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Representation Theory in Homotopy Algebraic Quantum Field Theory

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Abstract

Algebraic quantum field theory (AQFT) is an axiomatic approach aiming to provide a rigorous framework for the description of quantum field theories. Although it put forward several new insights and allowed for the development of quantum field theory on curved spacetimes, it became apparent that its axioms were too strict for the description of quantum gauge field theories. This motivated the initiation of the program of homotopy AQFT.

Homotopy AQFT attempts to combine the framework of the Batalin–Vilkovisky (BV) formalism with the insights gained from AQFT, with the goal to provide an axiomatic framework for quantum gauge theories. The main goal of this dissertation is to study representations of homotopical AQFTs.

An important aspect of this framework is that homotopical AQFTs come equipped with a notion of weak equivalences. Two weakly equivalent homotopical AQFTs are understood to be physically equivalent. Therefore, our main challenge is to introduce a concept of representations of homotopical AQFTs such that it is consistent with the weak equivalences between homotopical AQFTs. To this end, after defining representations of homotopical AQFTs by generalizing the usual concept of representations in AQFT, we introduce a notion of weak equivalences between them. These weak equivalences determine a homotopy category of representations (i.e. the category obtained by formally inverting these weak equivalences) for each homotopical AQFT. Then, we utilize the theory of model categories to prove that, given two weakly equivalent homotopical AQFTs, the associated homotopy categories of representations are equivalent, demonstrating this way the compatibility of the proposed concept of representations with the homotopy AQFT framework.

Having established the concept of representations for homotopical AQFTs, we aim next at constructing explicit representations for the homotopy AQFT associated with Maxwell p-forms. Assuming an ultrastatic spacetime with compact Cauchy surface, we illustrate that one can construct representations in this framework mimicking the usual procedure of constructing an appropriate two-point function for the global (differential graded) algebra

of observables, and then producing a representation of the global algebra of observables in a manner similar to the Gelfand–Naimark–Segal construction. Then, a representation for the homotopical AQFT is obtained by simply restricting the representation of the global algebra of observables to the local ones.

Contents

Introduction

Quantum field theory (QFT) has been one of the most successful physical theories. Combining quantum mechanics and the theory of special relativity, it led to a far reaching understanding of the structure of physics at subatomic scales, often achieving astonishing agreement between theoretical predictions and experiments. The pinnacle of quantum field theory has been the establishment of the Standard Model of particle physics. The Standard Model is a gauge theory, and more specifically a Yang–Mills theory with a $\mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(3)}$ gauge symmetry, that successfully unified the three of the four known fundamental forces and classified all known elementary particles. However, despite these successes, establishing the Standard Model as a mathematically complete and consistent quantum field theory remains an open question.

Algebraic quantum field theory (AQFT) is an axiomatic approach aiming to provide a rigorous mathematical framework for quantum field theory. Its origins lie in the seminal work of Haag and Kastler [HK64], who first realized that a quantum field theory can be described as an assignment of the algebra of local observables to each region of Minkowski spacetime. This assignment is subject to certain compatibility conditions motivated by physical principles.

One of the major successes of this framework was that it allowed for the development of QFT on curved spacetimes [Dim80], as those described in general relativity. This approach, known as locally covariant QFT, was formally axiomatized utilizing the language of category theory in [BFV03]. In this formalism, a quantum field theory is described as a covariant functor $\mathfrak{A}$ from the category **Loc** of globally hyperbolic Lorentzian manifolds with morphisms isometric embeddings to the category $\mathbf{C^*Alg}$ of unital C*-algebras with morphisms unital *-homomorphisms. This functor is subject to the following axioms:

(a) **Isotony:** Given a morphism $f\colon M_1 \to M_2$ in **Loc**, the morphism $\mathfrak{A}(f)\colon \mathfrak{A}(M_1) \to \mathfrak{A}(M_2)$ in $\mathbf{C^*Alg}$ is injective.

(b) **Einstein causality:** Given morphisms $f_1\colon M_1 \to M$ and $f_2\colon M_2 \to M$

in **Loc** such that their images in M are causally disjoint, the images of the corresponding algebras $\mathfrak{A}(f_1)(\mathfrak{A}(M_1))$ and $\mathfrak{A}(f_2)(\mathfrak{A}(M_2))$ commute as subalgebras of $\mathfrak{A}(M)$.

(c) **Time-slice axiom:** Given a Cauchy embedding, i.e. a morphism $f\colon M_1 \to M_2$ in **Loc** that contains a Cauchy surface of M_2, the morphism $\mathfrak{A}(f)\colon \mathfrak{A}(M_1) \to \mathfrak{A}(M_2)$ in $\mathbf{C^*Alg}$ is an isomorphism.

A QFT described by this framework is called a *locally covariant QFT*.

This framework proved to be very fruitful and flexible. Soon, concrete models of locally covariant QFTs associated with free fields were constructed [BFV03; DHP09; San10; Dap11]. Moreover, this framework was adapted to allow for the description of locally covariant conformal field theories [Pin09; CRV22; BGS22] by modifying the source category **Loc** so that morphisms are defined as conformal embeddings instead of isometric embeddings. Furthermore, this framework has also been adapted for the description of locally covariant perturbative algebraic quantum field theory [BF00; HW01; BDF09] by replacing the target category $\mathbf{C^*Alg}$ with the category of algebras over a ring of formal power series instead of the field of complex numbers.

However, it turned out that the axioms of AQFT are too strict for the description of quantum gauge field theories, which are of utmost interest in Physics, due to their integral role in the Standard Model of particle physics. It was first noticed in an attempt to quantize Maxwell's equations [DL12] that the Faraday tensor violates the isotony axiom. It was then understood that the root of this incompatibility is that some elements in the algebra of observables corresponding to the quantization of gauge field theories are sensitive to topological properties of the underlying spacetime [Ben+14; BDS14]. This realization motivated the initiation of the homotopy AQFT program [BS19a; BSW19a; BBS20].

In general, gauge field theories exhibit certain symmetries. By that, we mean that applying certain transformations (the so-called gauge transformations) to the mathematical model employed to describe a gauge field, leaves the physical system described by this model invariant, even if the mathematical model is not. In this sense, we say that two gauge fields connected via a gauge transformation are physically equivalent. These symmetries are called gauge symmetries.

Homotopy AQFT uses the Batalin–Vilkovisky (BV) formalism in order to describe quantum gauge theories. The BV formalism, developed by Batalin and Vilkovisky in [BV77; BV81; BV83a; BV83b], in order to describe the algebra of observables associated with a physical system, utilizes the category $\mathbf{DGA}_{\mathbb{C}}$ of differential graded algebras (DG-algebras) over the field of complex numbers $\mathbb{C}$, instead of the category $\mathbf{C^*Alg}$ of C*-algebras. DG-algebras are essentially cochain complexes equipped with a compatible algebra structure.

In order to describe the gauge symmetries, the BV formalism relies on the differential of the cochain complexes. Specifically, the gauge fields are represented by degree 0 elements, while gauge transformations (i.e. the ghost fields) correspond to degree -1 elements. Two gauge fields are physically equivalent when they belong to the same cohomology class, i.e. they differ by the differential of a gauge transformation. That implies that the physical information is encoded in the cohomology of the cochain complex rather than the cochain complex itself. Therefore, when two cochain complexes are connected via a quasi-isomorphism, i.e. a cochain map that preserves cohomology, then they are physically equivalent. It is precisely this fact that allows for the introduction of the auxiliary fields which are eventually responsible for the efficacy of this approach (see [Hol08; FR12; FR13] for applications in AQFT).

In order for AQFT to be able to employ the BV formalism for the description of gauge theories, it needs to recognize that the physical information associated with a local algebra of observables represented by a DG-algebra is encapsulated in its cohomology. In other words, two local algebras of observables represented by two quasi-isomorphic DG-algebras need to be treated as equivalent. In homotopy AQFT this is achieved by introducing an appropriate notion of weak equivalence between nets of DG-algebras and then employing homotopy theory [BS19a; BSW19a].

In more detail, in homotopy AQFT, a quantum field theory is described by a net of DG-algebras, i.e. a functor from the category $\mathbf{Loc}$ to $\mathbf{DGA}_{\mathbb{C}}$. In order to accommodate for the fact that weakly equivalent DG-algebras need to be treated as equivalent, the axioms of AQFT are relaxed and replaced by their homotopical analogues.

Specifically, the time-slice axiom is modified. The requirement for a QFT to assign an isomorphism between the local algebras of observables to a Cauchy embedding is relaxed to the requirement to assign a quasi-isomorphism instead. The isotony axiom is abandoned, since it is understood that local observables corresponding to the topological properties of the underlying manifold do not need to be mapped injectively to the local algebra of observables of a larger region [Ben+14; BDS14]. Finally, the Einstein causality axiom can also be relaxed, however that leads to an equivalent description of homotopy AQFT [BSW19a].

A QFT described in this framework is called a *homotopical algebraic quantum field theory* (homotopical AQFT).

The notion of weak equivalence between homotopical AQFTs is defined as follows. Given two homotopical AQFTs $\mathfrak{A}$ and $\mathfrak{B}$, a natural transformation

$\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is a weak equivalence if, for each $M \in \mathbf{Loc}$, the morphisms $\Phi_M\colon \mathfrak{A}(M) \to \mathfrak{B}(M)$ in $\mathbf{CoCh}_{\mathbb{C}}$ are quasi-isomorphisms.

Then, using homotopy theory, one can study homotopical AQFTs up to weak equivalences (i.e. independently of the choice of model among weakly equivalent ones), conforming with the principle that quasi-isomorphic DG-algebras must be treated as equivalent.

Thus, homotopy AQFT manages to combine the advantages of the BV formalism with the insights of AQFT in order to provide an axiomatic framework for the description of quantum gauge field theories. In fact, the first construction of a proper example of a gauge theory in this framework was achieved in [BBS20], where the QFT corresponding to the linear Yang–Mills theory was described.

But, the concept of representations in this framework had not been studied before [AB23]. This is the focus of this dissertation.

Goal of the dissertation

Historically, QFT on Minkowski spacetime was developed in the form of operators acting on a Hilbert space. One of the great insights that AQFT put forward was that this structure combines two related, but different, pieces of data, namely (1) the algebra of observables, whose non-commutative multiplication witnesses the quantum nature of the theory, and (2) a distinguished choice of representation of this algebra, whose role is to draw a bridge between the abstract algebra of observables and its familiar implementation by operators acting on a Hilbert space. One of the fundamental contributions of the algebraic approach to quantum field theory [HK64] was to recognize and isolate the two different and complementary roles played by the data (1) and (2). On the one hand, the general axioms of quantum field theory can be postulated abstractly, at the algebraic level; on the other hand, the (typically many) inequivalent representations provide incarnations of an algebra of observables corresponding to different states of the physical system.

So far, the program of homotopy AQFT focused mainly on datum (1) of a quantum field theory, i.e. the algebras of observables, as mentioned above. The main purpose of the present dissertation is to investigate also datum (2), the representations of the algebras of observables and, in particular, their homotopy theory.

Representations in the context of AQFT first appeared in [FH87]. A representation of a net of C*-algebras was defined as a family of Hilbert space representations of the unital C*-algebras subject to compatibility conditions [RV12a; RV12b]. The main goal of this dissertation is to provide a generalization of this concept suitable for QFTs described in the framework of homotopy

AQFT. In particular, representations of homotopical AQFTs should respect the weak equivalences of homotopical AQFTs. That is, representations associated with a homotopical AQFT should be insensitive to the choice of model among the various weakly equivalent ones.

Main results

In order to define representations in the context of homotopy AQFT, we adopted a more general setting, utilizing homotopy theory and the theory of modules over monoids in a monoidal category. This framework allowed for the generalization of the concept of representations of nets of C*-algebras to representations of nets of monoids over arbitrary monoidal categories.

Specifically, the category $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ of $\mathbf{M}$-valued nets of algebras, for some monoidal category $\mathbf{M}$ and a category $\mathbf{C}$ (interpreted as a category of spacetime regions), is identified with the category $\mathbf{Fun}(\mathbf{C}, \mathbf{Mon}(\mathbf{M}))$ of functors from $\mathbf{C}$ to the category of monoids in $\mathbf{M}$. Morphisms in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ are simply natural transformations. In this framework, we introduce a concise definition of the category $\mathbf{Rep}(\mathfrak{A})$ of representations of a net $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ as the category of modules over $\mathfrak{A}$ (see Section 3.2), and we illustrate how it encompasses the defining data and axioms of representations of nets of C*-algebras that usually appear in the literature [FH87; RV12a; RV12b].

Notice that this framework describes the nets of DG-algebras, which are used in homotopy AQFT, when $\mathbf{C} = \mathbf{Loc}$ and $\mathbf{M} = \mathbf{CoCh}_{\mathbb{C}}$. Namely, homotopical AQFTs are objects in $\mathbf{Net}_{\mathbf{Loc}}^{\mathbf{CoCh}_{\mathbb{C}}}$. Then, a representation of a homotopical AQFT is defined simply as a module over it (see Section 3.8).

We, then, need to examine the appropriateness of this definition for the purposes of homotopy AQFT. As we already explained, homotopical AQFTs come with a concept of weak equivalence, namely a natural transformation whose components are quasi-isomorphisms, i.e. the weak equivalences in the category of cochain complexes. In our abstract setting, this concept can be reproduced by assuming that the category $\mathbf{M}$ is equipped with a collection of weak equivalences. Then, the category $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ inherits a collection of weak equivalences from $\mathbf{M}$ in a natural way. A morphism Φ in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ is a weak equivalence if and only if, for any object $c \in \mathbf{C}$, the component Φ_c is a weak equivalence in $\mathbf{M}$.

As we have already argued, in order for the proposed definition of representations to be sensible in the context of homotopy AQFT, it is necessary that the categories of representations corresponding to weakly equivalent nets of algebras are equivalent.

However, this is not, in general, the case. Indeed, in Section 3.8.1 we present a counterexample. In Example 3.8.11, we show that for two weakly equivalent nets of algebras in $\mathbf{Net}_{\mathbf{Loc}}^{\mathbf{CoCh}_{\mathbb{C}}}$, both implementing the Klein–Gordon field, the corresponding categories of net representations are manifestly inequivalent in the ordinary categorical sense.

We argue that this problem arises because the notion of isomorphism is too strict to capture the notion of physically equivalent representations of a homotopical AQFT, and it needs to be replaced by an appropriate notion of weak equivalence. Passing, then, to the associated homotopy category of net representations, i.e. the category obtained by formally inverting these weak equivalences, should resolve the issue.

In fact, the category of representations of a homotopical AQFT inherits a collection of weak equivalences from the category of cochain complexes in a way completely analogous to the category of nets of algebras (see Definition 3.6.2). Our task was to show that this notion of weak equivalences between representations of homotopical AQFTs is compatible with the weak equivalences between homotopical AQFTs. Namely, our goal is to show that given a weak equivalence $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{Net}_{\mathbf{Loc}}^{\mathbf{CoCh}_{\mathbb{C}}}$, the homotopy categories $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$, defined by formally inverting these collections of weak equivalences, are equivalent.

In order to prove this, we relied on the theory of model categories, which is instrumental for the study of homotopy categories.

So, returning to our abstract setting, we now assume that $\mathbf{M}$ is a closed symmetric monoidal *model* category. Then, using transfer theorems from Schwede & Shipley [SS00] and Berger & Moerdijk [BM03], we proved that (under certain extra assumptions on the category $\mathbf{M}$, which are satisfied by the category $\mathbf{CoCh}_{\mathbb{C}}$ of cochain complexes) the categories of net representations inherit the structure of a model category, with weak equivalences the ones inherited from $\mathbf{M}$ (see Proposition 3.6.17). Moreover, we showed that, with respect to this model structure, any morphism $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ induces a Quillen adjunction between $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\mathfrak{B})$ (see Proposition 3.6.19). This entails that Φ gives rise to an adjunction between the homotopy categories $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$. Finally, in Proposition 3.6.20 we proved that the induced Quillen adjunction between $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\mathfrak{B})$ is a Quillen equivalence when Φ is a weak equivalence. From this, it follows that the homotopy categories $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$ are equivalent.

Having established the concept of net representations of homotopical AQFTs, we turned our attention to the construction of explicit net representations for the interesting example of the homotopical AQFT associated with Maxwell p-forms.

The Maxwell p-form field equations are a generalization of the Maxwell equations over curved spacetimes of arbitrary dimension. Specifically, the

p-form field for $p = 1$ is the electromagnetic vector potential and for $p = 0$ the massless Klein–Gordon field [HT86; HT92].

For this example, we consider a fixed oriented and time-oriented globally hyperbolic Lorentzian manifold M, which for simplicity we assume ultra-static and admitting a compact spacelike Cauchy surface. As explained in more detail below, we proceed constructing a net representation by first constructing a representation of the algebra of observables that corresponds to the whole spacetime M and then use it to produce a representation for the net of algebras.

The construction of a representation of the global algebra of observables involves two steps. In the first step, we construct a two-point function ω_2 as a cochain map defined on the complex of linear observables on M. The fact that ω_2 is a cochain map simultaneously encodes its compatibility both with the action of gauge transformations and with the equation of motion. (Incidentally, for $p = 1$ we observe that ω_2 recovers in degree 0 cohomology the Hadamard two-point function constructed in [FP03] for the electromagnetic vector potential, thus drawing a bridge between the approach we propose and well-established ones.) The second step mimics the first stage of the Gelfand-Naimark-Segal construction to define from the two-point function ω_2 a left module on the global algebra of observables.

Having now a representation of the global algebra of observables, we derive a constant (in the sense of Section 3.3) net representation, which essentially amounts to restricting the global algebra representation to the local algebras.

Finally, we took advantage of the homotopy theory of net representations developed in the first part to present a very explicit description of the data of all constant (in the sense of Section 3.3) net representations up to weak equivalence in the simplest scenario, i.e. $p = 1$ and M the two-dimensional flat Lorentz cylinder.

All these results have already been published in [AB23].

In summary, this dissertation follows the trend, initiated by the seminal work of Brunetti, Fredenhagen and Verch [BFV03], of formulating and expanding the foundations of Physics utilizing the language of category theory. More specifically, it takes advantage of the theory of model categories to develop a framework for the description of representations of nets of algebras appearing in homotopy AQFT, contributing, in this sense, to the quest for obtaining an axiomatic framework for the description of quantum gauge field theories.

Moreover, due to the abstractness of this approach, it provides a universal framework for the description of representations of nets valued in a wide range of algebras and over any category of spacetimes that is essentially small. For instance, it can be applied on locally covariant conformal quantum field theories by using as the source category **C** the category with the same objects as **Loc** but with morphisms the conformal embeddings.

Outline of the dissertation

Chapter 1 contains no original content. It serves as a short introduction to the theory of modules in a monoidal category **M** and it collects some facts and constructions related to modules that will be of use in the subsequent chapters. In particular, we recall how a morphism of monoids gives rise to a change-of-monoid adjunction between the corresponding module categories, and we review the **M**-tensoring, **M**-powering and **M**-enriched hom functors over categories of modules.

Chapter 2 also contains no original content. It provides a brief introduction in the theory of model categories. In particular, we review the concepts of homotopy category, model category, Quillen functor and symmetric monoidal model category, and we present a transfer theorem from [SS00] for transferring a model structure on a category of modules over monoids in a symmetric monoidal model category. In the last section we recall the symmetric monoidal model structure on the category $\mathbf{CoCh}_{\mathbb{K}}$ of cochain complexes over some field $\mathbb{K}$ of characteristic zero.

Chapter 3 recalls the categories of nets of algebras and of net representations for any domain category **C** and target symmetric monoidal category **M**, generalizing the standard concepts in a straightforward way. In passing, we present a construction of constant net representations as part of an adjunction relating representations of a net of algebras and left modules over a single monoid from the net. This construction will be used in Chapter 4 to construct a concrete net representation. Next, we move on to illustrate the change-of-net adjunction associated with a morphism of nets and we discuss the **M**-tensoring, **M**-powering and **M**-enriched hom on the category of net representations, as those will be useful tools when we endow the categories of net representations with a model structure. This is achieved in Section 3.6. For this purpose we assume **M** to be a suitable (in the sense of Setup 3.6.13) closed symmetric monoidal model category. This allows us to promote the change-of-net adjunction to a Quillen adjunction. In particular, we show that the change-of-net adjunction is a Quillen equivalence when it arises from a weak equivalence of nets of algebras (Proposition 3.6.20). In the last section of this chapter, we focus on homotopical AQFTs, which are in particular $\mathbf{CoCh}_{\mathbb{C}}$-valued nets of algebras over the category **Loc** or the category $\mathbf{CCO}(M)$ of causally convex open subsets of some fixed globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}$. We define their representations simply as representations of the underlying nets of algebras. Then, we provide an example illustrating that the categories of representations associated to weakly equivalent homotopical AQFTs need not be equivalent with respect to the usual notion of categorical equivalence. We conclude this section demonstrating how equipping the categories of representations with the appropriate model structures resolves the issue.

Chapter 4 has been borrowed with only minor modifications from [AB23]. It focuses on constructing concrete representations of the homotopical AQFT associated with Maxwell p-forms. First, we construct the net of algebras of Maxwell p-forms via CCR quantization (4.4.2) of the complex of linear observables for Maxwell p-forms from (4.1.7) endowed with the Poisson structure (4.4.1). Then, we verify that the resulting net of algebras actually satisfy the axioms of a homotopical algebraic quantum field theory on the category of globally hyperbolic Lorentzian manifolds. Then, restricting the underlying net of algebras to all causally convex open subsets of a fixed oriented and time-oriented globally hyperbolic Lorentzian manifold M, which for simplicity we assume ultra-static and admitting a compact spacelike Cauchy surface, we present a simple construction of a two-point function ω_2, that is the cochain map we use to define a concrete constant net representation. We conclude with a simple and explicit description of all constant net representations up to weak equivalence for Maxwell 1-forms on the flat Lorentz cylinder.

Appendix A contains a very short introduction to category theory. The main purpose is to establish notation and collect basic facts about functor categories, limits and adjunctions that are used throughout the text, usually with no mention.

Part I

Mathematical Preliminaries

Chapter 1

Modules over Monoids

In this chapter we collect basic results on modules over a monoid in a monoidal category. More detailed expositions on this topic can be found in [Mac78; Bor94; Eti+15].

After recalling the definitions of monoidal categories, monoids and modules, we present, in Section 1.4, the free module functor as an adjoint of the forgetful functor from the category of modules to the underlying monoidal category.

In Section 1.5, we recall the change-of-monoid adjunction induced by a morphism of monoids. In particular, we present in detail the extension and restriction of scalars functors involved in the change-of-monoid adjunction, which is going to be instrumental in the study of net representations in Chapter 3.

Lastly, in Section 1.6, we present in detail the tensoring, powering, and enriched hom functors on categories of modules. These functors will be useful in Section 3.5 for the description of the tensoring, powering and enriched hom functors on categories of net representations.

1.1 Monoidal categories

Definition 1.1.1. A **monoidal category** is a tuple $(\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ consisting of the following data:

(1) $\mathbf{M}$ is a category.

(2) $\otimes$ is a functor $\otimes\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$, called the **tensor product** or **monoidal product**.

(3) $\mathbb{1}$ is an object of $\mathbf{M}$, called the **monoidal unit**.

(4) α is a natural isomorphism $\alpha\colon (- \otimes -) \otimes (-) \xrightarrow{\cong} (-) \otimes (- \otimes -)$ called the **associator**,

(5) λ is a natural isomorphism $\lambda\colon \mathbb{1} \otimes (-) \xrightarrow{\cong} \mathrm{id}_{\mathbf{M}}$ called the **left unitor**.

(6) ρ is a natural isomorphism $\rho\colon (-) \otimes \mathbb{1} \xrightarrow{\cong} \mathrm{id}_{\mathbf{M}}$ called the **right unitor**.

These data are subject to the following axioms:

(i) **Unit Axiom**: For any $X, Y \in \mathrm{Ob}(\mathbf{M})$, the diagram

$$\begin{array}{ccc} (X \otimes \mathbb{1}) \otimes Y & \xrightarrow{\alpha_{X,\mathbb{1},Y}} & X \otimes (\mathbb{1} \otimes Y) \\ & {\scriptstyle \rho_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} & \\ & X \otimes Y & \end{array}$$

in $\mathbf{M}$ commutes.

(ii) **Pentagon Axiom**: For any $X, Y, Z, W \in \mathrm{Ob}(\mathbf{M})$, the diagram

$$\begin{array}{ccc} & ((X \otimes Y) \otimes Z) \otimes W & \\ & {\scriptstyle \alpha_{X,Y,Z} \otimes \mathrm{id}_W} \swarrow \qquad \searrow {\scriptstyle \alpha_{X \otimes Y, Z, W}} & \\ (X \otimes (Y \otimes Z)) \otimes W & & (X \otimes Y) \otimes (Z \otimes W) \\ \downarrow {\scriptstyle \alpha_{X, Y \otimes Z, W}} & & \downarrow {\scriptstyle \alpha_{X,Y,Z \otimes W}} \\ X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow{\mathrm{id}_X \otimes \alpha_{Y,Z,W}} & X \otimes (Y \otimes (Z \otimes W)) \end{array}$$

in $\mathbf{M}$ commutes.

Notation 1.1.2. Abusing notation, we will refer to a monoidal category $(\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ simply by $(\mathbf{M}, \otimes, \mathbb{1})$ or even by $\mathbf{M}$, omitting the extra structure when it is not explicitly used.

Moreover, to simplify the exposition, the associator will be often suppressed. Namely, given $X, Y, Z \in \mathrm{Ob}(\mathbf{M})$, we will often make no distinction between the objects $(X \otimes Y) \otimes Z \in \mathrm{Ob}(\mathbf{M})$ and $X \otimes (Y \otimes Z) \in \mathrm{Ob}(\mathbf{M})$, both of which we will denote by $X \otimes Y \otimes Z$.

Remark 1.1.3. Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a monoidal category and let $f_i \in \mathbf{M}(X_i, Y_i)$ and $g_i \in \mathbf{M}(Y_i, Z_i)$ for some $X_i, Y_i, Z_i \in \mathrm{Ob}(\mathbf{M})$ and $i \in \{1, 2\}$. It follows from the functoriality of the tensor product that $(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)$. $\triangle$

Very often, the tensor product of a monoidal category has a right adjoint. In this case, we have a closed monoidal category.

Definition 1.1.4. A **closed monoidal category** is a monoidal category $(\mathbf{M}, \otimes, \mathbb{1})$ equipped with a functor $[-,-]\colon \mathbf{M}^{\mathrm{op}} \times \mathbf{M} \to \mathbf{M}$, called **internal hom**, such that, for every $X \in \mathrm{Ob}(\mathbf{M})$, there exists an adjunction $(-) \otimes X \dashv [X,-]$ (see Definition A.5.1).

Remark 1.1.5. Notice that, by Proposition A.5.9, in a closed monoidal category $\mathbf{M}$, for any $X \in \mathrm{Ob}(\mathbf{M})$ the functor $(-) \otimes X \colon \mathbf{M} \to \mathbf{M}$ preserves colimits, whereas the functor $[X,-]$ preserves limits. △

Definition 1.1.6. Given a closed monoidal category $(\mathbf{M}, \otimes, \mathbb{1})$ we define for each $X \in \mathrm{Ob}(\mathbf{M})$ the **evaluation map on** X to be the natural transformation $\mathrm{ev}_X \colon [X,-] \otimes X \to \mathrm{Id}_{\mathbf{M}}$ with components as follows. Given $Y \in \mathrm{Ob}(\mathbf{M})$, we define $\mathrm{ev}_{X,Y} \colon [X,Y] \otimes X \to Y$ to be the adjunct of the identity morphism $\mathrm{id}_{[X,Y]}$ with respect to the adjunction $(-) \otimes X \dashv [X,-]$. Specifically, this adjunction determines an isomorphism $-^{\flat} \colon \mathbf{M}([X,Y],[X,Y]) \to \mathbf{M}([X,Y] \otimes X, Y)$, and $\mathrm{ev}_{X,Y} \coloneqq {\mathrm{id}_{[X,Y]}}^{\flat}$ (see Definition A.5.1(d)).

Next, we introduce the concept of a symmetric monoidal category. For this, we need the following definitions.

Definition 1.1.7. Given a monoidal category $(\mathbf{M}, \otimes, \mathbb{1})$ we define the functor $\mathrm{Rv}_{\mathbf{M}} \colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ as follows. To each $(X,Y) \in \mathrm{Ob}(\mathbf{M} \times \mathbf{M})$ it assigns the object $\mathrm{Rv}_{\mathbf{M}}(X,Y) \coloneqq Y \otimes X \in \mathrm{Ob}(\mathbf{M})$, and to each morphism $(f,g) \in \mathrm{Mor}(\mathbf{M} \times \mathbf{M})$ it assigns the morphism $\mathrm{Rv}_M(f,g) \coloneqq g \otimes f \in \mathrm{Mor}(\mathbf{M})$.

Definition 1.1.8. A **braided monoidal category** is a monoidal category $(\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ equipped with a natural isomorphism $\gamma \colon (-) \otimes (-) \to \mathrm{Rv}_{\mathbf{M}}$, called the **braiding** of $\mathbf{M}$, that satisfies the following conditions:

(i) For any $X, Y, Z \in \mathrm{Ob}(\mathbf{M})$, the diagram

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{\gamma_{X,Y \otimes Z}} & (Y \otimes Z) \otimes X \\ \downarrow{\scriptstyle \gamma_{X,Y} \otimes \mathrm{id}_Z} & & & & \downarrow{\scriptstyle \alpha_{Y,Z,X}} \\ (Y \otimes X) \otimes Z & \xrightarrow{\alpha_{Y,X,Z}} & Y \otimes (X \otimes Z) & \xrightarrow{\mathrm{id}_Y \otimes \gamma_{X,Z}} & Y \otimes (Z \otimes X) \end{array}$$

in $\mathbf{M}$ commutes.

(ii) For any $X, Y, Z \in \mathrm{Ob}(\mathbf{M})$, the diagram

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{\alpha^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\gamma_{X \otimes Y, Z}} & Z \otimes (X \otimes Y) \\ \downarrow{\scriptstyle \mathrm{id}_X \otimes \gamma_{Y,Z}} & & & & \downarrow{\scriptstyle \alpha^{-1}_{Z,X,Y}} \\ X \otimes (Z \otimes Y) & \xrightarrow{\alpha_{X,Y,Z}} & (X \otimes Z) \otimes Y & \xrightarrow{\gamma_{X,Z} \otimes \mathrm{id}_Y} & (Z \otimes X) \otimes Y \end{array}$$

in $\mathbf{M}$ commutes.

Now we can define a symmetric monoidal category as follows.

Definition 1.1.9. A **symmetric monoidal category** is a braided monoidal category with braiding γ such that for any $X, Y \in \mathrm{Ob}(\mathbf{M})$ the diagram

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\gamma_{X,Y}} & Y \otimes X \\ & {\scriptstyle \mathrm{id}_{X \otimes Y}} \searrow & \downarrow{\scriptstyle \gamma_{Y,X}} \\ & & X \otimes Y \end{array}$$

in $\mathbf{M}$ commutes.

Remark 1.1.10. Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a closed symmetric monoidal category with braiding γ and let X be an object in $\mathbf{M}$. The functor $X \otimes (-)\colon \mathbf{M} \to \mathbf{M}$ preserves colimits. This follows from the fact that $(-) \otimes X$ preserves colimits (see Remark 1.1.5) and the fact that the braiding γ gives rise to a natural isomorphism between the functors $X \otimes (-)$ and $(-) \otimes X$. △

Remark 1.1.11. In a closed symmetric monoidal category $(\mathbf{M}, \otimes, \mathbb{1})$ with internal hom $[-,-]$, there is a two-variable adjunction $\big(\otimes, [-,-], [-,-]\big)\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ (see Definition A.5.11). This follows directly from the definitions. △

1.2 Monoids

Definition 1.2.1. Let $(\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. A monoid in $\mathbf{M}$ is a triplet $(A, \mu, \mathbf{1})$ consisting of the following data:

(1) An object $A \in \mathrm{Ob}(\mathbf{M})$.

(2) A morphism $\mu \in \mathbf{M}(A \otimes A, A)$, called the **multiplication in** A.

(3) A morphism $\mathbf{1} \in \mathbf{M}(\mathbb{1}, A)$ called the **unit of** A.

These data are subject to the following axioms:

(i) **Associativity**: The diagram

$$\begin{array}{ccccc} (A\otimes A)\otimes A & & \xrightarrow{\alpha_{A,A,A}} & & A\otimes(A\otimes A) \\ \Big\downarrow{\scriptstyle \mu\otimes\mathrm{id}_A} & & & & \Big\downarrow{\scriptstyle \mathrm{id}_A\otimes\mu} \\ A\otimes A & & & & A\otimes A \\ & \searrow^{\mu} & & \swarrow_{\mu} & \\ & & A & & \end{array}$$

in $\mathbf{M}$ commutes.

(ii) **Unit Axiom**: The diagram

$$\begin{array}{ccccc} \mathbb{1}\otimes A & \xrightarrow{\mathbf{1}\otimes\mathrm{id}_A} & A\otimes A & \xleftarrow{\mathrm{id}_A\otimes\mathbf{1}} & A\otimes\mathbb{1} \\ & \searrow_{\lambda_A} & \Big\downarrow{\scriptstyle\mu} & \swarrow_{\rho_A} & \\ & & A & & \end{array}$$

in $\mathbf{M}$ commutes.

Notation 1.2.2. Let $\mathbf{M}$ be a monoidal category. Abusing notation, we often refer to a monoid $(A, \mu, \mathbf{1})$ in $\mathbf{M}$ simply by A.

Definition 1.2.3. Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a monoidal category. The **category of monoids** of $\mathbf{M}$, denoted by $\mathbf{Mon}(\mathbf{M})$ is the subcategory of $\mathbf{M}$ with objects the monoids in $\mathbf{M}$ and morphisms as follows. For $(A, \mu_A, \mathbf{1}_A)$ and $(B, \mu_B, \mathbf{1}_B)$ two monoids in $\mathbf{M}$, the morphisms in $\mathbf{Mon}(\mathbf{M})(A, B)$ are the morphisms $f \in \mathbf{M}(A, B)$ such that the diagrams

$$\begin{array}{ccc} A\otimes A & \xrightarrow{f\otimes f} & B\otimes B \\ \Big\downarrow{\scriptstyle\mu_A} & & \Big\downarrow{\scriptstyle\mu_B} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccccc} & & \mathbb{1} & & \\ & \swarrow^{\mathbf{1}_A} & & \searrow^{\mathbf{1}_B} & \\ A & & \xrightarrow[f]{} & & B \end{array}$$

in $\mathbf{M}$ commute.

1.3 Modules

Definition 1.3.1. Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a monoidal category and $(A, \mu, \mathbf{1})$ a monoid in $\mathbf{M}$. A **left module over** A (or **left** A**-module**) is a pair (L, ϕ) consisting of:

(1) An object $L \in \mathrm{Ob}(\mathbf{M})$,

(2) A morphism $\phi \in \mathbf{M}(A \otimes L, L)$, called the A**-action** on L, required to satisfy the following conditions:

(i) The diagram

$$\begin{array}{ccc} A \otimes A \otimes L & \xrightarrow{\mathrm{id}_A \otimes \phi} & A \otimes L \\ {\scriptstyle \mu \otimes \mathrm{id}_L}\downarrow & & \downarrow{\scriptstyle \phi} \\ A \otimes L & \xrightarrow{\quad\phi\quad} & L \end{array} \tag{1.3.1}$$

in $\mathbf{M}$ commutes.

(ii) The diagram

$$\begin{array}{ccc} \mathbb{1} \otimes L & \xrightarrow{\mathbf{1} \otimes \mathrm{id}_L} & A \otimes L \\ & {\scriptstyle \lambda_L}\searrow & \downarrow{\scriptstyle \phi} \\ & & L \end{array} \tag{1.3.2}$$

in $\mathbf{M}$ commutes.

Remark 1.3.2. There is an analogous definition of a **right** A**-module** (L, ϕ), with the difference that the monoid A acts on L from the right, i.e. the A-action is a morphism $\phi\colon L \otimes A \to L$ and Diagrams 1.3.1 and 1.3.2 are adjusted accordingly. △

Notation 1.3.3. Let $\mathbf{M}$ be a monoidal category and A a monoid in $\mathbf{M}$.

- Since in this text we will use almost exclusively left modules, we will often say simply A-module to mean *left* A-module.

- Abusing notation, we will often refer to an A-module (L, ϕ) simply by L.

Definition 1.3.4. Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a monoidal category and A a monoid in $\mathbf{M}$. The **category of left modules over** A, denoted by ${}_A\mathbf{Mod}$, is the category with objects the left A-modules and morphisms as follows. Let

(L_1, ϕ_1) and (L_2, ϕ_2) be left modules over A. The morphisms ${}_A\mathbf{Mod}(L_1, L_2)$ are the morphisms $f \in \mathbf{M}(L_1, L_2)$ such that the diagram

$$\begin{array}{ccc} A \otimes L_1 & \xrightarrow{\mathrm{id}_A \otimes f} & A \otimes L_2 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ L_1 & \xrightarrow{\quad f \quad} & L_2 \end{array} \tag{1.3.3}$$

in $\mathbf{M}$ commutes.

Remark 1.3.5. The category of right A-modules, denoted by $\mathbf{Mod}_A$, is defined analogously. The only difference is that Diagram 1.3.3 is adjusted so that the monoid A acts on the right. △

When a monoidal category $\mathbf{M}$ is cocomplete, then for each monoid $A \in \mathbf{M}$, there exists a useful functor from $\mathbf{Mod}_A \times {}_A\mathbf{Mod}$ to $\mathbf{M}$, defined as follows.

Definition 1.3.6. Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a cocomplete monoidal category and A a monoid in $\mathbf{M}$. We define the **relative tensor product over** A as the functor $\otimes_A \colon \mathbf{Mod}_A \times {}_A\mathbf{Mod} \to \mathbf{M}$ consisting of the following data:

(1) To each pair $(R, \psi) \in \mathbf{Mod}_A$ and $(L, \phi) \in {}_A\mathbf{Mod}$, it assigns the object $R \otimes_A L \in \mathbf{M}$ defined as the coequalizer (see Definition A.4.15(c)) of the diagram

$$R \otimes A \otimes L \underset{\mathrm{id}_R \otimes \phi}{\overset{\psi \otimes \mathrm{id}_L}{\rightrightarrows}} R \otimes L \tag{1.3.4}$$

in $\mathbf{M}$.

(2) To each pair of morphisms $f \colon (R_1, \psi_1) \to (R_2, \psi_2)$ in $\mathbf{Mod}_A$ and $g \colon (L_1, \phi_1) \to (L_2, \phi_2)$ in ${}_A\mathbf{Mod}$, it assigns the unique morphism $f \otimes_A g \colon R_1 \otimes_A L_1 \to R_2 \otimes_A L_2$ in $\mathbf{M}$ determined by the universal property of $R_2 \otimes_A L_2$ as a coequalizer and the commutativity of the diagram

$$\begin{array}{ccc} R_1 \otimes A \otimes L_1 & \underset{\mathrm{id}_{R_1} \otimes \phi_1}{\overset{\psi_1 \otimes \mathrm{id}_{L_1}}{\rightrightarrows}} & R_1 \otimes L_1 \\ {\scriptstyle f \otimes A \otimes g}\downarrow & & \downarrow{\scriptstyle f \otimes g} \\ R_2 \otimes A \otimes L_2 & \underset{\mathrm{id}_{R_2} \otimes \phi_2}{\overset{\psi_2 \otimes \mathrm{id}_{L_2}}{\rightrightarrows}} & R_2 \otimes L_2 \end{array} \tag{1.3.5}$$

in $\mathbf{M}$.

1.4 The free module functor and the forgetful functor

Definition 1.4.1. Given a monoidal category $(\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and a monoid $(A, \mu, \mathbf{1}) \in \mathbf{Mon}(\mathbf{M})$ we define:

(a) The **forgetful functor** $\underline{(-)} : {}_A\mathbf{Mod} \to \mathbf{M}$ as the functor that sends each $(L, \phi) \in {}_A\mathbf{Mod}$ to the underlying object $L \in \mathbf{M}$, and each morphism $f \in \mathrm{Mor}({}_A\mathbf{Mod})$ to the morphism $f \in \mathrm{Mor}(\mathbf{M})$.

(b) The **free A-module functor** $\mathrm{Fr}_A \colon \mathbf{M} \to \mathbf{Mon}(A)$ as follows. To each $L \in \mathrm{Ob}(\mathbf{M})$, it assigns the **free A-module** $\mathrm{Fr}_A(L) := (A \otimes L, \varphi_L) \in {}_A\mathbf{Mod}$, where the A-action $\varphi_L \colon A \otimes (A \otimes L) \to \mathbf{M} \otimes L$ is defined as the morphism

$$A \otimes (A \otimes L) \xrightarrow{\alpha^{-1}_{A,A,L}} (A \otimes A) \otimes L \xrightarrow{\mu \otimes \mathrm{id}_L} A \otimes L$$

in $\mathbf{M}$. The free A-module functor acts on morphisms as follows. To each morphism $f \in \mathbf{M}(L_1, L_2)$, it assigns the morphism $\mathrm{Fr}_A(f) := \mathrm{id}_L \otimes f \in {}_A\mathbf{Mod}(\mathrm{Fr}_A(L_1), \mathrm{Fr}_A(L_2))$.

Proposition 1.4.2. *[[Bor94, Vol. 2, Prop. 4.1.4(3)]] Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a monoidal category and $(A, \mu, \mathbf{1})$ a monoid in $\mathbf{M}$. There is an adjunction $\mathrm{Fr}_A \dashv \underline{(-)}$ between the categories $\mathbf{M}$ and ${}_A\mathbf{Mod}$.*

It follows from Proposition A.5.9 that the free A-module functor preserves colimits, whereas the forgetful functor preserves limits. Moreover, we have the following proposition.

Proposition 1.4.3 ([Bor94, Vol. 2, Prop. 4.3.1])**.** *Let $\mathbf{M}$ be a monoidal category and A a monoid in $\mathbf{M}$. The forgetful functor $\underline{(-)} \colon {}_A\mathbf{Mod} \to \mathbf{M}$ preserves and lifts all limits (see Definition A.4.20).*

In fact, in the case of a closed symmetric monoidal category, a similar statement holds for the colimits. This follows from the fact that, according to [Bor94, Vol. 2, Prop. 4.3.2], when the functor $\underline{(-)} \circ \mathrm{Fr}_A$ preserves colimits, then the forgetful functor $\underline{(-)}$ preserves and lifts colimits. But, it is easy to see that $\underline{(-)} \circ \mathrm{Fr}_A = A \otimes (-)$, which preserves colimits in the case of a closed symmetric monoidal category (see Remark 1.1.5). So, we have the following proposition.

Proposition 1.4.4. *Let $\mathbf{M}$ be a closed symmetric monoidal category and A a monoid in $\mathbf{M}$. The forgetful functor $\underline{(-)} \colon {}_A\mathbf{Mod} \to \mathbf{M}$ preserves and lifts all colimits.*

Propositions 1.4.3 and 1.4.4 yield the following corollary.

Corollary 1.4.5. *Let* $\mathbf{M}$ *be a closed symmetric monoidal category. If* $\mathbf{M}$ *is complete (resp. cocomplete) then the category* ${}_A\mathbf{Mod}$ *is also complete (resp. cocomplete) for any* $A \in \mathbf{Mon}(\mathbf{M})$.

1.5 The change-of-monoid adjunction

Given a cocomplete, closed symmetric monoidal category $\mathbf{M}$ and $A, B \in \mathbf{Mon}(\mathbf{M})$, any morphism $f\colon A \to B$ in $\mathbf{Mon}(\mathbf{M})$ gives rise to the extension and restriction of scalars functors, whose definitions we recall below.

Setup 1.5.1. Throughout this section we assume that:

- $(\mathbf{M}, \otimes, \mathbb{1})$ is a cocomplete, closed symmetric monoidal category,
- $(A, \mu_A, \mathbf{1}_A)$ and $(B, \mu_B, \mathbf{1}_B)$ are monoids in $\mathbf{M}$,
- $f\colon A \to B$ is a morphism in $\mathbf{Mon}(\mathbf{M})$.

Definition 1.5.2. The **restriction of scalars** along f is the functor $\mathrm{Res}_f\colon {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$ consisting of the following data:

(1) To each $(L, \phi) \in {}_B\mathbf{Mod}$, it assigns the A-module $\mathrm{Res}_f(L) \coloneqq (L, f^*\phi) \in {}_A\mathbf{Mod}$, where, with a slight abuse of notation, we denote by $f^*\phi$ the morphism in $\mathbf{M}$ given by the composition

$$A \otimes L \xrightarrow{f \otimes \mathrm{id}_L} B \otimes L \xrightarrow{\quad \phi \quad} L\,.$$

(2) To each morphism $g \in {}_B\mathbf{Mod}(L_1, L_2)$, for some B-modules L_1 and L_2, it assigns the morphism $\mathrm{Res}_f(g) \coloneqq g \in {}_A\mathbf{Mod}(\mathrm{Res}_f(L_1), \mathrm{Res}_f(L_2))$.

Notation 1.5.3. We will often denote the A-module $\mathrm{Res}_f(L) \in {}_A\mathbf{Mod}$ by $L|_A$, when there is no risk of confusion.

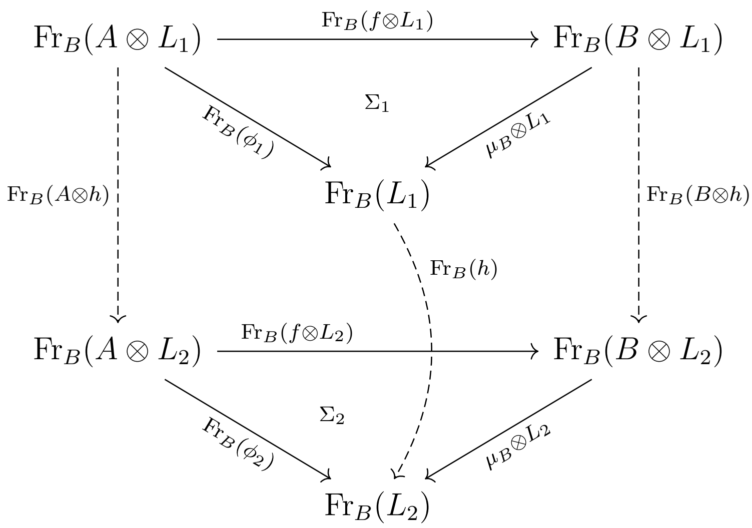
Definition 1.5.4. The **extension of scalars** along f is the functor $\mathrm{Ext}_f\colon {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ consisting of the following data:

(1) To each $(L, \phi) \in {}_A\mathbf{Mod}$, it assigns the B-module $\mathrm{Ext}_f(L) \in {}_B\mathbf{Mod}$ defined as the colimit of the diagram

$$\begin{array}{ccc} \mathrm{Fr}_B(A \otimes L) & \xrightarrow{\mathrm{Fr}_B(f \otimes \mathrm{id}_L)} & \mathrm{Fr}_B(B \otimes L) \\ & {\scriptstyle \mathrm{Fr}_B(\phi)} \searrow \quad \swarrow {\scriptstyle \mu_B \otimes \mathrm{id}_L} & \\ & \mathrm{Fr}_B(L) & \end{array} \tag{1.5.1}$$

in ${}_B\mathbf{Mod}$. Here, we denote by Fr_B the free B-module functor (see Definition 1.4.1(b)), while $\mu_B \otimes \mathrm{id}_L$ stands for the morphism $\mu_B \otimes \mathrm{id}_L \colon B \otimes B \otimes L \to B \otimes L$ in the underlying monoidal category $\mathbf{M}$.

(2) To each morphism $h \in {}_A\mathbf{Mod}(L_1, L_2)$, for some A-modules (L_1, ϕ_1) and (L_2, ϕ_2), it assigns the morphism $\mathrm{Ext}_f(h) \in {}_B\mathbf{Mod}(\mathrm{Ext}_f(L_1), \mathrm{Ext}_f(L_2))$ defined as follows. Notice that the morphism h gives rise to a natural transformation between the subdiagrams Σ_1 and Σ_2 of the diagram



(1.5.2)

with components the dashed arrows. The morphism $\mathrm{Ext}_f(h)$ is defined as the unique morphism between the colimits of the diagrams Σ_1 and Σ_2 determined by this natural transformation (see Proposition A.4.17).

Remark 1.5.5. The well-definedness of the extension of scalars functor follows from the following observations.

- The morphism $\mu_B \otimes \mathrm{id}_L$ in Diagram 1.5.1 is indeed a B-module morphism, since the diagram

$$\begin{array}{ccc} B \otimes B \otimes B \otimes L & \xrightarrow{\mathrm{id}_B \otimes \mu_B \otimes \mathrm{id}_L} & B \otimes B \otimes L \\ \downarrow{\scriptstyle \varphi_{B \otimes L}} & & \downarrow{\scriptstyle \varphi_L} \\ B \otimes B \otimes L & \xrightarrow[\mu_B \otimes \mathrm{id}_L]{} & B \otimes L \end{array} \tag{1.5.3}$$

in $\mathbf{M}$ commutes. The morphisms φ_L and $\varphi_{B \otimes L}$ in this diagram stand for the actions on the free B-modules generated by L and $B \otimes L$

respectively. Specifically, $\varphi_L := \mu_B \otimes \mathrm{id}_L$ and $\varphi_{B\otimes L} := \mu_B \otimes \mathrm{id}_{B\otimes L}$ (see Definition 1.4.1(b)). Taking also into account the associativity of μ_B it is easy to see that the above diagram commutes.

- The existence of the colimit of Diagram 1.5.1 is guaranteed by the fact that the category ${}_B\mathbf{Mod}$ is cocomplete. This follows from Corollary 1.4.5.

- The dashed arrows in Diagram 1.5.2 form a natural transformation, as one can verify as follows. Notice that $\mathrm{Fr}_B(\phi_2) \circ \mathrm{Fr}_B(A \otimes h) = \mathrm{Fr}_B(B \otimes h) \circ \mathrm{Fr}_B(\phi_1)$ due to the functoriality of Fr_B and the fact that h is an A-module morphism. Moreover, $\mathrm{Fr}_B(f \otimes L_2) \circ \mathrm{Fr}_B(A \otimes h) = \mathrm{Fr}_B(B \otimes h) \circ \mathrm{Fr}_B(f \otimes L_1)$ due to the functoriality of Fr_B and the composition rule of tensor products of morphisms (see Remark 1.1.3). Finally, using the definition of Fr_B (see Definition 1.4.1(b)) and again the composition rule of tensor products of morphisms, we obtain that $(\mu_B \otimes L_2) \circ \mathrm{Fr}_B(B \otimes h) = \mathrm{Fr}_B(h) \circ (\mu_B \otimes L_1)$. This concludes the naturality of the dashed arrows between diagrams Σ_1 and Σ_2.

$\triangle$

Given an A-module (L, ϕ), the underlying object in $\mathbf{M}$ of the B-module $\mathrm{Ext}_f(L)$ can also be described as a relative tensor product, whose definition we recall below.

Definition 1.5.6. Let (L, ϕ) an A-module. We define the **relative tensor product** as the functor $(-) \otimes_A L\colon {}_A\mathbf{Mod} \to \mathbf{M}$ consisting of the following data:

(1) To each $(N, \psi) \in {}_A\mathbf{Mod}$, it assigns the object $N \otimes_A L$ defined as the colimit of the diagram

$$\begin{array}{ccc} N \otimes (A \otimes L) & \xrightarrow{\;\cong\;} & (A \otimes N) \otimes L \\ & {\scriptstyle \mathrm{id}_N \otimes \phi} \searrow \quad \swarrow {\scriptstyle \psi \otimes \mathrm{id}_L} & \\ & N \otimes L & \end{array} \tag{1.5.4}$$

in $\mathbf{M}$, where the depicted isomorphism stands for the composition of the associator with the braiding of $\mathbf{M}$ (see Definitions 1.1.1(4) and 1.1.8).

(2) To each morphism $h\colon (N_1, \psi_1) \to (N_2, \psi_2)$ in ${}_A\mathbf{Mod}$, it assigns the morphism $h \otimes_A L\colon N_1 \otimes_A L \to N_2 \otimes_A L$ in $\mathbf{M}$ defined as follows. Notice that, due to the functoriality of the tensor product $\otimes$ of $\mathbf{M}$, h gives rise

to a natural transformation between the diagrams with respect to which $N_1 \otimes_A L$ and $N_2 \otimes_A L$ are defined as their colimits. The morphism $h \otimes_A L$ is defined as the unique morphism between the colimits of these diagrams determined by this natural transformation (see Proposition A.4.17).

Remark 1.5.7. Although we used the same notation, this functor is different from the relative tensor product over A from Definition 1.3.6. So, whenever we use this notation, we will make explicit which of the two definitions we refer to. △

Remark 1.5.8. Given an A-module (L, ϕ), the underlying object in M of the B-module $\mathrm{Ext}_f(L)$ is the object $B \otimes_A L$, where B is seen as an A-module equipped with the action

$$A \otimes B \xrightarrow{f \otimes \mathrm{id}_B} B \otimes B \xrightarrow{\mu_B} B\,. \tag{1.5.5}$$

This can be verified as follows.

