^3)$ to $\bar{\mathbf{u}}-\frac12\mathbf{w}\wedge\mathbf{x}$ as $j\to+\infty$, that $h_j\|\nabla(\bar{\mathbf{u}}_j-\frac12\mathbf{w}_j\wedge\mathbf{x})\|_{L^\infty(\Omega,\mathbb{R}^{3\times3})}\to0$ as $j\to\infty$, and that for every $j\in\mathbb{N}$ there holds

$$\int_\Omega|\mathbb{E}(\bar{\mathbf{u}}_j-\tfrac12\mathbf{w}_j\wedge\mathbf{x})|^2=\int_\Omega|\mathbb{E}\bar{\mathbf{u}}_j|^2\le M^2.$$

By (4.14), (4.13) and by Step 2 we get

$$\begin{aligned}\mathcal{F}_0(\mathbf{u}_*;\mathcal{L}) &\le \liminf_{j\to+\infty}\mathcal{F}_{h_j}(\mathbf{u}_j;\mathcal{L}) \le \limsup_{j\to+\infty}\mathcal{F}_{h_j}(\mathbf{u}_j;\mathcal{L})\\ &\le \limsup_{j\to+\infty}\mathcal{F}_{h_j}(\bar{\mathbf{u}}_j - \tfrac12\mathbf{w}_j\wedge\mathbf{x};\mathcal{L}) = \mathcal{F}_0(\bar{\mathbf{u}} - \tfrac12\mathbf{w}\wedge\mathbf{x};\mathcal{L}) = \mathcal{F}_0(\bar{\mathbf{u}};\mathcal{L}),\end{aligned}$$

where the last equality is due to the invariance of $\mathcal{F}_0(\cdot;\mathcal{L})$ by infinitesimal rigid displacements. By the arbitrariness of $\bar{\mathbf{u}}$, we deduce that $\mathbf{u}_*$ is a solution to the minimization problem in the right hand side of (4.10), and that indeed (4.10) holds. ■

Lemma 4.7. *Assume that* ($\boldsymbol{\mathcal{W}}$**1**), ($\boldsymbol{\mathcal{W}}$**2**), ($\boldsymbol{\mathcal{W}}$**3**), ($\boldsymbol{\mathcal{W}}$**4**), ($\boldsymbol{\mathcal{W}}$**5**), ($\boldsymbol{\mathcal{W}}$**6**) *hold true, that* $\mathcal{L}\in(H^1(\Omega,\mathbb{R}^3))^*$ *satisfies* (2.8) *and that*

$$\|\mathcal{L}\|_* \le \frac12\,\gamma(\Omega,C), \tag{4.15}$$

having defined

$$\gamma(\Omega,C) := \frac{C}{2K_\Omega(4CC_\Omega+8)},$$

where C_Ω *is the constant in* (2.10), K_Ω *is the constant in* (2.9), C *is the constant in* ($\boldsymbol{\mathcal{W}}$**4**). *Then there exists* $\bar h>0$ *such that if* $h\in(0,\bar h)$ *and if*

$$\mathbf{v}_h \in \operatorname{argmin}\left\{\mathcal{F}(\mathbf{v};h\mathcal{L}) : \mathbf{v}\in H^1(\Omega,\mathbb{R}^3),\ \int_\Omega|\mathbb{E}(\mathbf{v})|^2\le h^2,\ \int_\Omega \operatorname{curl}\mathbf{v} = \mathbf{0}\right\}, \tag{4.16}$$

then $\displaystyle\int_\Omega|\mathbb{E}\mathbf{v}_h|^2 < h^2.$

Proof. **Step 1.** Let us consider a minimizer $\mathbf{u}_*\in H^1(\Omega,\mathbb{R}^3)$ of the linear elastic problem (4.4), whose existence is ensured by the compatibility conditions (2.8) on $\mathcal{L}$. It satisfies the energy identity (4.5), and since by (4.15) we have $\|\mathcal{L}\|_*\le\frac{C}{2K_\Omega}$, by repeating the proof of Lemma 4.2 we deduce $\int_\Omega|\mathbb{E}\mathbf{u}_*|^2\le 1/4$. Let $h\in(0,1)$ and let $\bar{\mathbf{u}}_h := \mathbf{u}_* * \rho_h$, where $\rho_h(\mathbf{x}) = \delta_h^{-3}\rho(\mathbf{x}/\delta_h)$ and $\delta_h = h^{1/5}$, so that the argument in Step 2 of the proof of Lemma 4.6 yields $\bar{\mathbf{u}}_h\to\mathbf{u}_*$ in $H^1(\Omega,\mathbb{R}^3)$ along with $h\|\nabla\bar{\mathbf{u}}_h\|_{L^\infty(\Omega,\mathbb{R}^{3\times3})}\to 0$ as $h\to0$, and moreover

$$\lim_{h\to0}\frac{1}{h^2}\int_\Omega\mathcal{W}(\mathbf{x},\mathbf{I}+h\nabla\bar{\mathbf{u}}_h)\,d\mathbf{x} = \frac12\int_\Omega\mathbb{E}\mathbf{u}_*\,D^2\mathcal{W}(x,\mathbf{I})\,\mathbb{E}\mathbf{u}_*\,d\mathbf{x}.$$