First, notice that since the colimits in the category ${}_B\mathbf{Mod}$ are preserved and lifted by the forgetful functor $\underline{(-)}\colon {}_B\mathbf{Mod} \to \mathbf{M}$ (see Corollary 1.4.5, $\underline{\mathrm{Ext}_f(L)}$ can be computed as the colimit of the diagram

$$\begin{array}{ccc} B \otimes A \otimes L & \xrightarrow{\mathrm{id}_B \otimes f \otimes \mathrm{id}_L} & B \otimes B \otimes L \\ & {\scriptstyle \mathrm{id}_B \otimes \phi} \searrow \quad \swarrow {\scriptstyle \mu_B \otimes \mathrm{id}_L} & \\ & B \otimes L & \end{array} \tag{1.5.6}$$

in $\mathbf{M}$, i.e. Diagram 1.5.1 composed with the forgetful functor. However, it is easy to see that the colimit of Diagram 1.5.6 is also a colimit of the diagram

$$\begin{array}{ccc} B \otimes (A \otimes L) & \xrightarrow{\cong} & (A \otimes B) \otimes L \\ & {\scriptstyle \mathrm{id}_B \otimes \phi} \searrow \quad \swarrow {\scriptstyle \mu_B (f \otimes \mathrm{id}_B) \otimes \mathrm{id}_L} & \\ & B \otimes L & \end{array} \tag{1.5.7}$$

in $\mathbf{M}$. From the uniqueness of colimits (see Remark A.4.11), it follows that $\underline{\mathrm{Ext}_f(L)} \cong B \otimes_A L$. △

The following proposition is a special case of [GM20, Lemma 2.2].

Proposition 1.5.9. *Any morphism $f\colon A \to B$ as in Setup 1.5.1 induces an adjunction*

$$(\mathrm{Ext}_f \dashv \mathrm{Res}_f)\colon {}_A\mathbf{Mod} \to {}_B\mathbf{Mod} \tag{1.5.8}$$

that is called ***change-of-monoid*** *adjunction.*

Remark 1.5.10. The restriction of scalars along the identity morphism on A coincides with the identity functor on ${}_A\mathbf{Mod}$, i.e. $\mathrm{Res}_{\mathrm{id}_A} = \mathrm{Id}_{{}_A\mathbf{Mod}}$, by construction.

Similarly, the extension of scalars along the identity morphism on A is isomorphic to the identity functor on ${}_A\mathbf{Mod}$, i.e. $\mathrm{Ext}_{\mathrm{id}_A} \xrightarrow{\cong} \mathrm{Id}_{{}_A\mathbf{Mod}}$. This follows from the fact that $\mathrm{Id}_{{}_A\mathbf{Mod}} \dashv \mathrm{Res}_{\mathrm{id}_A}$, since any equivalence of categories can be promoted to an adjunction (see Proposition A.5.8), and the fact that the adjoint of a functor is unique up to isomorphism (see Remark A.5.6). △

Remark 1.5.11. Given two composable morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathbf{Mon}(\mathbf{M})$, by construction, the restriction along g composed with the restriction along f coincides with the restriction along $g \circ f$. In other words, $\mathrm{Res}_f \circ \mathrm{Res}_g = \mathrm{Res}_{g\circ f}\colon {}_C\mathbf{Mod} \to {}_A\mathbf{Mod}$.

Moreover, from the change-of-monoid adjunction (1.5.8), the composition of adjunctions (see Proposition A.5.10) and the uniqueness of adjoint functors (see Remark A.5.6), we deduce that there exists a natural isomorphism $\mathrm{Ext}_g \circ \mathrm{Ext}_f \cong \mathrm{Ext}_{g\circ f}\colon {}_A\mathbf{Mod} \to {}_C\mathbf{Mod}$. △

Remark 1.5.12. Given an isomorphism $f\colon A \cong B$ in $\mathbf{Mon}(\mathbf{M})$, the change-of-monoid adjunction $\mathrm{Ext}_f \dashv \mathrm{Res}_f$ is an adjoint equivalence (see Definition A.5.7).

This can be verified as follows. First, notice that when f is an isomorphism, then we have that $\mathrm{Res}_f \circ \mathrm{Res}_{f^{-1}} = \mathrm{Id}_{{}_A\mathbf{Mod}}$ and $\mathrm{Res}_{f^{-1}} \circ \mathrm{Res}_f = \mathrm{Id}_{{}_B\mathbf{Mod}}$, by Remarks 1.5.10 and 1.5.11. This implies that the functors Res_f and $\mathrm{Res}_{f^{-1}}$ constitute an equivalence of categories. By Proposition A.5.8, this equivalence gives rise to an adjoint equivalence with $\mathrm{Res}_{f^{-1}} \dashv \mathrm{Res}_f$. By the change-of-monoid adjunction (1.5.8) and the uniqueness of adjoint functors (see Remark A.5.6), it follows that $\mathrm{Ext}_f \cong \mathrm{Res}_{f^{-1}}$. That implies that the functors Ext_f and Res_f also constitute and equivalence of categories. From this and Proposition A.5.8, we deduce that there is an adjoint equivalence of the form $\mathrm{Ext}_f \dashv \mathrm{Res}_f$. △

1.6 Tensoring, powering and enriched hom on the category of modules

Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a complete closed symmetric monoidal category with internal hom $[-,-]\colon \mathbf{M} \times \mathbf{M}^{\mathrm{op}} \to \mathbf{M}$. Given a monoid $A \in \mathbf{Mon}(\mathbf{M})$, there exists a two variable adjunction consisting of the functors $\mathbf{M}$-tensoring, $\mathbf{M}$-powering and $\mathbf{M}$-enriched hom, whose definitions we recall below.

Definition 1.6.1. The **M-tensoring**, denoted by $(-) \otimes (-)\colon {}_A\mathbf{Mod} \times \mathbf{M} \to {}_A\mathbf{Mod}$, is defined as the functor consisting of the following data:

(1) To each pair $((L, \phi), X) \in {}_A\mathbf{Mod} \times \mathbf{M}$, the $\mathbf{M}$-tensoring assigns the A-module $(L \otimes X, \psi) \in {}_A\mathbf{Mod}$, where the A-action $\psi\colon A \otimes L \otimes X$ is defined by $\psi \coloneqq \phi \otimes \mathrm{id}_X$.

(2) To each morphism $(f, g)\colon ((L, \phi), X) \to ((L', \phi'), X')$ in ${}_A\mathbf{Mod} \times \mathbf{M}$, for some pairs $((L, \phi), X)$ and $((L', \phi'), X')$ in ${}_A\mathbf{Mod} \times \mathbf{M}$, the $\mathbf{M}$-tensoring assigns the morphism $f \otimes g\colon L \otimes X \to L' \otimes X'$, which can be easily verified to be an A-module morphism.

Definition 1.6.2. The **M-powering**, denoted by $(-)^{(-)}\colon {}_A\mathbf{Mod} \times \mathbf{M}^{\mathrm{op}} \to {}_A\mathbf{Mod}$, is defined as the functor consisting of the following data:

(1) To each pair $((L, \phi), X) \in {}_A\mathbf{Mod} \times \mathbf{M}^{\mathrm{op}}$, the $\mathbf{M}$-powering assigns the A-module $L^X \coloneqq ([X, L], \phi) \in {}_A\mathbf{Mod}$, where $[-,-]\colon \mathbf{M} \times \mathbf{M}^{\mathrm{op}} \to \mathbf{M}$ denotes the internal hom in $\mathbf{M}$ and the A-action $\phi\colon A \otimes [X, L] \to [X, L]$ is defined as the adjunct of the morphism

$$A \otimes [X, L] \otimes X \xrightarrow{\mathrm{id}_A \otimes \mathrm{ev}_{X,L}} A \otimes L \xrightarrow{\quad \phi \quad} L \tag{1.6.1}$$

in $\mathbf{M}$ with respect to the adjunction $(-) \otimes X \dashv [X, -]$ (see Definition 1.1.4). The morphism $\mathrm{ev}_{X,L}$ in this diagram stands for the evaluation map on X (see Definition 1.1.6).

(2) To each morphism $(f, g)\colon ((L_1, \phi_1), X_1) \to ((L_2, \phi_2), X_2)$ in ${}_A\mathbf{Mod} \times \mathbf{M}^{\mathrm{op}}$, for some pairs $((L_1, \phi_1), X_1)$ and $((L_2, \phi_2), X_2)$ in ${}_A\mathbf{Mod} \times \mathbf{M}^{\mathrm{op}}$, the $\mathbf{M}$-powering assigns the morphism $f^g \coloneqq [g, f]\colon L_1^{X_1} \to L_2^{X_2}$, which can be easily verified to be an A-module morphism.

Definition 1.6.3. The **M-enriched hom**, denoted by $[-,-]_A\colon {}_A\mathbf{Mod}^{\mathrm{op}} \times {}_A\mathbf{Mod} \to \mathbf{M}$, is defined as the functor consisting of the following data:

(1) To each pair of A-modules (L, ϕ) and (L', ϕ'), the $\mathbf{M}$-enriched hom assigns the equalizer

$$[L, L']_A \coloneqq \lim \Big([L, L'] \overset{[\phi, \mathrm{id}]}{\underset{\phi'_*}{\rightrightarrows}} [A \otimes L, L'] \Big) \in \mathbf{M}, \tag{1.6.2}$$

where by ϕ'_* we denote the adjunct of the morphism

$$[L, L'] \otimes A \otimes L \xrightarrow{\cong} A \otimes [L, L'] \otimes L \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_{L,L'}} A \otimes L' \xrightarrow{\phi'} L' \tag{1.6.3}$$

in $\mathbf{M}$ with respect to the adjunction $(-) \otimes (A \otimes L) \dashv [A \otimes L, -]$.

(2) To each morphism $(f, g)\colon ((L, \phi), (L', \phi')) \to ((N, \psi), (N', \psi'))$ in ${}_A\mathbf{Mod}^{\mathrm{op}} \times {}_A\mathbf{Mod}$, for some $(L, \phi), (L', \phi'), (N, \psi), (N', \psi') \in \mathbf{M}$, the $\mathbf{M}$-enriched hom assigns the morphism $[f, g]_A\colon [L, L']_A \to [N, N']_A$ in $\mathbf{M}$ defined as follows. Notice that, due to the functoriality of the tensor product and the internal hom of $\mathbf{M}$, the morphism (f, g) gives rise to a natural transformation between the diagrams

$$[L, L'] \overset{[\phi, \mathrm{id}]}{\underset{\phi'_*}{\rightrightarrows}} [A \otimes L, L'] \tag{1.6.4}$$

and

$$[N, N'] \overset{[\psi, \mathrm{id}]}{\underset{\psi'_*}{\rightrightarrows}} [A \otimes N, N'] \tag{1.6.5}$$

in $\mathbf{M}$, where ϕ'_* and ψ'_* are defined similarly to ϕ'_* in Diagram 1.6.2. We define $[f, g]_A$ as the unique morphism between the limits of these diagrams determined by this natural transformation (see Proposition A.4.17).

The following proposition is a special case of [Bor94, Vol. 2, Prop. 6.5.7].

Proposition 1.6.4. *There exists a two-variable adjunction* $\big(\otimes, (-)^{(-)}, [-, -]_A\big)\colon {}_A\mathbf{Mod} \times \mathbf{M} \to {}_A\mathbf{Mod}$ *(see Definition A.5.11) consisting of the* $\mathbf{M}$*-tensoring, the* $\mathbf{M}$*-powering and the* $\mathbf{M}$*-enriched hom functors. Specifically, for any* $L_1, L_2 \in {}_A\mathbf{Mod}$ *and* $V \in \mathbf{M}$*, there exist isomorphisms*

$${}_A\mathbf{Mod}(L_1 \otimes V, L_2) \cong {}_A\mathbf{Mod}(L_1, L_2^V) \cong \mathbf{M}(V, [L_1, L_2]_A), \tag{1.6.6}$$

natural in L_1, L_2 *and* V*.*

Corollary 1.6.5. *For any object* $V \in \mathbf{M}$*, there exists an adjunction*

$$(-) \otimes V \dashv (-)^V \tag{1.6.7}$$

Corollary 1.6.6. *For any A-module $L \in {}_A\mathbf{Mod}$ there exists an adjunction*

$$[-, L]_A \dashv L^{(-)}. \tag{1.6.8}$$

Remark 1.6.7. Given a morphism $f \colon A \to B$ in $\mathbf{Mon}(\mathbf{M})$ and an object $V \in \mathbf{M}$, the restriction of scalars Res_f and the $\mathbf{M}$-powering $(-)^V$ commute, in the sense that the diagram

$$\begin{array}{ccc} {}_B\mathbf{Mod} & \xrightarrow{\mathrm{Res}_f} & {}_A\mathbf{Mod} \\ {\scriptstyle (-)^V}\downarrow & & \downarrow{\scriptstyle (-)^V} \\ {}_B\mathbf{Mod} & \xrightarrow[\mathrm{Res}_f]{} & {}_A\mathbf{Mod} \end{array} \tag{1.6.9}$$

commutes. This follows directly from the definitions.

Moreover, when $\mathbf{M}$ is also cocomplete, the corresponding diagram of their left adjoints (with respect to the adjunctions (1.5.8) and (1.6.7)) commutes up to natural isomorphism, i.e. there exists a natural isomorphism $\mathrm{Ext}_f(- \otimes V) \cong \mathrm{Ext}_f(-) \otimes V$. This follows from the composition of adjunctions (see Proposition A.5.10) and the uniqueness of adjoint functors (see Remark A.5.6). △

Remark 1.6.8. Given a morphism $f \colon A \to B$ in $\mathbf{Mon}(\mathbf{M})$, there exists a natural transformation

$$[-, -]_B \longrightarrow [-, -]_A \circ (\mathrm{Res}_f \times \mathrm{Res}_f) \tag{1.6.10}$$

in $\mathbf{Fun}({}_B\mathbf{Mod}^{\mathrm{op}} \times {}_B\mathbf{Mod}, \mathbf{M})$.

Explicitly, its component for $L, L' \in {}_B\mathbf{Mod}$ is the morphism $[L, L']_B \to [L|_A, L'|_A]_A$ in $\mathbf{M}$ defined by the following three-step construction: (1) Consider the L-component $L \to L'^{[L,L']_B}$ in ${}_B\mathbf{Mod}$ of the unit of the adjunction $[-, L']_B \dashv L'^{(-)} : \mathbf{M}^{\mathrm{op}} \to {}_B\mathbf{Mod}$. (2) Construct from the latter the morphism $L|_A \to L'|_A{}^{[L,L']_B}$ in ${}_A\mathbf{Mod}$ by applying the restriction $\mathrm{Res}_f \colon {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$ along f and by recalling the commutative square in Diagram 1.6.9. (3) Recalling also the adjunction $[-, L'|_A]_A \dashv L'|_A{}^{(-)} : \mathbf{M}^{\mathrm{op}} \to {}_A\mathbf{Mod}$, define the morphism $[L|_A, L'|_A]_A \to [L, L']_B$ in the opposite category $\mathbf{M}^{\mathrm{op}}$, which is equivalent to the desired morphism $[L, L']_B \to [L|_A, L'|_A]_A$ in $\mathbf{M}$. (Note that we used here two different instances of the adjunction displayed in Equation (1.6.8).) △

Chapter 2

Model Categories

In this chapter we provide a brief introduction to model categories. For more comprehensive expositions we refer to [Hov99; Hir09; Bal21].

We begin by recalling, in Section 2.1, the notion of a localization of a category at a collection of morphisms and the notions of weak equivalences, homotopy categories and derived functors.

Then, we proceed to define model categories in Section 2.2 and describe the fibrant and cofibrant replacement functors, while in Section 2.3, we recall the fundamental theorem of model categories, which provides an alternative model for the homotopy category associated with a model category.

In the following sections, we recall the definitions of Quillen functors, cofibrantly generated model categories and symmetric monoidal model categories, which are going to be useful in Chapter 3.

In Section 2.8, we present a transfer theorem from [SS00], that endows the categories of modules over monoids in a symmetric monoidal model with model structures. This will be useful in endowing the categories of net representations with model structures in Section 3.6.

Lastly, in Section 2.9, we recall the standard symmetric monoidal model structure on the category $\mathbf{CoCh}_{\mathbb{K}}$ of cochain complexes over some field $\mathbb{K}$ of characteristic zero, and we collect some basic facts about it.

2.1 Localization of a category

There is often the need to raise certain morphisms of a category to isomorphisms. This is achieved by passing to the localization of a category defined below.

Definition 2.1.1. Let $\mathbf{C}$ be a category and $\mathcal{W}$ a collection of morphisms in $\mathbf{C}$. A **localization of $\mathbf{C}$ at $\mathcal{W}$** is a category $\mathbf{C}[\mathcal{W}^{-1}]$ along with a functor

$\mathit{\Gamma}\colon \mathbf{C} \to \mathbf{C}[\mathcal{W}^{-1}]$ satisfying the following properties:

(i) The functor $\mathit{\Gamma}$ sends morphisms in $\mathcal{W}$ to isomorphisms in $\mathbf{C}[\mathcal{W}^{-1}]$.

(ii) **Universal property:** Given another category $\mathbf{D}$ and a functor $\mathcal{G}\colon \mathbf{C} \to \mathbf{D}$ that sends morphisms in $\mathcal{W}$ to isomorphisms in $\mathbf{D}$, there exists a unique functor $\tilde{\mathcal{G}}\colon \mathbf{C}[\mathcal{W}^{-1}] \to \mathbf{D}$ such that $\mathcal{G} = \tilde{\mathcal{G}} \circ \mathit{\Gamma}$.

The functor $\mathit{\Gamma}$ is called a **localization functor**.

Remark 2.1.2. A localization of a category at some collection of morphisms, provided that it exists, is unique up to isomorphism, in the following sense. If $\mathit{\Gamma}\colon \mathbf{C} \to \mathbf{D}$ and $\mathit{\Gamma}'\colon \mathbf{C} \to \mathbf{D}'$ are two localizations of a category $\mathbf{C}$ at a collection of morphisms $\mathcal{W}$, then there exists a unique isomorphism of categories $\mathcal{G}\colon \mathbf{D} \to \mathbf{D}'$ such that $\mathcal{G} \circ \mathit{\Gamma} = \mathit{\Gamma}'$.

The existence of a unique functor $\mathcal{G}\colon \mathbf{D} \to \mathbf{D}'$ such that $\mathcal{G} \circ \mathit{\Gamma} = \mathit{\Gamma}'$ follows from the universal property of a localization. By symmetry, there exists also a functor $\mathcal{G}'\colon \mathbf{D} \to \mathbf{D}'$ such that $\mathcal{G}' \circ \mathit{\Gamma}' = \mathit{\Gamma}$. Moreover, $\mathcal{G}' \circ \mathcal{G} = \mathrm{Id}_{\mathbf{D}}$. This follows from the requirement for uniqueness in the universal property of localizations and the fact that both the diagrams

$$\begin{array}{ccc} \mathbf{C} & & \\ {\scriptstyle \mathit{\Gamma}}\downarrow & \searrow^{\mathit{\Gamma}} & \\ \mathbf{D} & \xrightarrow[\mathcal{G}'\circ\mathcal{G}]{} & \mathbf{D} \end{array} \qquad \begin{array}{ccc} \mathbf{C} & & \\ {\scriptstyle \mathit{\Gamma}}\downarrow & \searrow^{\mathit{\Gamma}} & \\ \mathbf{D} & \xrightarrow[\mathrm{Id}_{\mathbf{D}}]{} & \mathbf{D} \end{array} \tag{2.1.1}$$

commute. By symmetry, we conclude also that $\mathcal{G} \circ \mathcal{G}' = \mathrm{Id}'_{\mathbf{D}}$. Therefore, $\mathcal{G}$ is an isomorphism of categories.

Because of the uniqueness of a localization of a category up to isomorphism, we often refer to *the* localization of a category. △

There is a simple description of the localization of a category due to Gabriel and Zisman[GZ67].

Construction 2.1.3. Let $\mathbf{C}$ be a category and $\mathcal{W}$ a collection of morphisms in $\mathbf{C}$. A model for $\mathbf{C}[\mathcal{W}^{-1}]$ is the category that has the same objects as the category $\mathbf{C}$ and the morphisms are defined as follows.

Consider finite strings of the form $(f_1, f_2, \ldots, f_n)$, where each f_i denotes a morphism in $\mathbf{C}$ or the reverse morphism $w^{\mathrm{op}} \in \mathrm{Mor}(\mathbf{C}^{\mathrm{op}})$ of a morphism $w \in \mathcal{W}$ (Definition A.1.11(b)), and the source of f_{i+1} equals the target of f_i for each $i \in \{1, \ldots, n-1\}$. Next, we define the following relations between such strings.

- If, for some string $(f_1, \dots, f_n)$, the composite $f_{i+1} \circ f_i$ exists for some $i \in \{1, \dots, n-1\}$, then $(f_1, \dots, f_n) \sim (f_1, \dots, f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, \dots, f_n)$.

- If, for some string $(f_1, \dots, f_n)$, $f_i \in \mathcal{W}$ and $f_{i+1} = f_i^{\mathrm{op}}$ for some $i \in \{1, \dots, n-1\}$, then $(f_1, \dots, f_n) \sim (f_1, \dots, f_{i-1}, \mathrm{id}_{\mathrm{s}(f_i)}, f_{i+2}, \dots, f_n)$, where $\mathrm{s}(f_i)$ denotes the source of f_i.

- If, for some string $(f_1, \dots, f_n)$, $f_i \in \mathcal{W}$ and $f_{i-1} = f_i^{\mathrm{op}}$ for some $i \in \{2, \dots, n\}$, then $(f_1, \dots, f_n) \sim (f_1, \dots, f_{i-2}, \mathrm{id}_{\mathrm{t}(f_i)}, f_{i+1}, \dots, f_n)$, where $\mathrm{t}(f_i)$ denotes the target of f_i.

We take the morphisms of $\mathbf{C}[\mathcal{W}^{-1}]$ to be the collection of strings, described above, modulo the equivalence relation generated by these relations.

The composition of morphisms in $\mathbf{C}[\mathcal{W}^{-1}]$ is given by concatenation of strings, and the identity morphism on an object $C \in \mathbf{C}[\mathcal{W}^{-1}]$ is the string (id_C).

The localization functor $\Gamma \colon \mathbf{C} \to \mathbf{C}[\mathcal{W}^{-1}]$ is the functor that sends each object to itself and each morphism $f \in \mathrm{Mor}(\mathbf{C})$ to the one element string $\Gamma(f) \coloneqq (f) \in \mathbf{C}[\mathcal{W}^{-1}]$.

From now on, whenever we refer to the localization of a category we will have in mind this specific model, unless otherwise specified.

In practice, we are usually interested to localize a category over a collection of weak equivalences, which we define below.

Definition 2.1.4. Let $\mathbf{C}$ be a category. A **collection of weak equivalences in $\mathbf{C}$** is a collection of morphisms $\mathcal{W} \subseteq \mathrm{Mor}(\mathbf{C})$, satisfying the following properties:

(i) For any $C \in \mathrm{Ob}(\mathbf{C})$, $id_C \in \mathcal{W}$.

(ii) **2-out-of-3 property:** Given morphisms $f, g \in \mathrm{Mor}(\mathbf{C})$ such that $g \circ f$ exists, if two of the morphisms f, g and $g \circ f$ belong to $\mathcal{W}$, then so is the third.

This gives rise to the notions of a category with weak equivalences and the associated homotopy category.

Definition 2.1.5. A **category with weak equivalences** $(\mathbf{C}, \mathcal{W})$ consists of a category $\mathbf{C}$ and a collection of weak equivalences $\mathcal{W}$. Two objects $c_1, c_2 \in \mathrm{Ob}(\mathbf{C})$ are called **weakly equivalent** if there exists a zig-zag of weak equivalences from c_1 to c_2.

Remark 2.1.6. It is easy to see that any subcategory $\mathbf{D}$ of a category with weak equivalences $(\mathbf{C}, \mathcal{W})$ inherits a collection of weak equivalences from $\mathbf{C}$. Specifically, the collection of morphisms $\mathcal{W} \cap \mathrm{Mor}(\mathbf{D})$ is a collection of weak equivalences. △

Definition 2.1.7. Let $(\mathbf{C}, \mathcal{W})$ be a category with weak equivalences. The **homotopy category associated with** $(\mathbf{C}, \mathcal{W})$, denoted by $\mathrm{Ho}(\mathbf{C})$, is the category $\mathbf{C}[\mathcal{W}^{-1}]$, described in Construction 2.1.3.

Next, we define functors between categories with weak equivalences such that they induce functors between the associated homotopy categories.

Definition 2.1.8. Let $(\mathbf{C}, \mathcal{W}_{\mathbf{C}})$ and $(\mathbf{D}, \mathcal{W}_{\mathbf{D}})$ be categories with weak equivalences. A functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ is called a **homotopical functor** if it preserves weak equivalences, that is $\mathcal{F}(\mathcal{W}_{\mathbf{C}}) \subseteq \mathcal{W}_{\mathbf{D}}$.

Notice that, given a homotopical functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ between categories with weak equivalences $(\mathbf{C}, \mathcal{W}_{\mathbf{C}})$ and $(\mathbf{D}, \mathcal{W}_{\mathbf{D}})$, its precomposition with the localization functor $\Gamma_{\mathbf{D}}\colon \mathbf{D} \to \mathbf{D}[\mathcal{W}_{\mathbf{D}}^{-1}]$ from Construction 2.1.3 sends weak equivalences in $\mathbf{C}$ to isomorphisms in $\mathbf{D}[\mathcal{W}_{\mathbf{D}}^{-1}] = \mathrm{Ho}(\mathbf{D})$. From the universal property of the localization $\Gamma_{\mathbf{C}}\colon \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ it follows that there exists a unique functor $\tilde{\mathcal{F}}\colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ such that the diagram

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\mathcal{F}} & \mathbf{D} \\ \downarrow{\scriptstyle \Gamma_{\mathbf{C}}} & & \downarrow{\scriptstyle \Gamma_{\mathbf{D}}} \\ \mathrm{Ho}(\mathbf{C}) & \xrightarrow[\tilde{\mathcal{F}}]{} & \mathrm{Ho}(\mathbf{D}) \end{array} \tag{2.1.2}$$

commutes. This fact allows the following definition.

Definition 2.1.9. Let $(\mathbf{C}, \mathcal{W}_{\mathbf{C}})$ and $(\mathbf{D}, \mathcal{W}_{\mathbf{D}})$ be categories with weak equivalences and $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ a homotopical functor. We define the **homotopy functor associated with** $\mathcal{F}$ as the functor $\mathrm{Ho}(\mathcal{F})\colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ determined by the universal property of the localization $\Gamma\colon \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$.

Remark 2.1.10. Let $(\mathbf{C}, \mathcal{W}_{\mathbf{C}})$, $(\mathbf{D}, \mathcal{W}_{\mathbf{D}})$ and $(\mathbf{E}, \mathcal{W}_{E})$ be categories with weak equivalences. It is easy to see that, given two homotopical functors $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ and $\mathcal{F}\colon \mathbf{D} \to \mathbf{E}$, their composition preserves weak equivalences and consequently it is a homotopical functor. Moreover, it follows from the uniqueness of the universal property of the localization that $\mathrm{Ho}(\mathcal{G}) \circ \mathcal{F} = \mathrm{Ho}(\mathcal{G}) \circ \mathrm{Ho}(\mathcal{F})$. △

In practice, it is often the case that the functors of interest are not homotopical. In this case, it is still possible to obtain relevant functors between the associated homotopy categories when there is a suitable deformation, which we define below.

Definition 2.1.11. Let $(\mathbf{C}, \mathcal{W}_{\mathbf{C}})$ and $(\mathbf{D}, \mathcal{W}_{\mathbf{D}})$ be categories with weak equivalences and $\mathcal{F}, \mathcal{G}\colon \mathbf{C} \to \mathbf{D}$ functors between them. A **natural weak equivalence between $\mathcal{F}$ and $\mathcal{G}$** is a natural transformation $\alpha\colon \mathcal{F} \to \mathcal{G}$ such that $\alpha_C \in \mathcal{W}_{\mathbf{D}}$, for any $C \in \mathrm{Ob}(\mathbf{C})$.

Definition 2.1.12. Let $(\mathbf{C}, \mathcal{W}_{\mathbf{C}})$ be a category with weak equivalences.

(a) A **left deformation on $\mathbf{C}$** consists of an endofunctor $\mathcal{Q}\colon \mathbf{C} \to \mathbf{C}$, called a **left deformation functor**, and a natural weak equivalence $q\colon \mathcal{Q} \to \mathrm{Id}_{\mathbf{C}}$.

(b) A **right deformation on $\mathbf{C}$** consists of an endofunctor $\mathcal{R}\colon \mathbf{C} \to \mathbf{C}$, called a **right deformation functor**, and a natural weak equivalence $r\colon \mathrm{Id}_{\mathbf{C}} \to \mathcal{R}$.

Remark 2.1.13. A deformation functor is always a homotopical functor. This follows from the 2-out-of-3 property that the collection of weak equivalence satisfies and the fact that there is a natural weak equivalence between the deformation functor and the identity functor. △

This allows for the following definition.

Definition 2.1.14. Assume that

- $(\mathbf{C}, \mathcal{W}_{\mathbf{C}})$ is a category with weak equivalences,
- $(\mathcal{Q}, q)$ is a left deformation on $\mathbf{C}$,
- $(\mathbf{D}, \mathcal{W}_{\mathbf{D}})$ is a category with weak equivalences,
- $(\mathcal{R}, r)$ is a right deformation on $\mathbf{D}$,
- $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ is a functor such that it preserves all the weak equivalences between objects in a full subcategory containing the image of $\mathcal{Q}$,
- $\mathcal{G}\colon \mathbf{D} \to \mathbf{C}$ is a functor such that it preserves all the weak equivalences between objects in a full subcategory containing the image of $\mathcal{R}$.

We define:

(a) the **left derived functor of** $\mathcal{F}$, denoted by $\mathrm{L}\,\mathcal{F}$, as the composite

$$\mathrm{Ho}(\mathbf{C}) \xrightarrow{\mathrm{Ho}(\mathcal{Q})} \mathrm{Ho}(\mathcal{Q}(\mathbf{C})) \xrightarrow{\mathrm{Ho}(\mathcal{F})} \mathrm{Ho}(\mathbf{D})\,, \tag{2.1.3}$$

(b) the **right derived functor of** $\mathcal{G}$, denoted by $\mathrm{R}\,\mathcal{G}$, as the composite

$$\mathrm{Ho}(\mathbf{D}) \xrightarrow{\mathrm{Ho}(\mathcal{R})} \mathrm{Ho}(\mathcal{R}(\mathbf{D})) \xrightarrow{\mathrm{Ho}(\mathcal{G})} \mathrm{Ho}(\mathbf{C}). \tag{2.1.4}$$

Lastly, we explain some of the motivations for introducing model categories.

First, notice that even if Construction 2.1.3 proves the existence of homotopy categories, it doesn't guarantee that they are locally small (see Definition A.1.8(c)) even if the associated categories with weak equivalences are locally small.

Moreover, a morphism in a category with weak equivalences may be raised to an isomorphism in the associated homotopy category, even if it is not a weak equivalence. This implies that two objects in the homotopy category may be isomorphic even if there is no weak equivalence between them. This fact motivates the the following definition.

Definition 2.1.15. Let $\mathbf{C}$ be a category and $\mathcal{W}$ a collection of morphisms in $\mathbf{C}$. We say that the collection $\mathcal{W}$ is **saturated** if, for any $f \in \mathrm{Mor}(\mathbf{C})$, its image $\Gamma(f)$ under the localization $\Gamma\colon \mathbf{C} \to \mathbf{C}[\mathcal{W}^{-1}]$ is an isomorphism if and only if $f \in \mathcal{W}$.

Some of the reasons that model categories have been instrumental in the study of homotopy categories is that they resolve these issues. In fact, equipping a category with weak equivalences with a suitable model structure guarantees that the collection of weak equivalences is saturated and the associated homotopy category is locally small. On top of these, a model structure gives rise to both a left and a right deformation that allow not only for the construction of derived functors but also of derived adjunctions. We present all these well-known facts about model categories in Sections 2.3 and 2.4.

2.2 Model categories

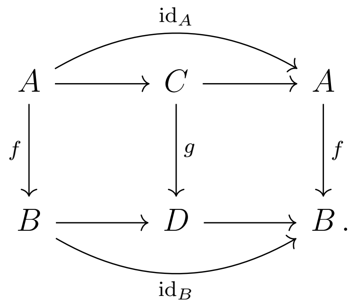
In order to define model categories, we will need first to introduce some preliminary concepts.

Definition 2.2.1. Let $i\colon A \to B$ and $p\colon C \to D$ be two morphisms in a category $\mathbf{C}$. We say that i has the **left lifting property with respect to** p and p has the **right lifting property with respect to** i if for any morphisms $f\colon A \to C$ and $g\colon B \to D$ in $\mathbf{C}$ such that the diagram

$$\begin{CD} A @>{f}>> C \\ @V{i}VV @VV{p}V \\ B @>>{g}> D \end{CD} \tag{2.2.1}$$

in $\mathbf{C}$ commutes, there exists a morphism $h\colon B \to C$ such that $h \circ i = f$ and $p \circ h = g$. The morphism h is called a **lift** in this diagram.

Definition 2.2.2. Let $f\colon A \to B$ and $g\colon C \to D$ be two morphisms in a category $\mathbf{C}$. We say that f is a retract of g if there exists a commutative diagram in $\mathbf{C}$ of the form



(2.2.2)

Definition 2.2.3. Let $\mathbf{C}$ be a category. A **functorial factorization** in $\mathbf{C}$ is an ordered pair of functors $(\alpha, \beta) \in \mathbf{C}^{\mathbf{C}} \times \mathbf{C}^{\mathbf{C}}$ such that $f = \beta(f) \circ \alpha(f)$ for any morphism $f \in \mathrm{Mor}(\mathbf{C})$.

Now we can define a model structure as follows.

Definition 2.2.4. Let $\mathbf{C}$ be a category. A model structure on $\mathbf{C}$ consists of two functorial factorizations (α, β) and (γ, δ), and three collections of morphisms:

(a) **weak equivalences**, denoted by $\mathcal{W}_{\mathbf{C}}$,

(b) **fibrations**, denoted by $\mathrm{Fib}_{\mathbf{C}}$,

(c) **cofibrations**, denoted by $\mathrm{Cof}_{\mathbf{C}}$.

A fibration (resp. cofibration) that is also a weak equivalence is called **acyclic fibration** (resp. **acyclic cofibration)**. These functorial factorizations and collections of morphisms are subject to the following axioms:

(i) The collections of weak equivalences, fibrations and cofibrations are closed under composition.

(ii) **2-out-of-3 axiom:** The collection of weak equivalences satisfies the 2-out-of-3 property (Definition 2.1.4).

(iii) **Retract axiom:** Let $f, g \in \mathrm{Mor}(\mathbf{C})$ such that f is a retract of g. If g is a weak equivalence, fibration or cofibration, then so is f.

(iv) **Lifting axiom:** Acyclic cofibrations have the left lifting property with respect to fibrations, and acyclic fibrations have the right lifting property with respect to cofibrations.

(v) **Factorization axiom:** For any $f \in \mathrm{Mor}(\mathbf{C})$, the factorization $f = \beta(f) \circ \alpha(f)$ consists of an acyclic fibration $\beta(f)$ and a cofibration $\alpha(f)$. On the other hand, the factorization $f = \delta(f) \circ \gamma(f)$ consists of a fibration $\delta(f)$ and an acyclic cofibration $\gamma(f)$.

Definition 2.2.5. Let $(\mathbf{C}, \mathcal{W})$ be a category with weak equivalences as in Definition 2.1.5. We say that a model structure on $\mathbf{C}$ is a **compatible** model structure on $(\mathbf{C}, \mathcal{W})$ when $\mathcal{W}_{\mathbf{C}} = \mathcal{W}$, where $\mathcal{W}_{\mathbf{C}}$ denotes the weak equivalences of the model structure on $\mathbf{C}$.

Definition 2.2.6. A **model category** is a locally small, complete and cocomplete category $\mathbf{C}$ equipped with a model structure.

Notation 2.2.7. In diagrams, we denote:

- a weak equivalence by $\xrightarrow{\approx}$
- a fibration by $\twoheadrightarrow$
- a cofibration by $\hookrightarrow$

Remark 2.2.8. The data of the model structure of a model category is superfluous, in the sense that the collection of fibrations can be determined by the collections of cofibrations and weak equivalences, and vice-versa, the collection of cofibrations can be determined by the collections of fibrations and weak equivalences, as implied by the following proposition. $\triangle$

Proposition 2.2.9 ([Hov99, Lem. 1.1.10])**.** *Let* $\mathbf{C}$ *be a model category. A morphism* $f \in \mathrm{Mor}(\mathbf{C})$ *is a fibration (resp. acyclic fibration) if and only if it has the right lifting property with respect to any acyclic cofibration (resp. cofibration). On the other hand, a morphism* $f \in \mathrm{Mor}(\mathbf{C})$ *is a cofibration (resp. acyclic cofibration) if and only if it has the left lifting property with respect to any acyclic fibration (resp. fibration).*

A consequence of Proposition 2.2.9 is that the isomorphisms of a model category belong to all three collections of weak equivalences, fibrations and cofibrations since an isomorphism has both the left and the right lifting properties with respect to any morphism as one can show using the inverse morphism (see Definition A.1.4(c)). In fact, the converse is also true.

Proposition 2.2.10 ([Bal21, Cor. 2.1.10]). *let* $\mathbf{C}$ *be a model category. A morphism in* $\mathbf{C}$ *belongs to all three collections of morphisms* $\mathcal{W}_{\mathbf{C}}$*,* $\mathrm{Fib}_{\mathbf{C}}$ *and* $\mathrm{Cof}_{\mathbf{C}}$ *if and only if it is an isomorphism.*

The crucial property of model categories is that every object in a model category can be replaced with a better behaved object. These objects are the fibrant, cofibrant and bifibrant objects, which we introduce next.

Remark 2.2.11. A model category has always an initial and a terminal object since, by definition, it is complete and cocomplete. △

This fact allows for the following definitions.

Definition 2.2.12. Let $\mathbf{C}$ be a model category. An object $C \in \mathrm{Ob}(\mathbf{C})$ is called:

(a) **fibrant** if the unique morphism $C \to *$ from C to the terminal object is a fibration,

(b) **cofibrant** if the unique morphism $\emptyset \to C$ from the initial object to C is a cofibration,

(c) **bifibrant** if it is both fibrant and cofibrant.

The categories of fibrant, cofibrant and bifibrant objects of a model category are defined as follows.

Definition 2.2.13. Let $\mathbf{C}$ be a model category.

(a) We define the **category of fibrant objects of** $\mathbf{C}$, denoted by $\mathbf{C}_{\mathrm{f}}$, as the full subcategory of $\mathbf{C}$ with objects all the fibrant objects of $\mathbf{C}$.

(b) We define the **category of cofibrant objects of** $\mathbf{C}$, denoted by $\mathbf{C}_{\mathrm{c}}$, as the full subcategory of $\mathbf{C}$ with objects all the cofibrant objects of $\mathbf{C}$.

(c) We define the **category of bifibrant objects of** $\mathbf{C}$, denoted by $\mathbf{C}_{\mathrm{cf}}$, as the full subcategory of $\mathbf{C}$ with objects all the bifibrant objects of $\mathbf{C}$.

Next, we give a precise meaning to the notion or replacing an object in a model category with a fibrant or cofibrant object.

Definition 2.2.14. Let $\mathbf{C}$ be a model category and C an object in $\mathbf{C}$.

(a) A **fibrant resolution** of C is a factorization of the morphism $C \to *$ in the form

$$C \xrightarrow[\approx]{f} X \xrightarrow{g} * \,, \tag{2.2.3}$$

for some $X \in \mathrm{Ob}(\mathbf{C})$ such that f is a weak equivalence and g a cofibration. The object X is called a **fibrant replacement of** C.

(b) A **cofibrant resolution** of C is a factorization of the morphism $\emptyset \to C$ of the form

$$\emptyset \xhookrightarrow{f} X \xrightarrow[\approx]{g} C \,, \tag{2.2.4}$$

for some $X \in \mathrm{Ob}(\mathbf{C})$ such that f is a cofibration and g a weak equivalence. The object X is called a **cofibrant replacement of** C.

Remark 2.2.15. Any fibrant replacement X of a fibrant object C is isomorphic to C △

Remark 2.2.16. The functorial factorizations that are part of the definition of a model structure (see Definition 2.2.4) provide fibrant and cofibrant resolutions for any object in a model category.

Specifically, let $\mathbf{C}$ be a model category with functorial factorizations (α, β) and (γ, δ) as in Definition 2.2.4. Given an object $C \in \mathrm{Ob}(\mathbf{C})$ and the unique morphisms $f_C \colon C \to *$ and $g_C \colon \emptyset \to C$ in $\mathbf{C}$, the factorization

$$C \xhookrightarrow[\approx]{\gamma(f_C)} \delta(C) \xrightarrow{\delta(f_C)} * \tag{2.2.5}$$

is a fibrant resolution of C, while the factorization

$$\emptyset \xhookrightarrow{\alpha(g_C)} \alpha(C) \xrightarrow[\approx]{\beta(g_C)} C \tag{2.2.6}$$

is a cofibrant resolution of C. △

Moreover, the fact that the factorizations in the definition of a model structure are functorial gives rise to the fibrant and cofibrant replacement functors defined below.

Definition 2.2.17. Let $\mathbf{C}$ be a model category with functorial factorizations (α, β) and (γ, δ) as in Definition 2.2.4.

(a) We define the **fibrant replacement functor** of $\mathbf{C}$, denoted by $\mathcal{R}_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{C}_{\mathrm{f}}$, as the functor that assigns to each object $C \in \mathrm{Ob}(\mathbf{C})$ the fibrant replacement $\mathcal{R}_{\mathbf{C}}(C) \coloneqq \delta(C) \in \mathrm{Ob}(\mathbf{C}_{\mathrm{f}})$, and to each morphism $h \colon C \to C'$ in $\mathbf{C}$, the morphism $\mathcal{R}_{\mathbf{C}}(h) \coloneqq \delta(h)$ in $\mathbf{C}_{\mathrm{f}}$.

(b) We define the **cofibrant replacement functor** of $\mathbf{C}$, denoted by $\mathcal{Q}_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}_{\mathrm{c}}$, as the functor that assigns to each object $C \in \mathrm{Ob}(\mathbf{C})$ the cofibrant replacement $\mathcal{Q}_{\mathbf{C}}(C) := \alpha(C) \in \mathrm{Ob}(\mathbf{C}_{\mathrm{c}})$, and to each morphism $h\colon C \to C'$ in $\mathbf{C}$, the morphism $\mathcal{Q}_{\mathbf{C}}(h) := \alpha(h)$ in $\mathbf{C}_{\mathrm{c}}$.

Notation 2.2.18. We use the following conventions:

- To simplify notation, we will make no distinction between the fibrant replacement functor of a model category $\mathbf{C}$, as defined above, and its composition with the inclusion functor $\mathcal{J}_{\mathrm{f}}\colon \mathbf{C}_{\mathrm{f}} \to \mathbf{C}$. That means that we will refer to the functor $\mathbf{C} \xrightarrow{\mathcal{R}_{\mathbf{C}}} \mathbf{C}_{\mathrm{f}} \xrightarrow{\mathcal{J}_{\mathrm{f}}} \mathbf{C}$ also as the fibrant replacement functor of $\mathbf{C}$, and we will denote it by $\mathcal{R}_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$.

  Similarly, we will make no distinction between the cofibrant replacement functor of a model category $\mathbf{C}$, as defined above, and its composition with the inclusion functor $\mathcal{J}_{\mathrm{c}}\colon \mathbf{C}_{\mathrm{c}} \to \mathbf{C}$. That means that we will refer to the functor $\mathbf{C} \xrightarrow{\mathcal{Q}_{\mathbf{C}}} \mathbf{C}_{\mathrm{c}} \xrightarrow{\mathcal{J}_{\mathrm{c}}} \mathbf{C}$ also as the cofibrant replacement functor of $\mathbf{C}$, and we will denote it by $\mathcal{Q}_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$.

- When there is no risk of confusion, we will suppress the index of the fibrant and cofibrant replacement functors. In other words, instead of $\mathcal{R}_{\mathbf{C}}$ we will write $\mathcal{R}$ and instead of $\mathcal{Q}_{\mathbf{C}}$ we will write $\mathcal{Q}$.

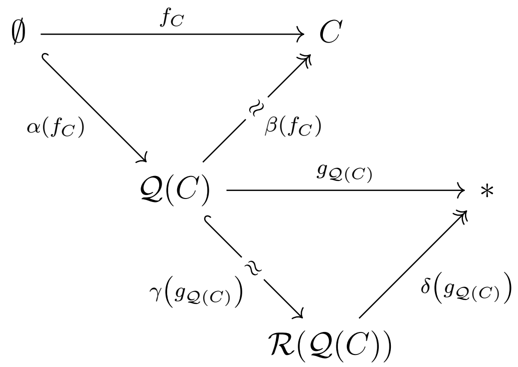
Remark 2.2.19. The fibrant and cofibrant replacement functors on a model category define a right and a left deformation on $\mathbf{C}$ respectively (see Definition 2.1.12).

Specifically, let $\mathbf{C}$ be a model category with functorial factorizations (α, β) and (γ, δ) as in Definition 2.2.4. The fibrant replacement functor $\mathcal{R}\colon \mathbf{C} \to \mathbf{C}$ defines a right deformation on $\mathbf{C}$ along with the natural weak equivalence $q\colon \mathrm{Id}_{\mathbf{C}} \to \mathcal{R}$ defined as follows. For each $C \in \mathrm{Ob}(\mathbf{C})$, let f_C be the morphism $C \to *$ in $\mathbf{C}$. We define the C component of q as $q_C := \gamma(f_C)\colon C \to \mathcal{R}(C)$. That q is a natural weak equivalence follows from the functoriality of γ and the fact that $\gamma(f)$ is by definition a weak equivalence for any $f \in \mathrm{Mor}(\mathbf{C})$.

Similarly, the cofibrant replacement functor $\mathcal{Q}\colon \mathbf{C} \to \mathbf{C}$ defines a left deformation on $\mathbf{C}$ along with the natural weak equivalence $r\colon \mathcal{Q} \to \mathrm{Id}_{\mathbf{C}}$ defined as follows. For each $C \in \mathrm{Ob}(\mathbf{C})$, let g_C be the morphism $\emptyset \to C$ in $\mathbf{C}$. We define the C component of r as $r_C := \beta(g_C)\colon \mathcal{Q}(C) \to C$. That r is a natural weak equivalence follows from the functoriality of β and the fact that $\beta(f)$ is by definition a weak equivalence for any $f \in \mathrm{Mor}(\mathbf{C})$. △

Remark 2.2.20. The composition $\mathcal{R} \circ \mathcal{Q}$ of the fibrant and cofibrant replacement functors of a model category $\mathbf{C}$ gives rise to a functor from $\mathbf{C}$ to $\mathbf{C}_{\mathrm{cf}}$. The fact that the image of $\mathcal{R} \circ \mathcal{Q}$ lies in $\mathbf{C}_{\mathrm{cf}}$ can be verified as follows. Let

(α, β) and (γ, δ) be the functorial factorizations of the model structure on $\mathbf{C}$ as in Definition 2.2.4. For each $C \in \mathrm{Ob}(\mathbf{C})$, consider the diagram



(2.2.7)

where $f_C \colon \emptyset \to C$ and $g_{\mathcal{Q}(C)} \colon \mathcal{Q}(C) \to *$. Since, by definition, $\alpha(f_C)$ and $\gamma(g_{\mathcal{Q}(C)})$ are cofibrations, $\gamma(g_{\mathcal{Q}(C)}) \circ \alpha(f_C) \colon \emptyset \to \mathcal{R}(\mathcal{Q}(C))$ is also a cofibration which proves that $\mathcal{R}(\mathcal{Q}(C))$ is a cofibrant object. Moreover, since, by definition, $\delta(g_{\mathcal{Q}(C)})$ is a fibration, $\mathcal{R}(\mathcal{Q}(C))$ is also a fibrant object. △

2.3 The homotopy category of a model category

Let $\mathbf{C}$ be a model category. It is easy to see that $(\mathbf{C}, \mathcal{W}_{\mathbf{C}})$, where $\mathcal{W}_{\mathbf{C}}$ is the collection of weak equivalences of the model structure on $\mathbf{C}$, is a category with weak equivalences in the sense of Definition 2.1.4 since $\mathcal{W}_{\mathbf{C}}$ satisfies the 2-out-of-3 property (see Definition 2.2.4) and it contains all the identity morphisms (see Proposition 2.2.10). Therefore, there is a homotopy category associated with each model category.

Definition 2.3.1. Let $\mathbf{C}$ be a model category. We define the **homotopy category** associated with $\mathbf{C}$ to be the homotopy category associated with the category with weak equivalences $(\mathbf{C}, \mathcal{W}_{\mathbf{C}})$, where $\mathcal{W}_{\mathbf{C}}$ denotes the collection of weak equivalences of the model structure on $\mathbf{C}$ (see Definition 2.2.4).

The goal of this section is to present the fundamental theorem of model category theory. That is, that the homotopy category associated with a model category is a locally small category. For a detailed derivation we refer the reader to [Hov99, Sec. 1.2].

First, notice that, given a model category $\mathbf{C}$, any full subcategory $\mathbf{C}'$ of $\mathbf{C}$ inherits a collection of weak equivalences from $\mathbf{C}$. Indeed, it is easy to see that the collection of morphisms $\mathcal{W}_{\mathbf{C}'} \coloneqq \mathcal{W}_{\mathbf{C}} \cap \mathrm{Mor}(\mathbf{C}')$ contains all the identity morphisms in $\mathbf{C}'$ and it satisfies the 2-out-of-3 property due to the fact that

$\mathbf{C}'$ is a full subcategory of $\mathbf{C}$. Therefore, $(\mathbf{C}', \mathcal{W}_{\mathbf{C}'})$ is a category with weak equivalences. Moreover, we notice that the inclusion functor $\mathcal{J}_{\mathbf{C}'} \colon \mathbf{C}' \to \mathbf{C}$ is a homotopical functor (see Definition 2.1.8), which means that it induces a homotopy functor $\mathrm{Ho}(\mathcal{J}_{\mathbf{C}'}) \colon \mathrm{Ho}(\mathbf{C}') \to \mathrm{Ho}(\mathbf{C})$ (see Definition 2.1.9).

This implies that the subcategories $\mathbf{C}_{\mathrm{f}}$, $\mathbf{C}_{\mathrm{c}}$ and $\mathbf{C}_{\mathrm{cf}}$ of a model category $\mathbf{C}$ inherit collections of weak equivalences $\mathcal{W}_{\mathbf{C}_{\mathrm{f}}}$, $\mathcal{W}_{\mathbf{C}_{\mathrm{c}}}$ and $\mathcal{W}_{\mathbf{C}_{\mathrm{cf}}}$ respectively, and the inclusion functors $\mathcal{J}_{\mathbf{C}_{\mathrm{f}}} \colon \mathbf{C}_{\mathrm{f}} \to \mathbf{C}$, $\mathcal{J}_{\mathbf{C}_{\mathrm{c}}} \colon \mathbf{C}_{\mathrm{c}} \to \mathbf{C}$ and $\mathcal{J}_{\mathbf{C}_{\mathrm{cf}}} \colon \mathbf{C}_{\mathrm{cf}} \to \mathbf{C}$ are homotopical functors.

Moreover, the fibrant and cofibrant replacement functors $\mathcal{R} \colon \mathbf{C} \to \mathbf{C}$ and $\mathcal{Q} \colon \mathbf{C} \to \mathbf{C}$ are also homotopical functors since they are deformation functors (see Remark 2.2.19) and deformation functors are homotopical (see Remark 2.1.13). As a result, the composite functor $\mathcal{R} \circ \mathcal{Q} \colon \mathbf{C} \to \mathbf{C}_{\mathrm{cf}}$ from Remark 2.2.20 is a homotopical functor as well (see Remark 2.1.10).

The first step towards the fundamental theorem about model categories is the following proposition.

Proposition 2.3.2 ([Hov99, Prop. 1.2.3]). *Let* $\mathbf{C}$ *be a model category,* $\mathbf{J}_{\mathbf{C}_{\mathrm{cf}}} \colon \mathbf{C}_{\mathrm{cf}} \to \mathbf{C}$ *the inclusion functor and* $\mathcal{R} \circ \mathcal{Q} \colon \mathbf{C} \to \mathbf{C}_{\mathrm{cf}}$ *the composite functor of the fibrant and cofibrant replacement functors of* $\mathbf{C}$*. The associated homotopy functors* $\mathrm{Ho}(\mathbf{J}_{\mathbf{C}_{\mathrm{cf}}})$ *and* $\mathrm{Ho}(\mathcal{R} \circ \mathcal{Q})$ *constitute an equivalence between the categories* $\mathrm{Ho}(\mathbf{C}_{\mathrm{cf}})$ *and* $\mathrm{Ho}(\mathbf{C})$ *(see Definition A.2.11).*

Taking advantage of this fact, we can focus on the homotopy category associated with $\mathbf{C}_{\mathrm{cf}}$ instead. This homotopy category accepts an alternative construction which lies at the center of the fundamental theorem about model categories. We summarize this construction now.

Definition 2.3.3. Let $\mathbf{C}$ be a model category, X an object in $\mathbf{C}$ and $\nabla \colon X \amalg X \to X$ be the codiagonal morphism, i.e. the unique morphism out of the coproduct $X \amalg X$ such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\iota_1} & X \amalg X & \xleftarrow{\iota_2} & X \\ & {\scriptstyle \mathrm{id}_X} \searrow & \downarrow {\scriptstyle \nabla} & \swarrow {\scriptstyle \mathrm{id}_X} & \\ & & X & & \end{array} \tag{2.3.1}$$

in $\mathbf{C}$ commutes, where ι_1 and ι_2 denote the inclusions to the coproduct $X \amalg X$. A **cylinder object** for X is an object $X' \in \mathrm{Ob}(\mathbf{C})$ such that there exists a factorization of the codiagonal map $\nabla \colon X \amalg X \to X$ of the form

$$X \amalg X \xhookrightarrow{(i_1, i_2)} X' \xrightarrow[\approx]{s} X \,, \tag{2.3.2}$$

where (i_1, i_2) is a cofibration and s a weak equivalence.