We let $\mathbf{u}_h^* := \bar{\mathbf{u}}_h - \frac12\mathbf{w}_h\wedge\mathbf{x}$, where $\mathbf{w}_h := |\Omega|^{-1}\int_\Omega\operatorname{curl}\bar{\mathbf{u}}_h$, and $\mathbf{w} := |\Omega|^{-1}\int_\Omega\operatorname{curl}\mathbf{u}_*$, so that also along $\mathbf{u}_h^*$ we have

$$\lim_{h\to0}\frac{1}{h^2}\int_\Omega\mathcal{W}(\mathbf{x},\mathbf{I}+h\nabla\mathbf{u}_h^*)\,d\mathbf{x} = \frac12\int_\Omega\mathbb{E}\mathbf{u}_*\,D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathbb{E}\mathbf{u}_*\,d\mathbf{x},$$

since the linear elastic energy is unaffected by the addition of infinitesimal rigid displacements, and similarly we have $\mathcal{L}(\mathbf{u}_h^*)\to\mathcal{L}(\mathbf{u}_*)$ as $h\to0$ thanks to (2.8). Moreover, since $\mathbf{u}_h^*\to\mathbf{u}_*-\frac12\mathbf{w}\wedge\mathbf{x}$ in $H^1(\Omega,\mathbb{R}^3)$ as $h\to0$, and since $\int_\Omega|\mathbb{E}\mathbf{u}_*|^2\le1/4$, we see that for small enough h there holds $\int_\Omega|h\mathbb{E}\mathbf{u}_h^*|^2 = \int_\Omega|h\mathbb{E}\bar{\mathbf{u}}_h|^2\le h^2$, so that $h\mathbf{u}_h^*$ is admissible for problem (4.16).

Step 2. For every $h\in(0,1)$, let $\mathbf{v}_h$ as in (4.16) and $\mathbf{u}_h := h^{-1}\mathbf{v}_h$. By Step 1 we get

$$\mathcal{F}(\mathbf{v}_h;h\mathcal{L}) \le \mathcal{F}(h\mathbf{u}_h^*;h\mathcal{L}) = \frac{h^2}{2}\int_\Omega\mathbb{E}\mathbf{u}_*\,D^2\mathcal{W}(x,\mathbf{I})\,\mathbb{E}\mathbf{u}_*\,d\mathbf{x} - h^2\mathcal{L}(\mathbf{u}_*) + o(h^2) \tag{4.17}$$

as $h \to 0$. On the other hand, by (2.10) and thanks to the Euler-Rodrigues formula $\mathbf{R}_h - \mathbf{I} = \sin\theta_h \mathbf{W}_h + (1-\cos\theta_h)\mathbf{W}_h^2$ and to the elementary inequality $|\mathrm{sym}\mathbf{F}| \le |\mathbf{F}|$ we have

$$\frac{1}{C_\Omega}\int_\Omega \left|\mathbb{E}\mathbf{v}_h - (1-\cos\theta_h)\mathbf{W}_h^2\right|^2\,d\mathbf{x} - h\mathcal{L}(\mathbf{v}_h) \le \frac{1}{C_\Omega}\int_\Omega |\mathbf{I}-\mathbf{R}_h+\nabla\mathbf{v}_h|^2\,d\mathbf{x} - h\mathcal{L}(\mathbf{v}_h)$$
$$\le \int_\Omega \mathcal{W}(\mathbf{x}, \mathbf{I}+\nabla\mathbf{v}_h)\,d\mathbf{x} - h\mathcal{L}(\mathbf{v}_h)$$

so that

$$\frac{1}{C_\Omega}\int_\Omega \left|\mathbb{E}\mathbf{v}_h - (1-\cos\theta_h)\mathbf{W}_h^2\right|^2\,d\mathbf{x} \le h\mathcal{L}(\mathbf{v}_h) + \mathcal{F}(\mathbf{v}_h; h\mathcal{L}). \tag{4.18}$$

From (4.16), (4.17) and (4.18), by (2.9) and by the energy identity (4.5) we deduce

$$\begin{aligned}
&\frac{1}{C_\Omega}\int_\Omega \left|\mathbb{E}\mathbf{v}_h - (1-\cos\theta_h)\mathbf{W}_h^2\right|^2\,d\mathbf{x} \\
&\le -\frac{h^2}{2}\int_\Omega \mathbb{E}\mathbf{u}_*\, D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathbb{E}\mathbf{u}_*\,d\mathbf{x} + K_\Omega h\|\mathcal{L}\|_*\,\|\mathbb{E}\mathbf{v}_h\|_{L^2(\Omega,\mathbb{R}^{3\times3})} + o(h^2) \\
&\le -\frac{h^2}{2}\int_\Omega \mathbb{E}\mathbf{u}_*\, D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathbb{E}\mathbf{u}_*\,d\mathbf{x} + K_\Omega h^2\|\mathcal{L}\|_* + o(h^2)
\end{aligned} \tag{4.19}$$

By (4.16) and (4.19), and by setting $\mathbf{A}_h := h^{-1}(1-\cos\theta_h)\mathbf{W}_h^2$, we deduce that

$$\int_\Omega |\mathbb{E}\mathbf{u}_h|^2\,d\mathbf{x} \le h^{-2}\int_\Omega |\mathbb{E}\mathbf{v}_h|^2\,d\mathbf{x} \le 1 \tag{4.20}$$

and that

$$\frac{1}{C_\Omega}\int_\Omega |\mathbb{E}\mathbf{u}_h - \mathbf{A}_h|^2 \le -\frac{1}{2}\int_\Omega \mathbb{E}\mathbf{u}_*\, D^2\mathcal{W}(x,\mathbf{I})\,\mathbb{E}\mathbf{u}_*\,d\mathbf{x} + K_\Omega\|\mathcal{L}\|_* + o(1) \tag{4.21}$$

as $h \to 0$. By (4.20) and (4.21) we deduce

$$\begin{aligned}
|\Omega||\mathbf{A}_h|^2 &\le 2\int_\Omega |\mathbf{A}_h - \mathbb{E}\mathbf{u}_h|^2 + 2\int_\Omega |\mathbb{E}\mathbf{u}_h|^2 \\
&\le 2C_\Omega K_\Omega\|\mathcal{L}\|_* + o(1) + 2 \le C\,C_\Omega + 2 + o(1)
\end{aligned}$$

as $h \to 0$, where the last inequality is due to $\|\mathcal{L}\|_* \le \frac{C}{2K_\Omega}$ which follows from the assumption (4.15). This implies that $\sup_{h\in(0,1)}|\mathbf{A}_h| < +\infty$ and

$$|\mathbf{A}_h|^2 \le C\,C_\Omega\,|\Omega|^{-1} + 2|\Omega|^{-1} + o(1) \quad \text{as } h \to 0. \tag{4.22}$$

Step 3. We end the proof by contradiction, assuming that there exists a vanishing sequence $(h_n)_{n\in\mathbb{N}} \subset (0,1)$ such that

$$h_n^2\int_\Omega |\mathbb{E}\mathbf{u}_{h_n}|^2 = \int_\Omega |\mathbb{E}\mathbf{v}_{h_n}|^2 = h_n^2 \tag{4.23}$$

for all $n \in \mathbb{N}$. Thanks to Lemma 4.6 (applied with $M=1$), along a not relabeled subsequence we have $\mathbb{E}\mathbf{u}_{h_n} \rightharpoonup \mathbb{E}\mathbf{u}_*$ weakly in $L^2(\Omega,\mathbb{R}^{3\times3})$: indeed, by (4.15) we have $\|\mathcal{L}\|_* \le \frac{C}{2K_\Omega}$ so that

$\int_\Omega |\mathbb{E}\mathbf{u}_*|^2 \le 1$ thanks to Lemma 4.2 (with $M = 1$). Thanks to (4.23) and to (4.21) we have for every $\alpha \in (0, C_\Omega^{-1}]$

$$\begin{aligned}\alpha\left(1 - 2\int_\Omega \mathbb{E}\mathbf{u}_{h_n} : \mathbf{A}_{h_n}\right) &= \alpha\left(\int_\Omega |\mathbb{E}\mathbf{u}_{h_n}|^2 - 2\int_\Omega \mathbb{E}\mathbf{u}_{h_n} : \mathbf{A}_{h_n}\right)\\ &\le \alpha \int_\Omega |\mathbb{E}\mathbf{u}_{h_n} - \mathbf{A}_{h_n}|^2 \le \frac{1}{C_\Omega}\int_\Omega |\mathbb{E}\mathbf{u}_{h_n} - \mathbf{A}_{h_n}|^2\\ &\le -\frac{1}{2}\int_\Omega \mathbb{E}\mathbf{u}_*\, D^2\mathcal{W}(x,\mathbf{I})\,\mathbb{E}\mathbf{u}_*\, d\mathbf{x} + K_\Omega \|\mathcal{L}\|_* + o(1)\end{aligned}$$

as $n \to +\infty$. We pass to the limit as $n \to +\infty$ and we obtain

$$\alpha\left(1 - \int_\Omega 2\mathbb{E}\mathbf{u}_* : \mathbf{A}_*\right) \le -\frac{1}{2}\int_\Omega \mathbb{E}\mathbf{u}_*\, D^2\mathcal{W}(x,\mathbf{I})\,\mathbb{E}\mathbf{u}_*\, d\mathbf{x} + K_\Omega\|\mathcal{L}\|_*$$

for every $\alpha \in (0, C_\Omega^{-1}]$. Therefore, by taking into account the basic ellipticity estimate $\mathrm{sym}\mathbf{F}\, D^2\mathcal{W}(\mathbf{x},\mathbf{I})\,\mathrm{sym}\mathbf{F} \ge C|\mathrm{sym}\mathbf{F}|^2$, we get

$$\alpha\left(1 - \int_\Omega 2\mathbb{E}\mathbf{u}_* : \mathbf{A}_*\right) + \frac{C}{2}\int_\Omega |\mathbb{E}\mathbf{u}_*|^2 \le K_\Omega\|\mathcal{L}\|_*,$$