Definition 2.3.4. Let $\mathbf{C}$ be a model category and $f, g \in \mathbf{C}(X, Y)$ for some $X, Y \in \mathrm{Ob}(\mathbf{C})$. Moreover, let $X' \in \mathrm{Ob}(\mathbf{C})$ be a cylinder object for X and $\nabla = s \circ (i_1, i_2)$ be a factorization of the codiagonal map $\nabla\colon X \amalg X \to X$ through X' as in Definition 2.3.3. A **left homotopy** from f to g is a morphism $H\colon X' \to Y$ such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X' & \xleftarrow{i_2} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle H} & \swarrow{\scriptstyle g} & \\ & & Y & & \end{array} \tag{2.3.3}$$

in $\mathbf{C}$ commutes.

The dual notions of a cylinder object and left homotopy are those of a path object and a right homotopy, defined below.

Definition 2.3.5. Let $\mathbf{C}$ be a model category, Y an object in $\mathbf{C}$ and $\Delta\colon Y \to Y \times Y$ be the diagonal morphism, i.e. the unique morphism into the product $Y \times Y$ such that the diagram

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle \mathrm{id}_Y}\swarrow & \downarrow{\scriptstyle \Delta} & \searrow{\scriptstyle \mathrm{id}_Y} & \\ Y & \xleftarrow[\pi_1]{} & Y \times Y & \xrightarrow[\pi_2]{} & Y \end{array} \tag{2.3.4}$$

in $\mathbf{C}$ commutes, where π_1 and π_2 denote the projections out of the product $Y \times Y$. A **path object** for Y is an object $Y' \in \mathrm{Ob}(\mathbf{C})$ such that there exists a factorization of the diagonal map $\Delta\colon Y \to Y \times Y$ of the form

$$Y \xrightarrow[\approx]{r} Y' \xrightarrow{(p_1, p_2)} Y \times Y\,, \tag{2.3.5}$$

where r is a weak equivalence and (p_1, p_2) a fibration.

Definition 2.3.6. Let $\mathbf{C}$ be a model category and $f, g \in \mathbf{C}(X, Y)$ for some $X, Y \in \mathrm{Ob}(\mathbf{C})$. Moreover, let $Y' \in \mathrm{Ob}(\mathbf{C})$ be a cylinder object for Y and $\Delta = (p_1, p_2) \circ r$ be a factorization of the diagonal map $\Delta\colon Y \to Y \times Y$ through Y' as in Definition 2.3.5.

A **right homotopy** from f to g is a morphism $H\colon X \to Y'$ such that the diagram

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle f}\swarrow & \downarrow{\scriptstyle H} & \searrow{\scriptstyle g} & \\ Y & \xleftarrow[p_1]{} & Y' & \xrightarrow[p_2]{} & Y \end{array} \tag{2.3.6}$$

in $\mathbf{C}$ commutes.

Now, we can define homotopy equivalences as follows.

Definition 2.3.7. Let $\mathbf{C}$ be a model category and $f, g \in \mathbf{C}(X, Y)$ for some $X, Y \in \mathrm{Ob}(\mathbf{C})$.

(a) We say f and g are **homotopic**, which we denote by $f \sim g$, if there exist both a left and a right homotopy between them.

(b) We say that f is a **homotopy equivalence** if there exists a morphism $h \in \mathrm{Mor}(\mathbf{C})$ such that $f \circ h \sim \mathrm{id}_X$ and $h \circ f \sim \mathrm{id}_Y$.

The key for the description of the homotopy category associated with the category of bifibrant objects of a model category is the following proposition.

Proposition 2.3.8 ([Hov99, Cor. 1.2.6])**.** *Let $\mathbf{C}$ be a model category, $X \in \mathrm{Ob}(\mathbf{C})$ a cofibrant object and $Y \in \mathrm{Ob}(\mathbf{C})$ a fibrant object. The property of two morphisms in $\mathbf{C}(X, Y)$ being homotopic defines an equivalence relation on $\mathbf{C}(X, Y)$, which is called **homotopy relation**.*

Corollary 2.3.9 ([Hov99, Cor. 1.2.7])**.** *Let $\mathbf{C}$ be a model category. The homotopy relation $\sim$ on morphisms of $\mathbf{C}_{\mathrm{cf}}$ is an equivalence relation compatible with compositions. Therefore, there exists the quotient category $\mathbf{C}_{\mathrm{cf}}/\sim$ (see Definition A.1.12).*

By construction, given a model category $\mathbf{C}$, the homotopy equivalences in $\mathbf{C}_{\mathrm{cf}}$ are raised to isomorphisms in the quotient category $\mathbf{C}_{\mathrm{cf}}/\sim$. One would like the weak equivalences to be raised to isomorphisms. This is resolved by the following proposition.

Proposition 2.3.10 ([Hov99, Prop. 1.2.8])**.** *Let $\mathbf{C}$ be a model category. A morphism in the model category $\mathbf{C}_{\mathrm{cf}}$ is a weak equivalence if and only if it is a homotopy equivalence.*

It follows from Corollary 2.3.9 and Proposition 2.3.10 that the weak equivalences in $\mathbf{C}_{\mathrm{cf}}$ are raised to isomorphisms in $\mathbf{C}_{\mathrm{cf}}/\sim$. In fact, one can show that $\mathbf{C}_{\mathrm{cf}}/\sim$ is a localization of $\mathbf{C}_{\mathrm{cf}}$ at $\mathcal{W}_{\mathbf{C}_{\mathrm{cf}}}$, which is the content of the following proposition.

Proposition 2.3.11 ([Hov99, Cor. 1.2.9])**.** *Let $\mathbf{C}$ be a model category. The category $\mathbf{C}_{\mathrm{cf}}/\sim$ is isomorphic to $\mathrm{Ho}(\mathbf{C}_{\mathrm{cf}})$.*

The fundamental theorem about model categories follows directly from Proposition 2.3.11 and Proposition 2.3.2.

Theorem 2.3.12 ([Hov99, Th. 1.2.10(i)])**.** *Let* $\mathbf{C}$ *be a model category. There exists an equivalence of categories between the associated homotopy category* $\mathrm{Ho}(\mathbf{C})$ *and the quotient category* $\mathbf{C}_{\mathrm{cf}}/\sim$*.*

Remark 2.3.13. An immediate consequence of Theorem 2.3.12 is that the homotopy category $\mathrm{Ho}(\mathbf{C})$ associated with a model category $\mathbf{C}$ is locally small. One can show this as follows.

Since, by definition, a model category is locally small, $\mathbf{C}_{\mathrm{cf}}$ must also be locally small, since it is a subcategory of $\mathbf{C}$. It is easy to see that $\mathbf{C}_{\mathrm{cf}}/\sim$ must also be locally small, as a quotient category of $\mathbf{C}_{\mathrm{cf}}$. Then, Theorem 2.3.12 implies that $\mathrm{Ho}(\mathbf{C})$ is equivalent to a locally small category. Since a category equivalence is witnessed by a fully faithful functor (see Proposition A.2.13) it follows that, given any two objects in $\mathrm{Ho}(\mathbf{C})$, the collection of morphisms between them is isomorphic to the collection of morphisms between their images in $\mathbf{C}_{\mathrm{cf}}$, which is a set. Therefore, $\mathrm{Ho}(\mathbf{C})$ is locally small. △

Corollary 2.3.14 ([Hov99, Th. 1.2.10(iv)])**.** *Let* $\mathbf{C}$ *be a model category. The collection of weak equivalences* $\mathcal{W}_{\mathbf{C}}$ *is saturated (see Definition 2.1.15).*

Corollary 2.3.15 ([Bal21, Rem. 2.2.11])**.** *Let* $\mathbf{C}$ *be a model category. Any morphism in* $\mathrm{Ho}(\mathbf{C})$ *can be represented by a string of morphisms (see Construction 2.1.3) of the form* $(w_1^{\mathrm{op}}, f, w_2^{\mathrm{op}})$*, for some morphisms* $w_1, w_2, f \in \mathrm{Mor}(\mathbf{C})$ *as in the diagram*

$$X \xleftarrow[\approx]{w_1} \mathcal{Q}(X) \xrightarrow{\quad f \quad} \mathcal{R}(Y) \xleftarrow[\approx]{w_2} Y\,, \tag{2.3.7}$$

with $w_1, w_2 \in \mathcal{W}_{\mathbf{C}}$*.*

Remark 2.3.16. Let $\mathbf{C}$ be a model category. Two objects $X, Y \in \mathrm{Ob}(\mathbf{C})$ are isomorphic in $\mathrm{Ho}(\mathbf{C})$ if and only if there exists a weak equivalence between the cofibrant replacement $\mathcal{Q}(X)$ of X and the fibrant replacement $\mathcal{R}(Y)$ of Y. This implies that X and Y are isomorphic in $\mathrm{Ho}(\mathbf{C})$ if and only if they are weakly equivalent in $\mathbf{C}$ (see Definition 2.1.5). This can be shown as follows.

Let $g\colon X \to Y$ be an isomorphism in $\mathrm{Ho}(\mathbf{C})$. By Corollary 2.3.15, the morphism g can be represented by a string of morphisms $(w_1^{\mathrm{op}}, f, w_2^{\mathrm{op}})$, as in Equation (2.3.7). It suffices to show that f is a weak equivalence in $\mathbf{C}$. Since g is an isomorphism, there exists a morphism $g^{-1}\colon Y \to X$ in $\mathrm{Ho}(\mathbf{C})$ such that $(w_1^{\mathrm{op}}, f, w_2^{\mathrm{op}}) \circ g^{-1} = \mathrm{id}_Y$ and $g^{-1} \circ (w_1^{\mathrm{op}}, f, w_2^{\mathrm{op}}) = \mathrm{id}_X$. It is easy to verify, using the relations in Construction 2.1.3, that the morphism $(w_2) \circ g^{-1} \circ (w_1)$ in $\mathrm{Ho}(\mathbf{C})$ is the inverse of (f) and consequently (f) is an isomorphism in $\mathrm{Ho}(\mathbf{C})$. We conclude from Corollary 2.3.14 that f is a weak equivalence in $\mathbf{C}$, which completes the proof. △

2.4 Quillen functors and equivalences

The fact that model categories are naturally equipped with both left and right deformations (see Remark 2.2.19) allows not only for the construction of derived functors but also derived adjunctions. That means that a suitable adjunction between model categories can be transferred to an adjunction between the associated homotopy categories. The notion of a Quillen adjunction, defined below, plays precisely this role.

Definition 2.4.1. Let $\mathbf{C}$ and $\mathbf{D}$ be two model categories.

(a) A functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ is called a **left Quillen functor** if it preserves cofibrations and acyclic cofibrations, and it has a right adjoint (see Definition A.5.1).

(b) A functor $\mathcal{U}\colon \mathbf{C} \to \mathbf{D}$ is called a **right Quillen functor** if it preserves fibrations and acyclic fibrations, and it has a left adjoint.

(c) A pair of functors $\mathcal{F}\colon \mathbf{C} \rightleftarrows \mathbf{D} :\!\mathcal{U}$ constitute a **Quillen adjunction** if $\mathcal{F} \dashv \mathcal{U}$ and $\mathcal{F}$ is a left Quillen functor.

Proposition 2.4.2 ([Hov99, Lem. 1.3.4])**.** *Let $\mathbf{C}$ and $\mathbf{D}$ be two model categories and let $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$ be an adjunction. The functor $\mathcal{F}$ is a left Quillen functor if and only if $\mathcal{U}$ is a right Quillen functor.*

Quillen functors give rise to adjunctions between the associated homotopy categories. This essentially follows from Ken Brown's Lemma, which we recall below.

Ken Brown's Lemma ([Hov99, Lem. 1.1.12])**.** *Let $\mathbf{C}$ and $\mathbf{D}$ be model categories and $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ a functor that sends acyclic cofibrations (resp. acyclic fibrations) between cofibrant (resp. fibrant) objects to weak equivalences. Then, the functor $\mathcal{F}$ sends any weak equivalence between cofibrant (resp. fibrant) objects to a weak equivalence.*

Remark 2.4.3 (Derived functors from Quillen functors)**.** Ken Brown's Lemma implies that, given model categories $\mathbf{C}$ and $\mathbf{D}$ and a Quillen adjunction $\mathcal{F}\colon \mathbf{C} \rightleftarrows \mathbf{D} :\!\mathcal{U}$, the left adjoint functor $\mathcal{F}$ preserves all the weak equivalences between objects in the full subcategory $\mathbf{C}_{\mathrm{c}}$ since, by definition, it preserves acyclic cofibrations between cofibrant objects. It follows that $\mathcal{F}$ gives rise to a left derived functor of $\mathrm{L}\,\mathcal{F}\colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ (see Definition 2.1.14) with respect to the left deformation defined by the cofibrant replacement functor of $\mathbf{C}$ (see Remark 2.2.19).

Similarly, the right adjoint functor $\mathcal{U}$ preserves weak equivalences between objects in the full subcategory $\mathbf{D}_{\mathrm{f}}$ and consequently it gives rise to a right derived functor $\mathrm{R}\mathcal{U}\colon \mathrm{Ho}(\mathbf{D}) \to \mathrm{Ho}(\mathbf{C})$ with respect to the right deformation defined by the fibrant replacement functor of $\mathbf{D}$. △

Proposition 2.4.4 ([Hov99, Lem. 1.3.10])**.** *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two model categories and let* $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$ *be a Quillen adjunction. Then, the pair of derived functors* $\mathrm{L}\mathcal{F}\colon \mathrm{Ho}(\mathbf{C}) \rightleftarrows \mathrm{Ho}(\mathbf{D}) : \mathrm{R}\mathcal{U}$ *induces a* ***derived adjunction*** $(\mathrm{L}\mathcal{F} \dashv \mathrm{R}\mathcal{U})\colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ *between the associated homotopy categories.*

The derived adjunction associated with a Quillen adjunction can be an adjoint equivalence (see Definition A.5.7) between the associated homotopy categories even if the Quillen adjunction is not an equivalence of categories. The concept of a Quillen equivalence, defined next, captures precisely when a Quillen adjunction induces a derived adjoint equivalence.

Definition 2.4.5. Let $\mathbf{C}$ and $\mathbf{D}$ be two model categories and let $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$ be a Quillen adjunction. This adjunction is called a **Quillen equivalence** if for any cofibrant object $X \in \mathrm{Ob}(\mathbf{C})$ and fibrant object $Y \in \mathrm{Ob}(\mathbf{D})$, a morphism $f\colon \mathcal{F}(X) \to Y$ is a weak equivalence in $\mathbf{D}$ if and only if its transpose $f^{\sharp}\colon X \to \mathcal{U}(Y)$ (see Definition A.5.1) is a weak equivalence in $\mathbf{C}$.

Proposition 2.4.6 ([Hov99, Prop. 1.3.13])**.** *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two model categories and let* $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$ *be a Quillen adjunction. Then,* $\mathcal{F} \dashv \mathcal{U}$ *is a Quillen equivalence if and only if* $(\mathrm{L}\mathcal{F} \dashv \mathrm{R}\mathcal{U})\colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ *is an adjoint equivalence.*

Remark 2.4.7. Note that, given two Quillen equivalences $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$ and $(\mathcal{F}' \dashv \mathcal{U}')\colon \mathbf{D} \to \mathbf{E}$, they can be composed to a Quillen equivalence $(\mathcal{F}' \circ \mathcal{F} \dashv \mathcal{U} \circ \mathcal{U}')\colon \mathbf{C} \to \mathbf{E}$.

However, if the Quillen equivalence between $\mathbf{D}$ and $\mathbf{E}$ was of the form $(\mathcal{U}' \dashv \mathcal{F}')\colon \mathbf{E} \to \mathbf{D}$ instead, then it is not possible, in general, to compose this Quillen equivalence with the Quillen equivalence $\mathcal{F} \dashv \mathcal{U}$ between $\mathbf{C}$ and $\mathbf{D}$ in order to obtain a Quillen equivalence between $\mathbf{C}$ and $\mathbf{E}$.

Therefore, the relation determined by the existence of a Quillen equivalence between two model categories is not a transitive relation. This is essentially a consequence of the fact that an adjunction between two categories is not symmetric.

Still, the Quillen equivalences $\mathcal{F} \dashv \mathcal{U}$ and $\mathcal{U}' \dashv \mathcal{F}'$ induce an equivalence of categories between the associated homotopy categories $\mathrm{Ho}(\mathbf{C})$ and $\mathrm{Ho}(\mathbf{E})$, since the property of categorical equivalence satisfies transitivity (see

Remark A.2.12). This implies that an equivalence between the associated homotopy categories of some model categories may be witnessed by a zig-zag of Quillen equivalences but not a Quillen equivalence. △

Remark 2.4.8. It is important to notice that the existence of a zig-zag of Quillen equivalences between two model categories is a strictly stronger condition than the existence of an equivalence of categories between the associated homotopy categories. That means that an equivalence between the associated homotopy categories of some model categories does not necessarily come from a zig-zag of Quillen equivalences between the model categories. For a couple of examples see [Pat13, Sec. 4]. △

2.5 Cofibrantly generated model categories

A special type of model categories is the cofibrantly generated model categories. In a cofibrantly generated model category all cofibrations (resp. acyclic cofibrations) can be generated by a small set of cofibrations (resp. acyclic cofibrations). This can be very helpful, since in this case it often suffices to verify a property on the generating small set of cofibrations (resp. acyclic cofibrations) in order to conclude for all the cofibrations (resp. acyclic cofibrations) of the model category.

Before we define cofibrantly generated model categories, we need to recall what it means for a collection of morphisms to permit the small object argument. In this section we follow the conventions in [Bal21] rather than [Hov99].

Definition 2.5.1. Let κ be a regular cardinal. An ordinal λ is **κ-filtered** if it is a limit ordinal and, given any subset $A \subseteq \lambda$ such that $|A| \leq \kappa$, it holds that $\sup A < \lambda$.

Definition 2.5.2. Let $\mathbf{C}$ be a cocomplete category and λ an ordinal. A **λ-sequence** in $\mathbf{C}$ is a colimit preserving functor $X\colon \alpha \to \mathbf{C}$, denoted by

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots \tag{2.5.1}$$

Definition 2.5.3. Let $\mathbf{C}$ be a cocomplete category, $\mathcal{I}$ a collection of morphisms in $\mathbf{C}$, $C \in \mathrm{Ob}(\mathbf{C})$ and κ a regular cardinal.

(a) We say that C is **κ-small relative to $\mathcal{I}$** if for any κ-filtered ordinal λ and any λ-sequence

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots , \tag{2.5.2}$$

such that each morphism $X_\beta \to X_{\beta+1}$ belongs to $\mathcal{I}$, the morphism

$$\coprod_{\beta\le\lambda} \mathbf{C}(C, X_\beta) \xrightarrow{\coprod_{\beta\le\lambda}(\iota_\beta)_*} \mathbf{C}(C, \coprod_{\beta\le\lambda} X_\beta) \tag{2.5.3}$$

is an isomorphism, where $\iota_\alpha\colon X_\alpha \to \coprod_{\beta\le\lambda} X_\beta$ is the canonical injection for each $\alpha \le \beta$.

(b) We say that C is **small relative to** $\mathcal{I}$ if it is μ-small relative to $\mathcal{I}$ for some regular cardinal μ.

(c) We say that C is **small** if it is small relative to $\mathrm{Mor}(\mathbf{C})$.

Definition 2.5.4. Let $\mathbf{C}$ be a cocomplete category, $\mathcal{I}$ a collection of morphisms in $\mathbf{C}$. We say that $\mathcal{I}$ **permits the small object argument** if given any morphism $f \in \mathcal{I}$, its source $\mathrm{s}(f) \in \mathrm{Ob}(\mathbf{C})$ is small relative to $\mathcal{I}$.

We need also to introduce what is a relative cell complex with respect to a collection of morphisms.

Definition 2.5.5. Let $\mathbf{C}$ be a category.

(a) A **pushout square** in $\mathbf{C}$ is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & X \end{array} \tag{2.5.4}$$

in $\mathbf{C}$ such that X is the pushout of the diagram $B \xleftarrow{f} A \xrightarrow{g} C$ in $\mathbf{C}$ (see Definition A.4.15(d)).

(b) Given a morphism $f\colon A \to B$ in $\mathbf{C}$, a **pushout of** f is a morphism $h \in \mathrm{Mor}(\mathbf{C})$ such that there exists a pushout square of the form

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ B & \longrightarrow & X \end{array} \tag{2.5.5}$$

in $\mathbf{C}$.

Definition 2.5.6. Let **C** be a cocomplete category and $\mathcal{I}$ a collection of morphisms in **C**. A **relative $\mathcal{I}$-cell complex** is a transfinite composition of pushouts of morphisms in $\mathcal{I}$.

Definition 2.5.7. Let **C** be a cocomplete category and $\mathcal{I}$ a collection of morphisms in **C**. We denote by:

(a) $\mathrm{cell}\,(\mathcal{I})$ the collection of relative $\mathcal{I}$-cell complexes.

(b) $\mathrm{cof}(\mathcal{I})$ the collection of retracts of morphisms in $\mathrm{cell}\,(\mathcal{I})$.

Definition 2.5.8. A model category **C** is called a **cofibrantly generated model category** if there exist sets of morphisms $\mathcal{I}, \mathcal{J} \subseteq \mathrm{Mor}(\mathbf{C})$ such that:

(i) $\mathrm{cof}(\mathcal{I})$ equals the collection of cofibrations of **C**,

(ii) $\mathrm{cof}(\mathcal{J})$ equals the collection of acyclic cofibrations of **C**,

(iii) $\mathrm{cof}(\mathcal{I})$ and $\mathrm{cof}(\mathcal{J})$ both permit the small object argument.

One of the advantages of working with a cofibrantly generated model category is that it is easier to verify that an adjunction is a Quillen adjunction, as indicated by the following proposition.

Proposition 2.5.9 ([Hov99, Lem. 2.1.20])**.** *Assume that:*

- **C** *is a cofibrantly generated model category with $\mathcal{I}$ a generating set of cofibrations and $\mathcal{J}$ a generating set of acyclic cofibrations.*
- **D** *is a model category.*
- *There is an adjunction $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$.*

Then, $\mathcal{F} \dashv \mathcal{U}$ is a Quillen adjunction if and only if:

(i) $\mathcal{F}(f) \in \mathrm{Cof}_{\mathbf{D}}$ *for any* $f \in \mathcal{I}$,

(ii) $\mathcal{F}(f) \in \mathcal{W}_{\mathbf{D}} \cap \mathrm{Cof}_{\mathbf{D}}$ *for any* $f \in \mathcal{J}$.

2.6 Symmetric monoidal model categories and M-model categories

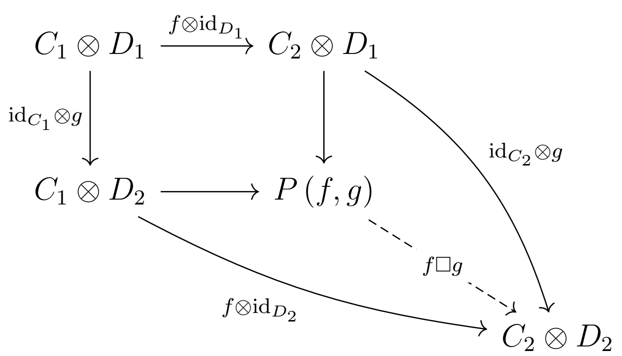
In this section we recall the concept of a symmetric monoidal model category, which is defined so that the associated homotopy category inherits the structure of a closed symmetric monoidal category.

First, we recall the definition of Quillen bifunctors, for which we need the concept of a pushout-product.

Definition 2.6.1. Assume that:

- $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ are categories/
- $\mathbf{E}$ has all pushouts (see Definition A.4.15(d)).
- $\otimes\colon \mathbf{C}\times\mathbf{D}\to\mathbf{E}$ is a functor.
- $f\colon C_1\to C_2$ in $\mathbf{C}$.
- $g\colon D_1\to D_2$ in $\mathbf{D}$.

Let $P(f,g)$ denote the pushout $(C_1\otimes D_2)\coprod_{C_1\otimes D_1}(C_2\otimes D_1)$ (see Notation A.4.16). We define the **pushout-product** of f and g with respect to $\otimes$ as the morphism $f\,\square\,g\colon P(f,g)\to C_2\otimes D_2$ in $\mathbf{E}$ determined by the universal property of the pushout and the commutativity of the diagram



(2.6.1)

in $\mathbf{E}$.

Definition 2.6.2. Let $\mathbf{C},\mathbf{D}$ and $\mathbf{E}$ be model categories.

(a) A **two-variable Quillen adjunction** is a two-variable adjunction $(\mathcal{F},\mathcal{G},\mathcal{H})\colon \mathbf{C}\times\mathbf{D}\to\mathbf{E}$ (see Definition A.5.11) such that, given cofibrations $f\in \mathrm{Cof}_{\mathbf{C}}$ and $g\in\mathrm{Cof}_{\mathbf{D}}$:

  (i) their pushout-product $f\,\square\,g$ with respect to the functor $\mathcal{F}$ is a cofibration in $\mathbf{E}$,

  (ii) if one of the cofibrations f and g is an acyclic cofibration, then their pushout-product $f\,\square\,g$ with respect to $\mathcal{F}$ is also an acyclic cofibration.

(b) A **Quillen bifunctor** is the left adjoint $\mathcal{F}$ of a two-variable Quillen adjunction $(\mathcal{F},\mathcal{G},\mathcal{H})\colon \mathbf{C}\times\mathbf{D}\to\mathbf{E}$.

It is easier to check that a functor $\mathcal{F}\colon \mathbf{C}\times\mathbf{D}\to\mathbf{E}$ is a Quillen bifunctor when $\mathbf{C}$ and $\mathbf{D}$ are both cofibrantly generated model categories, as indicated by the following proposition.

Proposition 2.6.3 ([Hov99, Cor. 4.2.5]). *Assume that:*

- $\mathbf{C}$ *is a cofibrantly generated model category, with* $\mathcal{I}_{\mathbf{C}}$ *a generating set of cofibrations and* $\mathcal{J}_{\mathbf{C}}$ *a generating set of acyclic cofibrations,*
- $\mathbf{D}$ *is a cofibrantly generated model category, with* $\mathcal{I}_{\mathbf{D}}$ *a generating set of cofibrations and* $\mathcal{J}_{\mathbf{D}}$ *a generating set of acyclic cofibrations,*
- $\mathbf{E}$ *is a model category,*
- $(\mathcal{F},\mathcal{G},\mathcal{H})\colon \mathbf{C}\times\mathbf{D}\to\mathbf{E}$ *a two-variable adjunction.*

The functor $\mathcal{F}$ *is a Quillen bifunctor if and only if:*

(i) $\mathcal{I}_{\mathbf{C}}\,\square\,\mathcal{I}_{\mathbf{D}}\subseteq \mathrm{Cof}_{\mathbf{E}}$ *and*

(ii) both $\mathcal{I}_{\mathbf{C}}\,\square\,\mathcal{J}_{\mathbf{D}}$ *and* $\mathcal{J}_{\mathbf{C}}\,\square\,\mathcal{I}_{\mathbf{D}}$ *consist of acyclic cofibrations.*

A two-variable Quillen adjunction induces a two-variable adjunction between the associated homotopy categories as described in the following proposition.

Proposition 2.6.4 ([Hov99, Prop. 4.3.1]). *Let* $\mathbf{C},\mathbf{D}$ *and* $\mathbf{E}$ *be model categories and* $(\mathcal{F},\mathcal{G},\mathcal{H})\colon \mathbf{C}\times\mathbf{D}\to\mathbf{E}$ *a two-variable Quillen adjunction. The induced derived functors give rise to a two-variable adjunction* $(\mathrm{L}\,\mathcal{F},\mathrm{R}\,\mathcal{G},\mathrm{R}\,\mathcal{H})\colon$ $\mathrm{Ho}(\mathbf{C})\times\mathrm{Ho}(\mathbf{D})\to\mathrm{Ho}(\mathbf{E})$ *between the associated homotopy categories.*

We now define a symmetric monoidal model category as follows.

Definition 2.6.5. A **symmetric monoidal model category** is a closed symmetric monoidal category $(\mathbf{M},\otimes,\mathbb{1})$ (see Definitions 1.1.4 and 1.1.9) equipped with a model structure such that the following compatibility conditions are satisfied:

(i) **Pushout-product axiom:** The tensor product $\otimes\colon \mathbf{M}\times\mathbf{M}\to\mathbf{M}$ is a Quillen bifunctor.

(ii) **Unit axiom:** For any cofibrant object $X\in\mathrm{Ob}(\mathbf{M})$, the morphism $\mathcal{Q}_{\mathbf{M}}(\mathbb{1})\otimes X\xrightarrow{q\otimes X}\mathbb{1}\otimes X$ is a weak equivalence in $\mathbf{M}$, where q denotes the weak equivalence of the cofibrant resolution $\emptyset\hookrightarrow\mathcal{Q}_{\mathbf{M}}(\mathbb{1})\overset{q}{\underset{\approx}{\twoheadrightarrow}}\mathbb{1}$ of $\mathbb{1}$, as in Remark 2.2.16.

Remark 2.6.6. Note that a symmetric monoidal model category is not merely a symmetric monoidal category but a *closed* symmetric monoidal category. In fact, the tensor product of a symmetric monoidal model category is part of the two-variable adjunction described in Remark 1.1.11. △

Remark 2.6.7. As noted in [Hov99] below Def. 4.2.6, when the monoidal unit is a cofibrant object, the unit axiom follows from the pushout-product. Therefore, whenever a closed symmetric monoidal category is equipped with a model structure such that the monoidal unit is cofibrant it suffices to check the pushout-product axiom in order to verify that it is a symmetric monoidal model category. △

As anticipated, the homotopy category associated with a symmetric monoidal model category inherits the structure of a closed symmetric monoidal category. This result is a direct corollary of Proposition 2.6.4.

Proposition 2.6.8 ([Hov99, Th. 4.3.2])**.** *Let* $(\mathbf{M}, \otimes, \mathbb{1})$ *be a symmetric monoidal model category. Then,* $(\mathrm{Ho}(\mathbf{M}), \otimes^{\mathrm{L}}, \mathcal{R}_{\mathbf{M}}(\mathbb{1}))$ *is a closed symmetric monoidal category, where* $\otimes^{\mathrm{L}}$ *is the derived functor of the tensor product and* $\mathcal{R}_{\mathbf{M}}(\mathbb{1})$ *the fibrant replacement of the tensor unit.*

2.7 M-model categories

Given a monoidal category $\mathbf{M}$, there exists the notion of an $\mathbf{M}$-module category, whose definition we recall below. When $\mathbf{M}$ is a symmetric monoidal model category, one can define $\mathbf{M}$-model categories such that their associated homotopy categories are $\mathbf{M}$-module categories.

We first recall the definition of a right $\mathbf{M}$-module category.

Definition 2.7.1. Let $(\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category as in Definition 1.1.1. A **right M-module category** is a quadruplet $(\mathbf{D}, \otimes_{\mathbf{D}}, \phi, \psi)$, consisting of the following data:

(1) A category $\mathbf{D}$.

(2) A functor $\otimes_{\mathbf{D}} \colon \mathbf{D} \times \mathbf{M} \to \mathbf{D}$, called the **M-action** on $\mathbf{D}$.

(3) A natural isomorphism $\phi \colon (- \otimes_{\mathbf{D}} -) \otimes_{\mathbf{D}} (-) \xrightarrow{\cong} (-) \otimes_{\mathbf{D}} (- \otimes -)$.

(4) A natural isomorphism $\psi \colon (-) \otimes_{\mathbf{D}} \mathbb{1} \xrightarrow{\cong} \mathrm{Id}_{\mathbf{D}}$.

These data are subject to the following conditions:

(i) For any $D \in \mathbf{D}$ and $X, Y, Z \in \mathbf{M}$, the diagram

$$
\begin{array}{ccc}
 & ((D \otimes_{\mathbf{D}} X) \otimes_{\mathbf{D}} Y) \otimes_{\mathbf{D}} Z & \\
\swarrow^{\phi_{D,X,Y} \otimes_{\mathbf{D}} Z} & & \searrow^{\phi_{D \otimes_{\mathbf{D}} X, Y, Z}} \\
(D \otimes_{\mathbf{D}} (X \otimes Y)) \otimes_{\mathbf{D}} Z & & (D \otimes_{\mathbf{D}} X) \otimes_{\mathbf{D}} (Y \otimes Z) \\
\downarrow{\scriptstyle \phi_{D, X \otimes Y, Z}} & & \downarrow{\scriptstyle \phi_{D, X, Y \otimes Z}} \\
D \otimes_{\mathbf{D}} ((X \otimes Y) \otimes Z) & \xrightarrow[D \otimes_{\mathbf{D}} \alpha_{X,Y,Z}]{} & D \otimes_{\mathbf{D}} (X \otimes (Y \otimes Z))
\end{array}
\tag{2.7.1}
$$

in $\mathbf{D}$ commutes.

(ii) For any $D \in \mathbf{D}$ and $X \in \mathbf{M}$, the diagrams

$$
\begin{array}{ccc}
(D \otimes_{\mathbf{D}} \mathbb{1}) \otimes_{\mathbf{D}} X & \xrightarrow{\phi_{D,\mathbb{1},X}} & D \otimes_{\mathbf{D}} (\mathbb{1} \otimes X) \\
 & \searrow_{\psi_D \otimes_{\mathbf{D}} X} & \downarrow{\scriptstyle D \otimes_{\mathbf{D}} \lambda} \\
 & & D \otimes_{\mathbf{D}} X
\end{array}
\tag{2.7.2a}
$$

$$
\begin{array}{ccc}
(D \otimes_{\mathbf{D}} X) \otimes_{\mathbf{D}} \mathbb{1} & \xrightarrow{\phi_{D,X,\mathbb{1}}} & D \otimes_{\mathbf{D}} (X \otimes \mathbb{1}) \\
 & \searrow_{\psi_{D \otimes_{\mathbf{D}} X}} & \downarrow{\scriptstyle D \otimes_{\mathbf{D}} \rho} \\
 & & D \otimes_{\mathbf{D}} X
\end{array}
\tag{2.7.2b}
$$

in $\mathbf{D}$ commute.

Notation 2.7.2. Given a monoidal category $\mathbf{M}$ we will denote a right $\mathbf{M}$-module category $(\mathbf{D}, \otimes_{\mathbf{D}}, \phi, \psi)$ simply by $\mathbf{D}$ omitting the rest of the data when they are not explicitly used.

Example 2.7.3. Let $(\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category, and let $A \in \mathbf{Mon}(\mathbf{M})$ be a monoid in $\mathbf{M}$ (see Definition 1.2.1). The category ${}_A\mathbf{Mod}$ of left A-modules (see Definition 1.3.4) is an $\mathbf{M}$-module category with the $\mathbf{M}$-action given by the $\mathbf{M}$-tensoring $\otimes_{{}_A\mathbf{Mod}} \colon {}_A\mathbf{Mod} \times \mathbf{M} \to {}_A\mathbf{Mod}$ from Definition 1.6.1.

Specifically, let:

(1) $\phi\colon (- \otimes_{{}_A\mathbf{Mod}} -) \otimes_{{}_A\mathbf{Mod}} (-) \xrightarrow{\cong} (-) \otimes_{{}_A\mathbf{Mod}} (- \otimes -)$ be the natural isomorphism with components $\phi_{L,X,Y} \coloneqq \alpha_{L,X,Y}$ for each $L \in {}_A\mathbf{Mod}$ and $X, Y \in \mathbf{M}$.

(2) $\psi\colon (-) \otimes_{\mathbf{D}} \mathbb{1} \xrightarrow{\cong} \mathrm{Id}_{\mathbf{M}}$ be the natural isomorphism with components $\psi_L \coloneqq \rho_L$ for each $L \in {}_A\mathbf{Mod}$.

It is easy to verify that $({}_A\mathbf{Mod}, \otimes_{{}_A\mathbf{Mod}}, \phi, \psi)$ is an $\mathbf{M}$-module category using the pentagon and unit axioms of the monoidal category $\mathbf{M}$ (see Definition 1.1.1). $\triangledown$

Definition 2.7.4. Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a symmetric monoidal model category and let $(\mathbf{D}, \otimes_{\mathbf{D}}, \phi_{\mathbf{D}}, \psi_{\mathbf{D}})$ and $(\mathbf{E}, \otimes_{\mathbf{E}}, \phi_{\mathbf{E}}, \psi_{\mathbf{E}})$ be two right $\mathbf{M}$-module categories. A functor $\mathcal{F}\colon \mathbf{D} \to \mathbf{E}$ is called an **M-module functor** if there exists a natural isomorphism $\mu\colon \mathcal{F}(-) \otimes_{\mathbf{E}} (-) \xrightarrow{\cong} \mathcal{F}(- \otimes_{\mathbf{D}} -)$ such that:

(i) For any $D \in \mathbf{D}$ and $X, Y \in \mathbf{M}$, the diagram

$$\begin{array}{ccc} & (\mathcal{F}(D) \otimes_{\mathbf{E}} X) \otimes_{\mathbf{E}} Y & \\ \swarrow^{\mu_{D,X} \otimes_{\mathbf{E}} Y} & & \searrow^{\phi_{\mathbf{E}\,\mathcal{F}(D),X,Y}} \\ \mathcal{F}(D \otimes_{\mathbf{D}} X) \otimes_{\mathbf{E}} Y & & \mathcal{F}(D) \otimes_{\mathbf{E}} (X \otimes Y) \\ \downarrow{\scriptstyle \mu_{D \otimes_{\mathbf{D}} X, Y}} & & \downarrow{\scriptstyle \mu_{D, X \otimes Y}} \\ \mathcal{F}\left((D \otimes_{\mathbf{D}} X) \otimes_{\mathbf{D}} Y\right) & \xrightarrow{\phi_{\mathbf{D}\,D,X,Y}} & \mathcal{F}\left(D \otimes_{\mathbf{D}} (X \otimes Y)\right) \end{array} \tag{2.7.3}$$

commutes.

(ii) For any $D \in \mathbf{D}$, the diagram

$$\begin{array}{ccc} \mathcal{F}(D) \otimes_{\mathbf{E}} \mathbb{1} & \xrightarrow{\mu_{D,\mathbb{1}}} & \mathcal{F}\left(D \otimes_{\mathbf{D}} \mathbb{1}\right) \\ & \searrow_{\psi_{\mathbf{E}\,\mathcal{F}(D)}} & \downarrow{\scriptstyle \mathcal{F}(\psi_{\mathbf{D}\,D})} \\ & & \mathcal{F}(D) \end{array} \tag{2.7.4}$$

commutes.

Now, we can define $\mathbf{M}$-model categories as follows.

Definition 2.7.5. Let $(\mathbf{M}, \otimes, \mathbb{1})$ be a symmetric monoidal model category. An **M-model category** is a right $\mathbf{M}$-module category $(\mathbf{D}, \otimes_{\mathbf{D}}, \phi, \psi)$ equipped with a model structure such that $\mathbf{D}$ is a model category and the following conditions are satisfied:

(i) The $\mathbf{M}$-action $\otimes_{\mathbf{D}} \colon \mathbf{D} \times \mathbf{M} \to \mathbf{D}$ is a Quillen bifunctor.

(ii) For any cofibrant object $X \in \mathbf{D}$, the morphism $X \otimes_{\mathbf{D}} \mathcal{Q}_{\mathbf{M}}(\mathbb{1}) \xrightarrow{X \otimes_{\mathbf{D}} q} X \otimes_{\mathbf{D}} \mathbb{1}$ is a weak equivalence in $\mathbf{D}$, where q denotes the weak equivalence of the cofibrant resolution $\emptyset \hookrightarrow \mathcal{Q}_{\mathbf{M}}(\mathbb{1}) \overset{q}{\underset{\approx}{\twoheadrightarrow}} \mathbb{1}$ of $\mathbb{1}$, as in Remark 2.2.16.

Remark 2.7.6. Similarly to the unit axiom in the definition of symmetric monoidal model categories, the second condition in the above definition follows from the first when the monoidal unit $\mathbb{1}$ is cofibrant (see Remark 2.6.7). △

Definition 2.7.7. Let $\mathbf{M}$ be a symmetric monoidal model category and let $\mathbf{D}$ and $\mathbf{E}$ be two $\mathbf{M}$-model categories. A functor $\mathcal{F} \colon \mathbf{D} \to \mathbf{E}$ is an **M-Quillen** functor if it is both a Quillen functor (see Definition 2.4.1) and an $\mathbf{M}$-module functor.

Proposition 2.7.8 ([Hov99, Th. 4.3.4(1)])**.** *Let* $\mathbf{M}$ *be a symmetric monoidal model category. For any right* $\mathbf{M}$*-model category* $\mathbf{D}$*, the associated homotopy category* $\mathrm{Ho}(\mathbf{D})$ *inherits the structure of a* $\mathrm{Ho}(\mathbf{M})$*-module category from* $\mathbf{D}$*.*

2.8 Model structure for modules over monoids

Let $\mathbf{M}$ be a monoidal category with a collection of weak equivalences (see Definition 2.1.4). Since the category of monoids $\mathbf{Mon}(\mathbf{M})$ (see Definition 1.2.3) as well as the categories ${}_A\mathbf{Mod}$ and $\mathbf{Mod}_A$ of left and right modules over some monoid $A \in \mathbf{Mon}(\mathbf{M})$ (see Definition 1.3.4 and Remark 1.3.2) are subcategories of $\mathbf{M}$, Remark 2.1.6 implies that they inherit collections of weak equivalences from $\mathbf{M}$ as follows.

Definition 2.8.1. Let $(\mathbf{M}, \mathcal{W})$ be a monoidal category with weak equivalences, and let $A \in \mathbf{Mon}(M)$. We define a morphism in $\mathbf{Mon}(\mathbf{M})$ (resp. ${}_A\mathbf{Mod}$ or $\mathbf{Mod}_A$) to be a **weak equivalence** if the underlying morphism in $\mathbf{M}$ is a weak equivalence.

The question that we address in this section is the following. If the monoidal category with weak equivalences $(\mathbf{M}, \mathcal{W})$ is equipped with a compatible model structure (see Definition 2.2.5), does the category ${}_A\mathbf{Mod}$ inherit a model structure compatible with the weak equivalences from Definition 2.8.1?

We present some results from [SS00] about transferring the model structure from a symmetric monoidal model category $\mathbf{M}$ to the the category of left modules ${}_A\mathbf{Mod}$ over some monoid $A \in \mathbf{Mon}(\mathbf{M})$.

An important condition for the application of these transfer theorems is that the monoidal model category $\mathbf{M}$ satisfies the monoid axiom, which we recall below.

Definition 2.8.2 (Monoid Axiom)**.** Let $\mathbf{M}$ be a symmetric monoidal model category. We say that $\mathbf{M}$ satisfies the **monoid axiom** if, for any $X \in \mathbf{M}$, we have that

$$\operatorname{cell}\left((\mathcal{W}_{\mathbf{M}} \cap \operatorname{Cof}_{\mathbf{M}}) \otimes \operatorname{id}_X\right) \subseteq \mathcal{W}_{\mathbf{M}}\,, \tag{2.8.1}$$

where cell denotes the collection of relative cell complexes (see Definition 2.5.7).

For the remainder of this section we assume that the category $\mathbf{M}$ has the following properties.

Setup 2.8.3. We assume that $(\mathbf{M}, \otimes, \mathbb{1})$ is a symmetric monoidal model category (see Definition 2.6.5) such that:

(i) $\mathbf{M}$ is cofibrantly generated by the set of cofibrations $\mathcal{I}$ and the set of acyclic cofibrations $\mathcal{J}$ (see Definition 2.5.8),

(ii) $\mathbf{M}$ satisfies the monoid axiom (see Definition 2.8.2),

(iii) X is a small object for any $X \in \operatorname{Ob}(\mathbf{M})$ (see Definition 2.5.3(c)).

Moreover, A will stand for a monoid in $\mathbf{M}$.

We define collections of weak equivalences and fibrations in the category of modules over A as follows.

Definition 2.8.4. A morphism in ${}_A\mathbf{Mod}$ is a **weak equivalence** (resp. **fibration**) if its underlying morphism in $\mathbf{M}$ is a weak equivalence (resp. fibration).

Proposition 2.8.5 ([SS00, Th. 4.1])**.** *The weak equivalences and the fibrations from Definition 2.8.4 determine the structure of a cofibrantly generated model category on the category* ${}_A\mathbf{Mod}$. *Specifically,* ${}_A\mathbf{Mod}$ *is cofibrantly generated by the set of cofibrations* $\operatorname{Fr}_A(\mathcal{I})$ *and the set of acyclic cofibrations* $\operatorname{Fr}_A(\mathcal{J})$*, where* Fr_A *stands for the free A-module functor (see Definition 1.4.1).*

Moreover, the model structure from Proposition 2.8.5 on ${}_A\mathbf{Mod}$ is compatible with the $\mathbf{M}$-tensoring, powering and enriched hom on ${}_A\mathbf{Mod}$ (see Section 1.6), as indicated by the following proposition.

Proposition 2.8.6. *Assume that the monoidal unit in* $\mathbb{1}$ *in* $\mathbf{M}$ *is cofibrant. Then, the category* ${}_A\mathbf{Mod}$ *equipped with the model structure from Proposition 2.8.5 and the* $\mathbf{M}$*-tensoring (see Definition 1.6.1) is an* $\mathbf{M}$*-model category.*

Proof. As we saw in Example 2.7.3, the category ${}_A\mathbf{Mod}$ is an $\mathbf{M}$-module category. Moreover, the monoidal unit $\mathbb{1}$ is, by assumption, cofibrant. Hence, on account of Remark 2.7.6, we only need to show that the $\mathbf{M}$-tensoring is a Quillen bifunctor (see Definition 2.6.2).

Given that ${}_A\mathbf{Mod}$ is a cofibrantly generated model category (see Proposition 2.8.5), it suffices, by Proposition 2.6.3, to check that the pushout-product $f \,\square\, g$ (see Definition 2.6.1) is a cofibration (resp. acyclic cofibration) in ${}_A\mathbf{Mod}$ whenever $f \in \mathrm{Fr}_A(\mathcal{I})$ and $g \in \mathcal{I}$ (resp. $f \in \mathrm{Fr}_A(\mathcal{J})$ and $g \in \mathcal{I}$, or $f \in \mathrm{Fr}_A(\mathcal{I})$ and $g \in \mathcal{J}$).

Since $f \in \mathrm{Fr}_A(\mathcal{I})$ ($f \in \mathrm{Fr}_A(\mathcal{J})$), it is of the form $f = \mathrm{id}_A \otimes h$, with $h \in \mathcal{I}$ (resp. $h \in \mathcal{J}$). Moreover, there is an isomorphism $P(f, g) \cong A \otimes P(h, g)$ in ${}_A\mathbf{Mod}$, since the colimits in ${}_A\mathbf{Mod}$ can be computed in $\mathbf{M}$ (see Proposition 1.4.4) and the tensor product in $\mathbf{M}$ preserves colimits (see Remark 1.1.5). This implies that $f \,\square\, g$ can be computed from $A \otimes (h \,\square\, g) = \mathrm{Fr}_A(h \,\square\, g)$, where $h \,\square\, g$ denotes the pushout-product of the morphisms h and g with respect to the tensor product in $\mathbf{M}$.

Since $\mathbf{M}$ is per hypothesis a closed symmetric monoidal model category, $h \,\square\, g$ is a cofibration (resp. acyclic cofibration) in $\mathbf{M}$. Furthermore, the free A-module functor $\mathrm{Fr}_A \colon \mathbf{M} \to {}_A\mathbf{Mod}$ is a left Quillen functor, since it is left adjoint to the forgetful functor $\underline{(-)} \colon {}_A\mathbf{Mod} \to \mathbf{M}$ (see Proposition 1.4.2), which preserves fibrations and weak equivalences by the definition of the model structure on ${}_A\mathbf{Mod}$. It follows that $f \,\square\, g$ is a cofibration (resp. acyclic cofibration) in ${}_A\mathbf{Mod}$, as desired. □

Next, we look at the change of monoid adjunction induced by a morphism $f \colon A \to B$ in $\mathbf{Mon}(\mathbf{M})$ (see Proposition 1.5.9). In fact, it is easy to see that the restriction along f functor $\mathrm{Res}_f \colon {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$ (see Definition 1.5.2) preserves the weak equivalences and fibrations with respect to the model structures on ${}_A\mathbf{Mod}$ and ${}_B\mathbf{Mod}$ described in Proposition 2.8.5. This means that the change-of-monoid adjunction is a Quillen adjunction.

Proposition 2.8.7. *Let* $f \colon A \to B$ *be a morphism in* $\mathbf{Mon}(\mathbf{M})$*. The induced change-of-monoid adjunction* $(\mathrm{Ext}_f \dashv \mathrm{Res}_f) \colon {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}$ *is a Quillen adjunction with respect to the model structures on* ${}_A\mathbf{Mod}$ *and* ${}_B\mathbf{Mod}$ *described in Proposition 2.8.5.*

Moreover, the change-of-monoid adjunction is a Quillen equivalence (see Definition 2.4.5) under the conditions specified in the next proposition.

Proposition 2.8.8. *[SS00, Th. 4.3] Assume that:*

- $\mathbf{M}$ *as in Setup 2.8.3.*
- $f\colon A \to B$ *is a morphism in* $\mathbf{Mon}(\mathbf{M})$.
- *The categories* ${}_A\mathbf{Mod}$ *and* ${}_B\mathbf{Mod}$ *are equipped with the model structure described in Proposition 2.8.5.*
- *The category* $\mathbf{Mod}_A$ *is equipped with the collection of weak equivalences inherited from* $\mathbf{M}$ *as in Definition 2.8.1.*
- *For each cofibrant left A-module* $L \in {}_A\mathbf{Mod}$, *the functor* $(-) \otimes_A L\colon \mathbf{Mod}_A \to \mathbf{M}$ *sends weak equivalences in* $\mathbf{Mod}_A$ *to weak equivalences in* $\mathbf{M}$. *Here, the symbol* $\otimes_A$ *stands for the relative tensor product over A from Definition 1.3.6.*

Then, if the underlying morphism $\underline{f} \in \mathbf{M}$ *is a weak equivalence in* $\mathbf{M}$, *the induced change-of-monoid adjunction* $\mathrm{Ext}_f \dashv \mathrm{Res}_f$ *is a Quillen equivalence.*

2.9 The projective model structure on the category of cochain complexes

In the context of homotopy algebraic quantum field theory, the monoidal category of interest is that of cochain complexes over some field $\mathbb{K}$ of characteristic zero. In this section, we recall the monoidal structure and the projective model structure on this category.

Definition 2.9.1. Given a field $\mathbb{K}$ of characteristic zero, we have the following definitions.

(a) A **cochain complex** V of $\mathbb{K}$-vector spaces consists of the following data.

  (1) A $\mathbb{K}$-vector space V^n for each $n \in \mathbb{Z}$.

  (2) A **differential** d_V defined as a collection of $\mathbb{K}$-linear maps $\{\mathrm{d}_V^n\colon V^n \to V^{n+1}\}_{n\in\mathbb{Z}}$, such that $\mathrm{d}_V^{n+1}\,\mathrm{d}_V^n = 0$ for any $n \in \mathbb{Z}$.

(b) Given a cochain complex $V \in \mathbf{CoCh}_{\mathbb{K}}$ and a vector $v \in V^n$ for some $n \in \mathbb{Z}$, we define the **degree** of v, denoted by $|v|$, to be the number n.

(c) Let V_1 and V_2 be two cochain complexes of $\mathbb{K}$-vector spaces. A **cochain map** $f\colon V_1 \to V_2$ in $\mathbf{CoCh}_{\mathbb{C}}$ is defined as a collection of linear maps $\{f^n\colon V_1^n \to V_2^n\}_{n\in\mathbb{Z}}$, such that $f^{n+1}\,\mathrm{d}_{V_1}^n = \mathrm{d}_{V_2}^n\, f^n$ for any $n \in \mathbb{Z}$.

(d) We define the **category of cochain complexes** of $\mathbb{K}$-vector spaces, denoted by $\mathbf{CoCh}_{\mathbb{K}}$, as the category with objects the cochain complexes of $\mathbb{K}$-vector spaces and morphisms the cochain maps.

Definition 2.9.2. Given $V \in \mathbf{CoCh}_{\mathbb{K}}$ and $k \in \mathbb{Z}$, the ***k*-shifted cochain complex** of V, denoted by $V[k] \in \mathbf{CoCh}_{\mathbb{K}}$, is defined as the cochain complex consisting of the following data.

(1) The vector space $V[k]^n := V^{n+k}$, for each $n \in \mathbb{Z}$.

(2) The differential $\{\mathrm{d}^n_{V[k]} := (-1)^k \, \mathrm{d}^{n+k}_V\}_{n\in\mathbb{Z}}$.

Next, we recall the projective model structure on the category of cochain complexes.

Definition 2.9.3. The **projective model structure** on $\mathbf{CoCh}_{\mathbb{K}}$ is defined as follows. We define:

(a) the **weak equivalences in** $\mathbf{CoCh}_{\mathbb{K}}$ as the *quasi-isomorphisms*, i.e. the cochain maps inducing isomorphisms in cohomology,

(b) the **fibrations in** $\mathbf{CoCh}_{\mathbb{K}}$ as the degree-wise surjective cochain maps,

(c) the **cofibrations in** $\mathbf{CoCh}_{\mathbb{K}}$ as the degree-wise injective cochain maps.