and since

$$2\alpha \int_\Omega \mathbb{E}\mathbf{u}_* : \mathbf{A}_* \le \frac{C}{2}\int_\Omega |\mathbb{E}\mathbf{u}_*|^2 + \frac{2}{C}\,\alpha^2|\Omega||\mathbf{A}_*|^2,$$

we obtain

$$\alpha - \frac{2\alpha^2|\Omega||\mathbf{A}_*|^2}{C} \le K_\Omega\|\mathcal{L}\|_* \tag{4.24}$$

for every $\alpha \in (0, C_\Omega^{-1}]$. By setting

$$\alpha_* := \frac{C}{4CC_\Omega + 8},$$

we have $\alpha^* < C_\Omega^{-1}$ and (4.22) entails

$$\frac{2\alpha_*^2|\Omega||\mathbf{A}_*|^2}{C} \le \frac{\alpha_*}{2}.$$

Choosing $\alpha = \alpha_*$ in (4.24), we get $\alpha_* \le 2K_\Omega\|\mathcal{L}\|_*$, which contradicts (4.15). ■

Corollary 4.8. *Suppose that the assumptions of* Theorem 2.5 *are satisfied. Let*

$$M_0 := \max\left\{1, \frac{2\|\mathcal{L}\|_*}{\gamma(\Omega, C)}\right\},$$

where $\gamma(\Omega, C)$ *is defined in* Lemma 4.7. *There exist* $h_0 > 0$ *such that if* $0 < h < h_0$ *and* $\mathbf{v}_h$ *is a solution to*

$$\min\left\{\mathcal{F}(\mathbf{v}; h\mathcal{L}) : \mathbf{v} \in H^1(\Omega,\mathbb{R}^3),\ \int_\Omega |\mathbb{E}(\mathbf{v})|^2 \le M_0^2 h^2,\ \int_\Omega \mathrm{curl}\,\mathbf{v} = \mathbf{0}\right\}, \tag{4.25}$$

then $\mathbf{v}_h$ *is a local minimizer for* $\mathcal{F}(\cdot; h\mathcal{L})$ *over* $H^1_{\mathbf{I}}(\Omega,\mathbb{R}^3)$ *and* $\mathbf{u}_h := h^{-1}\mathbf{v}_h$ *is a local minimizer for* $\mathcal{F}_h(\cdot;\mathcal{L})$ *over* $H^1_{\mathbf{I}}(\Omega,\mathbb{R}^3)$.

Proof. The existence of a solution to problem (4.25) is due to Lemma 4.3 and Remark 4.5. Since $\|M_0^{-1}\mathcal{L}\|_* \le \gamma(\Omega, C)/2$, we may apply Lemma (4.7) with $M_0^{-1}\mathcal{L}$ in place of $\mathcal{L}$, thus finding $h_0 > 0$ such that $\mathbf{v}_h$ satisfies $\int_\Omega |\mathbb{E}\mathbf{v}_h|^2 < M_0^2 h^2$ as soon as $h \in (0, h_0)$.

Let $\boldsymbol{\psi} \in H^1(\Omega, \mathbb{R}^3)$ be such that $\int_\Omega \operatorname{curl} \boldsymbol{\psi} = \mathbf{0}$. For any given $h \in (0, h_0)$, the continuity of the map $[0,1] \ni \varepsilon \mapsto \int_\Omega |\mathbb{E}\mathbf{v}_h + \varepsilon\boldsymbol{\psi}|^2$ shows that there exists $\varepsilon_0 = \varepsilon_0(\mathbf{v}_h, \boldsymbol{\psi})$ such that for every $\varepsilon < \varepsilon_0$ there holds $\int_\Omega |\mathbb{E}\mathbf{v}_h + \varepsilon\boldsymbol{\psi}|^2 < M_0^2 h^2$. Thus for every $\varepsilon < \varepsilon_0$ we have that $\mathbf{v}_h + \varepsilon\boldsymbol{\psi}$ is admissible for problem (4.25) and so $\mathcal{F}(\mathbf{v}_h; h\mathcal{L}) \le \mathcal{F}(\mathbf{v}_h + \varepsilon\boldsymbol{\psi}; h\mathcal{L})$ as claimed. The statement about $\mathbf{u}_h$ follows from Remark 2.3. ■

Proof of Theorem 2.5. Let M_0 and h_0 be as in Corollary 4.8 and let $(h_j)_{j\in\mathbb{N}} \subset (0, h_0)$ be a vanishing sequence. For every $j \in \mathbb{N}$, let $\mathbf{v}_j$ be a solution to problem (4.25) (with h_j in place of h), whose existence is ensured by Remark 4.5. By Corollary 4.8, for every $j \in \mathbb{N}$ we have that $\mathbf{v}_j$ is a local minimizer for $\mathcal{F}(\cdot; h_j\mathcal{L})$ over $H^1_{\mathbf{I}}(\Omega, \mathbb{R}^{3\times3})$, and $\mathbf{u}_j := h_j^{-1}\mathbf{v}_j$ is a local minimizer for $\mathcal{F}_{h_j}(\cdot; \mathcal{L})$ over $H^1_{\mathbf{I}}(\Omega, \mathbb{R}^{3\times3})$. Moreover, by Lemma 4.6 we get, up to subsequences, $\mathbb{E}(\mathbf{u}_j) = {h_j}^{-1}\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{u}_*)$ weakly in $L^2(\Omega, \mathbb{R}^{3\times3})$ as $j \to \infty$ and

$$\lim_{j\to\infty} \mathcal{F}_{h_j}(\mathbf{u}_j; \mathcal{L}) = \lim_{j\to+\infty} h_j^{-2}\mathcal{F}(\mathbf{v}_j; h_j\mathcal{L}) = \mathcal{F}_0(\mathbf{u}_*; \mathcal{L}),$$

where $\mathbf{u}_* \in H^1(\Omega, \mathbb{R}^3)$ is a solution to

$$\min_{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3)} \left\{ \mathcal{F}_0(\mathbf{u}; \mathcal{L}) : \int_\Omega |\mathbb{E}\mathbf{u}|^2 \le M_0^2 \right\}.$$

Since the choice of M_0 from Corollary 4.8 is such that $M_0^{-1}\|\mathcal{L}\|_* \le \gamma(\Omega, C)/2$, we obtain $\|\mathcal{L}\|_* \le \frac{CM_0}{K_\Omega}$, thus by Lemma 4.2 we deduce that $\mathbf{u}_*$ minimizes $\mathcal{F}(\cdot; \mathcal{L})$ over the whole $H^1(\Omega, \mathbb{R}^3)$. ■

Proof of Corollary 2.6. Since $\mathbf{R} \in \mathcal{S}^0_{\mathcal{L}}$, then $\mathbf{R}^T\mathcal{L}$ satisfies (2.8) as remarked in Section 2. Hence, by Theorem 2.5 there exist a sequence $\{h_j\}_{j\in\mathbb{N}} \subset (0,1)$ and local minimizers $\mathbf{v}_j$ for $\mathcal{F}(\cdot\,; h_j\mathbf{R}^T\mathcal{L})$ over $H^1_{\mathbf{I}}(\Omega, \mathbb{R}^3)$ such that

$$\lim_{j\to\infty} h_j^{-2}\mathcal{F}(\mathbf{v}_j\,; h_j\mathbf{R}^T\mathcal{L}) = \beta(\mathbf{R}),$$

where β is defined by (2.11). By setting $\mathbf{y}_j(\mathbf{x}) := \mathbf{R}\mathbf{x} + h_j\mathbf{R}\mathbf{v}_j$ we get $\int_\Omega \operatorname{curl} \mathbf{R}^T\mathbf{y}_j = 0$. By taking into account frame indifference and again that $\mathbf{R} \in \mathcal{S}^0_{\mathcal{L}}$ we get

$$\begin{aligned}
\mathcal{G}(\mathbf{y}_j\,; h_j\mathcal{L}) &= \int_\Omega \mathcal{W}(\mathbf{x}, \mathbf{R} + h_j\nabla\mathbf{R}\mathbf{v}_j)\,d\mathbf{x} - h_j\mathcal{L}(\mathbf{y}_j - \mathbf{x}) \\
&= \int_\Omega \mathcal{W}(\mathbf{x}, \mathbf{I} + h_j\nabla\mathbf{v}_j)\,d\mathbf{x} - h_j\mathcal{L}(\mathbf{y}_j - \mathbf{R}\mathbf{x}) \\
&= \int_\Omega \mathcal{W}(\mathbf{x}, \mathbf{I} + h_j\nabla\mathbf{v}_j)\,d\mathbf{x} - h_j^2\mathcal{L}(\mathbf{R}\mathbf{v}_j) = \mathcal{F}(\mathbf{v}_j\,; h_j\mathbf{R}^T\mathcal{L}).
\end{aligned}$$

Therefore $\mathbf{y}_j$ is a local minimizer for $\mathcal{G}(\cdot\,; h_j\mathcal{L})$ over $H^1_{\mathbf{R}}(\Omega, \mathbb{R}^3)$ and

$$\lim_{j\to\infty} h_j^{-2}\mathcal{G}(\mathbf{y}_j\,; h_j\mathcal{L}) = \lim_{j\to\infty} h_j^{-2}\mathcal{F}(\mathbf{v}_j\,; h_j\mathbf{R}^T\mathcal{L}) = \beta(\mathbf{R})$$

as claimed. ■

Proof of Corollary 2.7. Since $\mathcal{S}^0_{\mathcal{L}} \setminus \{\mathbf{I}\} \neq \emptyset$ and since β is not a constant map then $\beta(\mathcal{S}^0_{\mathcal{L}}) = [a, b]$ for some couple of reals $a, b,\ a < b$, and so for every $n \in \mathbb{N}$ there exists $\{\mathbf{R}_k : k = 1, 2...n\} \subset \mathcal{S}^0_{\mathcal{L}}$ such that $\beta(\mathbf{R}_k) \neq \beta(\mathbf{R}'_k)$ if $k \neq k'$. By Corollary 2.6 for every vanishing sequence $(h_j)_{j\in\mathbb{N}} \subset (0, 1)$ and for every $k = 1, 2....n$ there exist a subsequence $(h^{(k)}_j)_{j\in\mathbb{N}}$ and local minimizers $\mathbf{y}^{(k)}_j$ for $\mathcal{G}(\cdot\, ; h^{(k)}_j \mathcal{L})$ over $H^1_{\mathbf{R}_k}(\Omega, \mathbb{R}^3)$ such that

$$\lim_{j\to\infty} \left(h^{(k)}_j\right)^{-2} \mathcal{G}(\mathbf{y}^{(k)}_j ; h^{(k)}_j \mathcal{L}) = \beta(\mathbf{R}_k).$$