In the next proposition we collect some results from [Hov99, Sec. 2.3].

Proposition 2.9.4. *The category* $\mathbf{CoCh}_{\mathbb{K}}$ *of cochain complexes of* $\mathbb{K}$*-vector spaces equipped with the projective model structure is a cofibrantly generated model category where all objects are small and bifibrant.*

Monoidal structure on $\mathbf{CoCh}_{\mathbb{K}}$ The category of cochain complexes $\mathbf{CoCh}_{\mathbb{K}}$ can also be equipped with the structure of a closed symmetric monoidal category as follows.

The tensor product $V \otimes W \in \mathbf{CoCh}_{\mathbb{K}}$ of the cochain complexes $V, W \in \mathbf{CoCh}_{\mathbb{K}}$ is defined as the cochain complex consisting of the following data.

(1) For each $n \in \mathbb{Z}$, the vector space

$$(V \otimes W)^n := \coprod_{k\in\mathbb{Z}} \left(V^k \otimes W^{n-k}\right), \tag{2.9.1}$$

where the symbol $\otimes$ on the right hand side stands for the tensor product of vector spaces over $\mathbb{K}$.

(2) The differential given by the graded Leibniz rule $\mathrm{d}_\otimes := \mathrm{d}_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \mathrm{d}_W$.

The monoidal unit, which, abusing notation, we denote also by $\mathbb{K} \in \mathbf{CoCh}_\mathbb{K}$, is defined as the cochain complex which in degree 0 equals the field $\mathbb{K}$ and vanishes in any other degree.

The symmetric braiding $\gamma\colon (-) \otimes (-) \to \mathrm{Rv}_{\mathbf{CoCh}_\mathbb{K}}$ (see Definition 1.1.8) is defined as the natural transformation with the following components. Given $V, W \in \mathbf{CoCh}_\mathbb{K}$, the component $\gamma_{V,W}\colon V \otimes W \to W \otimes V$ is the morphism defined by the Koszul sign rule $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ for any $v \in V$ and $w \in W$.

Finally, the internal hom $[V, W] \in \mathbf{CoCh}_\mathbb{K}$ is defined degree-wise for all $n \in \mathbb{Z}$ by

$$[V, W]^n := \prod_{k \in \mathbb{Z}} \mathrm{Lin}_\mathbb{K}(V^k, W^{n+k}), \tag{2.9.2}$$

where $\mathrm{Lin}_\mathbb{K}(-,-)$ denotes the vector space of linear maps, with differential $\partial := [\mathrm{d}, -]$ given by the graded commutator with the original differentials.

Proposition 2.9.5 ([Hov99, Prop. 4.2.13])**.** *The category $\mathbf{CoCh}_\mathbb{K}$ equipped with the projective model structure is a symmetric monoidal model category.*

The next proposition follows from [SS00, Rem. 3.4] and the fact that every object in $\mathbf{CoCh}_\mathbb{K}$ is cofibrant (see Proposition 2.9.4).

Proposition 2.9.6. *The symmetric monoidal model category $\mathbf{CoCh}_\mathbb{K}$ satisfies the monoid axiom (see Definition 2.8.2).*

Finally, in view of the next proposition, the category $\mathbf{CoCh}_\mathbb{K}$ satisfies the assumptions of Proposition 2.8.8. The content of this proposition is stated in [SS00] in the paragraph above Th. 4.3.

Proposition 2.9.7. *For each monoid $A \in \mathbf{Mon}(\mathbf{CoCh}_\mathbb{K})$ and each cofibrant left A-module $L \in {}_A\mathbf{Mod}$ with respect to the model structure on ${}_A\mathbf{Mod}$ inherited from $\mathbf{CoCh}_\mathbb{K}$ (see Proposition 2.8.5), the relative tensor product $(-) \otimes_A L\colon \mathbf{Mod}_A \to \mathbf{CoCh}_\mathbb{K}$ (see Definition 1.3.6) sends weak equivalences in $\mathbf{Mod}_A$ (see Definition 2.8.1) to weak equivalences in $\mathbf{CoCh}_\mathbb{K}$.*

Part II

Representations in Homotopy Algebraic Quantum Field Theory

Chapter 3

Homotopy Theory of Net Representations

In this chapter we develop the homotopy theory of net representations following our published work [AB23].

In Section 3.1, we present a generalisation of the concept of nets of algebras as monoids in the functor category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ for a monoidal category $\mathbf{M}$ and a small category $\mathbf{C}$, and we demonstrate how this reproduces the usual definition.

In Section 3.2, net representations are defined as modules over a net of algebras and we demonstrate how to recover the usual concepts of net representations. Then, we proceed, in Section 3.3, to describe the construction of constant net representations as part of an adjunction.

In Section 3.4, we describe how a morphism of nets of algebras gives rise to a change-of-net adjunction between the respective categories of net representations, and in Section 3.5 we present in detail the $\mathbf{M}$-tensoring, $\mathbf{M}$-powering and $\mathbf{M}$-enrhiched hom over a category of nets representations. These functors will be useful in Section 3.6.

In Section 3.6, we assume that the category $\mathbf{M}$ is equipped with a cofibrantly generated model structure and we explore how to equip the categories of net representations with suitable model structures. We achieve this, by transferring the model structure of $\mathbf{M}$ along a suitable adjunction, and we conclude proving that, with respect to these model structures, the change-of-net adjunction along a weak equivalence of nets is a Quillen equivalence.

Then, in Section 3.7, we use some results from the theory of model categories to study the homotopy category associated with the model category of net representations described in Section 3.6.

Finally, in Section 3.8, we recall the definition of homotopical algebraic quantum field theories (homotopical AQFTs) and we define their representations. Then, we show that physically equivalent homotopical AQFTs do not correspond to equivalent categories of representations in the usual sense of categorical equivalence, by constructing a counter-example. We proceed to resolve this issue by replacing the notion of category equivalence with that of Quillen equivalence with respect to the model structures defined in Section 3.6.

3.1 Nets of algebras

Throughout this chapter, $\mathbf{C}$ will be a small category and $\mathbf{M} = (\mathbf{M}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ will be a complete and cocomplete, closed symmetric monoidal category with internal hom [-, -] (see Definition 1.1.4).

In order to define $\mathbf{M}$-valued nets of algebras over $\mathbf{C}$ and representations of nets of algebras we will use the fact that the functor category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ is also a complete and cocomplete, closed symmetric monoidal category. Indeed, the category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ inherits the structure of a closed symmetric monoidal category from $\mathbf{M}$ as follows.

The tensor product in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$, which, abusing notation, we also denote by $\otimes\colon \mathbf{Fun}(\mathbf{C}, \mathbf{M}) \times \mathbf{Fun}(\mathbf{C}, \mathbf{M}) \to \mathbf{Fun}(\mathbf{C}, \mathbf{M})$, is defined component-wise. Specifically, given a pair of functors $\mathfrak{A}, \mathfrak{B} \in \mathbf{Fun}(\mathbf{C}, \mathbf{M})$ we define $\mathfrak{A} \otimes \mathfrak{B} \in \mathbf{Fun}(\mathbf{C}, \mathbf{M})$ to be the functor that assigns to each object $c \in \mathbf{C}$, the object $(\mathfrak{A} \otimes \mathfrak{B})(c) \coloneqq \mathfrak{A}_c \otimes \mathfrak{B}_c \in \mathbf{M}$ and to each morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the morphism $(\mathfrak{A} \otimes \mathfrak{B})(\gamma) \coloneqq \mathfrak{A}(\gamma) \otimes \mathfrak{B}(\gamma)$ in $\mathbf{M}$. Given morphisms $\Phi_1\colon \mathfrak{A}_1 \to \mathfrak{B}_1$ and $\Phi_2\colon \mathfrak{A}_2 \to \mathfrak{B}_2$ in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$, we define $\Phi_1 \otimes \Phi_2\colon \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{B}_1 \otimes \mathfrak{B}_2$ to be the natural transformation with components $(\Phi_1 \otimes \Phi_2)(c) \coloneqq \Phi_1(c) \otimes \Phi_2(c)$ in $\mathbf{M}$.

The unit of the category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$, denoted by $\mathbb{1} \in \mathbf{Fun}(\mathbf{C}, \mathbf{M})$, is defined as the constant functor over the unit object $\mathbb{1} \in \mathbf{M}$ (see Example A.2.4). Notice that here we also abused notation denoting by $\mathbb{1}$ both the object in $\mathbf{M}$ and the constant functor over it.

The internal hom in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$, which, abusing notation again, we also denote by $[-,-]\colon \mathbf{Fun}(\mathbf{C}, \mathbf{M}) \times \mathbf{Fun}(\mathbf{C}, \mathbf{M}) \to \mathbf{Fun}(\mathbf{C}, \mathbf{M})$, is defined also component-wise. Specifically, given a pair of functors $\mathfrak{A}, \mathfrak{B} \in \mathbf{Fun}(\mathbf{C}, \mathbf{M})$ we define $[\mathfrak{A}, \mathfrak{B}] \in \mathbf{Fun}(\mathbf{C}, \mathbf{M})$ to be the functor that assigns to each object $c \in \mathbf{C}$, the object $[\mathfrak{A} \otimes \mathfrak{B}](c) \coloneqq [\mathfrak{A}_c, \mathfrak{B}_c] \in \mathbf{M}$ and to each morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the morphism $[\mathfrak{A}, \mathfrak{B}](\gamma) \coloneqq [\mathfrak{A}(\gamma), \mathfrak{B}(\gamma)] \in \mathrm{Mor}(\mathbf{M})$. Given morphisms $\Phi_1\colon \mathfrak{A}_1 \to \mathfrak{B}_1$ and $\Phi_2\colon \mathfrak{A}_2 \to \mathfrak{B}_2$ in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$, we define $[\Phi_1, \Phi_2]\colon [\mathfrak{A}_1, \mathfrak{A}_2] \to [\mathfrak{B}_1, \mathfrak{B}_2]$ to be the natural transformation with components $[\Phi_1, \Phi_2](c) \coloneqq [\Phi_1(c), \Phi_2(c)] \in \mathrm{Mor}(\mathbf{M})$.

It is easy to verify that the category $(\mathbf{Fun}(\mathbf{C}, \mathbf{M}), \otimes, \mathbb{1})$ with internal hom

$[-,-]\colon \mathbf{M}\times\mathbf{M}\to\mathbf{M}$ is a closed symmetric monoidal category (Definitions 1.1.9 and 1.1.4) by checking the criteria component-wise.

The fact that the category $\mathbf{Fun}(\mathbf{C},\mathbf{M})$ is also complete and cocomplete follows from the assumption that $\mathbf{M}$ is complete and cocomplete, and the fact that limits and a colimits in a functor category are computed component-wise (see Remark A.4.23).

Definition 3.1.1. Let $\mathbf{M}$ be a complete and cocomplete, closed symmetric category and $\mathbf{C}$ a small category. We define the **category of M-valued nets of algebras over C**, denoted by $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$, as the category $\mathbf{Mon}(\mathbf{Fun}(\mathbf{C},\mathbf{M}))$ of monoids in the monoidal category $\mathbf{Fun}(\mathbf{C},\mathbf{M})$ (see Definition 1.2.3).

Notation 3.1.2. We use the following conventions:

- In line with the convention about monoids (see Notation 1.2.2), we often refer to a net of algebras $(\mathfrak{A},\mu,\mathbf{1})\in\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ simply by $\mathfrak{A}$.

- In order to simplify notation, given a net of algebras $\mathfrak{A}\in\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ and an object $c\in\mathbf{C}$, we will denote by $\mathfrak{A}_c$ the object $\mathfrak{A}(c)\in\mathbf{M}$.

Remark 3.1.3. Given a complete and cocomplete, closed symmetric category $\mathbf{M}$ and a small category $\mathbf{C}$, the usual definition of an $\mathbf{M}$-valued net of algebras over $\mathbf{C}$ is that of a functor from the category $\mathbf{C}$ to the category $\mathbf{Mon}(\mathbf{M})$ of monoids in $\mathbf{M}$. In fact, this definition is equivalent to Definition 3.1.1 since there is an isomorphism between the categories $\mathbf{Fun}(\mathbf{C},\mathbf{Mon}(\mathbf{M}))$ and $\mathbf{Mon}(\mathbf{Fun}(\mathbf{C},\mathbf{M}))$.

Specifically, each $\mathfrak{A}\in\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ corresponds to a functor $\tilde{\mathfrak{A}}\in\mathbf{Fun}(\mathbf{C},\mathbf{Mon}(\mathbf{M}))$ as follows.

A net of algebras $\mathfrak{A}\in\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ consists of a triplet $(\mathfrak{A},\mu,\mathbf{1})$, where $\mathfrak{A}\colon\mathbf{C}\to\mathbf{M}$ is a functor, and $\mu\colon\mathfrak{A}\otimes\mathfrak{A}\to\mathfrak{A}$ and $\mathbf{1}\colon\mathbb{1}\to\mathfrak{A}$ are natural transformations subject to the associativity and unit axioms (see Definition 1.2.1).

The corresponding functor $\tilde{\mathfrak{A}}\colon\mathbf{C}\to\mathbf{Mon}(\mathbf{M})$ is the functor that assigns to each object $C\in\mathbf{C}$, the monoid $\tilde{\mathfrak{A}}(c):=(\mathfrak{A}_c,\mu_c,\mathbf{1}_c)$. The fact that μ and $\mathbf{1}$ satisfy the associativity and unit axioms means that their components μ_c and $\mathbf{1}_c$ satisfy them as well. Moreover, the functor $\tilde{\mathfrak{A}}$ assigns to each morphism $\gamma\colon c_1\to c_2$ in $\mathbf{C}$ the morphism $\tilde{\mathfrak{A}}(\gamma):=\mathfrak{A}(\gamma)$. Using the naturality of μ and $\mathbf{1}$ it is easy to verify that $\mathfrak{A}(\gamma)$ is a morphism of monoids (see Definition 1.2.3).

Moreover, each morphism $\Phi\colon(\mathfrak{A},\mu_{\mathfrak{A}},\mathbf{1}_{\mathfrak{A}})\to(\mathfrak{B},\mu_{\mathfrak{B}},\mathbf{1}_{\mathfrak{B}})$ in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ corresponds to a natural transformation $\tilde{\Phi}\colon\tilde{\mathfrak{A}}\to\tilde{\mathfrak{B}}$ defined as follows. For each $c\in\mathbf{C}$ let $\tilde{\Phi}_c:=\Phi_c$. By definition, $\Phi\,\mu_{\mathfrak{A}}=\mu_{\mathfrak{B}}\,(\Phi\times\Phi)\colon\mathfrak{A}\times\mathfrak{A}\to\mathfrak{B}$ and $\Phi\,\mathbf{1}_{\mathfrak{A}}=\mathbf{1}_{\mathfrak{B}}\colon\mathbb{1}\to\mathfrak{B}$. That means that the same hold for each of their components, which implies that $\Phi_c\in\mathbf{Mon}(\mathbf{M})$.

However, even if both definitions are equivalent, Definition 3.1.1 has the advantage that it facilitates many constructions over nets of algebras, as will become apparent in the following sections. △

3.2 Representations of nets of algebras

The representation of a net of algebras admits a very concise definition in terms of the monoidal category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ as follows.

Definition 3.2.1. Let $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ be an $\mathbf{M}$-valued net of algebras over $\mathbf{C}$.

(a) An **$\mathfrak{A}$-representation** is defined as a left $\mathfrak{A}$-module with respect to the monoidal category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ (see Definition 1.3.1).

(b) The **category of $\mathfrak{A}$-representations**, denoted by $\mathbf{Rep}(\mathfrak{A})$, is defined as the category ${}_{\mathfrak{A}}\mathbf{Mod}$ of left $\mathfrak{A}$-modules (see Definition 1.3.4).

Notation 3.2.2. Throughout this text we will denote a net representation and its underlying object in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ with the same symbol. Moreover, given a net representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ and an object $c \in \mathbf{C}$, we will denote by $\mathcal{L}_c$ the object $\mathcal{L}(c) \in \mathbf{M}$.

Remark 3.2.3. Notice that, since the monoidal category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ is complete and cocomplete, it follows by Corollary 1.4.5 that $\mathbf{Rep}(\mathfrak{A})$ is also complete and cocomplete for any $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$. △

3.2.1 Traditional description of net representations

Expanding the definition of a net representation, we see that given a net of algebras $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$, an $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ is a pair $(\mathcal{L}, \lambda) \in {}_{\mathfrak{A}}\mathbf{Mod}$, where $\mathcal{L}$ is an object in the underlying monoidal category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ and $\lambda\colon \mathfrak{A} \otimes \mathcal{L} \to \mathcal{L}$ in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ is an $\mathfrak{A}$-action.

More precisely, given a net of algebras $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$, an $\mathfrak{A}$-representation $\mathcal{L} = (\mathcal{L}, \lambda) \in \mathbf{Rep}(\mathfrak{A})$ consists of the following data:

(1) For each $c \in \mathbf{C}$, the $\mathfrak{A}_c$-module $(\mathcal{L}_c, \lambda_c) \in {}_{\mathfrak{A}_c}\mathbf{Mod}$. The fact that λ_c is an $\mathfrak{A}_c$-action follows from the fact that λ is an $\mathfrak{A}$-action.

(2) For each morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the morphism $\mathcal{L}(\gamma)\colon \mathcal{L}_{c_1} \to \mathcal{L}_{c_2}|_{\mathfrak{A}_{c_1}}$ in ${}_{\mathfrak{A}_{c_1}}\mathbf{Mod}$. Here, $|_{\mathfrak{A}_{c_1}}$ denotes the restriction of scalars along $\mathfrak{A}(\gamma)$ (see

Section 1.5). Notice that $\mathcal{L}(\gamma)$ is an $\mathfrak{A}_{c_1}$-module morphism because the diagram

$$\begin{tikzcd}
\mathfrak{A}_{c_1} \otimes \mathcal{L}_{c_1} \arrow[rr, "\mathfrak{A}_{c_1}\otimes\mathcal{L}(\gamma)"] \arrow[dd, "\lambda_{c_1}"'] \arrow[dr, "\mathfrak{A}(\gamma)\otimes\mathcal{L}(\gamma)"] & & \mathfrak{A}_{c_1} \otimes \mathcal{L}_{c_1} \arrow[dl, "\mathfrak{A}(\gamma)\otimes\mathcal{L}_{c_1}"'] \arrow[dd, "\mathfrak{A}(\gamma)\otimes\lambda_{c_2}"] \\
& \mathfrak{A}_{c_2} \arrow[dr, "\mathfrak{A}_{c_2}\oplus\lambda_{c_2}"] & \\
\mathcal{L}_{c_1} \arrow[rr, "\mathcal{L}(\gamma)"'] & & \mathcal{L}_{c_2}
\end{tikzcd} \tag{3.2.1}$$

in $\mathbf{M}$ commutes. In particular, the triangle on the left commutes because of the naturality of λ while the triangles on the top and right commute because of the composition rule for tensor products of morphisms (see Remark 1.1.3).

These data satisfy the following criteria.

(i) For any $c \in \mathbf{C}$, the morphism $\mathcal{L}(\mathrm{id}_c)\colon \mathcal{L}_c \to \mathcal{L}_c|_{\mathfrak{A}_c}$ in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ coincides with the identity morphism $\mathrm{id}_{\mathcal{L}_c}$. This follows directly from the fact that $\mathcal{L}$ is a functor.

(ii) For any pair of composable morphisms $\gamma_1\colon c_1 \to c_2$ and $\gamma_2\colon c_2 \to c_3$ in $\mathbf{C}$, the diagram

$$\begin{tikzcd}
\mathcal{L}_{c_1} \arrow[r, "\mathcal{L}(\gamma_1)"] \arrow[d, "\mathcal{L}(\gamma_2\gamma_1)"] & \mathcal{L}_{c_2}|_{\mathfrak{A}_{c_1}} \arrow[d, "\mathcal{L}(\gamma_2)|_{\mathfrak{A}_{c_1}}"] \\
\mathcal{L}_{c_3}|_{\mathfrak{A}_{c_1}} \arrow[r, "="] & \mathcal{L}_{c_3}|_{\mathfrak{A}_{c_2}}|_{\mathfrak{A}_{c_1}}
\end{tikzcd} \tag{3.2.2}$$

in ${}_{\mathfrak{A}_{c_1}}\mathbf{Mod}$ commutes. This follows from the functoriality of $\mathcal{L}$.

Remark 3.2.4. Notice that, when $\mathfrak{A}$ is a net of C*-algebras, then the structure presented above coincides with the usual concept of a representation of a net of C*-algebras (see [FH87; RV12b]). In particular, when $\mathcal{L}_c$ is a Hilbert space for each $c \in \mathbf{C}$, then we obtain what is called a Hilbert space representation of $\mathfrak{A}$.

Therefore, Definition 3.2.1 provides a very concise generalization of the usual concept of a representation of a net of C*-algebras for more general categories of algebras $\mathbf{Mon}(M)$. △

Now, let $(\mathcal{L}, \lambda)$ and $(\mathcal{L}', \lambda')$ be two $\mathfrak{A}$-representations and consider a morphism $\mathcal{F}\colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{A})$. By definition, $\mathcal{F}$ is a natural transformation between the underlying functors $\mathcal{L}, \mathcal{L}' \in \mathbf{Fun}(\mathbf{C}, \mathbf{M})$ such that $\mathcal{F} \circ \lambda = \lambda \circ (\mathfrak{A} \otimes \mathcal{F})$ in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$. Expanding this definition, we see that $\mathcal{F}$ consists of a morphism $\mathcal{F}_c\colon \mathcal{L}_c \to \mathcal{L}'_c$ in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ for each $c \in \mathbf{C}$, such that, given any morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the diagram

$$\begin{array}{ccc} \mathcal{L}_{c_1} & \xrightarrow{\mathcal{F}_{c_1}} & \mathcal{L}'_{c_1} \\ {\scriptstyle \mathcal{L}(\gamma)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{L}'(\gamma)} \\ \mathcal{L}_{c_2}|_{\mathfrak{A}_{c_1}} & \xrightarrow[\mathcal{F}_{c_2}|_{\mathfrak{A}_{c_1}}]{} & \mathcal{L}'_{c_2}|_{\mathfrak{A}_{c_1}} \end{array} \tag{3.2.3}$$

in ${}_{\mathfrak{A}_{c_1}}\mathbf{Mod}$ commutes.

3.2.2 Adjoint description of net representations

Notice that for each $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, there exists the change-of-monoid adjunction $(\mathrm{Ext}_{\mathfrak{A}(\gamma)} \dashv \mathrm{Res}_{\mathfrak{A}(\gamma)})\colon {}_{\mathfrak{A}_{c_1}}\mathbf{Mod} \to {}_{\mathfrak{A}_{c_2}}\mathbf{Mod}$, where $\mathrm{Ext}_{\mathfrak{A}(\gamma)}$ and $\mathrm{Res}_{\mathfrak{A}(\gamma)}$ denote the extension and restriction of scalars respectively along the morphism $\mathfrak{A}(\gamma)\colon \mathfrak{A}_{c_1} \to \mathfrak{A}_{c_2}$ in $\mathbf{Mon}(\mathbf{M})$ (see Section 1.5).

Using the change-of-monoid adjunctions generated by each $\gamma \in \mathrm{Mor}(\mathbf{C})$ we can provide an equivalent description of a net representation which will prove to be equally useful in the following sections.

So, we can equivalently describe an $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ with the following data.

(1) For each $c \in \mathbf{C}$, the $\mathfrak{A}_c$-module $(\mathcal{L}_c, \lambda_c) \in {}_{\mathfrak{A}_c}\mathbf{Mod}$.

(2) For each morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the morphism $\mathcal{L}(\gamma)^\flat\colon \mathrm{Ext}_{\mathfrak{A}(\gamma)}\mathcal{L}_{c_1} \to \mathcal{L}_{c_2}$ in ${}_{\mathfrak{A}_{c_2}}\mathbf{Mod}$, where $\mathcal{L}(\gamma)^\flat$ stands for the transpose of $\mathcal{L}(\gamma)$ (see Definition A.5.1) with respect to the adjunction $\mathrm{Ext}_{\mathfrak{A}(\gamma)} \dashv \mathrm{Res}_{\mathfrak{A}(\gamma)}$.

These data satisfy the following criteria.

(i) For any $c \in \mathbf{C}$, the morphism $\mathcal{L}(\mathrm{id}_c)^\flat\colon \mathrm{Ext}_{\mathfrak{A}(\mathrm{id}_c)}\mathcal{L}_c \,(= \mathrm{Ext}_{\mathrm{id}_{\mathfrak{A}(c)}}\mathcal{L}_c) \to \mathcal{L}_c$ in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ is the canonical isomorphism of $\mathfrak{A}_c$-modules induced by the left $\mathfrak{A}_c$-action as described in Remark 1.5.10.

(ii) For any pair of composable morphisms $\gamma_1\colon c_1 \to c_2$ and $\gamma_2\colon c_2 \to c_3$ in $\mathbf{C}$, the diagram

$$\begin{array}{ccc} \mathrm{Ext}_{\mathfrak{A}(\gamma_2)}\mathrm{Ext}_{\mathfrak{A}(\gamma_1)}\mathcal{L}_{c_1} & \xrightarrow{\mathrm{Ext}_{\mathfrak{A}(\gamma_2)}\mathcal{L}(\gamma_1)^\flat} & \mathrm{Ext}_{\mathfrak{A}(\gamma_2)}\mathcal{L}_{c_2} \\ \cong\downarrow & & \downarrow \mathcal{L}(\gamma_2)^\flat \\ \mathrm{Ext}_{\mathfrak{A}(\gamma_2\gamma_1)}\mathcal{L}_{c_1} & \xrightarrow[\mathcal{L}(\gamma_2\gamma_1)^\flat]{} & \mathcal{L}_{c_3} \end{array} \tag{3.2.4}$$

in ${}_{\mathfrak{A}_{c_3}}\mathbf{Mod}$ commutes.

A morphism $\mathcal{F}\colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{A})$ admits also an equivalent description with respect to the change-of-monoids adjunctions as follows. The morphism $\mathcal{F}$ consists of a morphism $\mathcal{F}_c\colon \mathcal{L}_c \to \mathcal{L}'_c$ in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ for each $c \in \mathbf{C}$, such that, given any morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the diagram

$$\begin{array}{ccc} \mathrm{Ext}_{\mathfrak{A}(\gamma)}\mathcal{L}_{c_1} & \xrightarrow{\mathrm{Ext}_{\mathfrak{A}(\gamma)}(\mathcal{F}_{c_1})} & \mathrm{Ext}_{\mathfrak{A}(\gamma)}\mathcal{L}'_{c_1} \\ \mathcal{L}(\gamma)^\flat\downarrow & & \downarrow \mathcal{L}'(\gamma)^\flat \\ \mathcal{L}_{c_2} & \xrightarrow[\mathcal{F}_{c_2}]{} & \mathcal{L}'_{c_2} \end{array} \tag{3.2.5}$$

in ${}_{\mathfrak{A}_{c_2}}\mathbf{Mod}$ commutes.

3.3 Constant net representations

In this section we describe the construction of the so-called constant net representations. Specifically, this construction is presented as a functor which we define as the right adjoint to the evaluation functor defined below.

Definition 3.3.1. Let $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ be a net of algebras and $\widetilde{c} \in \mathbf{C}$ be a fixed object in $\mathbf{C}$. We define the **evaluation functor of $\mathfrak{A}$-representations at $\widetilde{c}$**, denoted by $(-)_{\widetilde{c}}\colon \mathbf{Rep}(\mathfrak{A}) \to {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$, as follows. To each $\mathcal{L} = (\mathcal{L}, \lambda) \in \mathbf{Rep}(\mathfrak{A})$, it assigns the $\mathfrak{A}_{\widetilde{c}}$-module $(\mathcal{L}_{\widetilde{c}}, \lambda_{\widetilde{c}}) \in {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$, and to each morphism $\mathcal{F}\colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{A})$ it assigns the morphism $\mathcal{F}_{\widetilde{c}}\colon \mathcal{L}_{\widetilde{c}} \to \mathcal{L}'_{\widetilde{c}}$ in ${}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$.

Definition 3.3.2. Let $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ be a net of algebras and $\widetilde{c} \in \mathbf{C}$ be a fixed object in $\mathbf{C}$. We define

(a) a **constant $\mathfrak{A}$-representation functor under $\widetilde{c}$** to be any right adjoint to the evaluation functor of $\mathfrak{A}$-representations at $\widetilde{c}$,

(b) a **constant $\mathfrak{A}$-representation** to be any $\mathfrak{A}$-representation in the image of a constant $\mathfrak{A}$-representation functor.

By definition, the existence of constant $\mathfrak{A}$-representations for some $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ relies on the existence of a right adjoint functor to the evaluation functor of $\mathfrak{A}$-representations. In the following proposition we prove that the evaluation functor of $\mathfrak{A}$-representations has always a right adjoint by constructing a model for it. This construction relies on the completeness of the monoidal category $\mathbf{M}$ and the fact that $\mathbf{C}$ is a small category.

Proposition 3.3.3. *Given a net of algebras $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ and a fixed object $\widetilde{c} \in \mathbf{C}$, there exists a functor $(-)^{\widetilde{c}}\colon {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod} \to \mathbf{Rep}(\mathfrak{A})$ such that*

$$(-)_{\widetilde{c}} \dashv (-)^{\widetilde{c}}\,, \tag{3.3.1}$$

where $(-)_{\widetilde{c}}\colon \mathbf{Rep}(\mathfrak{A}) \to {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$ stands for the $\mathfrak{A}$-evaluation functor at $\widetilde{c}$.

Proof. We prove the existence of a right adjoint functor to the $\mathfrak{A}$-evaluation functor at $\widetilde{c}$, by explicitly constructing such a functor. In particular, it is convenient to define this functor using the description of $\mathfrak{A}$-representations presented in Section 3.2.1.

So, we define a functor $(-)^{\widetilde{c}}\colon {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod} \to \mathbf{Rep}(\mathfrak{A})$ as follows. To each $\mathfrak{A}_{\widetilde{c}}$-module $L \in {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$, it assigns the $\mathfrak{A}$-representation $L^{\widetilde{c}} \in \mathbf{Rep}(\mathfrak{A})$ consisting of the following data:

(1) For each $c \in \mathbf{C}$, the left $\mathfrak{A}_c$-module

$$(L^{\widetilde{c}})_c \coloneqq \prod_{\widetilde{\gamma} \in \mathbf{C}(c,\widetilde{c})} L|_{\mathfrak{A}_c} \in {}_{\mathfrak{A}_c}\mathbf{Mod}\,, \tag{3.3.2}$$

where $\prod_{\widetilde{\gamma} \in \mathbf{C}(c,\widetilde{c})} L|_{\mathfrak{A}_c}$ denotes the product (see Definition A.4.14(b)) of the diagram that assigns to each $\gamma \in \mathbf{C}(c,\widetilde{c})$ the ${}_{\mathfrak{A}_c}\mathbf{Mod}$-module $L|_{\mathfrak{A}_c}$. Notice that the indexing category of this diagram is the discrete category generated over the set $\mathbf{C}(c,\widetilde{c}) \in \mathbf{Set}$ and consequently it is small. The existence of such a limit is guaranteed by the completeness of $\mathbf{M}$ and Corollary 1.4.5.

(2) For each morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the morphism of left $\mathfrak{A}_{c_1}$-modules

$$L^{\widetilde{c}}(\gamma)\colon (L^{\widetilde{c}})_{c_1} = \prod_{\widetilde{\gamma}_1 \in \mathbf{C}(c_1,\widetilde{c})} L|_{\mathfrak{A}_{c_1}} \longrightarrow \prod_{\widetilde{\gamma}_2 \in \mathbf{C}(c_2,\widetilde{c})} L|_{\mathfrak{A}_{c_1}} = (L^{\widetilde{c}})_{c_2}|_{\mathfrak{A}_{c_1}} \tag{3.3.3}$$

in ${}_{\mathfrak{A}_{c_1}}\mathbf{Mod}$ defined by sending to the $\widetilde{\gamma}_2$-component of the codomain the $\widetilde{\gamma}_2\gamma$-component of the domain.

To each morphism of left $\mathfrak{A}_{\widetilde{c}}$-modules $F\colon L \to L'$ in ${}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$, the functor $(-)^{\widetilde{c}}$ assigns the morphism of $\mathfrak{A}$-representations $F^{\widetilde{c}}\colon L^{\widetilde{c}} \to L'^{\widetilde{c}}$ in $\mathbf{Rep}(\mathfrak{A})$ consisting of the morphism of left $\mathfrak{A}_c$-modules

$$(F^{\widetilde{c}})_c \coloneqq \prod_{\widetilde{\gamma}\in\mathbf{C}(c,\widetilde{c})} F|_{\mathfrak{A}_c}\colon (L^{\widetilde{c}})_c = \prod_{\widetilde{\gamma}\in\mathbf{C}(c,\widetilde{c})} L|_{\mathfrak{A}_c} \longrightarrow \prod_{\widetilde{\gamma}\in\mathbf{C}(c,\widetilde{c})} L'|_{\mathfrak{A}_c} = (L'^{\widetilde{c}})_c \tag{3.3.4}$$

in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ for each $c \in \mathbf{C}$.

It is straightforward to confirm that the above construction defines a functor $(-)^{\widetilde{c}}$. To check that this functor is right adjoint to $(-)_{\widetilde{c}}$, we exhibit the unit η and the counit ϵ of this adjunction (see Proposition A.5.5).

The unit is the natural transformation $\eta\colon \mathrm{Id}_{\mathbf{Rep}(\mathfrak{A})} \to ((-)_{\widetilde{c}})^{\widetilde{c}}$ whose component at the $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ is the morphism of $\mathfrak{A}$-representations $\eta_{\mathcal{L}}\colon \mathcal{L} \to (\mathcal{L}_{\widetilde{c}})^{\widetilde{c}}$ in $\mathbf{Rep}(\mathfrak{A})$ consisting of the left $\mathfrak{A}_c$-module morphism

$$(\eta_{\mathcal{L}})_c \coloneqq (\mathcal{L}_{\widetilde{\gamma}})_{\widetilde{\gamma}\in\mathbf{C}(c,\widetilde{c})}\colon \mathcal{L}_c \longrightarrow \prod_{\widetilde{\gamma}\in\mathbf{C}(c,\widetilde{c})} \mathcal{L}_{\widetilde{c}}|_{\mathfrak{A}_c} = ((\mathcal{L}_{\widetilde{c}})^{\widetilde{c}})_c \tag{3.3.5}$$

in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ for each $c \in \mathbf{C}$.

The counit is the natural transformation $\epsilon\colon ((-)^{\widetilde{c}})_{\widetilde{c}} \to \mathrm{Id}_{{}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}}$ whose component at the left $\mathfrak{A}_{\widetilde{c}}$-module $L \in {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$ is the morphism of left $\mathfrak{A}_{\widetilde{c}}$-modules

$$\epsilon_L \coloneqq \mathrm{pr}_{\mathrm{id}_{\widetilde{c}}}\colon (L^{\widetilde{c}})_{\widetilde{c}} = \prod_{\widetilde{\gamma}\in\mathbf{C}(\widetilde{c},\widetilde{c})} L|_{\mathfrak{A}_{\widetilde{c}}} \longrightarrow L \tag{3.3.6}$$

in ${}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$ given by the projection $\mathrm{pr}_{\mathrm{id}_{\widetilde{c}}}$ of the $\mathrm{id}_{\widetilde{c}}$-component. The triangle identities $(\epsilon_L)^{\widetilde{c}}\,\eta_{L^{\widetilde{c}}} = \mathrm{id}_{L^{\widetilde{c}}}$, for all $L \in {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$, and $\epsilon_{\mathcal{L}_{\widetilde{c}}}\,(\eta_{\mathcal{L}})_{\widetilde{c}} = \mathrm{id}_{\mathcal{L}_{\widetilde{c}}}$, for all $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$, are straightforward, which proves the adjunction (3.3.1). □

Remark 3.3.4. Since the adjoint to some functor is unique up to isomorphism (see Remark A.5.6), given a net of algebras $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ and an object $c \in \mathbf{C}$, we will refer to the functor $(-)^{\widetilde{c}}\colon {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod} \to \mathbf{Rep}(\mathfrak{A})$ as *the* constant $\mathfrak{A}$-representation functor under $\widetilde{c}$. △

Remark 3.3.5. Moreover, the uniqueness of an $\mathfrak{A}$-representation functor up to isomorphism implies that using the functor $(-)^{\widetilde{c}}$ we can construct any constant $\mathfrak{A}$-representation up to isomorphism. △

Remark 3.3.6. In the case of nets of algebras in the sense of Haag and Kastler, namely when $\mathbf{C} = \mathbf{CCO}(M)$, for some fixed lorentzian manifold M (see Definition 3.8.2), one recognizes that the constant $\mathfrak{A}$-representation functor

$(-)^M$ under M (see Remark 3.3.4), captures the well-known construction of constant $\mathfrak{A}$-representations from left $\mathfrak{A}_M$-modules over the global algebra of observables $\mathfrak{A}_M \in \mathbf{Mon}(\mathbf{M})$ of a net of algebras $\mathfrak{A} \in \mathbf{Net}^{\mathbf{M}}_{\mathbf{CCO}(M)}$ over $\mathbf{CCO}(M)$. This large class of net representations is the one that most frequently occurs in the literature (see [FH87] or [RV12a, Sec. 4.2]).

Contrary to what the name implies, the constant net representations are not trivial. We call them constant because in the C*-setting of algebraic quantum field theory the whole net of algebras is represented on the same Hilbert space (see [FH87, Th. 4.2]). △

The constant $\mathfrak{A}$-representation functor will be used in Chapter 4 to construct examples of representations for the theory of Maxwell p-forms in the framework of homotopy algebraic quantum field theory.

3.4 Change-of-net adjunction

In order to compare categories of net representations associated with different nets of algebras we will make extensive use of the *change-of-net* adjunction, i.e. the change-of-monoid adjunction (see Proposition 1.5.9) applied on the symmetric monoidal category $\mathbf{Net}^{\mathbf{M}}_{\mathbf{C}}$. More specifically, given a morphism of nets of algebras $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{Net}^{\mathbf{M}}_{\mathbf{C}}$, there is an associated **change-of-net** adjunction

$$(\mathrm{Ext}_\Phi \dashv \mathrm{Res}_\Phi)\colon \mathbf{Rep}(\mathfrak{A}) \to \mathbf{Rep}(\mathfrak{B})\,, \tag{3.4.1}$$

where $\mathrm{Ext}_\Phi\colon \mathbf{Rep}(\mathfrak{A}) \rightleftarrows \mathbf{Rep}(\mathfrak{B}) :\! \mathrm{Res}_\Phi$ denote the extension and restriction of scalars along Φ respectively (see Definitions 1.5.2 and 1.5.4). In this case, we will refer to these functors simply as **extension** and **restriction** functors respectively.

Remark 3.4.1. When $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is an isomorphism in $\mathbf{Net}^{\mathbf{M}}_{\mathbf{C}}$, the change-of-net adjunction is an adjoint equivalence. This follows from Remark 1.5.12. △

The extension Ext_Φ and restriction Res_Φ functors admit also an explicit description in terms of the more elementary extension and restriction of scalars along the components of Φ in the underlying category $\mathbf{M}$.

3.4.1 The restriction functor

The restriction functor $\mathrm{Res}_\Phi\colon \mathbf{Rep}(\mathfrak{B}) \to \mathbf{Rep}(\mathfrak{A})$ assigns to a $\mathfrak{B}$-representation $\mathcal{M} \in \mathbf{Rep}(\mathfrak{B})$ the $\mathfrak{A}$-representation $\mathrm{Res}_\Phi\mathcal{M} \in \mathbf{Rep}(\mathfrak{A})$ consisting of the following data (see Section 3.2.1).

(1) For each $c \in \mathbf{C}$, the left $\mathfrak{A}_c$-module

$$(\mathrm{Res}_\Phi \mathcal{M})_c \coloneqq \mathcal{M}_c|_{\mathfrak{A}_c} \in {}_{\mathfrak{A}_c}\mathbf{Mod} \tag{3.4.2a}$$

obtained restricting the left $\mathfrak{B}_c$-module $\mathcal{M}_c \in {}_{c}\mathbf{Mod}$ along $\Phi_c \colon \mathfrak{A}_c \to \mathfrak{B}_c$ in $\mathbf{Mon}(\mathbf{M})$ (see Definition 1.5.2 and Notation 1.5.3).

(2) For each morphism $\gamma \colon c_1 \to c_2$ in $\mathbf{C}$, the morphism of left $\mathfrak{A}_{c_1}$-modules

$$(\mathrm{Res}_\Phi \mathcal{M})(\gamma) \coloneqq \mathcal{M}(\gamma)|_{\mathfrak{A}_{c_1}} \colon (\mathrm{Res}_\Phi \mathcal{M})_{c_1} \longrightarrow (\mathrm{Res}_\Phi \mathcal{M})_{c_2}|_{\mathfrak{A}_{c_1}} \tag{3.4.2b}$$

in ${}_{\mathfrak{A}_{c_1}}\mathbf{Mod}$ defined by restricting the morphism of left $\mathfrak{B}_{c_1}$-modules $\mathcal{M}(\gamma) \colon \mathcal{M}_{c_1} \to \mathcal{M}_{c_2}|_{\mathfrak{B}_{c_1}}$ in ${}_{\mathfrak{B}_{c_1}}\mathbf{Mod}$ along $\Phi_{c_1} \colon \mathfrak{A}_{c_1} \to \mathfrak{B}_{c_1}$ in $\mathbf{Mon}(\mathbf{M})$ and observing that both iterated restrictions on the right hand side coincide with the restriction along the composition $\mathfrak{B}(\gamma)\, \Phi_{c_1} = \Phi_{c_2}\, \mathfrak{A}(\gamma) \colon \mathfrak{A}_{c_1} \to \mathfrak{B}_{c_2}$ in $\mathbf{Mon}(\mathbf{M})$ (see Remark 1.5.11).

Furthermore, Res_Φ assigns to a morphism $\mathcal{G} \colon \mathcal{M} \to \mathcal{M}'$ in $\mathbf{Rep}(\mathfrak{B})$ the morphism $\mathrm{Res}_\Phi \mathcal{G} \colon \mathrm{Res}_\Phi \mathcal{M} \to \mathrm{Res}_\Phi \mathcal{M}'$ in $\mathbf{Rep}(\mathfrak{A})$ consisting of the morphisms of left $\mathfrak{A}_c$-modules

$$(\mathrm{Res}_\Phi \mathcal{G})_c \coloneqq \mathcal{G}_c|_{\mathfrak{A}_c} \colon (\mathrm{Res}_\Phi \mathcal{M})_c \longrightarrow (\mathrm{Res}_\Phi \mathcal{M}')_c \tag{3.4.2c}$$

in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ obtained restricting the morphism of left $\mathfrak{B}_c$-modules $\mathcal{G}_c \colon \mathcal{M}_c \to \mathcal{M}'_c$ in ${}_{\mathfrak{B}_c}\mathbf{Mod}$ along $\Phi_c \colon \mathfrak{A}_c \to \mathfrak{B}_c$ in $\mathbf{Mon}(\mathbf{M})$, for all $c \in \mathbf{C}$.

Using the change-of-monoid adjunction from Proposition 1.5.9 and its compatibility with compositions (see Remark 1.5.11), it is straightforward to check that the data listed above fulfil the criteria for net representations and morphisms of net representations as described in Section 3.2.1.

3.4.2 The extension functor

The extension functor $\mathrm{Ext}_\Phi \colon \mathbf{Rep}(\mathfrak{A}) \to \mathbf{Rep}(\mathfrak{B})$ admits a similar description in terms of the extensions of scalars along the components of Φ. Specifically, the functor Ext_Φ assigns to an $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ the $\mathfrak{B}$-representation consisting of the following data:

(1) For each $c \in \mathbf{C}$, the left $\mathfrak{B}_c$-module

$$(\mathrm{Ext}_\Phi \mathcal{L})_c \coloneqq \mathrm{Ext}_{\Phi_c} \mathcal{L}_c \in {}_{\mathfrak{B}_c}\mathbf{Mod} \tag{3.4.3a}$$

obtained extending the left $\mathfrak{A}_c$-module $\mathcal{L}_c \in {}_{\mathfrak{A}_c}\mathbf{Mod}$ along $\Phi_c \colon \mathfrak{A}_c \to \mathfrak{B}_c$ in $\mathbf{Mon}(\mathbf{M})$ (see Definition 1.5.4).

(2) For each morphism $\gamma \colon c_1 \to c_2$ in $\mathbf{C}$, the morphism of left $\mathfrak{B}_{c_2}$-modules $(\mathrm{Ext}_\Phi \mathcal{L})(\gamma)^\flat \colon \mathrm{Ext}_{\mathfrak{B}(\gamma)}\big((\mathrm{Ext}_\Phi \mathcal{L})_{c_1}\big) \to (\mathrm{Ext}_\Phi \mathcal{L})_{c_2}$ in ${}_{\mathfrak{B}_{c_2}}\mathbf{Mod}$ defined by the diagram

$$\begin{array}{ccc} \mathrm{Ext}_{\mathfrak{B}(\gamma)} \mathrm{Ext}_{\Phi_{c_1}} \mathcal{L}_{c_1} & \xrightarrow{\ (\mathrm{Ext}_\Phi \mathcal{L})(\gamma)^\flat\ } & \mathrm{Ext}_{\Phi_{c_2}} \mathcal{L}_{c_2} \\ & {\scriptstyle \cong} \searrow \qquad \nearrow {\scriptstyle \mathrm{Ext}_{\Phi_{c_2}} \mathcal{L}(\gamma)^\flat} & \\ & \mathrm{Ext}_{\Phi_{c_2}} \mathrm{Ext}_{\mathfrak{A}(\gamma)} \mathcal{L}_{c_1} & \end{array} \tag{3.4.3b}$$

in ${}_{\mathfrak{B}_{c_2}}\mathbf{Mod}$, where we used that the iterated extensions on the left hand side are naturally isomorphic to the extension along the composition $\mathfrak{B}(\gamma)\,\Phi_{c_1} = \Phi_{c_2}\,\mathfrak{A}(\gamma) \colon \mathfrak{A}_{c_1} \to \mathfrak{B}_{c_2}$ in $\mathbf{Mon}(\mathbf{M})$, see Remark 1.5.11, and then we post-composed with the extension of the morphism of left $\mathfrak{A}_{c_2}$-modules $\mathcal{L}(\gamma)^\flat \colon \mathrm{Ext}_{\mathfrak{A}(\gamma)} \mathcal{L}_{c_1} \to \mathcal{L}_{c_2}$ in ${}_{\mathfrak{A}_{c_2}}\mathbf{Mod}$ along $\Phi_{c_2} \colon \mathfrak{A}_{c_2} \to \mathfrak{B}_{c_2}$ in $\mathbf{Mon}(\mathbf{M})$.

Furthermore, Ext_Φ assigns to a morphism $\mathcal{F} \colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{A})$ the morphism $\mathrm{Ext}_\Phi \mathcal{F} \colon \mathrm{Ext}_\Phi \mathcal{L} \to \mathrm{Ext}_\Phi \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{B})$ consisting of the morphisms of left $\mathfrak{B}_c$-modules

$$(\mathrm{Ext}_\Phi \mathcal{F})_c := \mathrm{Ext}_{\Phi_c} \mathcal{F}_c \colon (\mathrm{Ext}_\Phi \mathcal{L})_c \longrightarrow (\mathrm{Ext}_\Phi \mathcal{L}')_c \tag{3.4.3c}$$

in ${}_{\mathfrak{B}_c}\mathbf{Mod}$ obtained extending the morphism of left $\mathfrak{A}_c$-modules $\mathcal{F}_c \colon \mathcal{L}_c \to \mathcal{L}'_c$ in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ along $\Phi_c \colon \mathfrak{A}_c \to \mathfrak{B}_c$ in $\mathbf{Mon}(\mathbf{M})$, for all $c \in \mathbf{C}$. Once again, using the change-of-monoid adjunction from Proposition 1.5.9 and its compatibility with compositions (see Remark 1.5.11), it is straightforward to check that the above data fulfil the criteria of net representations and morphisms of net representations described in Section 3.2.2.

3.5 Tensoring, powering and enriched hom on categories of net representations

In this section we present a two-variable adjunction

$$(\otimes, (-)^{(-)}, [-,-]_{\mathfrak{A}}) \colon \mathbf{Rep}(\mathfrak{A}) \times \mathbf{M} \to \mathbf{Rep}(\mathfrak{A}) \tag{3.5.1}$$

consisting of the $\mathbf{M}$-tensoring, the $\mathbf{M}$-powering and the $\mathbf{M}$-enriched hom on the category $\mathbf{Rep}(\mathfrak{A})$ of representations of a net of algebras $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$. These concepts will be handy in endowing the category $\mathbf{Rep}(\mathfrak{A})$ with a model structure in Section 3.6.

Having defined the category of nets of algebras as the category of monoids over the closed symmetric monoidal functor category $\mathbf{Fun}(\mathbf{C},\mathbf{M})$, one obtains immediately from Section 1.6 a $\mathbf{Fun}(\mathbf{C},\mathbf{M})$-tensoring on the category $\mathbf{Rep}(\mathfrak{A})$. Then, restricting the latter $\mathbf{Fun}(\mathbf{C},\mathbf{M})$-tensoring to constant functors provides an $\mathbf{M}$-tensoring on $\mathbf{Rep}(\mathfrak{A})$. Working out the usual adjunctions leads also to the $\mathbf{M}$-powering and enrichement on $\mathbf{Rep}(\mathfrak{A})$. We present below the relevant constructions exploiting the $\mathbf{M}$-tensoring, powering and enriched hom on the category of left modules over a monoid (see Section 1.6).

3.5.1 M-tensoring

Let us start from the $\mathbf{M}$-tensoring

$$\otimes\colon \mathbf{Rep}(\mathfrak{A}) \times \mathbf{M} \longrightarrow \mathbf{Rep}(\mathfrak{A}). \tag{3.5.2a}$$

$\otimes$ assigns to an $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ and an object $V \in \mathbf{M}$ the $\mathfrak{A}$-representation $\mathcal{L} \otimes V \in \mathbf{Rep}(\mathfrak{A})$ consisting of the following data:

(1) For each $c \in \mathbf{C}$, the left $\mathfrak{A}_c$-module

$$(\mathcal{L} \otimes V)_c := \mathcal{L}_c \otimes V \in {}_{\mathfrak{A}_c}\mathbf{Mod} \tag{3.5.2b}$$

obtained evaluating the $\mathbf{M}$-tensoring $\otimes\colon {}_{\mathfrak{A}_c}\mathbf{Mod} \times \mathbf{M} \to {}_{\mathfrak{A}_c}\mathbf{Mod}$ on $\mathcal{L}_c \in {}_{\mathfrak{A}_c}\mathbf{Mod}$ and $V \in \mathbf{M}$ (see Definition 1.6.1).

(2) For each morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the morphism of left $\mathfrak{A}_{c_2}$-modules $(\mathcal{L} \otimes V)(\gamma)^\flat\colon \mathrm{Ext}_{\mathfrak{A}(\gamma)}\,(\mathcal{L} \otimes V)_{c_1} \to (\mathcal{L} \otimes V)_{c_2}$ in ${}_{\mathfrak{A}_{c_2}}\mathbf{Mod}$ defined by the diagram

$$\begin{array}{ccc} \mathrm{Ext}_{\mathfrak{A}(\gamma)}\Big(\mathcal{L}_{c_1} \otimes V\Big) & \xrightarrow{(\mathcal{L}\otimes V)(\gamma)^\flat} & \mathcal{L}_{c_2} \otimes V \\ & {\scriptstyle\cong}\searrow \quad \nearrow{\scriptstyle \mathcal{L}(\gamma)^\flat \otimes \mathrm{id}_V} & \\ & \Big(\mathrm{Ext}_{\mathfrak{A}(\gamma)}\,\mathcal{L}_{c_1}\Big) \otimes V & \end{array} \tag{3.5.2c}$$

in ${}_{\mathfrak{A}_{c_2}}\mathbf{Mod}$ as the composition of the natural isomorphism of left $\mathfrak{A}_{c_2}$-modules described in Remark 1.6.7 and the morphism of $\mathfrak{A}_{c_2}$-modules obtained by evaluating the $\mathbf{M}$-tensoring $\otimes\colon {}_{\mathfrak{A}_{c_2}}\mathbf{Mod} \times \mathbf{M} \to {}_{\mathfrak{A}_{c_2}}\mathbf{Mod}$ on $\mathcal{L}(\gamma)^\flat\colon \mathrm{Ext}_{\mathfrak{A}(\gamma)}\mathcal{L}_{c_1} \to \mathcal{L}_{c_2}$ in ${}_{\mathfrak{A}_{c_2}}\mathbf{Mod}$ and $\mathrm{id}_V\colon V \to V$ in $\mathbf{M}$.

Furthermore, the $\mathbf{M}$-tensoring $\otimes$ assigns to morphisms $\mathcal{F}\colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{A})$ and $\xi\colon V \to V'$ in $\mathbf{M}$ the morphism $\mathcal{F} \otimes \xi\colon \mathcal{L} \otimes V \to \mathcal{L}' \otimes V'$ in $\mathbf{Rep}(\mathfrak{A})$ consisting of the morphisms of left $\mathfrak{A}_c$-modules

$$(\mathcal{F} \otimes \xi)_c := \mathcal{F}_c \otimes \xi\colon (\mathcal{L} \otimes V)_c \longrightarrow (\mathcal{L}' \otimes V')_c \tag{3.5.2d}$$

in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ obtained evaluating the $\mathbf{M}$-tensoring $\otimes\colon {}_{\mathfrak{A}_c}\mathbf{Mod} \times \mathbf{M} \to {}_{\mathfrak{A}_c}\mathbf{Mod}$ on $\mathcal{F}_c\colon \mathcal{L}_c \to \mathcal{L}'_c$ in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ and $\xi\colon V \to V'$ in $\mathbf{M}$, for all $c \in \mathbf{C}$. Using the $\mathbf{M}$-tensoring for left modules (Definition 1.6.1) and its compatibility with the change-of-monoid adjunction (see Remark 1.6.7) it is straightforward to check that the above data fulfil the axioms of Section 3.2.2 and that the above assignment defines a functor $\otimes$.

Remark 3.5.1. The category $\mathbf{Rep}(\mathfrak{A})$ is a right $\mathbf{M}$-module category (see Definition 2.7.1) with the $\mathbf{M}$-action given by the $\mathbf{M}$-tensoring on $\mathbf{Rep}(\mathfrak{A})$, which we denote here by $\otimes_{\mathbf{Rep}(\mathfrak{A})}\colon \mathbf{Rep}(\mathfrak{A}) \times \mathbf{M} \to \mathbf{Rep}(\mathfrak{A})$, in order to distinguish it from the monoidal multiplication $\otimes\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$.

Specifically, let:

(1) $\phi\colon (- \otimes_{\mathbf{Rep}(\mathfrak{A})} -) \otimes_{\mathbf{Rep}(\mathfrak{A})} (-) \xrightarrow{\cong} (-) \otimes_{\mathbf{Rep}(\mathfrak{A})} (- \otimes -)$ be the natural isomorphism with components $\phi_{\mathcal{L},V,W} := (\alpha_{\mathcal{L}_c,V,W})_{c\in\mathbf{C}}$ for each $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ and $V, W \in \mathbf{M}$, where α denotes the associator of the monoidal category $\mathbf{M}$.

(2) $\psi\colon (-) \otimes_{\mathbf{Rep}(\mathfrak{A})} \mathbb{1} \xrightarrow{\cong} \mathrm{Id}_{\mathbf{M}}$ be the natural isomorphism with components $\psi_{\mathcal{L}} := (\rho_{\mathcal{L}_c})_{c\in\mathbf{C}}$ for each $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$, where ρ denotes the right unitor of the monoidal category $\mathbf{M}$.

It is easy to verify that $(\mathbf{Rep}(\mathfrak{A}), \otimes_{\mathbf{Rep}(\mathfrak{A})}, \phi, \psi)$ is an $\mathbf{M}$-module category using the pentagon and unit axioms of the monoidal category $\mathbf{M}$ (see Definition 1.1.1). △

3.5.2 M-Powering

The $\mathbf{M}$-powering

$$(-)^{(-)}\colon \mathbf{Rep}(\mathfrak{A}) \times \mathbf{M}^{\mathrm{op}} \longrightarrow \mathbf{Rep}(\mathfrak{A}) \tag{3.5.3a}$$

on $\mathbf{Rep}(\mathfrak{A})$ can be described explicitly as follows. $(-)^{(-)}$ assigns to an $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ and an object $V \in \mathbf{M}$ the $\mathfrak{A}$-representation $\mathcal{L}^V \in \mathbf{Rep}(\mathfrak{A})$ consisting of the following data:

(1) For each $c \in \mathbf{C}$, the left $\mathfrak{A}_c$-module

$$(\mathcal{L}^V)_c := (\mathcal{L}_c)^V \in {}_{\mathfrak{A}_c}\mathbf{Mod} \tag{3.5.3b}$$

obtained evaluating the powering $(-)^{(-)}\colon {}_{\mathfrak{A}_c}\mathbf{Mod} \times \mathbf{M}^{\mathrm{op}} \to {}_{\mathfrak{A}_c}\mathbf{Mod}$ on $\mathcal{L}_c \in {}_{\mathfrak{A}_c}\mathbf{Mod}$ and $V \in \mathbf{M}$ (see Definition 1.6.2).

(2) For each morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the morphism of left $\mathfrak{A}_{c_1}$-modules

$$(\mathcal{L}^V)(\gamma) := \mathcal{L}(\gamma)^{\mathrm{id}_V} \colon (\mathcal{L}^V)_{c_1} \longrightarrow (\mathcal{L}_{c_2}|_{\mathfrak{A}_{c_1}})^V = (\mathcal{L}^V)_{c_2}|_{\mathfrak{A}_{c_1}} \tag{3.5.3c}$$

in ${}_{\mathfrak{A}_{c_1}}\mathbf{Mod}$ defined as the morphism of $\mathfrak{A}_{c_1}$-modules obtained evaluating the powering $(-)^{(-)}\colon {}_{\mathfrak{A}_{c_1}}\mathbf{Mod} \times \mathbf{M}^{\mathrm{op}} \to {}_{\mathfrak{A}_{c_1}}\mathbf{Mod}$ on $\mathcal{L}(\gamma)\colon \mathcal{L}_{c_1} \to \mathcal{L}_{c_2}|_{\mathfrak{A}_{c_1}}$ in ${}_{\mathfrak{A}_{c_1}}\mathbf{Mod}$ and $\mathrm{id}_V\colon V \to V$ in $\mathbf{M}$, where we also used the commutative square (1.6.9) on the right hand side.

Furthermore, the $\mathbf{M}$-powering $(-)^{(-)}$ assigns to morphisms $\mathcal{F}\colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{A})$ and $\xi\colon V' \to V$ in $\mathbf{M}$ the morphism $\mathcal{F}^\xi\colon \mathcal{L}^V \to \mathcal{L}'^{V'}$ in $\mathbf{Rep}(\mathfrak{A})$ consisting of the morphisms of left $\mathfrak{A}_c$-modules

$$(\mathcal{F}^\xi)_c := (\mathcal{F}_c)^\xi \colon (\mathcal{L}^V)_c \longrightarrow (\mathcal{L}'^{V'})_c \tag{3.5.3d}$$

in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ obtained evaluating the powering $(-)^{(-)}\colon {}_{\mathfrak{A}_c}\mathbf{Mod} \times \mathbf{M}^{\mathrm{op}} \to {}_{\mathfrak{A}_c}\mathbf{Mod}$ on $\mathcal{F}_c\colon \mathcal{L}_c \to \mathcal{L}'_c$ in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ and $\xi\colon V' \to V$ in $\mathbf{M}$, for all $c \in C$. Using the $\mathbf{M}$-powering functor for left modules (Definition 1.6.2) and its compatibility with the change-of-monoid adjuction, see Remark 1.6.7, it is straightforward to check that the data listed above fulfil the axioms of Section 3.2.1 and that the above assignment defines a functor $(-)^{(-)}$.

For each object $V \in \mathbf{M}$, partial evaluations of the $\mathbf{M}$-tensoring and of the $\mathbf{M}$-powering on $\mathbf{Rep}(\mathfrak{A})$ give rise to an adjunction

$$(-) \otimes V \dashv (-)^V \tag{3.5.4}$$

natural with respect to V.

Remark 3.5.2. Note that the adjunction (3.5.4) is compatible with the adjunction (3.3.1) in the following sense. Given $\widetilde{c} \in \mathbf{C}$, $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ and $V \in \mathbf{M}$, the diagram of left adjoint functors

$$\begin{array}{ccc} \mathbf{Rep}(\mathfrak{A}) & \xrightarrow{(-)_{\widetilde{c}}} & {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod} \\ {\scriptstyle (-)\otimes V}\downarrow & & \downarrow{\scriptstyle (-)\otimes V} \\ \mathbf{Rep}(\mathfrak{A}) & \xrightarrow[(-)_{\widetilde{c}}]{} & {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod} \end{array} \tag{3.5.5}$$

commutes as a straightforward consequence of their definitions. △

Remark 3.5.3. The adjunction (3.5.4) is also compatible with the change-of-net adjunction (3.4.1) in the following sense. Given a morphism $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ and an object $V \in \mathbf{M}$, the diagram of right adjoint functors

$$\begin{array}{ccc} \mathbf{Rep}(\mathfrak{B}) & \xrightarrow{\mathrm{Res}_\Phi} & \mathbf{Rep}(\mathfrak{A}) \\ {\scriptstyle (-)^V}\downarrow & & \downarrow{\scriptstyle (-)^V} \\ \mathbf{Rep}(\mathfrak{B}) & \xrightarrow[\mathrm{Res}_\Phi]{} & \mathbf{Rep}(\mathfrak{A}) \end{array} \tag{3.5.6}$$

commutes as a straightforward consequence of their definitions. Therefore, the corresponding diagram of left adjoint functors commutes up to a unique natural isomorphism

$$\mathrm{Ext}_\Phi(- \otimes V) \cong \mathrm{Ext}_\Phi(-) \otimes V \tag{3.5.7}$$

△

Remark 3.5.4. Given a morphism $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$, the induced extension functor $\mathrm{Ext}_\Phi\colon \mathbf{Rep}(\mathfrak{A}) \to \mathbf{Rep}(\mathfrak{B})$ is an $\mathbf{M}$-module functor (see Definition 2.7.4). This follows from the compatibility of the extension functor and the tensoring (see Equation (3.5.7)). △

3.5.3 M-Enriched hom

We conclude describing the $\mathbf{M}$-enriched hom

$$[-,-]_{\mathfrak{A}}\colon \mathbf{Rep}(\mathfrak{A})^{\mathrm{op}} \times \mathbf{Rep}(\mathfrak{A}) \longrightarrow \mathbf{M} \tag{3.5.8a}$$

on $\mathbf{Rep}(\mathfrak{A})$. $[-,-]_{\mathfrak{A}}$ assigns to $\mathfrak{A}$-representations $\mathcal{L}, \mathcal{L}' \in \mathbf{Rep}(\mathfrak{A})$ the equalizer

$$[\mathcal{L}, \mathcal{L}']_{\mathfrak{A}} := \lim \Big(\prod_{c \in \mathbf{C}} [\mathcal{L}_c, \mathcal{L}'_c]_{\mathfrak{A}_c} \overset{\mathcal{L}^*}{\underset{\mathcal{L}'_*}{\rightrightarrows}} \prod_{\substack{c_1, c_2 \in \mathbf{C} \\ \gamma \in \mathbf{C}(c_1, c_2)}} [\mathcal{L}_{c_1}, \mathcal{L}'_{c_2}|_{\mathfrak{A}_{c_1}}]_{\mathfrak{A}_{c_1}} \Big) \in \mathbf{M}, \tag{3.5.8b}$$

where we used the $\mathbf{M}$-enriched hom $[-,-]_A\colon {}_A\mathbf{Mod}^{\mathrm{op}} \times {}_A\mathbf{Mod} \to \mathbf{M}$ from Definition 1.6.3. $\mathcal{L}^*$ above denotes the morphism in $\mathbf{M}$ defined via the universal property of the product by the diagram

$$\begin{array}{ccccc} \prod_{c \in \mathbf{C}} [\mathcal{L}_c, \mathcal{L}'_c]_{\mathfrak{A}_c} & & \xrightarrow{\quad \mathcal{L}^* \quad} & & \prod_{\substack{c_1, c_2 \in \mathbf{C} \\ \gamma \in \mathbf{C}(c_1, c_2)}} [\mathcal{L}_{c_1}, \mathcal{L}'_{c_2}|_{\mathfrak{A}_{c_1}}]_{\mathfrak{A}_{c_1}} \\ {\scriptstyle \mathrm{pr}_{c_2}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{pr}_{c_1, c_2, \gamma}} \\ [\mathcal{L}_{c_2}, \mathcal{L}'_{c_2}]_{\mathfrak{A}_{c_2}} & \xrightarrow[(1.6.10)]{} & [\mathcal{L}_{c_2}|_{\mathfrak{A}_{c_1}}, \mathcal{L}'_{c_2}|_{\mathfrak{A}_{c_1}}]_{\mathfrak{A}_{c_1}} & \xrightarrow[{[\mathcal{L}(\gamma), \mathrm{id}]_{\mathfrak{A}_{c_1}}}]{} & [\mathcal{L}_{c_1}, \mathcal{L}'_{c_2}|_{\mathfrak{A}_{c_1}}]_{\mathfrak{A}_{c_1}} \end{array} \tag{3.5.8c}$$

in $\mathbf{M}$, for each $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$. Whereas, $\mathcal{L}'_*$ above denotes the morphism in $\mathbf{M}$ defined via the universal property of the product by the diagram

$$\begin{array}{ccc} \prod_{c\in\mathbf{C}} [\mathcal{L}_c, \mathcal{L}'_c]_{\mathfrak{A}_c} & \xrightarrow{\mathcal{L}'_*} & \prod_{\substack{c_1,c_2\in\mathbf{C}\\ \gamma\in\mathbf{C}(c_1,c_2)}} [\mathcal{L}_{c_1}, \mathcal{L}'_{c_2}|_{\mathfrak{A}_{c_1}}]_{\mathfrak{A}_{c_1}} \\ {\scriptstyle \mathrm{pr}_{c_1}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{c_1,c_2,\gamma}} \\ [\mathcal{L}_{c_1}, \mathcal{L}'_{c_1}]_{\mathfrak{A}_{c_1}} & \xrightarrow[{[\mathrm{id}_{\mathcal{L}_{c_1}}, \mathcal{L}'(\gamma)]_{\mathfrak{A}_{c_1}}}]{} & [\mathcal{L}_{c_1}, \mathcal{L}'_{c_2}|_{\mathfrak{A}_{c_1}}]_{\mathfrak{A}_{c_1}} \end{array} \tag{3.5.8d}$$

in $\mathbf{M}$, for each $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$.

The action of the $\mathbf{M}$-enriched hom $[-,-]_{\mathfrak{A}}$ on morphisms is defined combining the universal property of the limit in (3.5.8), the $\mathbf{M}$-enriched hom $[-,-]_A\colon {}_A\mathbf{Mod}^{\mathrm{op}} \times {}_A\mathbf{Mod} \to \mathbf{M}$ (see Definition 1.6.3) and the restriction functor $(-)|_A\colon {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}$ (see Definition 1.5.2).

Remark 3.5.5. Let us provide a more explicit description of the $\mathbf{M}$-enriched hom $[\mathcal{L}, \mathcal{L}']_{\mathfrak{A}} \in \mathbf{M}$ from (3.5.8) when $\mathbf{M} = \mathbf{Vec}_{\mathbb{K}}$, i.e. the familiar closed symmetric monoidal category of vector spaces over a field $\mathbb{K}$. In this case the vector space $[\mathcal{L}, \mathcal{L}']_{\mathfrak{A}} \in \mathbf{Vec}_{\mathbb{K}}$ consists of collections $x = (x_c)_{c\in\mathbf{C}}$ of $\mathfrak{A}_c$-linear maps $x_c\colon \mathcal{L}_c \to \mathcal{L}'_c$, for all $c \in \mathbf{C}$, such that $x_{c_2}|_{\mathfrak{A}_{c_1}}\, \mathcal{L}(\gamma) = \mathcal{L}'(\gamma)\, x_{c_1} : \mathcal{L}_{c_1} \to \mathcal{L}'_{c_2}|_{\mathfrak{A}_{c_1}}$ coincide as $\mathfrak{A}_{c_1}$-linear maps, for all $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$. A similar description holds true for any concrete closed symmetric monoidal category, including the category of cochain complexes $\mathbf{M} = \mathbf{CoCh}_{\mathbb{K}}$, which is used in homotopy AQFT (see Section 3.8). △

For each object $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$, partial evaluations of the $\mathbf{M}$-tensoring and of the $\mathbf{M}$-enriched hom give rise to an adjunction

$$\mathcal{L} \otimes (-) \dashv [\mathcal{L}, -]_{\mathfrak{A}} \tag{3.5.9}$$

natural with respect to $\mathcal{L}$.

3.6 Model structure on the category of net representations

We turn our attention to the study of the categories of net representations in the case $\mathbf{M}$ is equipped with a collection of weak equivalences $\mathcal{W}$ (see Definition 2.1.4). It is easy to see that, in this case, the functor category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ inherits a collection of weak equivalence as follows.

Definition 3.6.1. Let $\mathbf{C}$ be a small category and $(\mathbf{M}, \mathcal{W})$ a monoidal category with weak equivalences (see Definition 2.1.5). A natural transformation $\Phi \in \mathbf{Fun}(\mathbf{C}, \mathbf{M})$ is called a **weak equivalence** if $\Phi_c \in \mathcal{W}$, for any $c \in \mathbf{C}$.

Since the category $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ of nets of algebras over $\mathbf{M}$ is simply the category of monoids in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$, it inherits weak equivalences from $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ as in Definition 2.8.1. Similarly, given a net of algebras $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$, since the category $\mathbf{Rep}(\mathfrak{A})$ of $\mathfrak{A}$-representations is just the category of modules over $\mathfrak{A}$, it also inherits weak equivalences from $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ as in Definition 2.8.1. In particular, we have the following definitions.

Definition 3.6.2. Let $\mathbf{C}$ be a small category, $(\mathbf{M}, \mathcal{W})$ a monoidal category with weak equivalences and $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ a net of algebras valued in $\mathbf{M}$. A natural transformation Φ in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ (resp. $\mathbf{Rep}(\mathfrak{A})$) is called a **weak equivalence** if $\Phi_c \in \mathcal{W}$, for any $c \in \mathbf{C}$.

These collections of weak equivalence give rise to the homotopy categories $\mathrm{Ho}(\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}})$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ (see Definition 2.1.7).

The first question that arises is if the homotopy category $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ is locally small (see Definition A.1.8(c)). As we discussed in Section 2.1, the category $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ doesn't need to be locally small even if the category $\mathbf{Rep}(\mathfrak{A})$ is.

Second, we are interested in the relation between the homotopy categories $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$ when there is a morphism $\Phi \colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$. Specifically, we would like to know if the induced change-of-net adjunction $(\mathrm{Ext}_\Phi \dashv \mathrm{Res}_\Phi) \colon \mathbf{Rep}(\mathfrak{A}) \to \mathbf{Rep}(\mathfrak{B})$ (see Section 3.4) gives rise to an adjunction between the associated homotopy categories $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$. And if this is the case, is this adjunction an adjoint equivalence (see Definition A.5.7) when Φ is a weak equivalence?

In order to address these questions we will rely on the theory of model categories. In particular, we wil assume that the monoidal category with weak equivalences $(\mathbf{M}, \mathcal{W})$ admits a compatible model structure (see Definition 2.2.5) and taking advantage of transfer techniques for cofibrantly generated model categories we prove that the category $\mathbf{Rep}(\mathfrak{A})$ with the weak equivalences from Definition 3.6.2 admits also a compatible model structure.

So, assuming that $\mathbf{M}$ is a symmetric monoidal model category we can define weak equivalences and fibrations on $\mathbf{Rep}(\mathfrak{A})$ as follows.

Definition 3.6.3 (Model structure on category of $\mathfrak{A}$-representations)**.** Let $\mathbf{C}$ be a small category and $\mathbf{M}$ a symmetric monoidal model category. We define a morphism $\mathcal{F} \colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{A})$ to be a **weak equivalence** (resp. **fibration**) if $\mathcal{F}_c \colon \mathcal{L}_c \to \mathcal{L}'_c$ is a weak equivalence (resp. fibration) in the underlying model category $\mathbf{M}$, for all $c \in \mathbf{C}$, The main goal of this section is to prove that under certain assumptions, the weak equivalences and fibrations in $\mathbf{Rep}(\mathfrak{A})$ from Definition 3.6.3 determine the structure of a model category on $\mathbf{Rep}(\mathfrak{A})$.

We distinguish two cases, depending on if the source category $\mathbf{C}$ admits finite coproducts.

C admits finite coproducts

When the source category $\mathbf{C}$ admits finite coproducts and the target category $\mathbf{M}$ meets the requirements of Setup 2.8.3, the functor category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ can be equipped with the structure of a symmetric monoidal model category satisfying the criteria of Proposition 2.8.5. Then, the categories of net representations inherit a model structure from $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$, as categories of modules over monoids in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$, according to Section 2.8.

So, we adopt the following setup.

Setup 3.6.4. We assume that:

- $\mathbf{C}$ is a small category that admits finite coproducts (see Definitions A.4.13 and A.4.15(b)).
- $(\mathbf{M}, \otimes, \mathbb{1})$ is a symmetric monoidal model category (see Definition 2.6.5) such that:
  - (i) $\mathbf{M}$ is cofibrantly generated by the set of cofibrations $\mathcal{I}$ and the set of acyclic cofibrations $\mathcal{J}$ (see Definition 2.5.8),
  - (ii) $\mathbf{M}$ satisfies the monoid axiom (see Definition 2.8.2),
  - (iii) X is a small object for any $X \in \mathrm{Ob}(\mathbf{M})$ (see Definition 2.5.3(c)).

The functor category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ will be equipped with the projective model structure, whose definition we recall below.

Definition 3.6.5. Let $\mathbf{M}$ be a model category. We define the projective model structure on $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ as follows:

(a) A morphism $\mathcal{F} \colon V \to V'$ in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ is a **weak equivalence** if the morphism $\mathcal{F}_c \colon V_c \to V'_c$ in $\mathbf{M}$ is a weak equivalence with respect to the model structure on $\mathbf{M}$, for all $c \in \mathbf{C}$.

(b) A morphism $\mathcal{F} \colon V \to V'$ in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ is a **fibration** if the morphism $\mathcal{F}_c \colon V_c \to V'_c$ in $\mathbf{M}$ is a fibration with respect to the model structure on $\mathbf{M}$, for all $c \in \mathbf{C}$.

(c) A morphism in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ is a **cofibration** if it has the left lifting property with respect to all the acyclic fibrations in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$.

In order to verify that the functor category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ meets the requirements of Proposition 2.8.5, we need the following well-known result (see for example the arXiv version of the proof of [PS18, Prop. 7.9]).

Proposition 3.6.6. *Let* $\mathbf{M}$ *be a cofibrantly generated symmetric monoidal model category, and let* $\mathbf{C}$ *be a small category that has finite coproducts. Then, the functor category* $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ *equipped with the projective model structure is also a cofibrantly generated symmetric monoidal model category.*

Sketch of proof. It is easy to see that $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ inherits from $\mathbf{M}$ the structures of a closed symmetric monoidal category (see Section 3.1) and of a cofibrantly generated model category (the projective model structure).

It only remains to check that these two structures are compatible in the sense that $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ forms a symmetric monoidal model category. It is straightforward to verify the unit axiom (see Definition 2.6.5(ii)) using the unit axiom in $\mathbf{M}$. Therefore, we only need to check that the pushout product the object-wise monoidal structure and the projective model structure interact nicely, i.e. that the pushout-product axiom holds (see Definition 2.6.5(i)). The latter follows from the assumption that the source category $\mathbf{C}$ admits finite coproducts, The core of this argument is the following.

Since the projective model structure on $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ is cofibrantly generated, it suffices to check the pushout-product axiom on the generating cofibrations (resp. acyclic cofibrations) $\coprod_{\gamma \in \mathbf{C}(c,-)} i$, where $c \in \mathbf{C}$ is any object and $i \in \mathcal{I}$ is any generating cofibration (resp. $i \in \mathcal{J}$ is any generating acyclic cofibration) of $\mathbf{M}$. Since colimits in $\mathbf{M}$ commute with each other and with the tensor product, the existence of finite coproducts in $\mathbf{C}$ entails that the pushout-product of $\coprod_{\gamma_1 \in \mathbf{C}(c_1,-)} i_1$ and $\coprod_{\gamma_2 \in \mathbf{C}(c_2,-)} i_2$ in $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ is given by $\coprod_{\gamma \in \mathbf{C}(c_1 \coprod c_2,-)} i_1 \Box i_2$, where $\Box$ denotes the pushout-product in $\mathbf{M}$. (Finite coproducts in $\mathbf{C}$ are responsible for the natural isomorphisms $\mathbf{C}(c_1,-) \times \mathbf{C}(c_2,-) \cong \mathbf{C}(c_1 \coprod c_2,-)$.) This shows that the pushout-product axiom of $\mathbf{M}$ entails that of $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$. □

Now, it is easy to see that, under the assumptions of Setup 3.6.4, the functor category $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ meets the requirements of Proposition 2.8.5. In particular, since all objects of $\mathbf{M}$ are small, all objects of $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ are small too. And since the object-wise symmetric monoidal structure from Section 3.1 and the projective model structure are suitably compatible (i.e. the pushout-product axiom is satisfied), $\mathbf{Fun}(\mathbf{C}, \mathbf{M})$ inherits the monoid axiom from $\mathbf{M}$.

Moreover, notice that a morphism in $\mathbf{Rep}(\mathfrak{A})$ is a weak equivalence (resp. fibration) if and only if its underlying morphism in $\mathbf{Fun}(\mathbf{C},\mathbf{M})$ is a weak equivalence (resp. fibration) with respect to the projective model structure on $\mathbf{Fun}(\mathbf{C},\mathbf{M})$. In other words, the weak equivalences and fibrations in $\mathbf{Rep}(\mathfrak{A})$ from Definition 3.6.3 coincide with the weak equivalences and fibrations that $\mathbf{Rep}(\mathfrak{A})$ inherits from $\mathbf{Fun}(\mathbf{C},\mathbf{M})$ according to Definition 2.8.4. Therefore, applying Proposition 2.8.5 we obtain the following proposition.

Proposition 3.6.7. *Under the assumptions of Setup 3.6.4, the weak equivalences and fibrations from Definition 3.6.3 determine the structure of a cofibrantly generated model category on* $\mathbf{Rep}(\mathfrak{A})$. *Specifically,* $\mathbf{Rep}(\mathfrak{A})$ *is cofibrantly generated by the set of cofibrations* $\mathrm{Fr}_{\mathfrak{A}}(\mathcal{I})$ *and the set of acyclic cofibrations* $\mathrm{Fr}_{\mathfrak{A}}(\mathcal{J})$*, where* $\mathrm{Fr}_{\mathfrak{A}}$ *stands for the free* $\mathfrak{A}$*-module functor (see Definition 1.4.1).*

Remark 3.6.8. In some cases of interest, the source category $\mathbf{C}$ admits finite coproducts. For example, nets of algebras in the sense of Haag and Kastler (see Remark 3.8.7) are defined on the source category $\mathbf{C} = \mathbf{CCO}(M)$ of causally convex open subsets of a fixed globally hyperbolic Lorentzian manifold M (see Definition 3.8.2). In this case, finite coproducts always exist and are given by the *causally convex hull* $U_1 \coprod U_2 \coloneqq \mathrm{I}^+_M(U_1 \cup U_2) \cap \mathrm{I}^-_M(U_1 \cup U_2)$ of $U_1, U_2 \in \mathbf{CCO}(M)$. (Here $\mathrm{I}^{\pm}_M$ denotes the chronological future/past of a subset in M.)

Unfortunately, finite coproducts do not exist in all the source categories of interest in the context of algebraic quantum field theory. For instance, for applications in the context of locally covariant quantum field theory (see Remark 3.8.7) one often takes the source category $\mathbf{C}$ to be the category $\mathbf{Loc}_m$, $m \geq 2$, of globally hyperbolic Lorentzian manifolds (see Definition 3.8.1). One easily realizes that in this category binary coproducts often fail to exist. △

C doesn't admit all finite coproducts

When finite coproducts fail to exist in the source category $\mathbf{C}$, the argument used in Proposition 3.6.6 to verify the pushout-product axiom in the functor category $\mathbf{Fun}(\mathbf{C},\mathbf{M})$ doesn't apply. That means that we cannot use the results about modules over monoids in symmetric monoidal model categories from Section 2.8. In this case, in order to equip the categories of net representations with appropriate model structures, we will add some extra assumptions on the target category $\mathbf{M}$, and make use of a transfer theorem from [BM03].

In preparation for the transfer theorem, we first recall the definition of a functorial path object.

Definition 3.6.9. Let $\mathbf{E}$ be a category with binary products and equipped with a collection of weak equivalences and a collection of fibrations. A **functorial path object** in $\mathbf{E}$ is a triplet (P, w, f) consisting of:

(1) a functor $P\colon \mathbf{E} \to \mathbf{E}$,

(2) a natural weak equivalence $w\colon \mathrm{Id}_{\mathbf{E}} \to P$ (see Definition 2.1.11),

(3) and a natural fibration $f\colon P \to (-)^{\times 2}$ (i.e. a natural transformation whose components $f_E\colon P(E) \to E \times E$ in $\mathbf{E}$ are fibrations for all $E \in \mathbf{E}$),

such that for each $E \in \mathbf{E}$, the object $P(E)$ is a path object for E with the factorization of the diagonal morphism $\Delta\colon E \to E \times E$ in $\mathbf{E}$ given by the weak equivalence w_E followed by the fibration f_E (see Definition 2.3.5).

The transfer theorem that we will use is the following.

Theorem 3.6.10 ([BM03, Sec. 2.5 and Sec. 2.6])**.** *Assume that:*

- $\mathbf{E}$ *is a complete and cocomplete category.*
- $\mathbf{D}$ *is a cofibrantly generated model category, whose objects are all fibrant (see Definition 2.2.14).*
- $(F \dashv U)\colon \mathbf{E} \to \mathbf{D}$ *is an adjunction.*

Define a morphism $f\colon E \to E'$ in $\mathbf{E}$ to be a weak equivalence (resp. fibration) if $U(f)\colon U(E) \to U(E')$ in $\mathbf{D}$ is a weak equivalence (resp. fibration). Then, these data determine a cofibrantly generated model structure on $\mathbf{E}$ under the following hypotheses:

(i) F preserves small objects (see Definition 2.5.3(c)),

(ii) $\mathbf{E}$ has a functorial path object (P, w, f).

Remark 3.6.11. Even though this is not stated explicitly in Theorem 3.6.10, it is straightforward to realize that, given a set $\mathcal{I}$ (resp. $\mathcal{J}$) of generating cofibrations (resp. acyclic cofibrations) for $\mathbf{D}$, $F(\mathcal{I})$ (resp. $F(\mathcal{J})$) is a set of generating cofibrations (resp. acyclic cofibrations) for $\mathbf{E}$. Indeed, by the adjunction $F \dashv U$, the right lifting property (see Definition 2.2.1) of a morphism $f\colon E \to E'$ in $\mathbf{E}$ against any morphism of $F(\mathcal{I})$ (resp. $F(\mathcal{J})$) is equivalent to the right lifting property of the morphism $U(f)\colon U(E) \to U(E')$ in $\mathbf{D}$ against any morphism of $\mathcal{I}$ (resp. $\mathcal{J}$). Since weak equivalences and fibrations in $\mathbf{E}$ are by definition detected by U, it follows that $F(\mathcal{I})$ (resp. $F(\mathcal{J})$) is a set of generating cofibrations (resp. acyclic cofibrations) for $\mathbf{E}$. △

In order to describe the setup that will be used, we also need to recall the definition of an interval object in a symmetric monoidal model category.

Definition 3.6.12. Let $(\mathbf{M}\otimes, \mathbb{1})$ be a symmetric monoidal model category. An **interval object** in $\mathbf{M}$, is a triplet (I, r, b) consisting of:

(1) a cylinder object $I \in \mathbf{M}$ for the monoidal unit $\mathbb{1}$ (see Definition 2.3.3),

(2) a cofibration $b : \mathbb{1} \amalg \mathbb{1} \to I$ in $\mathcal{M}$,

(3) a weak equivalence $r : I \to \mathbb{1}$ in $\mathcal{M}$,

such that $r \circ b\colon \mathbb{1} \amalg \mathbb{1} \to \mathbb{1}$ is a factorization of the codiagonal morphism $\nabla\colon \mathbb{1} \amalg \mathbb{1} \to \mathbb{1}$ (see Definition 2.3.3).

So, the setup that will be used in order to transfer a model structure from the target category $\mathbf{M}$ to the categories of net representations when the source category $\mathbf{C}$ fails to have finite coproducts, is the following.

Setup 3.6.13. Assume that:

- $\mathbf{C}$ is a small category.
- $(\mathbf{M}, \otimes, \mathbb{1})$ is a symmetric monoidal model category (see Definition 2.6.5) such that:
  (i) $\mathbf{M}$ is cofibrantly generated by the set of cofibrations $\mathcal{I}$ and the set of acyclic cofibrations $\mathcal{J}$ (see Definition 2.5.8),
  (ii) $\mathbf{M}$ satisfies the monoid axiom (see Definition 2.8.2),
  (iii) the monoidal unit $\mathbb{1}$ is cofibrant (see Definition 2.2.14),
  (iv) all objects of $\mathbf{M}$ are fibrant (see Definition 2.2.14),
  (v) $\mathbf{M}$ has an interval object (I, r, b).

For a given net of algebras $\mathfrak{A} \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ and using both Setup 3.6.13 and Theorem 3.6.10, we shall now go through a two-step procedure to transfer a model structure on the category $\mathbf{E} \coloneqq \mathbf{Rep}(\mathfrak{A})$ of $\mathfrak{A}$-representations from the product model category $\mathbf{D} \coloneqq \prod_{c\in\mathbf{C}} \mathbf{M}$. The latter is the model category whose weak equivalences, fibrations and cofibrations are defined componentwise, see [Hov99, Ex. 1.1.6]. Since per hypothesis $\mathbf{M}$ is cofibrantly generated and its objects are all fibrant, the same holds true for the product model category $\prod_{c\in\mathbf{C}} \mathbf{M}$.

Step 1 The first step presents the adjunction that will be used to transfer the model structure from $\prod_{c\in\mathbf{C}}\mathbf{M}$.

Consider the forgetful functor $U\colon \mathbf{Rep}(\mathfrak{A}) \to \prod_{c\in\mathbf{C}}\mathbf{M}$ that sends an $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ to its underlying collection $(\mathcal{L}_c)_c \in \prod_{c\in\mathbf{C}}\mathbf{M}$, where $\mathcal{L}_c \in \mathbf{M}$ denotes here the object of $\mathbf{M}$ underlying the left $\mathfrak{A}_c$-module $\mathcal{L}_c \in {}_{\mathfrak{A}_c}\mathbf{Mod}$, for all $c \in \mathbf{C}$. We emphasize that U preserves and lifts both limits and colimits because those can be computed component-wise in the product category $\prod_{c\in\mathbf{C}}\mathbf{M}$ by endowing the resulting collection with the induced structure of an $\mathfrak{A}$-representation. Furthermore, U is part of the adjunction

$$(F \dashv U)\colon \prod_{c\in\mathbf{C}}\mathbf{M} \rightleftarrows \mathbf{Rep}(\mathfrak{A}), \tag{3.6.1}$$

whose left adjoint F is defined as follows.

Given a collection $\underline{V} := (V_c)_c \in \prod_{c\in\mathbf{C}} M$, F assigns to it the $\mathfrak{A}$-representation $F(\underline{V}) \in \mathbf{Rep}(\mathfrak{A})$ consisting of the following data (see Definition 3.2.1).

(1) For each $c \in \mathbf{C}$, the free left $\mathfrak{A}_c$-module (see Definition 1.4.1)

$$(F(\underline{V}))_c := \mathrm{Fr}_{\mathfrak{A}_c}\left(\coprod_{\substack{\widetilde{c}\in\mathbf{C}\\ \widetilde{\gamma}\in\mathbf{C}(\widetilde{c},c)}} V_{\widetilde{c}}\right) = \mathfrak{A}_c \otimes \coprod_{\substack{\widetilde{c}\in\mathbf{C}\\ \widetilde{\gamma}\in\mathbf{C}(\widetilde{c},c)}} V_{\widetilde{c}} \in {}_{\mathfrak{A}_c}\mathbf{Mod}. \tag{3.6.2a}$$

(2) For each morphism $\gamma\colon c_1 \to c_2$ in $\mathbf{C}$, the morphism of left $\mathfrak{A}_{c_2}$-modules

$$(F(\underline{V}))(\gamma)\colon \mathrm{Ext}_\gamma(F(\underline{V}))_{c_1} \cong \mathfrak{A}_{c_2} \otimes \coprod_{\substack{\widetilde{c}_1\in\mathbf{C}\\ \widetilde{\gamma}\in\mathbf{C}(\widetilde{c}_1,c_1)}} V_{\widetilde{c}_1} \xrightarrow{\mathrm{id}_{\mathfrak{A}_{c_2}}\otimes\gamma_*} (F(\underline{V}))_{c_2} \tag{3.6.2b}$$

in ${}_{\mathfrak{A}_{c_2}}\mathbf{Mod}$, where $\gamma_*\colon \coprod_{\widetilde{c}_1,\widetilde{\gamma}_1} V_{\widetilde{c}_1} \to \coprod_{\widetilde{c}_2,\widetilde{\gamma}_2} V_{\widetilde{c}_2}$ in $\mathbf{M}$ denotes the morphism that sends the $(\widetilde{c}_1,\widetilde{\gamma}_1)$-component of the domain to the $(\widetilde{c}_1,\gamma\widetilde{\gamma}_1)$-component of the codomain. The depicted isomorphism comes from the isomorphism

$$\mathrm{Ext}_\gamma\mathfrak{A}_{c_1} = \mathfrak{A}_{c_2} \otimes_{\mathfrak{A}_{c_1}} \mathfrak{A}_{c_1} \cong \mathfrak{A}_{c_2}, \tag{3.6.2c}$$

where $\mathfrak{A}_{c_2} \otimes_{\mathfrak{A}_{c_1}} \mathfrak{A}_{c_1}$ denotes the relative tensor product, which coincides with $\mathrm{Ext}_\gamma\mathfrak{A}_{c_1}$ (see Definition 1.5.6 and Remark 1.5.8).

One easily checks that these data fulfil the axioms for an $\mathfrak{A}$-representation.

Furthermore, F sends a morphism $\underline{\xi} \coloneqq (\xi_c)_c \colon \underline{V} \to \underline{V}'$ in $\prod_{c\in\mathbf{C}} M$ to the morphism of $\mathfrak{A}$-representations $F(\underline{\xi})\colon F(\underline{V}) \to F(\underline{V}')$ in $\mathbf{Rep}(\mathfrak{A})$ consisting of the morphism of left $\mathfrak{A}_c$-modules

$$(F(\underline{\xi}))_c \coloneqq \mathrm{id}_{\mathfrak{A}_c} \otimes \coprod_{\substack{\widetilde{c}\in\mathbf{C}\\ \widetilde{\gamma}\in\mathbf{C}(\widetilde{c},c)}} \xi_{\widetilde{c}} \colon (F(\underline{V}))_c \longrightarrow (F(\underline{V}'))_c \tag{3.6.2d}$$

in ${}_{\mathfrak{A}_c}\mathbf{Mod}$, for each $c \in \mathbf{C}$. It is easy to check that these data indeed define a morphism of a $\mathfrak{A}$-representations and that F as defined above is a functor.

In order to show that (3.6.1) is an adjunction as claimed, we describe its unit η and its counit ϵ.

The unit is the natural transformation

$$\eta\colon \mathrm{Id}_{\prod_{c\in\mathbf{C}}\mathbf{M}} \longrightarrow UF, \tag{3.6.3a}$$

whose component $\eta_{\underline{V}}\colon \underline{V} \to UF(\underline{V})$ in $\prod_{c\in\mathbf{C}}\mathbf{M}$ at the collection $\underline{V} \in \prod_{c\in\mathbf{C}}\mathbf{M}$ consists of the morphisms in $\mathbf{M}$ defined by

$$(\eta_{\underline{V}})_c\colon V_c \cong \mathbb{1} \otimes V_c \xrightarrow{\mathbf{1}\otimes\iota_{c,\mathrm{id}_c}} (UF(\underline{V}))_c, \tag{3.6.3b}$$

for all $c \in \mathbf{C}$, where $\iota_{\widetilde{c},\widetilde{\gamma}}\colon V_{\widetilde{c}} \to \coprod_{\widetilde{c}',\widetilde{\gamma}'} V_{\widetilde{c}'}$ in $\mathbf{M}$ denotes the canonical morphism to the coproduct.

The counit is the natural transformation

$$\epsilon\colon FU \longrightarrow \mathrm{Id}_{\mathbf{Rep}(\mathfrak{A})}, \tag{3.6.4a}$$

whose component $\epsilon_{\mathcal{L}}\colon FU(\mathcal{L}) \to \mathcal{L}$ in $\mathbf{Rep}(\mathfrak{A})$ at the $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ consists, for each $c \in \mathbf{C}$, of the morphism of left $\mathfrak{A}_c$-modules $(\epsilon_{\mathcal{L}})_c\colon (FU(\mathcal{L}))_c \to \mathcal{L}_c$ in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ defined via the universal property of the coproduct by the diagram

$$\begin{array}{ccc} (FU(\mathcal{L}))_c & \xrightarrow{(\epsilon_{\mathcal{L}})_c} & \mathcal{L}_c \\ {\scriptstyle \mathrm{id}\otimes\iota_{\widetilde{c},\widetilde{\gamma}}}\uparrow & & \uparrow{\scriptstyle \mathcal{L}_{\widetilde{\gamma}}} \\ \mathfrak{A}_c \otimes \mathcal{L}_{\widetilde{c}} & \longrightarrow & \mathrm{Ext}_{\mathfrak{A}(\widetilde{\gamma})}\,\mathcal{L}_{\widetilde{c}} \end{array} \tag{3.6.4b}$$

in ${}_{\mathfrak{A}_c}\mathbf{Mod}$, for all $\widetilde{c} \in \mathbf{C}$ and $\widetilde{\gamma} \in \mathbf{C}(\widetilde{c},c)$. The bottom horizontal arrow denotes the canonical morphism to the coequalizer $\mathfrak{A}_c \otimes_{\mathfrak{A}_{\widetilde{c}}} \mathcal{L}_{\widetilde{c}} \in {}_{\mathfrak{A}_c}\mathbf{Mod}$ (see Definition 1.5.6), which coincides with $\mathrm{Ext}_{\mathfrak{A}(\widetilde{\gamma})}\,\mathcal{L}_{\widetilde{c}}$ (see Remark 1.5.8). The verification of the triangle identities $U(\epsilon)\,\eta_U = \mathrm{id}_U$ and $\epsilon_F\,F(\eta) = \mathrm{id}_F$ is straightforward, thus proving that (3.6.1) is an adjunction, as claimed.

Remark 3.6.14. Notice that a morphism $\mathcal{F}\colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{A})$ is a weak equivalence (resp. fibration) in $\mathbf{Rep}(\mathfrak{A})$ (see Definition 3.6.3) if and only if $U(\mathcal{F})\colon U(\mathcal{L}) \to U(\mathcal{L}')$ is a weak equivalence (resp. fibration) in $\prod_{c\in\mathbf{C}} \mathbf{M}$. Therefore, weak equivalences and fibrations on $\mathbf{Rep}(\mathfrak{A})$ are defined in accordance with Theorem 3.6.10. △

Remark 3.6.15. We observe incidentally that the right adjoint functor U preserves the $\mathbf{M}$-powerings on $\mathbf{Rep}(\mathfrak{A})$ and on $\prod_{c\in\mathbf{C}} \mathbf{M}$ respectively. The latter is obtained by acting component-wise with the internal hom of $\mathbf{M}$. It follows then, by the definition of the $\mathbf{M}$-powering functor on $\mathbf{Rep}(\mathfrak{A})$, see (3.5.3), that for each $V \in \mathbf{M}$, the diagram of functors

$$\begin{array}{ccc} \mathbf{Rep}(\mathfrak{A}) & \xrightarrow{U} & \prod_{c\in\mathbf{C}} \mathbf{M} \\ \downarrow{\scriptstyle (-)^V} & & \downarrow{\scriptstyle (-)^V} \\ \mathbf{Rep}(\mathfrak{A}) & \xrightarrow[U]{} & \prod_{c\in\mathbf{C}} \mathbf{M} \end{array} \tag{3.6.5}$$

commutes. As a consequence, the diagram of left adjoint functors commutes up to a unique natural isomorphism, i.e. there exists a unique natural isomorphism $F(-) \otimes V \cong F(- \otimes V)$ between functors from $\prod_{c\in\mathbf{C}} \mathbf{M}$ to $\mathbf{Rep}(\mathfrak{A})$, where the left hand side displays the $\mathbf{M}$-tensoring on $\mathbf{Rep}(\mathfrak{A})$ from (3.5.2) and the right hand side displays the $\mathbf{M}$-tensoring on $\prod_{c\in\mathbf{C}} \mathbf{M}$ defined component-wise in $\mathbf{C}$ by the tensor product of $\mathbf{M}$. △

Remark 3.6.16. Combining the model structure on $\mathbf{Rep}(\mathfrak{A})$ with the model structure on categories of left modules from Proposition 2.8.5, one easily recognizes that, for each $\widetilde{c} \in \mathbf{C}$, the adjunction $(-)_{\widetilde{c}} \dashv (-)^{\widetilde{c}}\colon {}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod} \to \mathbf{Rep}(\mathfrak{A})$ from (3.3.1) is a Quillen adjunction (see Definition 2.4.1). This follows from the fact that the right adjoint functor $(-)^{\widetilde{c}}$ sends (acyclic) fibrations in ${}_{\mathfrak{A}_{\widetilde{c}}}\mathbf{Mod}$, which are detected in $\mathbf{M}$, to (acyclic) fibrations in $\mathbf{Rep}(\mathfrak{A})$, which are detected in $\mathbf{M}$ component-wise in $\mathbf{C}$. Note that the Quillen adjunction (3.3.1) is compatible with the $\mathbf{M}$-tensoring, powering and enriched hom due to Remark 3.5.2. In other words, the left adjoint $(-)_{\widetilde{c}}$ is an $\mathbf{M}$-Quillen functor (see Definition 2.7.7). △

Note that all objects of $\mathbf{Rep}(\mathfrak{A})$ are fibrant because, as previously explained, all objects of $\prod_{c\in\mathbf{C}} \mathbf{M}$ are fibrant.

Step 2 The second step checks that the hypotheses of Theorem 3.6.10 are met. Hypothesis (i), i.e. that F preserves small objects, follows by recalling that, as explained in Step 1, U preserves colimits and F is left adjoint to U. We show that also hypothesis (ii) of Theorem 3.6.10 is met, i.e. we construct a functorial path object (P, w, f) in $\mathbf{Rep}(\mathfrak{A})$. This is achieved by a standard construction that uses the interval object (I, r, b) in $\mathbf{M}$, see Setup 3.6.13, the symmetric monoidal model structure on $\mathbf{M}$, the hypothesis that all objects of $\mathbf{M}$ are fibrant and that the unit $\mathbb{1}$ of $\mathbf{M}$ is cofibrant, and the $\mathbf{M}$-powering on $\mathbf{Rep}(\mathfrak{A})$ from (3.5.3). Explicitly, we consider the functor

$$P \coloneqq (-)^I \colon \mathbf{Rep}(\mathfrak{A}) \longrightarrow \mathbf{Rep}(\mathfrak{A}) \tag{3.6.6a}$$

and the natural transformations

$$w\colon \mathrm{Id}_{\mathbf{Rep}(\mathfrak{A})} \cong (-)^{\mathbb{1}} \xrightarrow{(-)^r} P, \qquad f\colon P \xrightarrow{(-)^b} (-)^{\mathbb{1}\amalg\mathbb{1}} \cong (-)^{\times 2}. \tag{3.6.6b}$$

By definition of the $\mathbf{M}$-powering (3.5.3) for each $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$, and up to the evident isomorphisms, $U(w_{\mathcal{L}})\colon U(\mathcal{L}) \to U(\mathcal{L}^I)$ consists of the components $[r, \mathcal{L}_c]\colon [\mathbb{1}, \mathcal{L}_c] \to [I, \mathcal{L}_c]$ in $\mathbf{M}$ for all $c \in \mathbf{C}$, and $U(f_{\mathcal{L}})\colon U(\mathcal{L}^I) \to U(\mathcal{L} \times \mathcal{L})$ consists of the components $[b, \mathcal{L}_c]\colon [I, \mathcal{L}_c] \to [\mathbb{1} \coprod \mathbb{1}, \mathcal{L}_c]$ in $\mathbf{M}$, for all $c \in \mathbf{C}$. (Here $[-,-]\colon \mathbf{M}^{\mathrm{op}} \times \mathbf{M} \to \mathbf{M}$ denotes the internal hom of the symmetric monoidal model category $\mathbf{M}$.) Since by Setup 3.6.13 all objects $V \in \mathbf{M}$ are fibrant, $[-, V]\colon \mathbf{M}^{\mathrm{op}} \to \mathbf{M}$ sends weak equivalences between cofibrant objects in $\mathbf{M}$ to weak equivalences in $\mathbf{M}$ and cofibrations in $\mathbf{M}$ to fibrations in $\mathbf{M}$, see [Hov99, Rem. 4.2.3]. Since $r\colon I \to \mathbb{1}$ in $\mathbf{M}$ is a weak equivalence between cofibrant objects and $b\colon \mathbb{1} \amalg \mathbb{1} \to I$ in $\mathbf{M}$ is a cofibration, recalling Definition 3.6.3 we conclude that $w_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{L}^I$ in $\mathbf{Rep}(\mathfrak{A})$ is a weak equivalence and $f_{\mathcal{L}}\colon \mathcal{L}^I \to \mathcal{L} \times \mathcal{L}$ in $\mathbf{Rep}(\mathfrak{A})$ is a fibration. Furthermore, since by Setup 3.6.13 the codiagonal morphism factors as $\langle \mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}} \rangle = r\, b\colon \mathbb{1} \amalg \mathbb{1} \to \mathbb{1}$ in $\mathbf{M}$, it follows that the diagonal morphism factors as $(\mathrm{id}_{\mathcal{L}}, \mathrm{id}_{\mathcal{L}}) = f_{\mathcal{L}}\, w_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{L} \times \mathcal{L}$ in $\mathbf{Rep}(\mathfrak{A})$, for all $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$. This shows that the functor P and the natural transformations w and f introduced in (3.6.6) define a functorial path object (P, w, f) in $\mathbf{Rep}(\mathfrak{A})$, hence also hypothesis (ii) of Theorem 3.6.10 is met. The next proposition summarizes the conclusions so far, taking into account also Remark 3.6.11.

Proposition 3.6.17. *Under the assumptions stated in Setup 3.6.13, the notions of weak equivalences and fibrations from Definition 3.6.3 determine a cofibrantly generated model structure on the category* $\mathbf{Rep}(\mathfrak{A})$ *of* $\mathfrak{A}$*-representations.*

Moreover, given a set of generating cofibrations $\mathcal{I}$ *(resp. acyclic cofibrations* $\mathcal{J}$*) for* $\mathbf{M}$*, it holds that* $F(\coprod_{c \in \mathbf{C}} \mathcal{I})$ *(resp.* $F(\coprod_{c \in \mathbf{C}} \mathcal{J})$*) is a set of generating cofibrations (resp. acyclic cofibrations) for* $\mathbf{Rep}(\mathfrak{A})$*.*

3.6.1 Categories of representations as M-model categories

Having established a model structure on the category $\mathbf{Rep}(\mathfrak{A})$, we now investigate its compatibility with the $\mathbf{M}$-tensoring, powering and enriched hom on $\mathbf{Rep}(\mathfrak{A})$ from Section 3.5. In other words, we check whether $\mathbf{Rep}(\mathfrak{A})$ is an $\mathbf{M}$-model category (see Definition 2.7.5).

Proposition 3.6.18 ($\mathbf{Rep}(\mathfrak{A})$ as an $\mathbf{M}$-model category)**.** *Under the assumptions stated in Setup 3.6.13, the model structure from Proposition 3.6.17 and the* $\mathbf{M}$*-tensoring from Section 3.5.1 endow the category* $\mathbf{Rep}(\mathfrak{A})$ *of* $\mathfrak{A}$*-representations with an* $\mathbf{M}$*-model structure.*

Proof. As we already pointed out in Remark 3.5.1, the category $\mathbf{Rep}(\mathfrak{A})$ is an $\mathbf{M}$-module category. Moreover, the monoidal unit $\mathbb{1}$ is cofibrant by assumption. Hence, in view of Remark 2.7.6, it only remains to show that the $\mathbf{M}$-tensoring (3.5.2) is a Quillen bifunctor (see Definition 2.6.2).

Since, by Proposition 3.6.17, the category $\mathbf{Rep}(\mathfrak{A})$ is a cofibrantly generated model category, on account of Proposition 2.6.3, it suffices to prove that the pushout-product morphism $F(\underline{\eta}) \,\square\, \xi$ is a cofibration (acyclic cofibration) when $F(\underline{\eta}) \in F(\prod_{c \in \mathbf{C}} \mathcal{I})$ and $\xi \in \mathcal{I}$ are both generating cofibrations (resp. $F(\underline{\eta}) \in F(\prod_{c \in \mathbf{C}} \mathcal{J})$ is a generating acyclic cofibration and $\xi \in \mathcal{I}$ is a generating cofibration, or $F(\underline{\eta}) \in F(\prod_{c \in \mathbf{C}} \mathcal{I})$ is a generating cofibration and $\xi \in \mathcal{J}$ is a generating acyclic cofibration).

Recalling that F preserves the $\mathbf{M}$-tensorings on $\prod_{c \in \mathbf{C}} \mathbf{M}$ and on $\mathbf{Rep}(\mathfrak{A})$, (see Remark 3.6.15), and mimicking the construction of the pushout-product morphism in ${}_A\mathbf{Mod}$ from Proposition 2.8.6, one finds that the pushout-product morphism $F(\underline{\eta}) \,\square\, \xi$ constructed in $\mathbf{Rep}(\mathfrak{A})$ coincides (up to the evident isomorphisms) with the image $F(\underline{\eta} \,\square\, \xi)$ under F of the pushout-product morphism $\underline{\eta} \,\square\, \xi$ constructed in $\prod_{c \in \mathbf{C}} \mathbf{M}$. (Here we used also that F preserves colimits because it is a left-adjoint functor.)