We let

$$\delta_n := \min\{|\beta(\mathbf{R}_k) - \beta(\mathbf{R}'_k)| : k \neq k',\ k, k' = 1, 2, ...n\}.$$

It is readily seen that for every $k = 1, 2....n$ there exists $h^{(k)}_0 > 0$ such that, for every $0 < h < h^{(k)}_0$, the functional $\mathcal{G}(\cdot\, ; h\mathcal{L})$ has a local minimizer $\mathbf{y}^{(k)}$ over $H^1_{\mathbf{R}_k}(\Omega, \mathbb{R}^3)$ satisfying

$$|\mathcal{G}(\mathbf{y}^{(k)} ; h\mathcal{L}) - \beta(\mathbf{R}_k)| < \delta_n/2.$$

The result follows by choosing $h_0 := \min\{h^{(k)}_0 : k = 1, 2...n\}$. ■

5. Constrained local minimizers at different energy levels: an example

In this Section we give an explicit example in which the situation described in Corollary 2.6 and Corollary 2.7 occurs. To this aim we shall consider the Yeoh type energy density defined by (2.1) and (2.4), with $\mathcal{W}_{\rm vol}(\mathbf{F}) = g(\det \mathbf{F})$, where $g : \mathbb{R}_+ \to \mathbb{R}$ is the convex C^2 function (satisfying (2.2) and (2.3) with $r = 2$) obtained by setting

$$g(t) = c(t^2 - 1 - 2 \log t)$$

for some $c > 0$. This is a usual choice, see [14].

By taking into account that for every $\mathbf{B} \in \mathbb{R}^{3\times3}_{\rm sym}$ there holds

$$\frac{|\mathbf{I} + \varepsilon\mathbf{B}|^2}{\det(\mathbf{I} + \varepsilon\mathbf{B})^{2/3}} - 3 = \varepsilon^2(2|\mathbf{B}|^2 - \tfrac{4}{3}|\mathrm{Tr}\mathbf{B}|^2) + o(\varepsilon^2)$$

as $\varepsilon \to 0$, and since

$$\mathcal{W}_{\rm vol}(\mathbf{I} + \varepsilon\mathbf{B}) = g(\det(\mathbf{I} + \varepsilon\mathbf{B})) = \frac{\varepsilon^2}{2} g''(1)|\mathrm{Tr}\mathbf{B}|^2 + o(\varepsilon^2) = 2c|\mathrm{Tr}\mathbf{B}|^2 + o(\varepsilon^2),$$

we get

$$\frac{1}{2}\mathbf{B}\, D^2\mathcal{W}(\mathbf{I})\, \mathbf{B} = 2c_1|\mathbf{B}|^2 + (2c - \tfrac{4}{3}c_1)|\mathrm{Tr}\mathbf{B}|^2$$

and we choose from now on $c_1 = 2,\ c = \frac{4}{3}$ so that

$$\frac{1}{2}\mathbf{B}\, D^2\mathcal{W}(\mathbf{I})\, \mathbf{B} = 4|\mathbf{B}|^2.$$

Let now

$$\Omega := \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1,\ 0 < z < 1\}, \tag{5.1}$$

$B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, let $\phi : B \to \mathbb{R}$ be the radial function whose radial profile (still denoted by ϕ) is

$$\phi(r) := \log r + r^2 - 3r + 2, \qquad r := \sqrt{x^2 + y^2}, \tag{5.2}$$

and let us consider a load functional of the form

$$\mathcal{L}(\mathbf{u}) = \int_\Omega \mathbf{f}\cdot\mathbf{u}\,d\mathbf{x} \qquad \text{where} \qquad \mathbf{f}(\mathbf{x}) = \mathbf{f}(x,y,z) := r^{-1}\phi'(r)(x,y,0). \tag{5.3}$$

It is readily seen that $\phi(1) = \phi'(1) = 0$ and that $\mathbf{f} \in L^p(\Omega,\mathbb{R}^3)$ if and only if $p < 2$ (so that assumption (1.5) is not satisfied) and thus $\mathcal{L} \in (H^1(\Omega,\mathbb{R}^3))^*$. It is also easy to see that $\mathcal{L}$ is equilibrated. We claim that $\mathcal{L}((\mathbf{R}-\mathbf{I})\mathbf{x}) = 0$ for every $\mathbf{R} \in SO(3)$, that is, condition (1.3) is satisfied and $\mathcal{S}^0_{\mathcal{L}} \equiv SO(3)$ (which implies that every direction in $\mathbb{R}^3$ is an axis of equilibrium of $\mathcal{L}$, because the astatic load is zero). Indeed if $\mathbf{R} \in SO(3)$ then by the Euler-Rodrigues formula there exist $\theta \in [0,2\pi)$ and $\mathbf{a} \in \mathbb{R}^3$, $|\mathbf{a}| = 1$, such that for every $\mathbf{x} \in \Omega$

$$\mathbf{R}\mathbf{x} = \mathbf{x} + \sin\theta\,(\mathbf{a}\wedge\mathbf{x}) + (\cos\theta - 1)\,\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{x}),$$

hence by setting $c(\mathbf{a},\theta) = (\cos\theta-1)(\pi(1-a_3^2)-1)$ we have

$$\begin{aligned}\int_\Omega \mathbf{f}(\mathbf{x})\cdot(\mathbf{R}-\mathbf{I})\mathbf{x}\,d\mathbf{x} &= \sin\theta\int_\Omega \mathbf{f}(\mathbf{x})\cdot(\mathbf{a}\wedge\mathbf{x})\,d\mathbf{x} + (\cos\theta-1)\int_\Omega \mathbf{f}(\mathbf{x})\cdot((\mathbf{a}\cdot\mathbf{x})\mathbf{a}-\mathbf{x})\,d\mathbf{x}\\ &= c(\mathbf{a},\theta)\int_0^1 r^2\phi'(r)\,dr = -2c(\mathbf{a},\theta)\int_0^1 r\phi(r)\,dr = 0\end{aligned}$$

as claimed.

Let us now consider the following rotation of $\pi/2$ around the z axis

$$\mathbf{R}_* := \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We claim that $\beta(\mathbf{R}_*) < \beta(\mathbf{I})$, that is, the map β defined in (2.11) takes two different values as required in Corollary 2.7. To this aim we need the following

Lemma 5.1. *Let Ω as in* (5.1)*, ϕ as in* (5.2) *and $\mathcal{L}$ as in* (5.3)*. Then*

$$\min_{H^1(\Omega;\mathbb{R}^3)} \mathcal{F}_0(\cdot\,;\mathbf{R}_*^T\mathcal{L}) \le \min_{u\in H^2(B)}\int_B 8u_{xy}^2 + 2(u_{yy}-u_{xx})^2 - \int_B (u_y\phi_y + u_x\phi_x) < 0.$$

Proof. Let B the unit ball in $\mathbb{R}^2$ and

$$\mathcal{K} := \{(u_y, -u_x, 0) : u \in H^2(B)\} \subset H^1(\Omega;\mathbb{R}^3).$$

It is easily seen that $\mathcal{K}$ is weakly closed in $H^1(\Omega,\mathbb{R}^3)$ so there are minimizers of $\mathcal{F}_0(\cdot\,;\mathbf{R}_*^T\mathcal{L})$ over $\mathcal{K}$ (recalling that $\mathbf{R}_*^T\mathcal{L}$ satisfies (2.8) since the z axis is an axis of equilibrium for $\mathcal{L}$). Therefore

$$\min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^3)} \mathcal{F}_0(\mathbf{v}\,;\mathbf{R}_*^T\mathcal{L}) \le \min_{u\in H^2(B)}\int_B 8u_{xy}^2 + 2(u_{yy}-u_{xx})^2 - \int_B (u_y\phi_y + u_x\phi_x) \tag{5.4}$$

If u is a minimizer of the right hand side of (5.4), $0 < \varepsilon < 1$, $B_\varepsilon := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < \varepsilon\}$ and $\zeta \in H^2_0(B\setminus B_\varepsilon)$, by taking into account that $\Delta\phi \in L^2(B\setminus B_\varepsilon)$, a first variation argument for the minimization problem in the right hand side of (5.4) yields

$$4\int_{B\setminus B_\varepsilon}(4u_{xy}\zeta_{xy} + u_{yy}\zeta_{yy} + u_{xx}\zeta_{xx} - u_{yy}\zeta_{xx} - u_{yy}\zeta_{xx}) = \int_{B\setminus B_\varepsilon}(\zeta_y\phi_y + \zeta_x\phi_x) = -\int_{B\setminus B_\varepsilon}\zeta\Delta\phi$$

and after integration by parts we obtain the Euler-Lagrange equation

$$4\Delta^2 u + \Delta\phi = 0 \qquad \text{in } \mathcal{D}'(B\setminus B_\varepsilon),$$

for every $0<\varepsilon<1$, where Δ^2 denotes the planar biharmonic operator. On the other hand if u is a minimizer then by using u as test function we easily get

$$\int_B 16u_{xy}^2+4(u_{yy}-u_{xx})^2+(u_y\phi_y+u_x\phi_x)=0. \tag{5.5}$$

In view of (5.4), and since $\mathcal{F}_0(\mathbf{0}\,;\mathbf{R}_*^T\mathcal{L})=0$, in order to conclude it is enough to show that $\min_{\mathbf{v}\in\mathcal{K}}\mathcal{F}_0(\mathbf{v}\,;\mathbf{R}_*^T\mathcal{L})\neq 0$. Assume by contradiction that $\min_{\mathbf{v}\in\mathcal{K}}\mathcal{F}_0(\mathbf{v}\,;\mathbf{R}_*^T\mathcal{L})=0$: then by (5.5) we get

$$\int_B 4u_{xy}^2+(u_{yy}-u_{xx})^2=0$$

that is $u_{xy}=u_{yy}-u_{xx}=0$ a.e. in B hence $\Delta^2 u=0$ a.e. in B and from the above Euler-Lagrange equation we deduce $\Delta\phi\equiv 0$ in $B\setminus B_\varepsilon$, a contradiction since $\Delta\phi=3-\frac{2}{r}$ in $B\setminus B_\varepsilon$.
■