Since $\mathbf{M}$ is a symmetric monoidal model category and the pushout-product in $\prod_{c \in \mathbf{C}} \mathbf{M}$ is computed component-wise, $\underline{\eta} \,\square\, \xi$ in $\prod_{c \in \mathbf{C}} \mathbf{M}$ is a cofibration (resp. acyclic cofibration). Taking into account that weak equivalences and fibrations in $\mathbf{Rep}(\mathfrak{A})$ are detected by U (see Remark 3.6.14), whose left adjoint is F, it follows that $F(\underline{\eta} \,\square\, \xi)$ in $\mathbf{Rep}(\mathfrak{A})$ has the left lifting property against all acyclic fibrations (resp. fibrations) in $\mathbf{Rep}(\mathfrak{A})$. This means that $F(\underline{\eta} \,\square\, \xi)$, and hence also $F(\underline{\eta}) \,\square\, \xi$, in $\mathbf{Rep}(\mathfrak{A})$ is a cofibration (resp. acyclic cofibration), which finishes the proof. $\square$

3.6.2 Compatibility with the change-of-monoid adjunction

We complete our analysis of the model structure on the categories of net representations investigating how these model structures interact with the change-of-net adjunction $\mathrm{Ext}_\Phi \dashv \mathrm{Res}_\Phi \colon \mathbf{Rep}(\mathfrak{B}) \to \mathbf{Rep}(\mathfrak{A})$ from (3.4.1) associated with a morphism of nets $\Phi \colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$.

In fact, when the source category $\mathbf{C}$ admits finite coproducts and the target category $\mathbf{M}$ satisfies the requirements in Setup 3.6.4, one can apply directly Proposition 2.8.5 to prove that the change-of-net adjunction is a Quillen adjunction (see Definition 2.4.1) with respect to the model structures on $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\mathfrak{B})$ from Proposition 3.6.7.

We obtain a similar result also when $\mathbf{C}$ fails to have finite coproducts.

Proposition 3.6.19 (Change-of-net as a Quillen adjunction)**.** *Assume Setup 3.6.13, and let $\Phi \colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ be a morphism of nets. Then, the change-of-net adjunction* $(\mathrm{Ext}_\Phi \dashv \mathrm{Res}_\Phi) \colon \mathbf{Rep}(\mathfrak{A}) \to \mathbf{Rep}(\mathfrak{B})$ *from* (3.4.1) *is a Quillen adjunction with respect to the model structures on* $\mathbf{Rep}(\mathfrak{A})$ *and* $\mathbf{Rep}(\mathfrak{B})$ *from Proposition 3.6.17.*

Moreover, the extension functor $\mathrm{Ext}_\Phi \colon \mathbf{Rep}(\mathfrak{A}) \to \mathbf{Rep}(\mathfrak{B})$ *is an* $\mathbf{M}$*-Quillen functor (see Definition 2.7.7).*

Proof. To prove that the change-of-net adjunction is a Quillen adjunction we just need to verify that Res_Φ preserves weak equivalences and fibrations.

Since weak equivalences and fibrations both in $\mathbf{Rep}(\mathfrak{A})$ and in $\mathbf{Rep}(\mathfrak{B})$ are defined component-wise (see Definition 3.6.3), and for each $c \in \mathbf{C}$, the c component of the restriction functor $(\mathrm{Res}_\Phi)_c = \mathrm{Res}_{\Phi_c} \colon {}_{\mathfrak{B}_c}\mathbf{Mod} \to {}_{\mathfrak{A}_c}\mathbf{Mod}$ is a right Quillen functor (see Proposition 2.8.7), it follows immediately that Res_Φ preserves both weak equivalences and fibrations, which entails that the change-of-net adjunction (3.4.1) is a Quillen adjunction.

Taking also into account that the extension functor Ext_Φ is an M-module functor (see Remark 3.5.4), it follows that Ext_Φ is an $\mathbf{M}$-Quillen functor. □

In fact, under one extra assumption, stated below, the change-of-net adjunction induced by a weak equivalence of nets is a Quillen equivalence. Specifically, we obtain the following proposition.

Proposition 3.6.20. *Let $\mathbf{C}$ and $\mathbf{M}$ be as in Setup 3.6.13, with the extra assumption on $\mathbf{M}$ that for each monoid $A \in \mathbf{Mon}(\mathbf{M})$ and each cofibrant left A-module $L \in {}_A\mathbf{Mod}$ with respect to the model structure on ${}_A\mathbf{Mod}$ inherited from $\mathbf{M}$ (see Proposition 2.8.5), the relative tensor product $(-) \otimes_A L \colon \mathbf{Mod}_A \to \mathbf{M}$ (see Definition 1.3.6) sends weak equivalences in $\mathbf{Mod}_A$ (see Definition 2.8.1) to weak equivalences in $\mathbf{M}$.*

Then, for any weak equivalence of nets $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ *in* $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ *(see Definition 3.6.2), the induced change-of-net adjunction* $\mathrm{Ext}_\Phi \dashv \mathrm{Res}_\Phi$ *is a Quillen equivalence with respect to the model structures on* $\mathbf{Rep}(\mathfrak{A})$ *and* $\mathbf{Rep}(\mathfrak{B})$ *from Proposition 3.6.17.*

Proof. By definition of the model structures on $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\mathfrak{B})$, the restriction functor Res_Φ detects weak equivalences and, as a consequence of Setup 3.6.13, all net representations are fibrant. Therefore, by [Hov99, Cor. 1.3.16], in order to conclude that the change-of-net adjunction $\mathrm{Ext}_\Phi \dashv \mathrm{Res}_\Phi$ is a Quillen equivalence, it suffices to show that the components of its unit $\mathcal{L} \to \mathrm{Res}_\Phi\, \mathrm{Ext}_\Phi \mathcal{L}$ in $\mathbf{Rep}(\mathfrak{A})$ are weak equivalences for all cofibrant $\mathfrak{A}$-representations $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$.

For this purpose we need to show that, for each $c \in \mathbf{C}$, the morphism $\mathcal{L}_c \to (\mathrm{Res}_\Phi \mathrm{Ext}_\Phi \mathcal{L})_c$ in ${}_{\mathfrak{A}_c}\mathbf{Mod}$ is a weak equivalence. (Refer to Section 2.8 for the model structure on ${}_{\mathfrak{A}_c}\mathbf{Mod}$.)

Recalling the construction of the change-of-net adjunction (3.4.1), the latter morphism is just the unit of the change-of-monoid adjunction $\mathrm{Ext}_{\Phi_c} \dashv \mathrm{Res}_{\Phi_c}$ associated with $\Phi_c\colon \mathfrak{A}_c \to \mathfrak{B}_c$ in $\mathbf{Mon}(\mathbf{M})$ (see Proposition 1.5.9). Since $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ is cofibrant by hypothesis and $(-)_c \dashv (-)^c\colon {}_{\mathfrak{A}_c}\mathbf{Mod} \to \mathbf{Rep}(\mathfrak{A})$ is a Quillen adjunction (see Remark 3.6.16), $\mathcal{L}_c \in {}_{\mathfrak{A}_c}\mathbf{Mod}$ is cofibrant too. Therefore, we would be able to finish the proof if the change-of-monoid adjunction associated with Φ_c were a Quillen equivalence, for all $c \in \mathbf{C}$. This result follows from Proposition 2.8.8, due to the assumption that, for each monoid $A \in \mathbf{Mon}(\mathbf{M})$ and each cofibrant left A-module L, the functor $(-) \otimes_A L\colon \mathbf{Mod}_A \to \mathbf{M}$ sends weak equivalences in $\mathbf{Mod}_A$ to weak equivalences in $\mathbf{M}$. □

Remark 3.6.21. We emphasize that the additional assumption on $\mathbf{M}$ in Proposition 3.6.20 holds true in many examples, see [SS00, Secs. 4 and 5], including the symmetric monoidal model category of cochain complexes $\mathbf{CoCh}_{\mathbb{K}}$ over a field $\mathbb{K}$. △

3.7 The homotopy category of net representations

As explained in Section 2.3, given a model category $\mathbf{C}$, there exists the associated homotopy category $\mathrm{Ho}(\mathbf{C})$ (see Definition 2.3.1), which is a model for the localization of $\mathbf{C}$ (see Definition 2.1.1) with respect to the weak equivalences $\mathcal{W}_{\mathbf{C}}$ of the model structure of $\mathbf{C}$.

In this section we explore the properties of the homotopy categories associated with the model categories of representations from Proposition 3.6.17, taking advantage of basic results from the theory of model categories, mentioned in Chapter 2.

So, throughout this section we assume that the source category $\mathbf{C}$ and the target category $\mathbf{M}$ meet the assumptions of Setup 3.6.13, and that the categories of net representations are model categories according to Proposition 3.6.17.

Just from the fact that the category $\mathbf{Rep}(\mathfrak{A})$ is a model category, one obtains the following two corollaries.

First, applying Remark 2.3.13 on a model category of net representations, we obtain the following.

Corollary 3.7.1. *Let $A \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ be a net of algebras. The category $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ is a locally small category.*

Second, from Remark 2.3.16, we deduce the following.

Corollary 3.7.2. *Let $A \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ be a net of algebras. Two net representations $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ are isomorphic in $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ if and only if they are weakly equivalent in $\mathbf{Rep}(\mathfrak{A})$.*

Moreover, since by Proposition 3.6.18 the categories of net representations have the structure of an $\mathbf{M}$-model category, Proposition 2.7.8 yields the following corollary.

Corollary 3.7.3. *Let $A \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ be a net of algebras. The category $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ has the structure of a $\mathrm{Ho}(\mathbf{M})$-module category (see Definition 2.7.1), where $\mathrm{Ho}(\mathbf{M})$ denotes the homotopy category associated with the model category $\mathbf{M}$.*

Now, we turn our attention to the relations between the homotopy categories associated with categories of net representations.

Corollary 3.7.4. *Let $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ be a morphism in $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$. There exists an adjunction*

$$\big(\mathrm{L}(\mathrm{Ext}_\Phi) \dashv \mathrm{R}(\mathrm{Res}_\Phi)\big) \colon \mathrm{Ho}(\mathbf{Rep}(\mathfrak{A})) \to \mathrm{Ho}(\mathbf{Rep}(\mathfrak{B})), \tag{3.7.1}$$

where $\mathrm{L}(\mathrm{Ext}_\Phi)$ is the left derived functor of Ext_Φ described in Remark 2.4.3 and $\mathrm{R}(\mathrm{Res}_\Phi)$ is the right derived functor of Res_Φ also from Remark 2.4.3.

Proof. By Proposition 3.6.19, the change-of-net adjunction induced by Φ is a Quillen adjunction. Hence, Proposition 2.4.4 yields the derived adjunction (3.7.1). $\square$

Finally, by direct application of Propositions 2.4.6 and 3.6.20, we obtain the following corollary.

Corollary 3.7.5. *Assume that for each monoid* $A \in \mathbf{Mon}(\mathbf{M})$ *and each cofibrant left* A*-module* $L \in {}_A\mathbf{Mod}$ *with respect to the model structure on* ${}_A\mathbf{Mod}$ *inherited from* $\mathbf{M}$ *(see Proposition 2.8.5), the relative tensor product* $(-) \otimes_A L \colon \mathbf{Mod}_A \to \mathbf{M}$ *(see Definition 1.3.6) sends weak equivalences in* $\mathbf{Mod}_A$ *(see Definition 2.8.1) to weak equivalences in* $\mathbf{M}$*.*

Then, for any weak equivalence $\Phi \colon \mathfrak{A} \to \mathfrak{B}$ *in* $\mathbf{Net}_{\mathbf{C}}^{\mathbf{M}}$ *(see Definition 3.6.2), the derived change-of-net adjunction* $\mathrm{L}(\mathrm{Ext}_\Phi) \dashv \mathrm{R}(\mathrm{Res}_\Phi)$ *from Corollary 3.7.4 is an adjoint equivalence between the categories* $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ *and* $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$*.*

3.8 Representations in homotopy algebraic quantum field theory

In the context of homotopy algebraic quantum field theory we take $\mathbf{M}$ to be the category closed symmetric monoidal category $\mathbf{CoCh}_{\mathbb{C}}$ of cochain complexes over the field of complex numbers equipped with the standard monoidal structure (see Section 2.9), whereas we take $\mathbf{C}$ to be some category of spacetimes. The relevant categories of spacetimes are the category $\mathbf{Loc}$ of globally hyperbolic Lorentzian manifolds and the category $\mathbf{CCO}(M)$ of causally convex opens subsets of some $M \in \mathbf{Loc}$. We recall the definitions of these categories below.

Definition 3.8.1. We define the category $\mathbf{Loc}_m$, $m \geq 2$, to be the category with objects the oriented and time-oriented m-dimensional globally hyperbolic Lorentzian manifolds and morphisms the orientation and time-orientation preserving isometric open embeddings whose image is causally convex.

Definition 3.8.2. Let M be an oriented and time-oriented globally hyperbolic Lorentzian manifold. We define the category $\mathbf{CCO}(M)$ to be the category with objects the causally convex open subsets of M and morphisms the subset inclusions.

For the remainder of this section, $\mathbf{C}$ will denote either $\mathbf{Loc}_m$, $m \geq 2$ or $\mathbf{CCO}(M)$ for some fixed $M \in \mathbf{Loc}_m$, $m \geq 2$.

For the definition of a homotopical algebraic quantum field theory we need the notion of a commutator in a differential graded algebra, whose definition we recall below.

Definition 3.8.3. We define:

(a) A **differential graded algebra over** $\mathbb{C}$ to be a monoid in the monoidal category of cochain complexes $\mathbf{CoCh}_{\mathbb{C}}$ over $\mathbb{C}$.

(b) The **category of differential graded algebras over** $\mathbb{C}$, denoted by $\mathbf{DGA}_{\mathbb{C}}$, to be the category $\mathbf{Mon}(\mathbf{CoCh}_{\mathbb{C}})$ of monoids in the monoidal category $\mathbf{CoCh}_{\mathbb{C}}$.

Definition 3.8.4. Let $(A, \mu, \mathbf{1}) \in \mathbf{DGA}_{\mathbb{C}}$. We define the **commutator in** A, to be the morphism $[-,-]_A : A \otimes A \to A$ in $\mathbf{CoCh}_{\mathbb{C}}$ given by $[-,-]_A \coloneqq \mu - \mu\, \gamma_{A,A}$, where γ denotes the braiding of $\mathbf{CoCh}_{\mathbb{C}}$.

Following [BBS20] we now define homotopical algebraic quantum field theories as follows.

Definition 3.8.5. We define a **homotopical algebraic quantum field theory** (homotopical AQFT) $\mathfrak{A}$ to be a $\mathbf{CoCh}_{\mathbb{C}}$-valued net of algebras $(\mathfrak{A}, \mu, \mathbf{1}) \in \mathbf{Net}_{\mathbf{C}}^{\mathbf{CoCh}_{\mathbb{C}}}$ over $\mathbf{C}$, subject to the following axioms.

(i) **Einstein causality:** Given a pair of morphisms $f_1 \colon M_1 \to N$ and $f_2 \colon M_2 \to N$ in $\mathbf{C}$ with causally disjoint images, the cochain map

$$[f_1, f_2]_{\mathfrak{A}_N} : \mathfrak{A}_{M_1} \otimes \mathfrak{A}_{M_2} \to \mathfrak{A}_N \tag{3.8.1}$$

in $\mathbf{CoCh}_{\mathbb{C}}$ vanishes, where $[-,-]_{\mathfrak{A}_N}$ denotes the commutator in $\mathfrak{A}_N$ (see Definition 3.8.4). Here, $\mathfrak{A}_N$ is seen as a differential graded algebra over $\mathbb{C}$, i.e. a monoid in $\mathbf{CoCh}_{\mathbb{C}}$, using the isomorphism described in Remark 3.1.3.

(ii) **Time-slice axiom:** For any Cauchy morphism $f \colon M_1 \to M_2$ in $\mathbf{C}$, the cochain map $\mathfrak{A}(f) \colon \mathfrak{A}_{M_1} \to \mathfrak{A}_{M_2}$ in $\mathbf{CoCh}_{\mathbb{C}}$ is a quasi-isomorphism.

Remark 3.8.6. Notice that, in [BBS20], homotopical algebraic quantum field theories are defined in the category $\mathbf{Fun}(\mathbf{C}, \mathbf{DGA}_{\mathbb{C}})$ instead of $\mathbf{Net}_{\mathbf{C}}^{\mathbf{CoCh}_{\mathbb{C}}}$, but in light of Remark 3.1.3, the two categories are isomorphic. △

Remark 3.8.7. Quantum field theories defined over the category $\mathbf{Loc}_m$, $m \geq 2$, as in the definition above, are known as locally covariant quantum field theories. On the other hand, quantum field theories in the sense of Haag–Kastler are defined over the category $\mathbf{CCO}(M)$ instead, for some $M \in \mathbf{Loc}_m$. △

Definition 3.8.8. We define the **category of homotopical AQFTs** over $\mathbf{C}$, denoted by $\mathbf{hAQFT}_{\mathbf{C}}$, to be the full subcategory (see Example A.2.8) of $\mathbf{Net}_{\mathbf{C}}^{\mathbf{CoCh}_{\mathbb{C}}}$ with objects the homotopical AQFTs.

Given that representations of algebraic quantum field theories are usually defined as net representations according to the description in Section 3.2.1 (see [RV12a; RV12b]), it makes sense to define a representation of a homotopical algebraic quantum field theory as follows.

Definition 3.8.9. We define a **representation** of a homotopical algebraic quantum field theory $\mathfrak{A} \in \mathbf{hAQFT_C}$ to be an $\mathfrak{A}$-representation, as in Definition 3.2.1.

3.8.1 Inequivalent categories of net representations

We now turn our attention to the question of when two representations are equivalent in the context of homotopy AQFT. First, notice that the category $\mathbf{Net}_{\mathbf{C}}^{\mathbf{CoCh}_{\mathbb{C}}}$ inherits a collection of weak equivalences from the projective model structure on the category $\mathbf{CoCh}_{\mathbb{C}}$, as described in Definition 3.6.2. Since the category $\mathbf{hAQFT_C}$ is a subcategory of $\mathbf{Net}_{\mathbf{C}}^{\mathbf{CoCh}_{\mathbb{C}}}$, it inherits a collection of weak equivalences from $\mathbf{Net}_{\mathbf{C}}^{\mathbf{CoCh}_{\mathbb{C}}}$ according to Remark 2.1.6. Specifically, weak equivalences in $\mathbf{hAQFT_C}$ are defined as follows.

Definition 3.8.10. A morphism $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{hAQFT_C}$ is a **weak equivalence** in $\mathbf{hAQFT_C}$ if, for any $N \in \mathbf{C}$, the morphism $\Phi_N\colon \mathfrak{A}_N \to \mathfrak{B}_N$ in $\mathbf{CoCh}_{\mathbb{C}}$ is a weak equivalence in $\mathbf{CoCh}_{\mathbb{C}}$ with respect to the projective model structure (see Definition 2.9.3), i.e. Φ_N is a quasi-isomorphism.

Two homotopical algebraic quantum field theories that are weakly equivalent (see Definition 2.1.5) are understood to be *physically equivalent*, i.e. they describe the same physical system. Consequently, the categories of representations corresponding to weakly equivalent homotopical algebraic quantum field theories must also be physically equivalent. In particular, it would be desirable that the change-of-net adjunction associated to a weak equivalence between homotopical algebraic quantum field theories determines some form of equivalence between the associated categories of net representations. However, as we demonstrate in the following example of the Klein–Gordon field, which we has been taken verbatim from [AB23], the change-of-net adjunction associated to a weak equivalence, in general, doesn't need to be an equivalence of categories (see Definition A.2.11).

Example 3.8.11 (Ordinary categorical equivalence fails for weakly equivalent nets of algebras)**.** We examine the category of net representations for the Klein–Gordon field of mass $m \geq 0$ using two different, yet equivalent, descriptions.

More precisely, we shall construct two $\mathbf{CoCh}_{\mathbb{C}}$-valued nets of algebras $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$ on $\mathbf{Loc}_m$ describing the Klein–Gordon field and the evident weak equivalence $\Phi\colon \widetilde{\mathfrak{A}} \to \mathfrak{A}$ in $\mathbf{Net}_{\mathbf{Loc}_m}^{\mathbf{CoCh}_\mathbb{C}}$ relating them. ($\mathfrak{A}$ will denote the standard Klein–Gordon net [FV15, Sec. 4.2], regarded as a $\mathbf{CoCh}_\mathbb{C}$-valued net of algebras concentrated in degree 0, while $\widetilde{\mathfrak{A}}$ will denote the Klein–Gordon net from the Batalin–Vilkovisky formalism [BBS20].) From the explicit description of the associated net representations categories $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\widetilde{\mathfrak{A}})$, it will be manifest that the restriction functor $\mathrm{Res}_\Phi\colon \mathbf{Rep}(\widetilde{\mathfrak{A}}) \to \mathbf{Rep}(\mathfrak{A})$ is not even essentially surjective and hence it fails to be an ordinary categorical equivalence. This explains why the flexibility of the Batalin–Vilkovisky formalism forces one to endow the net representations categories with suitable model structures, that come along with the more flexible concept of Quillen equivalence. In this way, although being genuinely different in the ordinary categorical sense, the net representations categories $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\widetilde{\mathfrak{A}})$, associated with the two equivalent descriptions $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$ of the Klein–Gordon field become the same in the model categorical sense due to Proposition 3.6.19.

Let us start from the standard Klein–Gordon net of algebras $\mathfrak{A} \in \mathbf{Net}_{\mathbf{Loc}_m}^{\mathbf{CoCh}_\mathbb{C}}$. To $M \in \mathbf{Loc}_m$ it assigns the unital associative differential graded algebra

$$\mathfrak{A}_M := T_\mathbb{C}\big(\mathcal{V}(M)\big)/J \in \mathbf{DGA}_\mathbb{C} := \mathbf{Mon}(\mathbf{CoCh}_\mathbb{C}) \tag{3.8.2a}$$

that is freely generated over $\mathbb{C}$ by the cochain complex $\mathcal{V}(M) := C^\infty_\mathrm{c}(M)/PC^\infty_\mathrm{c}(M) \in \mathbf{CoCh}_\mathbb{R}$ concentrated in degree 0, modulo the two-sided ideal $J \subseteq T_\mathbb{C}(\mathcal{V}(M))$ generated by the canonical commutation relations

$$[\varphi_1] \otimes [\varphi_2] - [\varphi_2] \otimes [\varphi_1] - i \int_M \varphi_1\, G\varphi_2 \,\mathrm{vol}_M\, \mathbf{1}, \tag{3.8.2b}$$

for all $[\varphi_1], [\varphi_2] \in \mathcal{V}(M)^0$. Here $P := \Box - m^2\colon C^\infty(M) \to C^\infty(M)$ denotes the Klein–Gordon operator, $G\colon C^\infty_\mathrm{c}(M) \to C^\infty(M)$ denotes the associated retarded-minus-advanced propagator (which vanishes on $PC^\infty_\mathrm{c}(M)$ and is formally skew-adjoint, see [BGP07, Sec. 3.4]) and vol_M denotes the volume form on M. The push-forward of compactly supported smooth functions along morphisms in $\mathbf{Loc}_m$ turns the assignment $M \in \mathbf{Loc}_m \mapsto \mathfrak{A}_M \in \mathbf{DGA}_\mathbb{C}$ into a net of algebras $\mathfrak{A} \in \mathbf{Net}_{\mathbf{Loc}_m}^{\mathbf{CoCh}_\mathbb{C}}$. The Klein–Gordon net $\widetilde{\mathfrak{A}} \in \mathbf{Net}_{\mathbf{Loc}_m}^{\mathbf{CoCh}_\mathbb{C}}$ from the Batalin–Vilkovisky formalism is defined in a similar fashion. Explicitly, to $M \in \mathbf{Loc}_m$ it assigns the differential graded algebra

$$\widetilde{\mathfrak{A}}_M := T_\mathbb{C}\big(\widetilde{\mathcal{V}}(M)\big)/I \in \mathbf{DGA}_\mathbb{C} \tag{3.8.3a}$$

that is freely generated by the cochain complex

$$\widetilde{\mathcal{V}}(M) := \big(\, C^\infty_\mathrm{c}(M) \xrightarrow{\;P\;} C^\infty_\mathrm{c}(M)\,\big) \in \mathbf{CoCh}_\mathbb{R}, \tag{3.8.3b}$$

resolving the quotient $\mathcal{V}(M)$ and concentrated in degrees -1 and 0, modulo the two-sided ideal $I \subseteq \mathcal{V}(M)$ generated by the (graded) canonical commutation relations

$$\varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1 - i \int_M \varphi_1 \, G\varphi_2 \, \mathrm{vol}_M \, \mathbf{1}, \tag{3.8.3c}$$

$$\varphi \otimes \varphi^\ddagger - \varphi^\ddagger \otimes \varphi, \tag{3.8.3d}$$

$$\varphi_1^\ddagger \otimes \varphi_2^\ddagger + \varphi_2^\ddagger \otimes \varphi_1^\ddagger, \tag{3.8.3e}$$

for all $\varphi_1, \varphi_2, \varphi \in \widetilde{\mathcal{V}}(M)^0$ and $\varphi^\ddagger, \varphi_1^\ddagger, \varphi_2^\ddagger \in \widetilde{\mathcal{V}}(M)^{-1}$. (The only non-trivial commutation relations involve 0-cochains.) There is an evident morphism

$$\Phi \colon \widetilde{\mathfrak{A}} \longrightarrow \mathfrak{A} \tag{3.8.4}$$

in $\mathbf{Net}_{\mathbf{Loc}_m}^{\mathbf{CoCh}_\mathbb{C}}$, whose M-component $\Phi_M \colon \widetilde{\mathfrak{A}}_M \to \mathfrak{A}_M$ is defined on generators by $\Phi_M(\varphi) := [\varphi]$ and $\Phi_M(\varphi^\ddagger) := 0$, for all $M \in \mathbf{Loc}_m$, $\varphi \in \widetilde{\mathcal{V}}(M)^0$ and $\varphi^\ddagger \in \widetilde{\mathcal{V}}(M)^{-1}$. In [BBS20, Rem. 6.20] it is shown that (3.8.4) is a weak equivalence.

Consider now an $\widetilde{\mathfrak{A}}$-representation $\widetilde{\mathcal{L}} \in \mathbf{Rep}(\widetilde{\mathfrak{A}})$. To each $M \in \mathbf{Loc}_m$, $\widetilde{\mathcal{L}}$ assigns a left $\widetilde{\mathfrak{A}}_M$-module $\widetilde{\mathcal{L}}(M) \in {}_{\widetilde{\mathfrak{A}}_M}\mathbf{Mod}$. One can equivalently encode its left $\mathfrak{A}_M$-action $\widetilde{\lambda}_M \colon \widetilde{\mathfrak{A}}_M \otimes \widetilde{\mathcal{L}}(M) \to \widetilde{\mathcal{L}}(M)$ in $\mathbf{CoCh}_\mathbb{C}$ as a morphism $\widetilde{\lambda}_M \colon \widetilde{\mathfrak{A}}_M \to [\widetilde{\mathcal{L}}(M), \widetilde{\mathcal{L}}(M)]$ in $\mathbf{DGA}_\mathbb{C}$. (Recall that the internal hom $[\widetilde{\mathcal{L}}(M), \widetilde{\mathcal{L}}(M)] \in \mathbf{CoCh}_\mathbb{C}$ carries a canonical monoid structure whose multiplication is the composition and whose unit is the identity. Below we will denote its differential by ∂.) It follows from (3.8.3) that $\widetilde{\lambda}_M$ assigns (compatibly with the (graded) canonical commutation relations) to each $\varphi \in \widetilde{\mathcal{V}}(M)^0$ a 0-cocycle $\widetilde{\lambda}_M(\varphi) \in Z^0([\widetilde{\mathcal{L}}(M), \widetilde{\mathcal{L}}(M)])$ and to each $\varphi^\ddagger \in \widetilde{\mathcal{V}}(M)^{-1}$ a (-1)-cochain $\widetilde{\lambda}_M(\varphi^\ddagger) \in [\widetilde{\mathcal{L}}(M), \widetilde{\mathcal{L}}(M)]^{-1}$ such that $\partial(\widetilde{\lambda}_M(\varphi^\ddagger)) = \widetilde{\lambda}_M(P\varphi^\ddagger)$.

Compared to an $\widetilde{\mathfrak{A}}$-representation, an $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ consists of fewer data, as explained below. Recalling (3.8.2), for each $M \in \mathbf{Loc}_m$, the underlying left $\mathfrak{A}_M$-action $\lambda_M \colon \mathfrak{A}_M \to [\mathcal{L}(M), \mathcal{L}(M)]$ in $\mathbf{DGA}_\mathbb{C}$ only assigns (compatibly with the canonical commutation relations) to each $[\varphi] \in \mathcal{V}(M)^0$ a 0-cocycle $\lambda_M([\varphi]) \in Z^0([\mathcal{L}(M), \mathcal{L}(M)])$.

Recall now that the restriction functor $\mathrm{Res}_\Phi \colon \mathbf{Rep}(\mathfrak{A}) \to \mathbf{Rep}(\widetilde{\mathfrak{A}})$ simply restricts the left actions, retaining all other data. Given an $\widetilde{\mathfrak{A}}$-representation $\widetilde{\mathcal{L}} \in \mathbf{Rep}(\widetilde{\mathfrak{A}})$ such that $\widetilde{\lambda}_M(\varphi^\ddagger) \neq 0 \in [\widetilde{\mathcal{L}}(M), \widetilde{\mathcal{L}}(M)]^{-1}$ for some $M \in \mathbf{Loc}_m$ and $\varphi^\ddagger \in \widetilde{\mathcal{V}}(M)^0$ (for instance take $\widetilde{\mathcal{L}} = \widetilde{\mathfrak{A}}$ to be the net of algebras $\widetilde{\mathfrak{A}}$ itself regarded as an $\widetilde{\mathfrak{A}}$-representation), one immediately realizes that there cannot exist an $\mathfrak{A}$-representation $\mathcal{L} \in \mathbf{Rep}(\mathfrak{A})$ and an isomorphism $\mathcal{F} \colon \mathrm{Res}_\Phi \mathcal{L} \to \widetilde{\mathfrak{A}}$ in $\mathbf{Rep}(\widetilde{\mathfrak{A}})$. Indeed, by construction of Φ, the left actions $\lambda_M \circ \Phi_M$ of $\mathrm{Res}_\Phi \mathcal{L}$

are such that $\lambda_M(\Phi_M(\varphi^\ddagger)) = 0 \in [\mathcal{L}(M), \mathcal{L}(M)]^{-1}$ vanishes for all $M \in \mathbf{Loc}_m$ and $\varphi^\ddagger \in \widetilde{\mathcal{V}}(M)^{-1}$; moreover, any $\widetilde{\mathfrak{A}}$-representation that is isomorphic to $\mathrm{Res}_\Phi \mathfrak{A}$ shares this feature. It follows that Res_Φ is not essentially surjective and hence $\mathrm{Ext}_\Phi \dashv \mathrm{Res}_\Phi$ cannot be an ordinary categorical equivalence. $\triangledown$

The above example indicates that the categories of net representations associated with physically equivalent homotopical AQFTs may fail to be equivalent in the ordinary categorical sense.

3.8.2 Equivalence of homotopical AQFT representations

The fact that the categories of net representations of equivalent homotopical AQFTs may be inequivalent in the plain categorical sense suggests that the notion of isomorphism between net representations is not the right notion of equivalence in this context, and it must be replaced by an appropriate concept of weak equivalence.

In fact, given $\mathfrak{A} \in \mathbf{hAQFT_C}$, the category $\mathbf{Rep}(\mathfrak{A})$ of $\mathfrak{A}$-representations inherits a collection of weak equivalences from $\mathbf{CoCh}_\mathbb{C}$ as in Definition 3.6.2. Specifically, we obtain the following definition.

Definition 3.8.12. Let $\mathfrak{A} \in \mathbf{hAQFT_C}$. A morphism $\mathcal{F} \colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{Rep}(\mathfrak{A})$ is a **weak equivalence** in $\mathbf{Rep}(\mathfrak{A})$, if for any $N \in \mathbf{C}$, the morphism $\mathcal{F}_N \colon \mathcal{L} \to \mathcal{L}'$ in $\mathbf{CoCh}_\mathbb{C}$ is a quasi-isomorphism (see Definition 2.9.3).

Furthermore, according to Proposition 3.6.17, the category $\mathbf{Rep}(\mathfrak{A})$ inherits a model structure from the projective model structure on $\mathbf{CoCh}_\mathbb{C}$ (see Definition 2.9.3), compatible (in the sense of Definition 2.2.5) with the weak equivalences from Definition 3.8.12.

Indeed, the category $\mathbf{CoCh}_\mathbb{C}$ meets the requirements of Proposition 3.6.17. Specifically, $\mathbf{CoCh}_\mathbb{C}$, equipped with the projective model structure, is a symmetric monoidal model category (see Proposition 2.9.5) such that:

(i) $\mathbf{CoCh}_\mathbb{C}$ is a cofibrantly generated model category, by Proposition 2.9.4,

(ii) $\mathbf{CoCh}_\mathbb{C}$ satisfies the monoid axiom, by Proposition 2.9.6,

(iii) the monoidal unit $\mathbb{1}$ of $\mathbf{CoCh}_\mathbb{C}$ is cofibrant, since every object in $\mathbf{CoCh}_\mathbb{C}$ is cofibrant (see Proposition 2.9.4),

(iv) all objects of $\mathbf{CoCh}_\mathbb{C}$ are fibrant, by Proposition 2.9.4,

(v) $\mathbf{CoCh}_{\mathbb{C}}$ has an interval object (I, r, b), for instance, the cochain complex $I \in \mathbf{CoCh}_{\mathbb{K}}$, consisting of $\mathbb{K}$ in degree -1 and $\mathbb{K} \oplus \mathbb{K}$ in degree 0, with differential defined by $\mathrm{d}(1) = 1 \oplus (-1)$.

Therefore, the category $\mathbf{Rep}(\mathfrak{A})$ is a model category, with the model structure determined by the weak equivalences from Definition 3.8.12 and fibrations the morphisms $\mathcal{F} \in \mathrm{Mor}(\mathbf{Rep}(\mathfrak{A}))$ such that for any $N \in \mathbf{C}$, the morphism $\mathcal{F}_N \in \mathrm{Mor}(\mathbf{CoCh}_{\mathbb{C}})$ is a degree-wise surjective map.

Furthermore, by Proposition 2.9.7, the category $\mathbf{CoCh}_{\mathbb{C}}$ fulfils the assumptions of 3.6.20. This implies that a weak equivalence of homotopical algebraic quantum field theories, induces a Quillen equivalence between the corresponding categories of net representations.

This indicates that the right notion of equivalence between net representations in the context of homotopy AQFT is that of weak equivalence from Definition 3.8.12, whereas the right notion of equivalence of categories of net representations is that of a Quillen equivalence with respect to the model structures from Proposition 3.6.17.

In other words, two weakly equivalent representations $\mathcal{L}, \mathcal{L}' \in \mathbf{Rep}(\mathfrak{A})$, for some $\mathfrak{A} \in \mathbf{hAQFT}_{\mathbf{C}}$, should be understood to be physically equivalent. Moreover, if two categories of net representations $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\mathfrak{B})$, for some $\mathfrak{A}, \mathfrak{B} \in \mathbf{hAQFT}_{\mathbf{C}}$, are connected via a zig-zag of Quillen equivalences, then they describe the same physical system.

Remark 3.8.13. The reason that we use zig-zags of Quillen equivalences instead of simply Quillen equivalences is that the relation determined by the existence of a Quillen equivalence between two model categories is not transitive. Namely, the existence of a zig-zag of Quillen equivalences between $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\mathfrak{B})$ does not imply the existence of a Quillen equivalence between them (see Remark 2.4.7). But the notion of two descriptions of a physical system being equivalent is transitive. Therefore, two categories of net representations of homotopical AQFTs need to be physically equivalent not only when there is a Quillen adjunction between them, but also when there is a zig-zag of Quillen adjunctions between them. △

With the understanding that categories of net representations connected via a zig-zag of Quillen equivalences are physically equivalent, one recovers consistency in the physical description associated with homotopy AQFT. Namely, given two physically equivalent homotopical AQFTs, the associated categories of net representations are also physically equivalent.

Passing to the homotopy categories

Since weakly equivalent homotopical AQFTs are understood to be physically equivalent, one could say that the physical information contained in the category $\mathbf{hAQFT_C}$ is reflected in the homotopy category $\mathrm{Ho}(\mathbf{hAQFT_C})$ associated with $\mathbf{hAQFT_C}$ with respect to the weak equivalences from Definition 3.8.10. Therefore, one can treat the category $\mathrm{Ho}(\mathbf{hAQFT_C})$ as the category of homotopical AQFTs. Then, the statement that two weakly equivalents nets in $\mathbf{hAQFT_C}$ are physically equivalent is equivalent to the statement that two isomorphic nets in $\mathrm{Ho}(\mathbf{hAQFT_C})$ are physically equivalent.

Similarly, since two weakly equivalent net representations are physically equivalent, the physical information contained in $\mathbf{Rep}(\mathfrak{A})$, for some $A \in \mathbf{hAQFT_C}$, is reflected in the homotopy category $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$. In other words, we can treat $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ as the category of $\mathfrak{A}$-representations.

From this perspective, the content of this section could be summarized as follows. The statement that two weakly equivalent representations in $\mathbf{Rep}(\mathfrak{A})$, for some $\mathfrak{A} \in \mathbf{hAQFT_C}$ are physically equivalent translates to the statement that two isomorphic representations in $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ are physically equivalent This follows from the fact that two objects in $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ are isomorphic if and only if they are weakly equivalent in $\mathbf{Rep}(\mathfrak{A})$ (see Corollary 3.7.2).

Finally, two categories of net representations $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$ that are equivalent in the ordinary categorical sense, describe the same physical system.

From this perspective, Corollary 3.7.5 implies that, given two isomorphic homotopical algebraic quantum field theories $\mathfrak{A}, \mathfrak{B} \in \mathrm{Ho}(\mathbf{hAQFT_C})$, the associated categories of net representations $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$ are equivalent. In other words, the categories of net representations associated with physically equivalent homotopical AQFTs are also physically equivalent.

Remark 3.8.14. Note that the condition that $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$ are equivalent may not be equivalent to the condition that $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\mathfrak{B})$ are connected via a zig-zag of Quillen equivalences. As we pointed out in Remark 2.4.8, in general, the condition of two model categories being connected via a zig-zag of Quillen equivalences is strictly stronger than the condition of their homotopy categories being equivalent. Namely, the homotopy categories associated with some model categories may be equivalent even if there is no zig-zag of Quillen equivalences between the model categories.

However, it is not clear to us if this could happen with the model categories obtained as categories of net representations of homotopical AQFTs. If that is the case, then two categories $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{A}))$ and $\mathrm{Ho}(\mathbf{Rep}(\mathfrak{B}))$ could be equivalent even if there is not a zig-zag of Quillen equivalences between $\mathbf{Rep}(\mathfrak{A})$ and $\mathbf{Rep}(\mathfrak{B})$, i.e. even if the associated nets $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic in Ho($\mathbf{hAQFT_C}$). That would imply that either the category Ho($\mathbf{hAQFT_C}$) contains physically irrelevant information that makes two equivalent descriptions of a physical system to appear different, or that the category Ho($\mathbf{Rep}(\mathfrak{A})$) doesn't contain the full information about some physical systems resulting in the same description for two different physical systems. $\triangle$

Chapter 4

Net Representations for Maxwell p-forms

We now move on to the problem of constructing explicit examples of representations of a homotopical algebraic quantum field theory. As an instructive example we shall consider the net of algebras associated with Maxwell p-forms [HT86; HT92], which we construct and study in detail through Sections 4.1, 4.2 and 4.4. The actual construction of an explicit net representation for Maxwell p-forms is presented in Section 4.5 and goes through the construction of a two-point function ω_2, which for our purposes consists of a cochain map: it is the preservation of differentials that encodes the compatibility both with the equation of motion and with the action of gauge transformations. Note that Maxwell 1-forms recover linear Yang-Mills theory, i.e. the electromagnetic vector potential, and in this case our ω_2 extends (in a sense made precise by Remark 4.5.4) a Hadamard two-point function constructed in [FP03] for the gauge invariant on-shell linear observables of the electromagnetic vector potential.

This chapter has appeared with only minor modifications in [AB23].

4.1 Cochain complexes of solutions and linear observables

Let M be an oriented and time-oriented globally hyperbolic Lorentzian manifold of dimension $m \geq 2$. We consider the field complex $\mathcal{F}(M) \in \mathbf{CoCh}_{\mathbb{R}}$ of p-forms on M, for $p < m$, defined by

$$\mathcal{F}(M) := \left(\overset{(-p)}{\Omega^0(M)} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \overset{(n)}{\Omega^{p+n}(M)} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \overset{(0)}{\Omega^p(M)} \right). \quad (4.1.1)$$

(Our convention is that the non-displayed degrees, in this case $n < -p$ and $n > 0$, and differentials vanish.) In degree 0 sit the gauge fields $A \in \mathcal{F}(M)^0 = \Omega^p(M)$, in degree -1 sit the ghosts $g^{-1} \in \mathcal{F}(M)^{-1} = \Omega^{p-1}(M)$ (gauge transformations) and, more generally, in lower degrees $n < -1$ sit higher ghosts $g^n \in \mathcal{F}(M)^n = \Omega^{p+n}(M)$. For $p = 1$, $\mathcal{F}(M)$ is the well-known complex of linear Yang–Mills theory, with connection 1-forms sitting in degree 0 and gauge transformations sitting in degree -1, whose action on connections is determined by the de Rham differential d. The dynamics is encoded by the linear differential operator

$$\delta \mathrm{d} \colon \mathcal{F}(M)^0 \to \mathcal{F}(M)^0. \tag{4.1.2}$$

One can interpret this linear differential operator as the equation of motion arising from the variation of the (formal) action functional

$$S(A) \coloneqq \frac{1}{2} \int_M A \wedge * \, \delta \mathrm{d} A, \tag{4.1.3}$$

where $*$ denotes the Hodge star operator defined by the metric and the orientation of M and $\delta \coloneqq (-1)^k *^{-1} \mathrm{d} \, *$ denotes the codifferential on $\Omega^k(M)$. The action functional S is manifestly gauge-invariant since $\delta \mathrm{d}\,\mathrm{d} = 0$ and $\delta \mathrm{d}$ is formally self-adjoint with respect to the integral pairing $\int (-) \wedge *(-)$ displayed above. Varying S leads to a section $\delta^{\mathrm{v}} S \colon \mathcal{F}(M) \to T^*\mathcal{F}(M)$ in $\mathbf{CoCh}_{\mathbb{R}}$ of the "cotangent bundle" $T^*\mathcal{F}(M) \in \mathbf{CoCh}_{\mathbb{R}}$ over $\mathcal{F}(M)$, defined as the product

$$T^*\mathcal{F}(M) \coloneqq \mathcal{F}(M) \times \mathcal{F}_{\mathrm{c}}(M)^* \tag{4.1.4}$$

of the cochain complex $\mathcal{F}(M) \in \mathbf{CoCh}_{\mathbb{R}}$, interpreted as the base, and the cochain complex

$$\mathcal{F}_{\mathrm{c}}(M)^* \coloneqq \left(\overset{(0)}{\Omega^p(M)} \xrightarrow{-\delta} \cdots \xrightarrow{-\delta} \overset{(n)}{\Omega^{p-n}(M)} \xrightarrow{-\delta} \cdots \xrightarrow{-\delta} \overset{(p)}{\Omega^0(M)} \right), \tag{4.1.5}$$

interpreted as the fiber. (The notation $\mathcal{F}_{\mathrm{c}}(M)^*$ is motivated by the existence of an evaluation pairing $\mathrm{ev} \colon \mathcal{F}_{\mathrm{c}}(M)^* \otimes \mathcal{F}_{\mathrm{c}}(M) \to \mathbb{R}$ in $\mathbf{CoCh}_{\mathbb{R}}$ against the cochain complex $\mathcal{F}_{\mathrm{c}}(M) \in \mathbf{CoCh}_{\mathbb{R}}$ of compactly supported fields, which is defined by $\mathrm{ev}(\Phi \otimes \varphi) \coloneqq (-1)^{\lfloor n/2 \rfloor} \int_M \Phi \wedge * \varphi$, for all $n = 0, \dots, p$, $\Phi \in (\mathcal{F}_{\mathrm{c}}(M)^*)^n$ and $\varphi \in \mathcal{F}_{\mathrm{c}}(M)^{-n}$, where $\lfloor - \rfloor$ denotes the floor function.) The derived critical locus of S, defined as the homotopy pull-back of $\delta^{\mathrm{v}} S \colon \mathcal{F}(M) \to T^*\mathcal{F}(M)$ in $\mathbf{CoCh}_{\mathbb{R}}$ along the zero-section of $T^*\mathcal{F}(M) \in \mathbf{CoCh}_{\mathbb{R}}$, determines the *solution complex* $\mathfrak{Sol}(M) \in \mathbf{CoCh}_{\mathbb{R}}$. Computing the latter explicitly as in [BS19a, Sec. 3.4] leads to the cochain complex

$$\begin{aligned}\mathfrak{Sol}(M) \coloneqq \Big(\overset{(-p)}{\Omega^0(M)} \xrightarrow{\;\mathrm{d}\;} \cdots \xrightarrow{\;\mathrm{d}\;} \overset{(0)}{\Omega^p(M)} &\\ \swarrow_{\delta \mathrm{d}} \quad\quad &\\ \overset{(1)}{\Omega^p(M)} \xrightarrow{\;\delta\;} \cdots \xrightarrow{\;\delta\;} \overset{(p+1)}{\Omega^0(M)} \Big). &\end{aligned} \tag{4.1.6}$$

The *observable complex* $\mathcal{L}(M) \in \mathbf{CoCh}_{\mathbb{R}}$ is defined as the 1-shifted compactly supported analog of $\mathfrak{Sol}(M)$. Explicitly, this reads as

$$\begin{aligned}\mathcal{L}(M) \coloneqq \Big(\overset{(-p-1)}{\Omega^0_{\mathrm{c}}(M)} \xrightarrow{\;-\mathrm{d}\;} \cdots \xrightarrow{\;-\mathrm{d}\;} \overset{(-1)}{\Omega^p_{\mathrm{c}}(M)} &\\ \swarrow_{-\delta \mathrm{d}} \quad\quad &\\ \overset{(0)}{\Omega^p_{\mathrm{c}}(M)} \xrightarrow{\;-\delta\;} \cdots \xrightarrow{\;-\delta\;} \overset{(p)}{\Omega^0_{\mathrm{c}}(M)} \Big). &\end{aligned} \tag{4.1.7}$$

The cochains in $\mathcal{L}(M)$ are interpreted as linear functionals on the solution complex $\mathfrak{Sol}(M)$ via the evaluation pairing

$$\mathrm{ev} \colon \mathfrak{Sol}(M) \otimes \mathcal{L}(M) \longrightarrow \mathbb{R} \tag{4.1.8a}$$

in $\mathbf{CoCh}_{\mathbb{R}}$, whose only non vanishing degree

$$\mathrm{ev}^0 \colon \bigoplus_{k \in \mathbb{Z}} \big(\mathfrak{Sol}(M)^k \otimes \mathcal{L}(M)^{-k} \big) \longrightarrow \mathbb{R} \tag{4.1.8b}$$

is defined component-wise for all $k \in \mathbb{Z}$ by

$$\mathrm{ev}^0_k(\Phi \otimes \varphi) \coloneqq (-1)^{\lfloor k/2 \rfloor} \int_M \Phi \wedge *\varphi, \tag{4.1.8c}$$

for all $\Phi \in \mathfrak{Sol}(M)^k$ and $\varphi \in \mathcal{L}(M)^{-k}$.

4.2 Retarded and advanced trivializations

We equip the observable complex with a Poisson structure following the approach of [BBS20], see also [BMS23] for a related, yet more conceptual, approach. In order to achieve this goal, let us consider the analog of the solution complex with an appropriate support restriction. Explicitly, we write

$\Omega^k_{\mathrm{sc}}(M)$ for the vector space of k-forms whose support is spacelike compact, i.e. contained in the causal shadow $J_M(K) := J^+_M(K) \cup J^-_M(K) \subseteq M$ of some compact subset $K \subseteq M$. Implementing the same support restriction on the solution complex $\mathfrak{Sol}(M)$, we define the solution complex with spacelike compact support $\mathfrak{Sol}_{\mathrm{sc}}(M) \in \mathbf{CoCh}_{\mathbb{R}}$.

Since by definition $K \subseteq J_M(K)$, there are obvious inclusions $\Omega^k_{\mathrm{c}}(M) \subseteq \Omega^k_{\mathrm{sc}}(M)$, for all $k = 0, \dots, p$. These inclusions assemble to form the cochain map $j \colon \mathcal{L}(M) \to \mathfrak{Sol}_{\mathrm{sc}}(M)[1]$ in $\mathbf{CoCh}_{\mathbb{R}}$ to the 1-shifted solution complex with spacelike compact support. Equivalently, this is a 1-cocycle $j \in [\mathcal{L}(M), \mathfrak{Sol}_{\mathrm{sc}}(M)]^1$ in the internal hom. As it is shown below, it turns out that j is homotopic to 0 in two inequivalent ways. Explicitly, this means that there exist 0-cochains $\Lambda_\pm \in [\mathcal{L}(M), \mathfrak{Sol}_{\mathrm{sc}}(M)]^0$ in the internal hom, called *retarded and advanced trivializations*, whose differentials $\partial \Lambda_\pm = \mathrm{d}_{\mathfrak{Sol}} \Lambda_\pm - \Lambda_\pm \mathrm{d}_{\mathcal{L}} = j$ trivialize the 1-cocycle j. Let us construct these retarded and advanced trivializations $\Lambda_\pm$ for Maxwell p-forms. To achieve this goal we consider the retarded and advanced Green's operators $G^{(k)}_\pm$ for the d'Alembert operator $\Box := \delta \mathrm{d} + \mathrm{d} \delta \colon \Omega^k(M) \to \Omega^k(M)$ on k-forms, $k = 0, \dots, p$. Recall from [BGP07; Bär15] that a *retarded/advanced Green's operator* $G^{(k)}_\pm \colon \Omega^k_{\mathrm{c}}(M) \to \Omega^k(M)$ for $\Box$ is a linear map, such that, for all $\omega \in \Omega^k_{\mathrm{c}}(M)$, it holds that (i) $\Box\, G^{(k)}_\pm\, \omega = \omega$, (ii) $G^{(k)}_\pm\, \Box\, \omega = \omega$ and (iii) $\mathrm{supp}(G^{(k)}_\pm\, \omega) \subseteq J^\pm_M(\mathrm{supp}(\omega))$. It is well-known that retarded and advanced Green's operators for $\Box$ exist, are unique and commute both with the de Rham differential $\mathrm{d}\, G^k_\pm = G^{k+1}_\pm\, \mathrm{d}$ and codifferential $\delta\, G^k_\pm = G^{k-1}_\pm\, \delta$. With these preparations, we define the *retarded and advanced trivializations* $\Lambda_\pm \in [\mathcal{L}(M), \mathfrak{Sol}_{\mathrm{sc}}(M)]^0$ degree-wise by

$$\Lambda^n_\pm := \begin{cases} G^{(p+n)}_\pm\, \delta, & n = -p, \dots, -1, \\ G^{(p)}_\pm, & n = 0, \\ \mathrm{d}\, G^{(p-n)}_\pm, & n = 1, \dots, p. \end{cases} \tag{4.2.1}$$

(All other components $\Lambda^n_\pm$, $n \leq -p-1$ and $n \geq p+1$, necessarily vanish.) Note that property (iii) ensures that the output of $\Lambda_\pm$ has the required support. Furthermore, direct inspection using properties (i) and (ii) and the commutation rules of $G^{(k)}_\pm$ with d and δ shows that $\partial \Lambda_\pm = j$.

Taking the difference of the retarded and the advanced trivializations from (4.2.1) defines the cochain map

$$\Lambda := \Lambda_+ - \Lambda_- \colon \mathcal{L}(M) \longrightarrow \mathfrak{Sol}_{\mathrm{sc}}(M) \tag{4.2.2}$$

in $\mathbf{CoCh}_{\mathbb{R}}$. ($\Lambda$ is indeed a cochain map because $\partial \Lambda = j - j = 0 \in [\mathcal{L}(M), \mathfrak{Sol}_{\mathrm{sc}}(M)]^1$ vanishes when regarded as a 1-cochain in the internal hom.)