We can now conclude by proving the claim, i.e., by proving that $\beta(\mathbf{R}_*)<\beta(\mathbf{I})$. Let $\mathbf{v}=(v_1,v_2,v_3)\in H^1(\Omega,\mathbb{R}^3)$, $\tilde{\mathbf{v}}:=(v_1,v_2)$ and let $\tilde{\mathbf{u}}\in H^1(B,\mathbb{R}^2)$ be defined by $\tilde{\mathbf{u}}(x_1,x_2):=\int_0^1\tilde{\mathbf{v}}(x_1,x_2,x_3)\,dx_3$. By Jensen inequality it is readily seen that

$$\mathcal{F}_0(\mathbf{v}\,;\mathcal{L})\geq 4\int_B|\widetilde{\mathbb{E}}(\tilde{\mathbf{u}})|^2-\int_B\nabla\phi\cdot\tilde{\mathbf{u}}$$

where $\widetilde{\mathbb{E}}(\cdot)$ is the upper-left 2×2 submatrix of $\mathbb{E}(\cdot)$. By arguing as in the proof of Theorem 2.7 of [17], if

$$\eta_*(r)=-\frac{1}{16}\,r\phi(r)+\frac{1}{16r}\int_0^r t^2\phi'(t)\,dt$$

then the radial function defined by $w(r):=\int_0^r\eta_*(t)\,dt$ belongs to $H^2(B)$ and ∇w minimizes

$$\mathcal{J}(\tilde{\mathbf{u}}):=4\int_B|\widetilde{\mathbb{E}}(\tilde{\mathbf{u}})|^2-\int_B\nabla\phi\cdot\tilde{\mathbf{u}}$$

over $H^1(B)$, hence w minimizes

$$4\int_B|D^2v|^2-\int_B\nabla\phi\cdot\nabla v$$

among all v in $H^2(B)$, where D^2 denotes the Hessian in the x,y variables. Therefore for every $0<\varepsilon<1$ and for every $\psi\in H_0^2(B\setminus B_\varepsilon)$

$$8\int_{B\setminus B_\varepsilon}D^2v\cdot D^2\psi-\int_{B\setminus B_\varepsilon}\nabla\phi\cdot\nabla\psi=0,$$

that is, w solves the biharmonic equation $8\Delta^2 w=-\Delta\phi$ in $B\setminus B_\varepsilon$ and since $\Delta\phi$ is not identically zero then Δw is not identically zero as well. This implies by Young inequality

$$\int_B 8w_{xy}^2+2(w_{yy}-w_{xx})^2-\int_B\nabla w\cdot\nabla\phi<\int_B 8w_{xy}^2+4w_{yy}^2+4w_{xx}^2-\int_B\nabla w\cdot\nabla\phi. \tag{5.6}$$

Notice that the inequality is strict, since the Young inequality $2(w_{yy}-w_{xx})^2\leq 4w_{xx}^2+4w_{yy}^2$ holds with equality if and only if $w_{xx}=-w_{yy}$, and we have just checked that Δw does not vanish identically on B. In particular, $2(w_{xx}-w_{yy})^2 < 4w_{xx}^2+4w_{yy}^2$ on a set of positive measure in B. From Lemma 5.1 and from (5.6) we infer

$$\beta(\mathbf{R}_*) = \min_{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3)} \mathcal{F}(\mathbf{u}\,;\mathbf{R}_*^T\mathcal{L}) \le \min_{v\in H^2(B)} \int_B 8v_{xy}^2 + 2(v_{yy}-v_{xx})^2 - \int_B \nabla v\cdot\nabla\phi$$
$$\le \int_B 8w_{xy}^2 + 2(w_{yy}-w_{xx})^2 - \int_B \nabla w\cdot\nabla\phi < \int_B 8w_{xy}^2 + 4w_{xx}^2 + 4w_{yy}^2 - \int_B \nabla w\cdot\nabla\phi$$
$$= \min_{v\in H^2(B)} \int_B 8v_{xy}^2 + 4v_{xx}^2 + 4v_{yy}^2 - \int_B \nabla v\cdot\nabla\phi \le \min_{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3)} \mathcal{F}(\mathbf{u}\,;\mathcal{L}) = \beta(\mathbf{I})$$

as claimed. In particular, Corollary 2.7 applies to this example.

Acknowledgements. The authors acknowledge support from the MIUR-PRIN project No 2017TEXA3H. The authors are members of the GNAMPA group of the Istituto Nazionale di Alta Matematica (INdAM).

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