Remark 4.2.1. The cochain map Λ in (4.2.2) is a quasi-isomorphism since it induces the isomorphism $H(\Lambda)\colon H(\mathcal{L}(M)) \to H(\mathfrak{Sol}_{\mathrm{sc}}(M))$ in cohomology, which in terms of the de Rham cohomologies of M with compact support $H_{\mathrm{dR,c}}(M)$ and with spacelike compact support $H_{\mathrm{dR,sc}}(M)$ reads as

$$\begin{cases} H^{p+1+n}_{\mathrm{dR,c}}(M) \cong H^{p+n}_{\mathrm{dR,sc}}(M), & n = -p-1, \dots, -1, \\ \Omega^p_{\mathrm{c}\,\delta}(M)/\delta\mathrm{d}\Omega^p_{\mathrm{c}}(M) \cong \Omega^p_{\mathrm{sc}\,\delta\mathrm{d}}(M)/\mathrm{d}\Omega^{p-1}_{\mathrm{sc}}(M), & n = 0, \\ H^{m-p+n}_{\mathrm{dR,c}}(M) \cong H^{m-p-1+n}_{\mathrm{dR,sc}}(M), & n = 1, \dots, p+1. \end{cases} \tag{4.2.3}$$

(Note that the Hodge star operator $*$ has been implicitly used to identify the cohomology of the codifferential δ with the more familiar de Rham cohomology.) We refer to [Kha16] for the definition of the de Rham cohomologies with spacelike compact support $H^k_{\mathrm{dR,sc}}(M)$ of an m-dimensional oriented and time-oriented globally hyperbolic Lorentzian manifold M and for the isomorphisms in degrees $n \neq 0$, while we refer to [Ben16] for the isomorphism in degree $n = 0$ between "gauge invariant linear observables modulo equations of motion" $\Omega^p_{\mathrm{c}\,\delta}(M)/\delta\mathrm{d}\Omega^p_{\mathrm{c}}(M)$ and "spacelike compact on-shell fields modulo gauge transformations" $\Omega^p_{\mathrm{sc}\,\delta\mathrm{d}}(M)/\mathrm{d}\Omega^{p-1}_{\mathrm{sc}}(M)$. See [BMS23] for a more conceptual proof of the fact that Λ is a quasi-isomorphism. △

4.3 Initial data complex

For any choice of a spacelike Cauchy surface $\Sigma \subseteq M$, let us also consider the compactly supported *initial data complex*

$$\mathfrak{D}_{\mathrm{c}}(\Sigma) \coloneqq \left(\begin{array}{c} \overset{(-p)}{\Omega^0_{\mathrm{c}}(\Sigma)} \xrightarrow{\mathrm{d}_\Sigma} \cdots \xrightarrow{\mathrm{d}_\Sigma} \overset{(-1)}{\Omega^{p-1}_{\mathrm{c}}(\Sigma)} \\ \swarrow (\mathrm{d}_\Sigma, 0) \\ \overset{(0)}{\Omega^p_{\mathrm{c}}(\Sigma)^2} \\ \swarrow \delta_\Sigma \mathrm{pr}_2 \\ \overset{(1)}{\Omega^{p-1}_{\mathrm{c}}(\Sigma)} \xrightarrow{\delta_\Sigma} \cdots \xrightarrow{\delta_\Sigma} \overset{(p)}{\Omega^0_{\mathrm{c}}(\Sigma)} \end{array} \right). \tag{4.3.1}$$

(Here the notation d_Σ and $\delta_\Sigma \coloneqq (-1)^k *_\Sigma^{-1} \mathrm{d}_\Sigma *_\Sigma$ is used to emphasize that these differential operators are defined with respect to the geometry of Σ.) The compactly supported initial data complex is related to the spacelike compactly supported solution complex via the initial data map $\mathbf{data} : \mathfrak{Sol}_{\mathrm{sc}}(M) \to \mathfrak{D}_{\mathrm{c}}(\Sigma)$

in $\mathbf{CoCh}_{\mathbb{R}}$ defined degree-wise by

$$\mathbf{data}^n := \begin{cases} \iota^*, & n = -p, \dots, -1, \\ (\iota^*, *_\Sigma^{-1}\, \iota^* * \mathrm{d}), & n = 0, \\ (-1)^n *_\Sigma^{-1}\, \iota^* *, & n = 1, \dots, p. \end{cases} \tag{4.3.2}$$

where $\iota\colon \Sigma \to M$ denotes the embedding of the chosen spacelike Cauchy surface. (All other components $\mathbf{data}^n$, $n \leq -p-1$ and $n \geq p+1$, necessarily vanish.)

Remark 4.3.1. The cochain map **data** in (4.3.2) is a quasi-isomorphism since it induces the cohomology isomorphism $H(\mathbf{data})\colon H(\mathfrak{Sol}_{\mathrm{sc}}(M)) \xrightarrow{\cong} H(\mathfrak{D}_{\mathrm{c}}(M))$, which in terms of the de Rham cohomology of M with spacelike compact support and the de Rham cohomology of Σ with compact support reads as

$$\begin{cases} H^{p+n}_{\mathrm{sc}}(M) \cong H^{p+n}_{\mathrm{c}}(\Sigma), & n = -p, \dots, -1, \\ \Omega^p_{\mathrm{sc}\,\delta\mathrm{d}}(M)/\mathrm{d}\Omega^{p-1}_{\mathrm{sc}}(M) \cong \big(\Omega^p_{\mathrm{c}}(\Sigma)/\mathrm{d}_\Sigma\Omega^{p-1}_{\mathrm{c}}(\Sigma)\big) \times \Omega^p_{\mathrm{c}\,\delta_\Sigma}(\Sigma), & n = 0, \\ H^{m-p-1+n}_{\mathrm{sc}}(M) \cong H^{m-p-1+n}_{\mathrm{c}}(\Sigma), & n = 1, \dots, p+1. \end{cases} \tag{4.3.3}$$

We refer to [Ben16; Kha16] for the isomorphisms in degrees $n \neq 0$ and to [SDH14] for the isomorphism in degree $n = 0$, where it is stated in the form of the well-posed initial value problem for gauge classes of on-shell Maxwell p-forms. This perspective suggests the interpretation of the quasi-isomorphism **data** as a refinement of this well-posed initial value problem.

Combining the quasi-isomorphisms **data** and Λ from Remarks 4.2.1 and 4.3.1, we obtain a quasi-isomorphism $\mathbf{data}\,\Lambda\colon \mathcal{L}(M) \to \mathfrak{D}_{\mathrm{c}}(\Sigma)$, which provides an explicit computation of the cohomology of the observable complex $\mathcal{L}(M)$ for Maxwell p-forms in terms of the compactly supported de Rham cohomology of a spacelike Cauchy surface $\Sigma \subseteq M$ and of the initial data for gauge classes of on-shell Maxwell p-forms. △

4.4 Poisson structure and quantization

This section is devoted to the construction of a homotopical algebraic quantum field theory, associated with Maxwell p-forms, over the category $\mathbf{Loc}_m$ of oriented and time-oriented m-dimensional globally hyperbolic Lorentzian manifolds (see Definition 3.8.1). The first step constructs a Poisson structure $\tau_M\colon \mathcal{L}(M)^{\wedge 2} \to \mathbb{R}$ in $\mathbf{CoCh}_{\mathbb{R}}$ on the observable complex $\mathcal{L}(M) \in \mathbf{CoCh}_{\mathbb{R}}$, for each $M \in \mathbf{Loc}_m$. This defines a Poisson complex $(\mathcal{L}(M), \tau_M)$. The second step quantizes this Poisson complex using canonical commutation relations (CCR). The third step exhibits the net structure. Incidentally, we will observe that the resulting net of algebras is actually a homotopical algebraic quantum field theory, i.e. it fulfils both the Einstein's causality axiom and the homotopy time-slice axiom, (see Definition 3.8.5). Since we will work with the symmetric monoidal model category of cochain complexes $\mathbf{M} = \mathbf{CoCh}_{\mathbb{C}}$, we shall replace the term "monoid" with the more familiar "differential graded algebra". Accordingly, we shall adopt the familiar notation $\mathbf{DGA}_{\mathbb{C}} \coloneqq \mathbf{Mon}(\mathbf{CoCh}_{\mathbb{C}})$.

Combining (4.1.8) and (4.2.2) allows us to equip the observable complex $\mathcal{L}(M)$ with the *Poisson structure*

$$\tau_M \coloneqq -\mathrm{ev}(\Lambda \otimes \mathrm{id})\colon \mathcal{L}(M)^{\wedge 2} \longrightarrow \mathbb{R} \tag{4.4.1}$$

in $\mathbf{CoCh}_{\mathbb{R}}$. (The obvious inclusion $\mathfrak{Sol}_{\mathrm{sc}}(M) \to \mathfrak{Sol}(M)$ in $\mathbf{CoCh}_{\mathbb{R}}$ is implicit in the definition above.) Note that τ_M is graded anti-symmetric, as implicitly claimed in (4.4.1). This follows because, with respect to the pairing $\int_M (-) \wedge *(-)$, the codifferential δ is the formal adjoint of the de Rham differential d , $\Box$ is formally self-adjoint and, as a consequence, also the retarded and advanced Green's operators are each the formal adjoint of the other.

We quantize the Poisson complex $(\mathcal{L}(M), \tau_M)$, consisting of the observable complex $\mathcal{L}(M) \in \mathbf{CoCh}_{\mathbb{R}}$ from (4.1.7) and the Poisson structure $\tau_M\colon \mathcal{L}(M)^{\wedge 2} \to \mathbb{R}$ in $\mathbf{CoCh}_{\mathbb{R}}$ from (4.4.1), applying the CCR quantization functor of [BBS20]. Explicitly, we consider the free differential graded algebra

$$T_{\mathbb{C}}\big(\mathcal{L}(M)\big) \coloneqq \bigoplus_{m \geq 0} \mathcal{L}(M)_{\mathbb{C}}^{\otimes m} \in \mathbf{DGA}_{\mathbb{C}} \tag{4.4.2a}$$

generated by the complexification $\mathcal{L}(M)_{\mathbb{C}} \coloneqq \mathcal{L}(M) \otimes_{\mathbb{R}} \mathbb{C} \in \mathbf{CoCh}_{\mathbb{C}}$ of the observable complex $\mathcal{L}(M) \in \mathbf{CoCh}_{\mathbb{R}}$. The multiplication μ is defined by juxtaposition $\varphi_1 \otimes \cdots \otimes \varphi_k \otimes \psi_1 \otimes \cdots \otimes \psi_m$ of words $\varphi_1 \otimes \cdots \otimes \varphi_k, \psi_1 \otimes \cdots \otimes \psi_m$ in $T_{\mathbb{C}}(\mathcal{L}(M))$ and the unit $\mathbf{1}$ corresponds to the length 0 word $1 \in \mathbb{C} \subseteq T_{\mathbb{C}}(\mathcal{L}(M))$. Taking the quotient by the two-sided ideal $I_\tau \subseteq T_{\mathbb{C}}(\mathcal{L}(M))$ generated by the elements

$$\varphi_1 \otimes \varphi_2 - (-1)^{|\varphi_1||\varphi_2|}\, \varphi_2 \otimes \varphi_1 - i\tau_M(\varphi_1, \varphi_2)\mathbf{1}, \tag{4.4.2b}$$

for all homogeneous cochains $\varphi_1, \varphi_2 \in \mathcal{L}(M)$ defines the differential graded algebra

$$\mathfrak{A}_M \coloneqq T_{\mathbb{C}}\big(\mathcal{L}(M)\big)/I_\tau \in \mathbf{DGA}_{\mathbb{C}}. \tag{4.4.2c}$$

Remark 4.4.1. It is straightforward to endow the differential graded algebra $\mathfrak{A}_M \in \mathbf{DGA}_{\mathbb{C}}$ with a multiplication reversing $*$-involution arising from complex conjugation $\overline{(-)}$ on $\mathbb{C}$. (We refer to [Jac12; BSW19b] for a more detailed discussion on reversing $*$-monoids in an involutive symmetric monoidal category.) Indeed, one defines $*$ on the free differential graded algebra $T_{\mathbb{C}}(\mathcal{L}(M)) \in \mathbf{DGA}_{\mathbb{C}}$ as the unique multiplication reversing graded $\mathbb{C}$-antilinear map extending the complex conjugation $\varphi \mapsto \overline{\varphi}$ on length 1 words φ in $\mathcal{L}(M)_{\mathbb{C}} \subseteq T_{\mathbb{C}}(\mathcal{L}(M))$. The graded anti-symmetry of τ entails that $I_\tau \subseteq T_{\mathbb{C}}(\mathcal{L}(M))$ is a two-sided $*$-ideal, therefore $*$ descends to the quotient $\mathfrak{A}_M \in \mathbf{DGA}_{\mathbb{C}}$. In other words, $\mathfrak{A}_M$ is a differential graded unital and associative $*$-algebra. △

We focus now on the construction of the net structure. Recall that differential forms with compact support can be extended by zero along open embeddings $f\colon M \to N$. Let us denote the corresponding extension-by-zero map by $f_*\colon \Omega^k_{\mathrm{c}}(M) \to \Omega^k_{\mathrm{c}}(N)$. Since all morphisms $f\colon M \to N$ in $\mathbf{Loc}_m$ are in particular open embeddings, one defines the cochain maps

$$\mathcal{L}(f)\colon \mathcal{L}(M) \longrightarrow \mathcal{L}(N) \tag{4.4.3}$$

in $\mathbf{CoCh}_{\mathbb{R}}$ degree-wise by $\mathcal{L}(f)^n := f_*$ for $n = -p-1, \dots, p$ and $\mathcal{L}(f)^n := 0$ else. Together with the assignment of the observable complex $M \in \mathbf{Loc}_m \mapsto \mathcal{L}(M) \in \mathbf{CoCh}_{\mathbb{R}}$, this defines the functor $\mathcal{L}\colon \mathbf{Loc}_m \to \mathbf{CoCh}_{\mathbb{R}}$. The naturality of the de Rham differential d and of the Hodge star operator $*$ with respect to morphisms in $\mathbf{Loc}_m$ entails the naturality of the linear differential operator $\Box\colon \Omega^k(M) \to \Omega^k(M)$. From the uniqueness of the associated retarded and advanced Green's operators $G^{(k)}_\pm\colon \Omega^k_{\mathrm{c}}(M) \to \Omega^k(M)$, it follows that $f^*\, G^{(k)}_\pm\, f_* = G^{(k)}_\pm$, for all $f\colon M \to N$ in $\mathbf{Loc}_m$, where $f^*\colon \Omega^k(N) \to \Omega^k(M)$ denotes the pull-back of k-forms. This fact entails that also the Poisson structure is natural, i.e. $\tau_N\,(\mathcal{L}(f) \otimes \mathcal{L}(f)) = \tau_M$, for all $f\colon M \to N$ in $\mathbf{Loc}_m$. In other words, one obtains a functor $(\mathcal{L}, \tau)$ that assigns to each $M \in \mathbf{Loc}_m$ the Poisson complex $(\mathcal{L}(M), \tau_M)$ and to each morphism $f\colon M \to N$ in $\mathbf{Loc}_m$ the Poisson structure preserving cochain map $\mathcal{L}(f)\colon (\mathcal{L}(M), \tau_M) \to (\mathcal{L}(N), \tau_N)$. The CCR quantization recalled in (4.4.2) is manifestly functorial (see [BBS20]). As a result, composing the functor $(\mathcal{L}, \tau)$ and the CCR quantization functor we obtain a functor $\mathfrak{A}\colon \mathbf{Loc}_m \to \mathbf{DGA}_{\mathbb{C}}$, i.e. a net of algebras

$$\mathfrak{A} \in \mathbf{Net}^{\mathbf{CoCh}_{\mathbb{C}}}_{\mathbf{Loc}_m}, \tag{4.4.4}$$

see Remark 3.1.3, associated with Maxwell p-forms.

Remark 4.4.2. We observe incidentally that the net $\mathfrak{A} \in \mathbf{Net}^{\mathbf{CoCh}_{\mathbb{C}}}_{\mathbf{Loc}_m}$ constructed above, endowed with the $*$-involution from Remark 4.4.1, is actually a homotopical algebraic quantum field theory, as described in Definition 3.8.5. In other words, this net fulfils both the Einstein's causality axiom and the homotopy time-slice axiom. The Einstein's causality axiom, i.e. the fact that the graded commutator

$$[\mathfrak{A}(f_1), \mathfrak{A}(f_2)] = 0 \colon \mathfrak{A}_{M_1} \otimes \mathfrak{A}_{M_2} \longrightarrow \mathfrak{A}_N \tag{4.4.5}$$

in $\mathbf{CoCh}_{\mathbb{C}}$ vanishes for all pairs of morphisms $f_1 \colon M_1 \to N \leftarrow M_2 \colon f_2$ in $\mathbf{Loc}_m$ with causally disjoint images, is a straightforward consequence of the support properties of the retarded and advanced Green's operators that enter the definition of the Poisson structure through the cochain map Λ, see (4.2.1), (4.2.2) and (4.4.1).

It is slightly more involved to check the homotopy time-slice axiom, i.e. that the morphism $\mathfrak{A}(f) \colon \mathfrak{A}_M \to \mathfrak{A}_N$ in $\mathbf{DGA}_{\mathbb{C}}$ is a weak equivalence whenever $f \colon M \to N$ in $\mathbf{Loc}_m$ is a Cauchy morphism. Recall first that weak equivalences in $\mathbf{DGA}_{\mathbb{C}}$ are just quasi-isomorphisms between the underlying cochain complexes. The fact that the cochain map underlying $\mathfrak{A}(f)$ is indeed a quasi-isomorphism can be best understood taking a closer look at the cochain map $\mathcal{L}(f) \colon \mathcal{L}(M) \to \mathcal{L}(N)$ in $\mathbf{CoCh}_{\mathbb{R}}$ from (4.4.3). Recalling that per hypothesis f is a Cauchy morphism, i.e. the image $f(M) \subseteq N$ contains a spacelike Cauchy surface $\Sigma \subseteq f(M)$ of the codomain N, one obtains the commutative diagram

$$\begin{array}{ccc} \mathcal{L}(M) & \xrightarrow{\mathcal{L}(f)} & \mathcal{L}(N) \\ & {\scriptstyle \mathbf{data}\,\Lambda} \searrow \quad \swarrow {\scriptstyle \mathbf{data}\,\Lambda} & \\ & \mathfrak{D}_{\mathrm{c}}(\Sigma) & \end{array} \tag{4.4.6}$$

in $\mathbf{CoCh}_{\mathbb{R}}$ involving the passage to the initial data complex $\mathfrak{D}_{\mathrm{c}}(\Sigma)$, see (4.3.1) and (4.3.2). (Note that we identified the spacelike Cauchy surface $\Sigma \subseteq N$ with its preimage in M via f, which is automatically a spacelike Cauchy surface of M.) As observed in Remarks 4.2.1 and 4.3.1, both Λ and **data** are quasi-isomorphisms. The fact that quasi-isomorphisms are closed under composition and fulfil the two-out-of-three property entails that $\mathcal{L}(f)$ is a quasi-isomorphism too. Summing up, $\mathcal{L}(f) \colon (\mathcal{L}(M), \tau_M) \to (\mathcal{L}(N), \tau_N)$ is a Poisson structure preserving quasi-isomorphism. By [BBS20, Prop. 5.3], see also [BS19b], the CCR quantization functor, which enters the construction of the Maxwell p-forms net $\mathfrak{A}$ from (4.4.4), maps Poisson structure preserving quasi-isomorphisms to weak equivalences. Therefore, it follows that $\mathfrak{A}(f) \colon \mathfrak{A}_M \to \mathfrak{A}_N$ in $\mathbf{DGA}_{\mathbb{C}}$ is a weak equivalence for all Cauchy morphisms $f \colon M \to N$ in $\mathbf{Loc}_m$. This proves that the Maxwell p-forms net $\mathfrak{A}$ fulfils also the homotopy time-slice axiom holds and hence it defines a homotopical algebraic quantum field theory in the sense of [BBS20; BSW19a; BS19a]. △

4.5 Construction of a net representation

The goal of the remaining sections is to construct a representation of the $\mathbf{CoCh}_{\mathbb{C}}$-valued (Haag-Kastler) net

$$\mathfrak{A}_M := \mathfrak{A}\,\iota_M \in \mathbf{Net}^{\mathbf{CoCh}_{\mathbb{C}}}_{\mathbf{CCO}(M)} \tag{4.5.1}$$

on $\mathbf{CCO}(M)$ (see Definition 3.8.2) that describes quantized Maxwell p-form fields on a fixed oriented and time-oriented globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$. The net $\mathfrak{A}_M$ is obtained by restricting the (generally covariant) net $\mathfrak{A} \in \mathbf{Net}^{\mathbf{CoCh}_{\mathbb{C}}}_{\mathbf{Loc}_m}$ from (4.4.4) along the functor $\iota_M \colon \mathbf{CCO}(M) \to \mathbf{Loc}_m$ that sends a causally convex open subset $U \in \mathbf{CCO}(M)$ to the oriented and time-oriented globally hyperbolic Lorentzian manifold $U \in \mathbf{Loc}_m$ defined by endowing U with the restriction of the geometry of M. We shall obtain a representation of the net $\mathfrak{A}_M$ from a two-point function $\omega_2 \colon \mathcal{L}(M) \otimes \mathcal{L}(M) \to \mathbb{C}$ in $\mathbf{CoCh}_{\mathbb{R}}$ on the observable complex $\mathcal{L}(M)$ from (4.1.7), i.e. a cochain map whose anti-symmetric part agrees with the Poisson structure τ_M from (4.4.1). Mimicking the usual construction of a quasi-free state from a two-point function (see e.g. [KM15, Sec. 5.2.4]), we shall use ω_2 to define a linear functional $\omega \colon \mathfrak{A}_M \to \mathbb{C}$ in $\mathbf{CoCh}_{\mathbb{C}}$. The first step of the Gelfand-Naimark-Segal construction applied to ω shall then provide a representation of the global algebra of observables $\mathfrak{A}_M = \mathfrak{A}_M(M)$. The latter defines a constant net representation of $\mathfrak{A}_M$ via the constant representation functor in Proposition 3.3.3.

Remark 4.5.1. Usually two-point functions are not only required to have anti-symmetric part matching the Poisson structure of interest, but are also required to be bisolutions of the equation of motion that governs the field theoretic model of interest. In our framework the latter requirement is replaced and generalized by the condition that ω_2 is a cochain map. Notably, this condition takes also care of the compatibility with the action of gauge transformations. △

Geometric setup In order to simplify the presentation and to better highlight the main features of our construction, we shall assume that $M \in \mathbf{Loc}_m$ is of the form

$$M = \mathbb{R} \times \Sigma, \tag{4.5.2a}$$

with Σ a *compact* spacelike Cauchy surface, and that the metric g is ultra-static, i.e.

$$g = -dt^2 + h, \tag{4.5.2b}$$

with h a Riemannian metric on Σ (constant with respect to $t \in \mathbb{R}$).

Remark 4.5.2. The construction that follows is applicable also when Σ is not compact, but it would require a more refined Hodge decomposition (see e.g. [MS24]) than the one we use in Equation (4.5.5). △

Decomposition of forms The explicit construction of the two-point function $\omega_2\colon \mathcal{L}(M)^{\otimes 2} \to \mathbb{C}$ in $\mathbf{CoCh}_{\mathbb{R}}$ partly follows the lines of [BCD17]. First, since $M = \mathbb{R} \times \Sigma$ is a product, k-forms on M decompose into sections of the pullbacks along the projection $\mathrm{pr}_2\colon M \to \Sigma$ of the bundles $\Lambda^k\Sigma$ of k-forms and $\Lambda^{k-1}\Sigma$ of $(k-1)$-forms over Σ. By abuse of notation, we denote the pullback bundles over M again by $\Lambda^k\Sigma$ and $\Lambda^{k-1}\Sigma$, so that the above mentioned decomposition takes the form

$$\Omega^k(M) = \Gamma(M, \Lambda^k\Sigma) \oplus dt \wedge \Gamma(M, \Lambda^{k-1}\Sigma), \qquad \alpha = \alpha_S + dt \wedge \alpha_T. \tag{4.5.3}$$

The first summand corresponds to k-forms on M having only "spatial legs", while the second summand corresponds to k-forms on M having precisely one "time leg". We shall use the subscripts $_S$ and $_T$ to refer to the space $\alpha_S \in \Gamma(M, \Lambda^k\Sigma)$ and respectively time $\alpha_T \in \Gamma(M, \Lambda^{k-1}\Sigma)$ parts of a k-form $\alpha \in \Omega^k(M)$. Since the ultra-static metric $g = -dt^2 + h$ decomposes into time and space parts, the de Rham differential d, the codifferential δ and the d'Alembert operator $\Box$ acting on k-forms on M admit a decomposition compatible with (4.5.3). Explicitly, denoting with d_S the de Rham differential, with δ_S the codifferential and with $\triangle \coloneqq \delta_S \mathrm{d}_S + \mathrm{d}_S \delta_S$ the Laplace operator, all acting on differential forms on Σ, and with ∂_t the time-derivative, one finds

$$\mathrm{d}\alpha = \mathrm{d}_S\alpha_S + dt \wedge (\partial_t \alpha_S - \mathrm{d}_S \alpha_T), \tag{4.5.4a}$$

$$\delta\alpha = (\partial_t \alpha_T + \delta_S \alpha_S) - dt \wedge \delta_S \alpha_T, \tag{4.5.4b}$$

$$\delta\mathrm{d}\alpha = (\partial_t^2 \alpha_S - \mathrm{d}_S \partial_t \alpha_T + \delta_S \mathrm{d}_S \alpha_S) + dt \wedge (\delta_S \mathrm{d}_S \alpha_T - \delta_S \partial_t \alpha_S), \tag{4.5.4c}$$

$$\Box\alpha = (\partial_t^2 \alpha_S + \triangle \alpha_S) + dt \wedge (\partial_t^2 \alpha_T + \triangle \alpha_T), \tag{4.5.4d}$$

for all $\alpha \in \Omega^k(M)$.

Retarded-minus-advanced propagator on k-forms Combining (4.5.3) and (4.5.4) with the Hodge decomposition for differential k-forms on Σ

$$\Omega^k(\Sigma) = \mathcal{H}^k(\Sigma) \oplus \mathcal{H}^k_\perp(\Sigma), \qquad \sigma = \sigma_{\mathcal{H}} + \sigma_\perp, \tag{4.5.5}$$

into (spatially) harmonic $\sigma_{\mathcal{H}} \in \mathcal{H}^k(\Sigma)$ ($\triangle \sigma_{\mathcal{H}} = 0$) and orthogonal $\sigma_\perp \in \mathcal{H}^k_\perp(\Sigma)$ ($\sigma_\perp = \mathrm{d}_S \sigma_1 + \delta_S \sigma_2$) parts, one obtains an explicit formula for the retarded-minus-advanced propagator

$$G^{(k)} \coloneqq G^{(k)}_+ - G^{(k)}_- \colon \Omega^k_{\mathrm{c}}(M) \longrightarrow \Omega^k(M) \tag{4.5.6a}$$

associated with $\Box\colon \Omega^k(M) \to \Omega^k(M)$, see Section 4.2, given by

$$G^{(k)}\alpha = G\alpha_S + dt \wedge G\alpha_T, \tag{4.5.6b}$$

for all $\alpha \in \Omega^k_{\mathrm{c}}(M)$, where

$$(G\beta)(t,\cdot) \coloneqq \int_{\mathbb{R}} dt'\,(t-t')\beta_{\mathcal{H}}(t',\cdot) + \int_{\mathbb{R}} dt'\,\Big(\triangle^{-\frac{1}{2}}\sin\big(\triangle^{\frac{1}{2}}(t-t')\big)\beta_{\perp}\Big)(t',\cdot), \tag{4.5.6c}$$

with $\beta \in \Gamma_{\mathrm{c}}(M, \Lambda^j\Sigma)$, $j = k, k-1$. (Note that the Laplacian $\triangle$ on the orthogonal part $\mathcal{H}^j_{\perp}(\Sigma)$ has strictly positive spectrum, hence one can consider $\triangle^{\frac{1}{2}}$, as well as its inverse $\triangle^{-\frac{1}{2}}$. Incidentally, we observe that G, is the retarded-minus-advanced propagator for the normally hyperbolic linear differential operator $\partial_T^2 + \triangle\colon \Gamma(M,\Lambda^j\Sigma) \to \Gamma(M,\Lambda^j\Sigma)$.) (4.5.6) can be used to compute Λ from (4.2.2) more explicitly, which leads to an explicit formula for the Poisson structure τ_M from (4.4.1) that makes the so-called "positive and negative frequency contributions" manifest.

Construction of a two-point function With these preparations, we construct a two-point function ω_2 by mimicking the usual prescription, which amounts to selecting the "positive frequency contribution" in (4.5.6). Explicitly, we introduce

$$\omega_2\colon \mathcal{L}(M)^{\otimes 2} \longrightarrow \mathbb{C} \tag{4.5.7a}$$

in $\mathbf{CoCh}_{\mathbb{R}}$ defining the only non-vanishing component

$$\omega_2^0\colon \bigoplus_{m=-p}^{p} \big(\mathcal{L}(M)^m \otimes \mathcal{L}(M)^{-m}\big) \longrightarrow \mathbb{C} \tag{4.5.7b}$$

component-wise for all $m = -p, \dots, p$ by

$$(\omega_2^0)_m \coloneqq \begin{cases} (-1)^{\lfloor m/2 \rfloor}\, W^{(p+m)}\,(\delta \otimes \mathrm{id}), & m = -p, \dots, -1, \\ W^{(p)}, & m = 0, \\ (-1)^{\lfloor m/2 \rfloor}\, W^{(p-m)}\,(\mathrm{id} \otimes \delta), & m = 1, \cdots, p, \end{cases} \tag{4.5.7c}$$

where the linear map

$$W^{(k)}\colon \Omega^k_{\mathrm{c}}(M)^{\otimes 2} \longrightarrow \mathbb{C}, \tag{4.5.8a}$$

for $k = 0, \dots, p$, is defined for all $\alpha, \alpha' \in \Omega^k_{\mathrm{c}}(M)$ by

$$W^{(k)}(\alpha \otimes \alpha') \coloneqq W_{\mathcal{H}}(\alpha_S \otimes \alpha'_S) + W_{\perp}(\alpha_S \otimes \alpha'_S) - W_{\mathcal{H}}(\alpha_T \otimes \alpha'_T) - W_{\perp}(\alpha_T \otimes \alpha'_T), \tag{4.5.8b}$$

with

$$W_{\mathcal{H}}(\beta \otimes \beta') := \frac{1}{2} \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \left\langle \beta_{\mathcal{H}}(t, \cdot), i(t - t')\beta'_{\mathcal{H}}(t', \cdot) \right\rangle, \tag{4.5.8c}$$

$$W_{\perp}(\beta \otimes \beta') := \frac{1}{2} \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \left\langle \beta_{\perp}(t, \cdot), \left(\triangle^{-\frac{1}{2}} \exp\left(i\triangle^{\frac{1}{2}}(t - t')\right)\beta'_{\perp}\right)(t', \cdot) \right\rangle, \tag{4.5.8d}$$

for all $\beta, \beta' \in \Gamma_{\mathrm{c}}(M, \Lambda^j \Sigma)$. Here the pairing $\langle \cdot, \cdot \rangle$ denotes the usual scalar product between $\mathbb{C}$-valued j-forms on Σ for $j = k, k-1$. (As before, $\triangle^{\frac{1}{2}}$ and $\triangle^{-\frac{1}{2}}$ are well defined since the Laplacian $\triangle$ on the orthogonal part $\mathcal{H}^j_{\perp}(\Sigma)$ has strictly positive spectrum.)

Incidentally, we observe that G, is the retarded-minus-advanced propagator for the normally hyperbolic linear differential operator $\partial_T^2 + \triangle \colon \Gamma(M, \Lambda^j \Sigma) \to \Gamma(M, \Lambda^j \Sigma)$.)

Now it remains to verify that ω_2 is a cochain map whose anti-symmetric part matches the Poisson structure of interest, as specified in Remark 4.5.1.

First, observe that $W^{(k)}$ is a $\square$-bisolution for all $k = 0, \dots, p$, i.e.

$$W^{(k)}(\square \otimes \mathrm{id}) = 0 = W^{(k)}(\mathrm{id} \otimes \square), \tag{4.5.9}$$

which follows from the fact that both $W_{\mathcal{H}}$ and $W_{\perp}$ are by construction $(\partial_t^2 + \triangle)$-bisolutions. Furthermore, combining (4.5.4) and (4.5.8) one checks that

$$W^{(k)}(\mathrm{d} \otimes \mathrm{id}) = W^{(k-1)}(\mathrm{id} \otimes \delta), \quad W^{(k)}(\mathrm{id} \otimes \mathrm{d}) = W^{(k-1)}(\delta \otimes \mathrm{id}), \tag{4.5.10}$$

for all $k = 1, \dots, p$.

Let us now confirm that ω_2 is indeed a cochain map. Explicitly, this is equivalent to the conditions

$$\begin{cases} W^{(0)}(\delta\mathrm{d} \otimes \mathrm{id}) = 0, & m = -p, \\ W^{(p+m)}(\delta\mathrm{d} \otimes \mathrm{id}) + W^{(p+m-1)}(\delta \otimes \delta) = 0, & m = -p+1, \dots, -1, \\ W^{(p)}(\delta\mathrm{d} \otimes \mathrm{id}) + W^{(p-1)}(\delta \otimes \delta) = 0, & m = 0, \\ W^{(p-1)}(\delta \otimes \delta) + W^{(p)}(\mathrm{id} \otimes \delta\mathrm{d}) = 0, & m = 1, \\ W^{(p-m)}(\delta \otimes \delta) + W^{(p-m+1)}(\mathrm{id} \otimes \delta\mathrm{d}) = 0, & m = 2, \dots, p, \\ W^{(p)}(\mathrm{id} \otimes \delta\mathrm{d}) = 0, & m = p+1. \end{cases} \tag{4.5.11}$$

These conditions are a direct consequence of (4.5.9) and (4.5.10), hence ω_2 is a cochain map, as claimed.

Moreover, one can verify that the anti-symmetric part of ω_2 agrees with τ_M, namely

$$\omega_2(\varphi_1 \otimes \varphi_2) - (-1)^{|\varphi_1||\varphi_2|} \omega_2(\varphi_2 \otimes \varphi_1) = i\, \tau_M(\varphi_1 \otimes \varphi_2), \tag{4.5.12}$$

for all homogeneous cochains $\varphi_1, \varphi_2 \in \mathcal{L}(M)$. This follows directly from (4.5.7) and (4.5.8) recalling (4.2.2), (4.4.1) and (4.5.6). This property will ensure that the cochain map $\omega\colon \mathfrak{A}_M \to \mathbb{C}$ in $\mathbf{CoCh}_{\mathbb{C}}$ defined in (4.5.13) is compatible with the canonical commutation relations (4.4.2) and hence well-defined.

Remark 4.5.3. Note that in (4.5.8) one could add any (time-constant) symmetric operator A_j acting on harmonic j-forms on Σ by replacing $i(t-t')$ with $i(t-t') + A_j$. Since Σ is assumed to be compact, harmonic j-forms on Σ form a finite dimensional Hilbert space. Therefore A_j is just any symmetric matrix. △

Remark 4.5.4. We emphasize that the integral kernel of $W_\perp$ from (4.5.8) is a bidistribution fulfilling the microlocal spectrum condition, see e.g. [KM15, Sec. 5.3.4]. In particular, this entails that the induced two-point function $H^0(\omega_2)\colon H^0(\mathcal{L}(M)) \otimes H^0(\mathcal{L}(M)) \to \mathbb{C}$ on the degree 0 cohomology of the observable complex $\mathcal{L}(M)$ fulfils the microlocal spectrum condition. Note that in degree 0 cohomology and for $p = 1$ our construction reproduces gauge-invariant on-shell linear observables for the electromagnetic vector potential. Indeed, the two-point function $H^0(\omega_2)\colon H^0(\mathcal{L}(M)) \otimes H^0(\mathcal{L}(M)) \to \mathbb{C}$ agrees with the Hadamard two-point function of [FP03, Sec. IV.C]. △

Linear functional from two-point function The next step uses ω_2 from (4.5.7) to define a linear functional

$$\omega\colon \mathfrak{A}_M \longrightarrow \mathbb{C} \tag{4.5.13a}$$

in $\mathbf{CoCh}_{\mathbb{C}}$ on (the cochain complex underlying) the quantized differential graded algebra $\mathfrak{A}_M$ from (4.4.2). The latter is specified on words of arbitrary length $m \geq 0$ ($m = 0$ corresponds to the unit $\mathbf{1} \in \mathfrak{A}_M$) by

$$\omega(\varphi_1 \otimes \cdots \otimes \varphi_m) := \begin{cases} 1, & m = 0, \\ 0, & m = 2k-1, k \geq 1, \\ \displaystyle\sum_{\sigma \in P} \operatorname{sign}(\sigma; |\varphi_1|, \dots, |\varphi_{2k}|) \cdot \\ \quad \cdot \prod_{i=1}^k \omega_2(\varphi_{\sigma_{i1}} \otimes \varphi_{\sigma_{i2}}), & m = 2k, k \geq 1, \end{cases} \tag{4.5.13b}$$

for all homogeneous cochains $\varphi_1, \dots, \varphi_m \in \mathcal{L}(M)$, see [KM15, Sec. 5.2.4]. In the previous formula P denotes the set of all partitions $\sigma = \{\sigma_1, \dots, \sigma_k\}$ of the ordered set $\{1 < \dots < 2k\}$ into k ordered pairs $\sigma_i = \{\sigma_{i1} < \sigma_{i2}\}$, $i = 1, \dots, k$. Furthermore, $\operatorname{sign}(\sigma; |\varphi_1|, \dots, |\varphi_{2k}|)$ denotes the Koszul sign obtained by permuting the letters $\varphi_1, \dots, \varphi_{2k}$ according to the permutation associated with σ. Let us confirm that ω is a well-defined cochain map. First, combining (4.5.12) and the explicit formula in (4.5.13), it follows that ω vanishes on the ideal generated by the canonical commutation relations (4.4.2) and hence it descends to the quotient $\mathfrak{A}_M$. Furthermore, the fact that ω_2 is a cochain map entails that ω is a cochain map too. (To this end Koszul signs play a crucial role.)

Remark 4.5.5. Since ω is a cochain map, passing to degree 0 cohomology one obtains a linear functional $H^0(\omega)\colon H^0(\mathfrak{A}_M) \to \mathbb{C}$. It is straightforward to check that the algebra $H^0(\mathfrak{A}_M)$ contains the CCR algebra $\mathfrak{A}^{\mathrm{inv}}(M) \subseteq H^0(\mathfrak{A}_M)$ associated with the degree 0 cohomology $H^0(\mathcal{L}(M))$ of the observable complex, consisting of gauge-invariant on-shell linear observables, endowed with the induced Poisson structure $H^0(\tau_M)\colon H^0(\mathcal{L}(M))^{\wedge 2} \to \mathbb{R}$. Recalling Remark 4.5.4, one observes that the restriction $\omega^{\mathrm{inv}}\colon \mathfrak{A}^{\mathrm{inv}}(M) \to \mathbb{C}$ of $H^0(\omega)$ is a Hadamard state. In particular, for $p = 1$, this produces a Hadamard state on the CCR algebra $\mathfrak{A}^{\mathrm{inv}}(M)$ of gauge-invariant on-shell linear observables for the electromagnetic vector potential. As we shall explain in Remark 4.6.2, the algebra $H^0(\mathfrak{A}_M)$ is often strictly larger than the algebra $\mathfrak{A}^{\mathrm{inv}}(M)$. As a consequence $H^0(\omega)$ is often richer than its restriction ω^{inv}. △

Construction of the net representation The second step of our construction of a net representation mimics the first part of the Gelfand-Naimark-Segal construction. This yields a representation of the differential graded algebra $\mathfrak{A}_M(M) = \mathfrak{A}_M \in \mathbf{DGA}_{\mathbb{C}}$ induced by ω from (4.5.13). Introducing the subcomplex $R_\omega \subseteq \mathfrak{A}_M(M) \in \mathbf{CoCh}_{\mathbb{C}}$ defined degree-wise for all $n \in \mathbb{Z}$ by

$$R^n_\omega \coloneqq \big\{ a \in \mathfrak{A}_M(M)^n : : \omega(ba) = 0,\ \forall b \in \mathfrak{A}_M(M)^{-n} \big\}, \tag{4.5.14a}$$

one immediately observes that R_ω is a left $\mathfrak{A}_M(M)$-ideal. Therefore, the quotient of $\mathfrak{A}_M(M)$ by R_ω defines the left $\mathfrak{A}_M(M)$-module

$$V_\omega \coloneqq \mathfrak{A}_M(M)/R_\omega \in {}_{\mathfrak{A}_M(M)}\mathbf{Mod}. \tag{4.5.14b}$$

In other words, V_ω defines a representation of the global algebra of observables $\mathfrak{A}_M(M)$.

Finally, in order to obtain a net representation of $\mathfrak{A}_M$ we exploit Proposition 3.3.3. Evaluating the right adjoint functor $(-)^M\colon {}_{\mathfrak{A}_M(M)}\mathbf{Mod} \to \mathbf{Rep}(\mathfrak{A}_M)$ on V_ω defines the constant net representation

$$\mathfrak{V}_\omega \coloneqq V_\omega^M \in \mathbf{Rep}(\mathfrak{A}_M) \tag{4.5.15}$$

of the net $\mathfrak{A}_M \in \mathbf{Net}^{\mathbf{CoCh}}_{\mathbf{CCO}(M)}$ from (4.5.1). Explicitly, for each causally convex open subset $U \in \mathbf{CCO}(M)$, $\mathfrak{V}_\omega(U) = V_\omega|_{\mathfrak{A}_M(U)} \in {}_{\mathfrak{A}_M(U)}\mathbf{Mod}$ is just the restriction of the left $\mathfrak{A}_M(M)$-module V_ω along the morphism $\mathfrak{A}_M(\iota_U^M)\colon \mathfrak{A}_M(U) \to \mathfrak{A}_M(M)$ in $\mathbf{DGA}_\mathbb{C}$ associated with the inclusion $\iota_U^M : U \to M$ in $\mathbf{CCO}(M)$.

4.6 2-dimensional example

In this section we shall compute explicitly the global algebra of observables $\mathfrak{A}_M(M)$ of the net $\mathfrak{A}_M \in \mathbf{Net}^{\mathbf{CoCh}_\mathbb{C}}_{\mathbf{CCO}(M)}$ from (4.5.1) and describe its constant net representations. We shall do so in the simplest scenario, namely setting $m = 2$, $p = 1$ and choosing as oriented and time-oriented globally hyperbolic Lorentzian manifold the 2-dimensional flat Lorentz cylinder $M \in \mathbf{Loc}_2$ consisting of the manifold $\mathbb{R} \times S^1$ endowed with the constant metric $g = -dt^2 + d\theta^2$, with the time-orientation determined by the vector field ∂_t and with the counter-clockwise orientation on the unit length circle S^1. Concretely, we shall construct a differential graded algebra $A \in \mathbf{DGA}_\mathbb{C}$ that is weakly equivalent to the original one $\mathfrak{A}_M(M)$, but has the advantages of having finitely many generators and trivial differential. These features make its category of left modules ${}_A\mathbf{Mod}$ easier to describe than the category ${}_{\mathfrak{A}_M(M)}\mathbf{Mod}$, which is Quillen equivalent to it on account of Proposition 2.8.8, but practically less accessible. This fact will allow us to obtain a very explicit description of all the constant net representations of $\mathfrak{A}_M$ up to weak equivalence.

To construct the differential graded algebra $A \in \mathbf{DGA}_\mathbb{C}$, weakly equivalent to $\mathfrak{A}_M(M)$, we proceed in three steps. First, we shall construct a cochain complex $L \in \mathbf{CoCh}_\mathbb{C}$ with trivial differential and a quasi-isomorphism $c\colon L \to \mathcal{L}(M)$ in $\mathbf{CoCh}_\mathbb{R}$ to the observable complex $\mathcal{L}(M)$ from (4.1.7). Second, we shall endow L with the Poisson structure $\widetilde{\tau} := \tau_M(c \otimes c)$ induced by τ_M from (4.4.1). This defines a new Poisson complex $(L, \widetilde{\tau})$ and a quasi-isomorphism $c\colon (L, \widetilde{\tau}) \to (\mathcal{L}(M), \tau_M)$ that by construction preserves the Poisson structures. In the third and last step we shall quantize the Poisson complex $(L, \widetilde{\tau})$ by means of the canonical commutation relations, as in (4.4.2). Since by [BBS20, Prop. 5.3] the CCR quantization functor preserves weak equivalences, from $c\colon (L, \widetilde{\tau}) \to (\mathcal{L}(M), \tau_M)$ we obtain a differential graded algebra $A \in \mathbf{DGA}_\mathbb{C}$ and a weak equivalence $q\colon A \to \mathfrak{A}_M(M)$ in $\mathbf{DGA}_\mathbb{C}$, to the original differential graded algebra $\mathfrak{A}_M(M) \in \mathbf{DGA}_\mathbb{C}$.

We consider the cochain complex with vanishing differential

$$L := \left(\underset{e^\ddagger}{\overset{(-1)}{\mathcal{H}^0(S^1)}} \xrightarrow{\;0\;} \underset{a_1}{\mathcal{H}^1(S^1)} \overset{(0)}{\oplus} \underset{a_2}{\mathcal{H}^1(S^1)} \xrightarrow{\;0\;} \underset{e}{\overset{(1)}{\mathcal{H}^0(S^1)}} \right) \in \mathbf{CoCh}_\mathbb{R}, \quad (4.6.1)$$

which is generated as a graded vector space by the harmonic forms $e^\ddagger := 1, a_1 := d\theta, a_2 := d\theta, e := 1$. Furthermore, choosing any compactly supported function $f \in C^\infty_{\mathrm{c}}(\mathbb{R})$ such that $\int_{\mathbb{R}} dt\, f = 1$, we define the cochain map

$$c \colon L \longrightarrow \mathcal{L}(M) \tag{4.6.2a}$$

in $\mathbf{CoCh}_{\mathbb{R}}$ degree-wise by

$$c^{-1}(e^{\ddagger}) \coloneqq f(t)\, dt, \tag{4.6.2b}$$

$$c^{0}(a_1) \coloneqq f(t)\, d\theta + \Big(\int_{\mathbb{R}} ds\, s\, f(s) \Big)\, f'(t)\, d\theta, \quad c^0(a_2) \coloneqq -f'(t)\, d\theta, \tag{4.6.2c}$$

$$c^{1}(e) \coloneqq f(t). \tag{4.6.2d}$$

Recalling (4.1.7) one easily checks that those on the right hand sides are cocycles in $\mathcal{L}(M)$ and therefore c is a well-defined cochain map, as claimed. Even more, c is a quasi-isomorphism. To check this fact, we consider also the cochain map

$$\widetilde{c} \colon \mathcal{L}(M) \longrightarrow L \tag{4.6.3a}$$

in $\mathbf{CoCh}_{\mathbb{R}}$ defined degree-wise by

$$\widetilde{c}^{-2}(\varepsilon^{\ddagger}) \coloneqq 0, \tag{4.6.3b}$$

$$\widetilde{c}^{-1}(\alpha^{\ddagger}) \coloneqq \Big(\int_{\mathbb{R}} dt\, \alpha^{\ddagger}_{T\mathcal{H}} \Big)\, e^{\ddagger}, \tag{4.6.3c}$$

$$\widetilde{c}^{0}(\alpha) \coloneqq \int_{\mathbb{R}} dt\, \alpha_{S\mathcal{H}} \oplus \int_{\mathbb{R}} dt\, t\, \alpha_{S\mathcal{H}}, \tag{4.6.3d}$$

$$\widetilde{c}^{1}(\varepsilon) \coloneqq \Big(\int_{\mathbb{R}} dt\, \varepsilon_{\mathcal{H}} \Big)\, e, \tag{4.6.3e}$$

for all $\varepsilon^{\ddagger} \in \mathcal{L}(M)^{-2} = \Omega^0_{\mathrm{c}}(M)$, $\alpha^{\ddagger} \in \mathcal{L}(M)^{-1} = \Omega^1_{\mathrm{c}}(M)$, $\alpha \in \mathcal{L}(M)^0 = \Omega^1_{\mathrm{c}}(M)$ and $\varepsilon \in \mathcal{L}^1(M) = \Omega^0_{\mathrm{c}}(M)$. Note that the above definition of $\widetilde{c}$ uses also the decompositions in (4.5.3) and (4.5.5). Using (4.5.4) to compute the relevant differentials, one checks that $\widetilde{c}$ as defined above is indeed a cochain map. Furthermore, one immediately realizes that $\widetilde{c}\, c = \mathrm{id}$, hence $H(\widetilde{c})\, H(c) = \mathrm{id}$ in cohomology. On the other hand direct inspection shows also that $H(c)\, H(\widetilde{c}) = \mathrm{id}$. Therefore $H(c) \colon H(L) \to H(\mathcal{L}(M))$ is an isomorphism, i.e. c is a quasi-isomorphism.

Remark 4.6.1. The previous argument can be refined by constructing a homotopy η that exhibits $\widetilde{c}$ as a quasi-inverse of c. More in detail, $\eta \in [\mathcal{L}(M), \mathcal{L}(M)]^{-1}$ is a (-1)-cochain in the internal hom, whose differential $\partial\eta = c\, \widetilde{c} - \mathrm{id}$ controls to what extent $c\, \widetilde{c}$ differs from id. The homotopy η and the equations $\widetilde{c}\, c = \mathrm{id}$ and $c\, \widetilde{c} = \mathrm{id} + \partial\eta$ witness that $\widetilde{c}$ is a quasi-inverse of c. We do not provide an explicit choice of η as it is not needed in the sequel. Let us just mention that it can be constructed for instance using the decompositions (4.5.3) and (4.5.5) in a way similar to the construction of $\widetilde{c}$ in (4.6.3). △

We now endow the cochain complex L with the transferred Poisson structure

$$\widetilde{\tau} := \tau_M\,(c \otimes c) \colon L^{\wedge 2} \longrightarrow \mathbb{R}, \tag{4.6.4a}$$

which reads explicitly as

$$\widetilde{\tau}(e, e^{\dagger}) = 1, \qquad \widetilde{\tau}(a_2, a_1) = 1, \qquad \widetilde{\tau}(a_i, e^{(\ddagger)}) = 0. \tag{4.6.4b}$$

This result follows from the definition of the Poisson structure τ_M in (4.4.1) and the explicit formula for the retarded-minus-advanced propagator $G^{(k)}$ from (4.5.6). (One also uses that S^1 has unit length and that f is such that $\int_{\mathbb{R}} dt\, f = 1$.)

Summarizing, $c \colon (L, \widetilde{\tau}) \to (\mathcal{L}(M), \tau_M)$ is a quasi-isomorphism that preserves the Poisson structures. Recalling [BBS20, Prop. 5.3], one finds that the CCR quantization functor sends c to the weak equivalence of differential graded algebras $q \colon A \to \mathfrak{A}_M(M)$ in $\mathbf{DGA}_{\mathbb{C}}$, where

$$A := T_{\mathbb{C}}(L)/I_{\widetilde{\tau}} \in \mathbf{DGA}_{\mathbb{C}} \tag{4.6.5a}$$

is the differential graded algebra generated by L and subject to the canonical commutation relations encoded by the two-sided ideal $I_{\widetilde{\tau}}$ generated by

$$e \otimes e^{\dagger} + e^{\ddagger} \otimes e - i, \qquad a_2 \otimes a_1 - a_1 \otimes a_2 - i, \qquad a_i\, e^{(\ddagger)} - e^{(\ddagger)}\, a_i, \tag{4.6.5b}$$

and q extends c from generators. (Compare the above construction of A to (4.4.2) and note that q is indeed well-defined because c preserves the Poisson structures.)

Remark 4.6.2. Recalling Remark 4.5.5, one realizes that the CCR algebra $\mathfrak{A}^{\mathrm{inv}}(M)$ generated by gauge-invariant on-shell linear observables is isomorphic to the subalgebra of A generated by a_1, a_2 only. In contrast, the cohomology $H(\mathfrak{A}_M(M))$ of the global algebra $\mathfrak{A}_M(M)$ is isomorphic to A (seen as a graded algebra, i.e. forgetting its differential). In particular, the degree 0 cohomology algebra $H^0(\mathfrak{A}_M(M)) \cong A^0$ is strictly larger than $\mathfrak{A}^{\mathrm{inv}}(M)$. Indeed, along with the generators a_1 and a_2, which are also contained in $\mathfrak{A}^{\mathrm{inv}}(M)$, A^0 has an additional generator $e^{\ddagger}\, e$ that commutes with both a_1 and a_2. The latter is a composite observable formed by an antifield observable $e^{\dagger}$ and a ghost observable e. Those may be regarded as “gauge-invariant” observables in the broader sense of being cohomologically non-trivial. △

While being Quillen equivalent to ${}_{\mathfrak{A}_M(M)}\mathbf{Mod}$ due to Proposition 2.8.8, the category of left A-modules ${}_A\mathbf{Mod}$ is considerably easier to describe. As it is always the case, a left A-module $V = (V, \nu) \in {}_A\mathbf{Mod}$ consists of a cochain complex $V \in \mathbf{CoCh}_{\mathbb{C}}$ and of a left A-action $\nu \colon A \otimes V \to V$ in $\mathbf{CoCh}_{\mathbb{C}}$, which is the same datum as a morphism $\nu \colon A \to [V, V]$ in $\mathbf{DGA}_{\mathbb{C}}$ to the internal endomorphism algebra[1] $[V, V] \in \mathbf{DGA}_{\mathbb{C}}$. The specific form of A, however, allows us to describe ν in terms of very few concrete data. Since $A = T_{\mathbb{C}}(L)/I_{\widetilde{\tau}}$ is defined as the differential graded algebra that is generated by the (-1)-cocycle $e^{\ddagger} \in L^{-1}$, the 0-cocycles $a_1, a_2 \in L^0$ and the 1-cocycle $e \in L^1$ and that is subject to the canonical commutation relations from (4.6.5), it follows that the left A-action ν is pinned down by the (-1)-cocycle $\nu(e^{\ddagger}) \in [V, V]^{-1}$, the 0-cocycles $\nu(a_1), \nu(a_2) \in [V, V]^0$ and the 1-cocycle $\nu(e) \in [V, V]^1$ in the internal endomorphism algebra $[V, V] \in \mathbf{DGA}_{\mathbb{C}}$. Furthermore, these cocycles must fulfil the relations $\nu(e)\, \nu(e^{\ddagger}) + \nu(e^{\ddagger})\, \nu(e) = i\, \mathrm{id}_V$, $\nu(a_2)\, \nu(a_1) - \nu(a_1)\, \nu(a_2) = i\, \mathrm{id}_V$ and $\nu(a_i)\, \nu(e^{(\ddagger)}) = \nu(e^{(\ddagger)})\, \nu(a_i)$. Similarly, a left A-module morphism $F \colon V \to V'$ in ${}_A\mathbf{Mod}$ consists of a cochain map $F \colon V \to V'$ in $\mathbf{CoCh}_{\mathbb{C}}$ that preserves the cocycles above, i.e. $F\, \nu(e^{\ddagger}) = \nu'(e^{\ddagger})\, F$, $F\, \nu(a_1) = \nu'(a_1)\, F$, $F\, \nu(a_2) = \nu'(a_2)\, F$ and $F\, \nu(e) = \nu'(e)\, F$.

The above very explicit description of left A-modules gives us a handle to construct all constant representations of the net $\mathfrak{A}_M$ up to weak equivalence. The latter form the essential image of the total right derived functor of the right Quillen functor $(-)^M \colon {}_{\mathfrak{A}_M(M)}\mathbf{Mod} \to \mathbf{Rep}(\mathfrak{A}_M)$ from Proposition 3.3.3 and Remark 3.6.16. Since $q \colon A \to \mathfrak{A}_M(M)$ in $\mathbf{DGA}_{\mathbb{C}}$ is a weak equivalence, it follows by Proposition 2.8.8 that $(\mathrm{Ext}_q \dashv \mathrm{Res}_q) \colon {}_{\mathfrak{A}_M(M)}\mathbf{Mod} \to {}_A\mathbf{Mod}$ is a Quillen equivalence and hence induces an equivalence of homotopy categories. Combining these facts, the essential image of the total right derived functor $(-)^M$ is equivalent to the homotopy category of ${}_A\mathbf{Mod}$. In other words, left A-modules provide all constant representations of the net $\mathfrak{A}_M$ up to weak equivalence. (For the notions of homotopy category associated with a model category and of total left (right) derived functor associated with a left (respectively right) Quillen functor we refer the reader to [Hov99, Ch. 1].)

[1]Given any cochain complex $V \in \mathbf{CoCh}_{\mathbb{C}}$, the internal hom $[V, V] \in \mathbf{CoCh}_{\mathbb{C}}$ carries a natural differential graded algebra structure consisting of the usual internal hom differential complemented with the multiplication given by composition and the unit given by id_V. When endowed with this structure we refer to $[V, V] \in \mathbf{DGA}_{\mathbb{C}}$ as the internal endomorphism algebra of $V \in \mathbf{CoCh}_{\mathbb{C}}$.

Appendix A

Category Theory

In this appendix we recall the foundations of category theory. The content of this chapter can be found in any introductory textbook on category theory (see [Mac78; Bor94; Rie16]).

A.1 Categories

Definition A.1.1. A **category** $\mathbf{C}$ consists of the following data:

(1) A collection of **objects** $\mathrm{Ob}(\mathbf{C})$.

(2) A collection of **morphisms** $\mathrm{Mor}(\mathbf{C})$.

(3) For each morphism $f \in \mathrm{Mor}(\mathbf{C})$, an object $\mathrm{s}_{\mathbf{C}}(f) \in \mathrm{Ob}(\mathbf{C})$, called the **source** (or **domain**) of f, and an object $\mathrm{t}_{\mathbf{C}}(f) \in \mathrm{Ob}(\mathbf{C})$, called the **target** (or **codomain**) of f.

(4) For each object $C \in \mathrm{Ob}(\mathbf{C})$, a morphism $\mathrm{id}_C \in \mathrm{Mor}(\mathbf{C})$ called the **identity on** C.

(5) For each pair of morphisms $f, g \in \mathrm{Mor}(\mathbf{C})$ such that $\mathrm{t}_{\mathbf{C}}(f) = \mathrm{s}_{\mathbf{C}}(g)$, a **composite** morphism $g \circ f \in \mathrm{Mor}(\mathbf{C})$.

These data are required to satisfy the following criteria:

(i) For every object $C \in \mathrm{Ob}(\mathbf{C})$, it holds that $\mathrm{s}_{\mathbf{C}}(\mathrm{id}_C) = \mathrm{t}_{\mathbf{C}}(\mathrm{id}_C) = C$.

(ii) For every pair of morphisms $f, g \in \mathrm{Mor}(\mathbf{C})$ such that $\mathrm{t}_{\mathbf{C}}(f) = \mathrm{s}_{\mathbf{C}}(g)$, it holds that $\mathrm{s}_{\mathbf{C}}(g \circ f) = \mathrm{s}_{\mathbf{C}}(f)$ and $\mathrm{t}_{\mathbf{C}}(g \circ f) = \mathrm{t}_{\mathbf{C}}(g)$.

(iii) **Unity**: For every morphism $f \in \mathrm{Mor}(\mathbf{C})$, it holds that $\mathrm{id}_{\mathrm{t}_{\mathbf{C}}(f)} \circ f = f \circ \mathrm{id}_{\mathrm{s}_{\mathbf{C}}(f)} = f$.

(iv) **Associativity**: For every $f, g, h \in \mathrm{Mor}(\mathbf{C})$ such that $h \circ (g \circ f)$ exists, it holds that $h \circ (g \circ f) = (h \circ g) \circ f$.

Notation A.1.2. The following notations and conventions will be used:

- We denote by $\mathbf{C}(C_1, C_2)$ the collection of morphisms with source object C_1 and target object C_2.
- We write $f\colon C_1 \to C_2$ to denote that $f \in \mathbf{C}(C_1, C_2)$.
- Given a morphism $f \in \mathrm{Mor}(\mathbf{C})$, we denote its source $\mathrm{s}_{\mathbf{C}}(f)$ by $\mathrm{s}(f)$ and its target $\mathrm{t}_{\mathbf{C}}(f)$ by $\mathrm{t}(f)$ when there is no risk of confusion.
- Abusing notation, we write $C \in \mathbf{C}$ to denote that $C \in \mathrm{Ob}(\mathbf{C})$.

A useful example of a category is the category of sets defined below.

Example A.1.3. The category of sets, denoted by **Set**, is defined as the category with objects the sets and morphisms the functions between sets. For each morphism $f \in \mathrm{Mor}(\mathbf{Set})$, the source of f is the domain of f and the target of f is the codomain of f. The identity morphism on each set is simply the identity function on that set, and the composition of two morphisms is the composition of them as functions. $\triangledown$

Next, we define some important concepts in category theory.

Definition A.1.4. Let $\mathbf{C}$ be a category and $f \in \mathrm{Mor}(\mathbf{C})$.

(a) f is called a **monomorphism** if for every pair of morphisms $g, h \in \mathrm{Mor}(\mathbf{C})$ such that $f \circ g = f \circ h$, it holds that $g = h$.

(b) f is called an **epimorphism** if for every pair of morphisms $g, h \in \mathrm{Mor}(\mathbf{C})$ such that $g \circ f = h \circ f$, it follows that $g = h$.

(c) f is called an **isomorphism** if there exists a morphism $g \in \mathrm{Mor}(\mathbf{C})$ such that $f \circ g = \mathrm{id}_{\mathrm{t}(f)}$ and $g \circ f = \mathrm{id}_{\mathrm{s}(f)}$. In this case g is called the **inverse** of f and it is denoted by f^{-1}.

Definition A.1.5. Let $\mathbf{C}$ be a category and $C_1, C_2 \in \mathrm{Ob}(\mathbf{C})$. We say that C_1 and C_2 are **isomorphic**, denoted by $C_1 \cong C_2$, if there exists an isomorphism $f \in \mathbf{C}(C_1, C_2)$.

Definition A.1.6. Let $\mathbf{C}$ be a category and C be an object of $\mathbf{C}$.

(a) C is called an **initial** object of $\mathbf{C}$ if for every object $C' \in \mathrm{Ob}(\mathbf{C})$ there exists a unique morphism $f \in \mathbf{C}(C, C')$.

(b) C is called a **terminal** object of $\mathbf{C}$ if for every object $C' \in \mathrm{Ob}(\mathbf{C})$ there exists a unique morphism $f \in \mathbf{C}(C', C)$.

Remark A.1.7. An initial (or terminal) object is unique up to isomorphism. That is to say, if there are two initial objects in a category $\mathbf{C}$ then there is a unique isomorphism between them. △

Definition A.1.8. A category $\mathbf{C}$ is called:

(a) **finite** when $\mathrm{Mor}(\mathbf{C})$ is a finite set,

(b) **small** when $\mathrm{Mor}(\mathbf{C})$ is a set,

(c) **locally small** when for every $C_1, C_2 \in \mathrm{Ob}(\mathbf{C})$ the collection $\mathbf{C}(C_1, C_2)$ is a set,

(d) **discrete** when it is small and contains only the identity morphisms.

Definition A.1.9. Let $\mathbf{C}$ be a category. A **subcategory of C** is a category $\mathbf{D}$ such that $\mathrm{Ob}(\mathbf{D}) \subseteq \mathrm{Ob}(\mathbf{C})$, $\mathrm{Mor}(\mathbf{D}) \subseteq \mathrm{Mor}(\mathbf{C})$, and the identity morphisms, the source and the target of a morphism and the composition of two morphisms are the same as in $\mathbf{C}$.

Finally, we present some basic constructions of categories that appear very frequently.

Definition A.1.10. Let $\mathbf{C}$, $\mathbf{D}$ be two categories. The **product category** $\mathbf{C} \times \mathbf{D}$ is the category consisting of the following data. The objects are the pairs (C, D) where $C \in \mathrm{Ob}(\mathbf{C})$ and $D \in \mathrm{Ob}(\mathbf{D})$, and the morphisms are the pairs (f, g) where $f \in \mathrm{Mor}(\mathbf{C})$ and $g \in \mathrm{Mor}(\mathbf{D})$. The source of (f, g) is $\mathrm{s}_{\mathbf{C}\times\mathbf{D}}(f, g) := (\mathrm{s}_{\mathbf{C}}(f), \mathrm{s}_{\mathbf{D}}(g)) \in \mathrm{Ob}(\mathbf{C} \times \mathbf{D})$, whereas its target is $\mathrm{t}_{\mathbf{C}\times\mathbf{D}}(f, g) := (\mathrm{t}_{\mathbf{C}}(f), \mathrm{t}_{\mathbf{D}}(g)) \in \mathrm{Ob}(\mathbf{C} \times \mathbf{D})$. For every object $(C, D) \in \mathbf{C} \times \mathbf{D}$, the identity morphism is $\mathrm{id}_{(C,D)} = (\mathrm{id}_C, \mathrm{id}_D)$. The composition of two morphisms is defined component-wise.

Definition A.1.11. Let $\mathbf{C}$ be a category and $\circ_{\mathbf{C}}$ be the composition in $\mathbf{C}$.

(a) The **opposite category of C**, denoted by $\mathbf{C}^{\mathrm{op}}$, is the category that consists of the same collections of objects and morphisms as well as the same identity morphisms as $\mathbf{C}$. The remaining data are as follows:

  (1) For each morphism $f \in \mathrm{Mor}(\mathbf{C}^{\mathrm{op}})$, the source and target objects are reversed, i.e. $\mathrm{s}_{\mathbf{C}^{\mathrm{op}}}(f) = \mathrm{t}_{\mathbf{C}}(f)$ and $\mathrm{t}_{\mathbf{C}^{\mathrm{op}}}(f) = \mathrm{s}_{\mathbf{C}}(f)$.

  (2) For each pair of morphisms $f, g \in \mathrm{Mor}(\mathbf{C}^{\mathrm{op}})$ such that $\mathrm{s}(g) = \mathrm{t}(f)$, the composite morphism is $g \circ f = f \circ_{\mathbf{C}} g$.

(b) Given a morphism $f \in \mathrm{Mor}(\mathbf{C})$, we define the **reverse morphism of** f, denoted by f^{op}, to be the corresponding morphism in $\mathbf{C}^{\mathrm{op}}$. That is $f^{\mathrm{op}} \coloneqq f \in \mathbf{C}^{\mathrm{op}}$.

Definition A.1.12. Let $\mathbf{C}$ be a locally small category. For each $X, Y \in \mathrm{Ob}(\mathbf{C})$, let $R_{X,Y}$ be an equivalence relation on $\mathbf{C}(X,Y)$ such that the equivalence relations respect the composition of morphisms. That means that if $(f_1, f_2) \in R_{X,Y}$ and $(g_1, g_2) \in R_{Y,Z}$ then $(g_1 \circ f_1, g_2 \circ f_2) \in R_{X,Z}$. The **quotient category**, denoted by $\mathbf{C}/R$, is the category that has the same objects as $\mathbf{C}$ and morphisms the equivalence classes of morphisms of $\mathbf{C}$, i.e. $(\mathbf{C}/R)(X,Y) \coloneqq C(X,Y)/R_{X,Y}$ for each $X, Y \in \mathbf{C}/R$.

A.2 Functors and natural transformations

Definition A.2.1. Let $\mathbf{C}$, $\mathbf{D}$ be two categories.

(a) A **functor** (also **covariant functor**) $\mathcal{F}$ from $\mathbf{C}$ to $\mathbf{D}$, denoted $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$, consists of the following data:

   (1) For each object $C \in \mathrm{Ob}(\mathbf{C})$, an object $\mathcal{F}(C) \in \mathrm{Ob}(\mathbf{D})$.

   (2) For each morphism $f \in \mathrm{Mor}(\mathbf{C})$, a morphism $\mathcal{F}(f)$.

   These data are required to satisfy the following properties:

   (i) For each morphism $f \in \mathrm{Mor}(\mathbf{C})$, it holds that $\mathrm{s}(\mathcal{F}(f)) = \mathcal{F}(\mathrm{s}(f))$ and $\mathrm{t}(\mathcal{F}(f)) = \mathcal{F}(\mathrm{t}(f))$.

   (ii) For each object $C \in \mathrm{Ob}(\mathbf{C})$, it holds that $\mathcal{F}(\mathrm{id}_C) = \mathrm{id}_{\mathcal{F}(C)}$.

   (iii) For each pair of morphisms $f, g \in \mathrm{Mor}(\mathbf{C})$ with $\mathrm{s}(g) = \mathrm{t}(f)$, it holds that $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

(b) A **contravariant functor** $\mathcal{F}$ from $\mathbf{C}$ to $\mathbf{D}$, is a functor from the opposite category $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{D}$, and is denoted $\mathcal{F}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$.

Some commonly encountered functors are the following.

Example A.2.2. Given a category $\mathbf{C}$, we define the **identity functor** on $\mathbf{C}$, denoted by $\mathrm{Id}_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$, as the functor that assigns to each object $C \in \mathrm{Ob}(\mathbf{C})$ the object $\mathrm{Id}_{\mathbf{C}}(C) \coloneqq C \in \mathrm{Ob}(\mathbf{C})$ and to each morphism $f \in \mathrm{Mor}(\mathbf{C})$ the morphism $\mathrm{Id}_{\mathbf{C}}(f) \coloneqq f \in \mathrm{Mor}(\mathbf{C})$. $\triangledown$

Example A.2.3. Given a subcategory $\mathbf{D}$ of a category $\mathbf{C}$ (see Definition A.1.9), the **inclusion functor of D**, denoted by $\mathcal{J}_{\mathbf{D}}\colon \mathbf{D} \to \mathbf{C}$ is defined as the functor that assigns to each object $D \in \mathrm{Ob}(\mathbf{D})$ the object $\mathcal{J}_{\mathbf{D}}(D) \coloneqq D \in \mathrm{Ob}(\mathbf{C})$ and to each morphism $f \in \mathrm{Mor}(\mathbf{D})$ the morphism $\mathcal{J}_{\mathbf{D}}(f) \coloneqq f \in \mathrm{Mor}(\mathbf{C})$. ▽

Example A.2.4. Given two categories $\mathbf{C}$ and $\mathbf{D}$ and an object $D \in \mathbf{D}$ we define the **constant functor over** D, denoted by ΔD, as the functor that assigns to each object $C \in \mathrm{Ob}(\mathbf{C})$ the object D and to each morphism $f \in \mathrm{Mor}(\mathbf{C})$ the identity morphism on D. ▽

Example A.2.5. Let $\mathbf{C}$ be a locally small category. For each object $C \in \mathrm{Ob}(\mathbf{C})$ there exists a functor $\mathrm{Hom}_{\mathbf{C}}(C,-)\colon \mathbf{C} \to \mathbf{Set}$, where $\mathbf{Set}$ denotes the category of sets (see Example A.1.3), consisting of the following data:

(1) For each object $C' \in \mathrm{Ob}(\mathbf{C})$, the set of morphisms $\mathrm{Hom}_{\mathbf{C}}(C,C') \coloneqq \mathbf{C}(C,C')$.

(2) For each morphism $f\colon C' \to C''$, the morphism $\mathrm{Hom}_{\mathbf{C}}(C,f) \coloneqq f_*$, where $f_*\colon \mathbf{C}(C,C') \to \mathbf{C}(C,C'')$ denotes the post-composition with f.

Similarly there is a functor $\mathrm{Hom}_{\mathbf{C}}(-,C)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ consisting of the following data:

(1) For each object $C' \in \mathrm{Ob}(\mathbf{C})$, the set of morphisms $\mathrm{Hom}_{\mathbf{C}}(C',C) \coloneqq \mathbf{C}(C',C)$.

(2) For each morphism $f\colon C' \to C''$, the morphism $\mathrm{Hom}_{\mathbf{C}}(C,f) \coloneqq f^*$, where $f^*\colon \mathbf{C}(C'',C) \to \mathbf{C}(C',C)$ denotes the pre-composition with f.

▽

Remark A.2.6. The collection of locally small categories forms a category $\mathbf{CAT}$ with the functors as morphisms. The composition of functors is defined as follows. Given functors $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ and $\mathcal{G}\colon \mathbf{D} \to \mathbf{E}$, the composite functor $\mathcal{G} \circ \mathcal{F}\colon \mathbf{C} \to \mathbf{E}$ assigns to each object $C \in \mathrm{Ob}(\mathbf{C})$ the object $(\mathcal{G} \circ \mathcal{F})(C) \coloneqq \mathcal{G}(\mathcal{F}(C)) \in \mathrm{Ob}(\mathbf{E})$, and to each morphism $f \in \mathrm{Mor}(\mathbf{C})$ the morphism $(\mathcal{G} \circ \mathcal{F})(f) \coloneqq \mathcal{G}(\mathcal{F}(f)) \in \mathrm{Mor}(\mathbf{E})$. The identity morphisms in $\mathbf{CAT}$ are simply the identity functors. △

Definition A.2.7. Let $\mathbf{C}$, $\mathbf{D}$ be locally small categories and $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ a functor.

(a) $\mathcal{F}$ is called **full** if the function $\mathcal{F}\colon \mathbf{C}(C_1,C_2) \to \mathbf{D}(\mathcal{F}(C_1),\mathcal{F}(C_2))$ is surjective for every $C_1, C_2 \in \mathrm{Ob}(\mathbf{C})$.

(b) $\mathcal{F}$ is called **faithful** if the function $\mathcal{F}\colon \mathbf{C}(C_1, C_2) \to \mathbf{D}(\mathcal{F}(C_1), \mathcal{F}(C_2))$ is injective for every $C_1, C_2 \in \mathrm{Ob}(\mathbf{C})$.

(c) $\mathcal{F}$ is called **fully faithful** if it is full and faithful.

(d) $\mathcal{F}$ is called **essentially surjective** if for every object $D \in \mathrm{Ob}(\mathbf{D})$ there exists an object $C \in \mathrm{Ob}(\mathbf{C})$ such that $\mathcal{F}(C) \cong D$.

Example A.2.8. Given a subcategory $\mathbf{D}$ of a category $\mathbf{C}$, the inclusion functor $\mathcal{J}_{\mathbf{D}}\colon \mathbf{D} \to \mathbf{C}$ from Example A.2.3 is a faithful functor. If on top of that $\mathcal{J}_{\mathbf{D}}$ is full then $\mathbf{D}$ is called a **full subcategory of C**. ▽

Definition A.2.9. Let $\mathbf{C}$, $\mathbf{D}$ be two categories and let $\mathcal{F}, \mathcal{G}\colon \mathbf{C} \to \mathbf{D}$ be two functors between them.

(a) A **natural transformation** α from $\mathcal{F}$ to $\mathcal{G}$, denoted $\alpha\colon \mathcal{F} \to \mathcal{G}$, assigns to each object $C \in \mathrm{Ob}(\mathbf{C})$ a morphism $\alpha_C \in \mathbf{D}(\mathcal{F}(C), \mathcal{G}(C))$ such that for every morphism $f \in \mathbf{C}(C, C')$, the diagram

$$\begin{array}{ccc} \mathcal{F}(C) & \xrightarrow{\alpha_C} & \mathcal{G}(C) \\ \big\downarrow{\scriptstyle \mathcal{F}(f)} & & \big\downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(C') & \xrightarrow{\alpha_{C'}} & \mathcal{G}(C') \end{array}$$

in $\mathbf{D}$ commutes.

(b) A **natural isomorphism** is a natural transformation such that for each $C \in \mathrm{Ob}(\mathbf{C})$, $\alpha_C \in \mathrm{Mor}(\mathbf{D})$ is an isomorphism. It is denoted by $\alpha\colon \mathcal{F} \xrightarrow{\cong} \mathcal{G}$.

Definition A.2.10. Let $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$ be categories $\mathcal{G}, \mathcal{H}\colon \mathbf{D} \to \mathbf{E}$ functors and $\alpha\colon \mathcal{G} \to \mathcal{H}$ a natural transformation.

(a) Given a functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$, we define the natural transformation $\alpha_{\mathcal{F}}\colon \mathcal{G} \circ \mathcal{F} \to \mathcal{H} \circ \mathcal{F}$ as follows. For each $C \in \mathbf{C}$, we define $(\alpha_{\mathcal{F}})_C \coloneqq \alpha_{\mathcal{F}(C)}$.

(b) Given a functor $\mathcal{F}\colon \mathbf{E} \to \mathbf{F}$, we define the natural transformation $\mathcal{F}\alpha\colon \mathcal{F} \circ \mathcal{G} \to \mathcal{F} \circ \mathcal{H}$ as follows. For each $D \in \mathbf{D}$, we define $(\mathcal{F}\alpha)_D \coloneqq \mathcal{F}(\alpha_D)$.

Definition A.2.11. An **equivalence** between two categories $\mathbf{C}$ and $\mathbf{D}$ consists of a pair of functors $\mathcal{F}\colon \mathbf{C} \rightleftarrows \mathbf{D} :\mathcal{G}$ such that there exist natural isomorphisms $\alpha\colon \mathcal{G}\circ\mathcal{F} \xrightarrow{\cong} \mathrm{Id}_{\mathbf{C}}$ and $\beta\colon \mathcal{F}\circ\mathcal{G} \xrightarrow{\cong} \mathrm{Id}_{\mathbf{D}}$, where $\mathrm{Id}_{\mathbf{C}}$ and $\mathrm{Id}_{\mathbf{D}}$ are the identity functors on $\mathbf{C}$ and $\mathbf{D}$ respectively. In this case, the functors $\mathcal{F}$ and $\mathcal{G}$ are also called **equivalences** and the categories $\mathbf{C}$ and $\mathbf{D}$ are called **equivalent**. We denote that two categories $\mathbf{C}$ and $\mathbf{D}$ are equivalent by $\mathbf{C} \simeq \mathbf{D}$.

Remark A.2.12 ([Rie16, Lem. 1.5.5])**.** The property of two categories being equivalent defines an equivalence relation. In particular, it satisfies transitivity, i.e. if $\mathbf{C} \simeq \mathbf{D}$ and $\mathbf{D} \simeq \mathbf{E}$ for some categories $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$, then $\mathbf{C} \simeq \mathbf{E}$. △

Proposition A.2.13 ([Rie16, Th. 1.5.9])**.** *Let $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ be a functor. The following conditions are equivalent:*

(1) $\mathcal{F}$ is an equivalence of categories.

(2) $\mathcal{F}$ is fully faithful and essentially surjective.

A.3 Functor categories and the Yoneda Lemma

Definition A.3.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. The **functor category $\mathbf{D}^{\mathbf{C}}$** is the category with objects the functors from $\mathbf{C}$ to $\mathbf{D}$ and morphisms the natural transformations. Given two natural transformations $\alpha\colon \mathcal{F} \to \mathcal{G}$ and $\beta\colon \mathcal{G} \to \mathcal{H}$, where $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{D}^{\mathbf{C}}$, the composite morphism $\beta \circ \alpha\colon \mathcal{F} \to \mathcal{H}$ is the natural transformation with components defined as follows. For every $C \in \mathbf{C}$, $(\beta\circ\alpha)_C \coloneqq \beta_C \circ \alpha_C$.

Notation A.3.2. For clarity, we often denote the functor category $\mathbf{D}^{\mathbf{C}}$ by $\mathbf{Fun}(\mathbf{C}, \mathbf{D})$.

Remark A.3.3. In general, when $\mathbf{C}$ and $\mathbf{D}$ are locally small, $\mathbf{D}^{\mathbf{C}}$ doesn't need to be locally small. However, if $\mathbf{C}$ is small and $\mathbf{D}$ is locally small then $\mathbf{D}^{\mathbf{C}}$ is also locally small. △

Remark A.3.4. A natural isomorphism is just an isomorphism in the relevant functor category. △

Definition A.3.5. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We define the **evaluation functor** $\mathrm{Ev}\colon \mathbf{C} \times \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$ as the functor that assigns to each pair $(C, \mathcal{F}) \in \mathbf{C} \times \mathbf{D}^{\mathbf{C}}$ the object $\mathrm{Ev}(C, \mathcal{F}) \coloneqq \mathcal{F}(C) \in \mathbf{D}$ and to each morphism $(f, \alpha)\colon (C_1, \mathcal{F}_1) \to (C_2, \mathcal{F}_2)$ in $\mathbf{C} \times \mathbf{D}^{\mathbf{C}}$, the morphism $\mathrm{Ev}(f, \alpha) \coloneqq \alpha_{C_2} \circ \mathcal{F}_1(f)\colon \mathcal{F}_1(C_1) \to \mathcal{F}_2(C_2)$ in $\mathbf{D}$. Moreover, given an object $C \in \mathbf{C}$, we denote by Ev_C the functor $\mathrm{Ev}(C, -)\colon \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$.

Yoneda Lemma. *Yoneda Lemma Let* $\mathbf{C}$ *be a locally small category and* $\mathcal{F}\colon \mathbf{C} \to \mathbf{Set}$ *a functor. Then, for any* $C \in \mathrm{Ob}(\mathbf{C})$*, there is a bijection of sets*

$$\upsilon_{C,\mathcal{F}}\colon \mathbf{Set}^{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(C,-),\mathcal{F}) \xrightarrow{\cong} \mathcal{F}(C)\,, \tag{A.3.1}$$

which is natural in C *and* $\mathcal{F}$*, i.e. for each morphism* $f\colon C \to C'$ *in* $\mathbf{C}$ *and each natural transformation* $\alpha\colon \mathcal{F} \to \mathcal{G}$*, the diagram*

$$\begin{array}{ccc} \mathbf{Set}^{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(C,-),\mathcal{F}) & \xrightarrow{\upsilon_{C,\mathcal{F}}} & \mathcal{F}(C) \\ \Big\downarrow{\scriptstyle \mathbf{Set}^{\mathbf{C}}(f^*,\alpha_C)} & & \Big\downarrow{\scriptstyle \alpha_{C'}\circ\mathcal{F}(f)} \\ \mathbf{Set}^{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(C',-),\mathcal{G}) & \xrightarrow{\upsilon_{C',\mathcal{G}}} & \mathcal{G}(C') \end{array} \tag{A.3.2}$$

in $\mathbf{Set}$ *commutes, where by* $\mathbf{Set}^{\mathbf{C}}(f^*,\alpha_C)$ *we denote the function that maps each natural transformation* $\beta\colon \mathrm{Hom}_{\mathbf{C}}(C,-) \to \mathcal{F}$ *to the natural transformation* $\mathbf{Set}^{\mathbf{C}}(f^*,\alpha_C)(\beta)$ *with components as follows. For each* $C'' \in \mathbf{C}$*,* $(\mathbf{Set}^{\mathbf{C}}(f^*,\alpha_C)(\beta))_{C''}$ *maps each* $g \in \mathrm{Hom}_{\mathbf{C}}(C',C'')$ *to* $\alpha_C(\beta_{C''}(g \circ f))$*.*

Remark A.3.6. Notice that in general, given a locally small category $\mathbf{C}$ and two functors $\mathcal{F},\mathcal{G}\colon \mathbf{C} \to \mathbf{Set}$, the collection of natural transformations $\mathbf{Set}^{\mathbf{C}}(\mathcal{F},\mathcal{G})$ doesn't need to be a set. However, it follows from the Yoneda Lemma that $\mathbf{Set}^{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(C,-),\mathcal{F})$ is a set. This implies that we can define a functor $\mathbf{Set}^{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(-,-),-)\colon \mathbf{C} \times \mathbf{Set}^{\mathbf{C}} \to \mathbf{Set}$ that sends each pair $(C,\mathcal{F}) \in \mathbf{C} \times \mathbf{Set}^{\mathbf{C}}$ to $\mathbf{Set}^{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(C,-),\mathcal{F}) \in \mathbf{Set}$. Then, we can more concisely express the remaining content of the Yoneda Lemma as follows. For any locally small category $\mathbf{C}$ there is a natural isomorphism $\upsilon\colon \mathbf{Set}^{\mathbf{C}}(\mathrm{Hom}_{\mathbf{C}}(-,-),-) \xrightarrow{\cong} \mathrm{Ev}$, where $\mathrm{Ev}\colon \mathbf{C} \times \mathbf{Set}^{\mathbf{C}} \to \mathbf{Set}$ is the evaluation functor (see Definition A.3.5). △

Definition A.3.7. Let $\mathbf{C}$ be a locally small category. The **Yoneda embedding** $\mathcal{Y}\colon \mathbf{C} \to \mathbf{Set}^{\mathbf{C}}$ is the functor which assigns to each object $C \in \mathrm{Ob}(\mathbf{C})$ the functor $\mathrm{Hom}_{\mathbf{C}}(C,-)$ and to each morphism $f \in \mathrm{Mor}(\mathbf{C})$ the post-composition morphism $f_* \in \mathrm{Mor}(\mathbf{Set}^{\mathbf{C}})$.

Corollary A.3.8. *Let* $\mathbf{C}$ *be a locally small category. The Yoneda embedding* $\mathcal{Y}\colon \mathbf{C} \to \mathbf{Set}^{\mathbf{C}}$ *is a fully faithful functor.*

A.4 Limits

Definition A.4.1. Let $\mathbf{J}$ be a small category and $\mathbf{C}$ a category.

(a) A **J-shaped diagram in C** (or simply **diagram**) is a functor $\mathcal{J}$ from the small category **J** to the category **C**. The source category **J** is called the **indexing** category of the diagram $\mathcal{J}$.

(b) Given a **J**-shaped diagram $\mathcal{J}$ in **C** and a morphism $f \in \mathbf{J}(j_1, j_2)$ for some $j_1, j_2 \in \mathrm{Ob}(\mathbf{J})$, we call the morphism $\mathcal{J}(f)\colon \mathcal{J}(j_1) \to \mathcal{J}(j_2)$ a **path** from $\mathcal{J}(j_1)$ to $\mathcal{J}(j_2)$.

Notation A.4.2. Let $\mathcal{J}$ be a **J**-shaped diagram in **C** for some category **C** and a small category **J**, and let $j \in \mathrm{Ob}(\mathbf{J})$. We denote the object $\mathcal{J}(j)$ by $\mathcal{J}_j$.

Definition A.4.3. Let **J** be a small category and **C** a category. A **J**-shaped diagram is called **finite** when the indexing category **J** is a finite category.

Definition A.4.4. Let **C** and **D** be two categories. We define the **C-ary diagonal functor of D** to be the functor $\Delta\colon \mathbf{D} \to \mathbf{D}^{\mathbf{C}}$ that assigns to each object $D \in \mathbf{D}$ the constant functor ΔD and to each morphism $f \in \mathrm{Mor}(\mathbf{D})$ the natural transformation whose every component is the morphism f.

Definition A.4.5. Let **J** be a small category and **C** a category.

(a) A **J**-shaped diagram in **C** is called **constant** when it is constant as a functor (see Definition A.2.4).

(b) Let $C \in \mathrm{Ob}(\mathbf{C})$ and $\mathcal{J}$ a **J**-shaped diagram in **C**. A **cone** over $\mathcal{J}$ with **tip** C is a natural transformation from the constant diagram ΔC to $\mathcal{J}$.

(c) Let $C \in \mathrm{Ob}(\mathbf{C})$ and $\mathcal{J}$ a **J**-shaped diagram in **C**. A **cocone** under $\mathcal{J}$ with **tip** C is a natural transformation from $\mathcal{J}$ to the constant diagram ΔC.

Definition A.4.6. Let **C** be a category and $\mathcal{F}$ a functor (or a contravariant functor) from **C** to **Set**. We say that $\mathcal{F}$ is **representable** if there exists an object $C \in \mathrm{Ob}(\mathbf{C})$ and a natural isomorphism $\alpha\colon \mathrm{Hom}_{\mathbf{C}}(C, -) \xrightarrow{\cong} \mathcal{F}$ ($\alpha\colon \mathrm{Hom}_{\mathbf{C}}(-, C) \xrightarrow{\cong} \mathcal{F}$ respectively). In this case we say that $\mathcal{F}$ is **represented** by the object C. We call the natural transformation α a **representation** of $\mathcal{F}$.

Remark A.4.7. A representation of a functor is unique up to isomorphism, in the following sense. Let $\alpha\colon \mathrm{Hom}_{\mathbf{C}}(C, -) \xrightarrow{\cong} \mathcal{F}$ and $\beta\colon \mathrm{Hom}_{C}(C', -) \xrightarrow{\cong} \mathcal{F}$ be two representations of $\mathcal{F}$. Then, there exists a unique isomorphism $f\colon C \to C'$ such that $\beta = \alpha \circ f^*$. This follows from Corollary A.3.8. △

Definition A.4.8. Given a functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{Set}$, we define the **category of elements** of $\mathcal{F}$, denoted by $\int\mathcal{F}$ to be the category with:

(1) objects: the pairs (C, x), where $C \in \mathbf{C}$ and $x \in \mathcal{F}(C)$,

(2) morphisms: for each pair of objects $(C_1, x_1), (C_2, x_2) \in \mathrm{Ob}(\int \mathcal{F})$: the morphisms $f\colon C_1 \to C_2$ in $\mathbf{C}$ such that $\mathcal{F}(f)(x_1) = x_2$.

Remark A.4.9. Notice that, given a representation of a covariant (resp. contravariant) functor $\mathcal{F}$, the Yoneda Lemma assigns to it a unique object in the category $\int \mathcal{F}$, i.e. the category of elements of $\mathcal{F}$. This element is called a **universal** element of $\mathcal{F}$. In particular, it follows from the Yoneda Lemma that a universal element is an initial (resp. terminal) object in the category $\int \mathcal{F}$. △

Definition A.4.10. Let $\mathbf{J}$ be a small category, $\mathbf{C}$ a locally small category and $\mathcal{J} \in \mathbf{C}^{\mathbf{J}}$ a diagram.

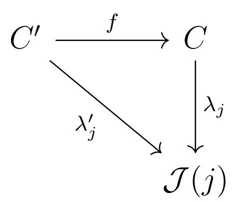
(a) A **limit** of $\mathcal{J}$, denoted by $\lim \mathcal{J}$, is a representation of the contravariant functor $\mathrm{Hom}_{\mathbf{C}^{\mathbf{J}}}(\Delta -, \mathcal{J})\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

(b) A **colimit** of $\mathcal{J}$, denoted by $\operatorname{colim} \mathcal{J}$, is a representation of the covariant functor $\mathrm{Hom}_{\mathbf{C}^{\mathbf{J}}}(\mathcal{J}, \Delta -)\colon \mathbf{C} \to \mathbf{Set}$.

Here, Δ denotes the $\mathbf{C}$-ary diagonal functor (see Definition A.4.4).

Remark A.4.11. It follows from Remark A.4.7 that limits (resp. colimits) are unique up to isomorphism whenever they exist. In this sense we often refer to *the* limit (resp. colimit) of a diagram. △

Remark A.4.12. It follows from Remark A.4.9 that given a diagram $\mathcal{J}$ from a small category $\mathbf{J}$ to a category $\mathbf{C}$ and a limit (resp. colimit) of it, the Yoneda Lemma assigns to it a universal object in the category of elements of the functor $\mathrm{Hom}_{\mathbf{C}^{\mathbf{J}}}(\Delta -, \mathcal{J})$ (resp. $\mathrm{Hom}_{\mathbf{C}^{\mathbf{J}}}(\mathcal{J}, \Delta -)$). More specifically, to each limit (resp. colimit) of $\mathcal{J}$, there corresponds a unique cone λ over $\mathcal{J}$ (resp. cocone under $\mathcal{J}$) with some tip $C \in \mathbf{C}$ such that, given any other cone λ' over $\mathcal{J}$ (resp. cocone under $\mathcal{J}$) with some tip $\mathbf{C}' \in \mathbf{C}$, there exists a unique morphism $f \in \mathbf{C}(C', C)$ (resp. $f \in \mathbf{C}(C, C')$) such that $\lambda' = \lambda \circ f$ (resp. $\lambda' = f \circ \lambda$), where f is seen as a natural transformation between the respective constant diagrams on C and C'. In other words, for each $j \in \mathrm{Ob}(\mathbf{J})$ the diagram



in **C** commutes. We usually identify each limit (resp. colimit) of $\mathcal{J}$ with the respective universal cone (resp. cocone), which we also call the **limit cone** (resp. **limit cocone**) of $\mathcal{J}$. △

Definition A.4.13. Let $\mathcal{J}$ be a diagram. A limit (or colimit) of $\mathcal{J}$ is called **finite** if $\mathcal{J}$ is a finite diagram.

In the following definitions we introduce some very common limits and colimits that will be used throughout the text.

Definition A.4.14. (a) A **terminal object**, denoted by $*$, is a limit of the diagram indexed by the empty category.

(b) A **product**, denoted by $\prod_{j\in\mathbf{J}} \mathcal{J}_j$, is a limit of a diagram $\mathcal{J}$ indexed by a discrete category **J**.

(c) An **equalizer** is a limit of a diagram of shape $\{\bullet \rightrightarrows \bullet\}$.

(d) A **pullback** is a limit of a diagram of shape $\{\bullet \to \bullet \leftarrow \bullet\}$.

Dually, the following colimits are defined.

Definition A.4.15. (a) An **initial object**, denoted by $\emptyset$, is a colimit of the diagram indexed by the empty category.

(b) A **coproduct**, denoted by $\coprod_{j\in\mathbf{J}} \mathcal{J}_j$, is a colimit of a diagram $\mathcal{J}$ indexed by a discrete category **J**.

(c) A **coequalizer** is a colimit of a diagram of shape $\{\bullet \rightrightarrows \bullet\}$.

(d) A **pushout** is a colimit of a diagram of shape $\{\bullet \leftarrow \bullet \to \bullet\}$.

Notation A.4.16. Let **C** be a category and $A, B, C \in \mathrm{Ob}(\mathbf{C})$. We denote:

- a pullback of a diagram $\{B \to A \leftarrow C\}$ by $B \times_A C$,
- a pushout of a diagram $\{B \xleftarrow{f} A \xrightarrow{g} C\}$ by $B \coprod_A C$.

Given a small category **J** and a category **C**, the limits of **J**-shaped diagrams in **C** are functorial as long as they exist, in the sense of the following propositions.

Proposition A.4.17. *Let* $\mathbf{J}$ *be a small category and* $\mathcal{J}$ *and* $\mathcal{J}'$ *be two* $\mathbf{J}$*-shaped diagrams in* $\mathbf{C}$*, for some category* $\mathbf{C}$*. Moreover, let* λ *be a limit cone for the diagram* $\mathcal{J}$ *with tip* $\lim \mathcal{J}$ *and* λ' *a limit cone of the diagram* $\mathcal{J}'$ *with tip* $\lim \mathcal{J}'$*. To each natural transformation* $\alpha\colon \mathcal{J} \to \mathcal{J}'$*, there corresponds a unique morphism* $f \in \mathbf{C}(\lim \mathcal{J}, \lim \mathcal{J}')$ *such that the diagram*

$$\begin{array}{ccc} \lim \mathcal{J} & \xrightarrow{\lambda_j} & \mathcal{J}_j \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \alpha_j} \\ \lim \mathcal{J}' & \xrightarrow{\lambda'_j} & \mathcal{J}'_j \end{array} \tag{A.4.1}$$

in $\mathbf{C}$ *commutes for any* $j \in \mathbf{J}$*.*

There is an analogous proposition for colimits.

Proposition A.4.18. *Let* $\mathbf{J}$ *be a small category and* $\mathbf{C}$ *a category such that every* $\mathbf{J}$*-shaped diagram in* $\mathbf{C}$ *has a limit (resp. colimit). Then, there exists a functor* $\lim\colon \mathbf{C}^{\mathbf{J}} \to \mathbf{C}$ *(resp.* $\operatorname{colim}\colon \mathbf{C}^{\mathbf{J}} \to \mathbf{C}$*) that assigns to each* $\mathbf{J}$*-shaped diagram in* $\mathbf{C}$ *a limit (resp. colimit) of it.*

Definition A.4.19. A category $\mathbf{C}$ is called:

(a) **complete** if every diagram in it has a limit,

(b) **cocomplete** if every diagram in it has a colimit.

In order to prove the existence of limits in a category we often rely on transferring them from other categories via suitable functors. For this purpose the following notions will be useful.

Definition A.4.20. Let $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ be a functor and $\mathcal{J}\colon \mathbf{J} \to \mathbf{C}$ a diagram. We say that $\mathcal{F}$:

(a) **preserves limits** of $\mathcal{J}$ if, for any limit cone λ of $\mathcal{J}$, the cone $\mathcal{F}(\lambda) \in \mathbf{D}^{\mathbf{J}}(\Delta(\mathcal{F}(C)), \mathcal{F} \circ \mathcal{J})$ is a limit cone of the diagram $\mathcal{F} \circ \mathcal{J}\colon \mathbf{J} \to \mathbf{D}$,

(b) **reflects limits** of $\mathcal{J}$ if, for any cone λ of $\mathcal{J}$ such that $\mathcal{F}(\lambda)$ is a limit cone of the diagram $\mathcal{F} \circ \mathcal{J}$ in $\mathbf{D}$, the cone λ is a limit cone of $\mathcal{J}$,

(c) **lifts limits** of $\mathcal{J}$ if, for any limit cone λ of $\mathcal{F} \circ \mathcal{J}$, there exists a limit cone λ' of $\mathcal{J}$ such that $\mathcal{F}(\lambda')$ is isomorphic to λ,

(d) **reflects the existence** of limits of $\mathcal{J}$ if, whenever $\mathcal{F} \circ \mathcal{J}$ has a limit in $\mathbf{D}$, then $\mathcal{J}$ has also a limit,

(e) **creates limits** of $\mathcal{J}$ if it preserves, reflects and reflects the existence of limits of $\mathcal{J}$.

The above definitions have analogues for colimits, where we simply substitute "colimit" for "limit" and "cocone" for "cone".

Remark A.4.21. Notice that the concepts (a) through (d) in Definition A.4.20 are independent from each other with the exception of the concepts (c) and (d). In fact, it is easy to see that if a functor lifts the limits of some diagram then it also reflects the existence of the limits of this diagram. △

Proposition A.4.22. *Let* $\mathbf{C}$ *be a category and* $\mathbf{D}$ *be a complete (resp. cocomplete) category. If there exists a functor* $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ *such that* $\mathcal{F}$ *reflects the existence of all limits (resp. colimits), then the category* $\mathbf{C}$ *is also complete (resp. cocomplete).*

Remark A.4.23 ([Rie16, Prop. 3.3.9])**.** Let $\mathbf{C}$ be a small category and $\mathbf{D}$ any category. The functor category $\mathbf{D}^{\mathbf{C}}$ has all the limits and colimits that $\mathbf{D}$ has in the following sense. Let $\mathcal{J}\colon \mathbf{J} \to \mathbf{D}^{\mathbf{C}}$ be a diagram. If, for any $C \in \mathbf{C}$, the diagram $\mathrm{Ev}_C \circ \mathcal{J}\colon \mathbf{J} \to \mathbf{D}$ has a limit, where Ev_C is the evaluation functor (see Definition A.3.5), then there exists a unique limit $\lim \mathcal{J} \in \mathbf{D}^{\mathbf{C}}$ of the diagram $\mathcal{J}$ such that, for each $C \in \mathbf{C}$, $(\lim \mathcal{J})(C) = \lim(\mathrm{Ev}_C \circ \mathcal{J})$. This implies that the limits in the category $\mathbf{D}^{\mathbf{C}}$ can be computed component-wise. The statement is analogous for colimits. For a proof see [Rie16] Proposition 3.3.9. △

A.5 Adjunctions

Definition A.5.1. Let $\mathbf{C}$, $\mathbf{D}$ be locally small categories.

(a) An **adjunction** between $\mathbf{C}$ and $\mathbf{D}$, denoted by $(\mathcal{F},\mathcal{U},\phi)\colon \mathbf{C} \to \mathbf{D}$, consists of a pair of functors $\mathcal{F}\colon \mathbf{C} \rightleftarrows \mathbf{D} :\!\mathcal{U}$ and a natural isomorphism

$$\phi\colon \mathbf{D}(\mathcal{F}(-),(-)) \xrightarrow{\cong} \mathbf{C}((-),\mathcal{U}(-))\,, \tag{A.5.1}$$

where $\mathbf{D}(\mathcal{F}(-),(-))$ and $\mathbf{C}((-),\mathcal{U}(-))$ are seen as functors from the category $\mathbf{C}^{\mathrm{op}} \times \mathbf{D}$ to the category $\mathbf{Set}$.

(b) The functor $\mathcal{F}$ is called the **left adjoint** of $\mathcal{U}$, and the functor $\mathcal{U}$ is called the **right adjoint** of $\mathcal{F}$. We denote this relation by $\mathcal{F} \dashv \mathcal{U}$.

(c) Let $f \in \mathbf{D}(\mathcal{F}(C), D)$ for some $C \in \mathrm{Ob}(\mathbf{C})$ and $D \in \mathrm{Ob}(\mathbf{D})$. The morphism $\phi_{C,D}(f) \in \mathbf{C}(C,\mathcal{U}(D))$ is called the **transpose** (or **adjunct**) of f and is denoted by $f^\sharp$.

(d) Let $g \in \mathbf{C}(C,\mathcal{U}(D))$ for some $C \in \mathrm{Ob}(\mathbf{C})$ and $D \in \mathrm{Ob}(\mathbf{D})$. The morphism $\phi^{-1}_{C,D}(g) \in \mathbf{D}(\mathcal{F}(C), D)$ is called the **transpose** (or **adjunct**) of g and is denoted by $g^\flat$.

Notation A.5.2. We denote an adjunction $(\mathcal{F},\mathcal{U},\phi)\colon \mathbf{C} \to \mathbf{D}$ simply by $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$, whenever the natural isomorphism ϕ is not explicitly used.

Definition A.5.3. Let $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$ be an adjunction.

(a) The **unit** of the adjunction is the natural transformation $\eta\colon \mathrm{Id}_{\mathbf{C}} \to \mathcal{U}\circ\mathcal{F}$ with components as follows. For each $C \in \mathbf{C}$, $\eta_C := {\mathrm{id}_{\mathcal{F}(C)}}^\sharp$.

(b) The **counit** of the adjunction is the natural transformation $\epsilon\colon \mathcal{F}\circ\mathcal{U} \to \mathrm{Id}_{\mathbf{D}}$ with components as follows. For each $D \in \mathbf{D}$, $\epsilon_D := {\mathrm{id}_{\mathcal{U}(D)}}^\flat$.

Proposition A.5.4 ([Rie16, Prop. 4.2.6])**.** *Let $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$ be an adjunction. Then, the unit $\eta\colon \mathrm{Id}_{\mathbf{C}} \to \mathcal{U}\circ\mathcal{F}$ and the counit $\epsilon\colon \mathcal{F}\circ\mathcal{U} \to \mathrm{Id}_{\mathbf{D}}$ of the adjunction satisfy the so-called **triangle identities**, i.e. the diagrams*

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\mathcal{F}\eta} & \mathcal{F}\circ\mathcal{U}\circ\mathcal{F} \\
& {\scriptstyle \mathrm{id}_{\mathcal{F}}} \searrow & \downarrow {\scriptstyle \epsilon_{\mathcal{F}}} \\
& & \mathcal{F}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\eta_{\mathcal{U}}} & \mathcal{U}\circ\mathcal{F}\circ\mathcal{U} \\
& {\scriptstyle \mathrm{id}_{\mathcal{U}}} \searrow & \downarrow {\scriptstyle \mathcal{U}\epsilon} \\
& & \mathcal{U}
\end{array}
\tag{A.5.2}
$$

in $\mathbf{D}^{\mathbf{C}}$ and $\mathbf{C}^{\mathbf{D}}$, respectively, commute, where $\mathcal{F}\eta$, $\epsilon_{\mathcal{F}}$, $\eta_{\mathcal{U}}$ and $\mathcal{U}\epsilon$ denote the natural transformations defined in Definition A.2.10.

Proposition A.5.5 ([Rie16, Prop. 4.2.6])**.** *Let $\mathcal{F}\colon \mathbf{C} \rightleftarrows \mathbf{D} :\mathcal{U}$ be two functors. The following two conditions are equivalent:*

(1) $\mathcal{F} \dashv \mathcal{U}$.

(2) There exist natural transformations $\eta\colon \mathrm{Id}_{\mathbf{C}} \to \mathcal{U}\circ\mathcal{F}$ and $\epsilon\colon \mathcal{F}\circ\mathcal{U} \to \mathrm{Id}_{\mathbf{D}}$ that satisfy the triangle identities, i.e. $\epsilon_{\mathcal{F}} \circ \mathcal{F}\eta = \mathrm{id}_{\mathcal{F}}$ and $\mathcal{U}\epsilon \circ \eta_{\mathcal{U}} = \mathrm{id}_{\mathcal{U}}$.

Remark A.5.6 ([Rie16, Prop. 4.4.1])**.** Adjoint functors are unique up to isomorphism. That means that if there are functors $\mathcal{F},\mathcal{G}\colon \mathbf{C} \to \mathbf{D}$ and $\mathcal{H}\colon \mathbf{D} \to \mathbf{C}$ such that $\mathcal{F} \dashv \mathcal{H}$ and $\mathcal{G} \dashv \mathcal{H}$ then $\mathcal{F} \cong \mathcal{G}$. Similarly, if $\mathcal{H} \dashv \mathcal{F}$ and $\mathcal{H} \dashv \mathcal{G}$ then $\mathcal{F} \cong \mathcal{G}$. △

Definition A.5.7. Let $\mathbf{C}, \mathbf{D}$ be locally small categories. An **adjoint equivalence** between $\mathbf{C}$ and $\mathbf{D}$ is an adjunction between $\mathbf{C}$ and $\mathbf{D}$ such that the unit and the counit of the adjunction are natural isomorphisms.

Proposition A.5.8 ([Rie16, Prop. 4.4.5])**.** *Let the functors* $\mathcal{F}\colon \mathbf{C} \rightleftarrows \mathbf{D} :\mathcal{U}$ *witness an equivalence of categories (see Definition A.2.11). Then, there is an adjoint equivalence between the categories* $\mathbf{C}$ *and* $\mathbf{D}$ *with* $\mathcal{F} \dashv \mathcal{U}$*.*

Proposition A.5.9 ([Rie16, Ths. 4.5.2 & 4.5.3])**.** *Let* $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$ *be an adjunction. Then, the left adjoint* $\mathcal{F}$ *preserves colimits, while the right adjoint* $\mathcal{U}$ *preserves limits.*

Proposition A.5.10 ([Rie16, Prop. 4.4.4])**.** *Let* $(\mathcal{F} \dashv \mathcal{U})\colon \mathbf{C} \to \mathbf{D}$ *and* $(\mathcal{F}' \dashv \mathcal{U}')\colon \mathbf{D} \to \mathbf{E}$ *be two adjunctions. Then, there is an adjunction* $(\mathcal{F}' \circ \mathcal{F} \dashv \mathcal{U} \circ \mathcal{U}')\colon \mathbf{C} \to \mathbf{E}$*.*

Lastly, a concept used extensively throughout the text is that of a two-variable adjunction, defined below.

Definition A.5.11. Let $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ be locally small categories. A **two-variable adjunction**, denoted by $(\mathcal{F}, \mathcal{G}, \mathcal{H}, \phi, \psi)\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$, consists of three functors $\mathcal{F}\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$, $\mathcal{G}\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{C}$ and $\mathcal{H}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{D}$ and natural isomorphisms

$$\mathbf{E}(\mathcal{F}(-,-),-) \overset{\phi}{\cong} \mathbf{C}(-,\mathcal{G}(-,-)) \overset{\psi}{\cong} \mathbf{D}(-,\mathcal{H}(-,-))\,. \tag{A.5.3}$$

The functor $\mathcal{F}$ is called the **left adjoint** of the two-variable adjunction.

Notation A.5.12. We will often denote to a two-variable adjunction simply by $(\mathcal{F}, \mathcal{G}, \mathcal{H})\colon \mathbf{C} \times \mathbf{D} \to \mathbf{E}$, omitting the natural isomorphisms whenever they are not explicitly used.

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