LMI Design Procedure for Incremental Input/Output-to-State Stability in Nonlinear Systems

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***Abstract*—This letter deals with the investigation of an important property that characterizes the detectability of nonlinear systems, namely the incremental Exponential Input/Output-to-State Stability (i-EIOSS). While such a property is easy to check for linear systems, for nonlinear systems it is a hard task. On the other hand, the i-EIOSS property is well-suited for the development of robust estimators. In this letter, we propose a novel numerical design procedure ensuring the computation of the i-EIOSS-related coefficients which are necessary to tune the parameters of the robust estimators. We first introduce a general simple but useful Lyapunov-based method, then we develop a new Linear Matrix Inequality (LMI) condition guaranteeing the computation of the i-EIOSS coefficients. The proposed design method is easily tractable by numerical software and may be used for several real-world applications. Compared to the existing literature, the proposed method is simpler, provides a finite number of LMIs to be solved, and does not need to convert the system into a new one with linear outputs leading to LMI conditions.**

***Index Terms*—Incremental input/output-to-state stability (i-EIOSS), observer design, Lyapunov functions, LMI approach.**

I. Introduction

The i-EIOSS property has a long story in control theory since the pioneering work by Sontag [1]. This important property, which characterizes the detectability of nonlinear systems, is recently investigated in a deeper way by [2], [3], [4] and the references therein. A considerable attention has been paid to the relation between the existence of a robust estimator and the i-EIOSS property of a given system. Significant results and elegant arguments have been presented in [3] on the existence of i-EIOSS Lyapunov functions. Such a property has also been extensively exploited in the context of the Moving Horizon Estimation (MHE) problem by [4] and related papers. Although all the above results are interesting and constructive, they need a simple and useful method to design the coefficients of the i-EIOSS property of a system before designing the estimation scheme, since the estimation scheme often needs the explicit knowledge of those coefficients.

The objective of this letter consists of establishing a simple and useful design method that can be easily exploited by numerical software and may be used for the design of the tuning parameters of any robust estimation scheme. The i-EIOSS notion has been investigated only recently in the discrete-time setting and, to our knowledge, up to now no constructive method has been proposed to find the parameters involved in the i-EIOSS upper bound formulation. This motivated us to propose a novel technique, which ensures not only the i-EIOSS property of a system but more importantly allows the explicit computation of the i-EIOSS related coefficients while optimizing their values by using Linear Matrix Inequalities (LMIs). The result is based on the use of a convenient mathematical tool for stability analysis, which allowed us to develop a novel Lyapunov function-based criterion. Hence, the combination of a quadratic Lyapunov function and the convexity principle led to new LMI conditions. Due to a lack of space, the feasibility analysis of the proposed LMI conditions is not addressed in detail; however, some comments on its conservatism and feasibility are provided.

The proposed LMI-based design procedure guaranteeing the i-EIOSS property plays an important role in designing robust estimators. The main motivation for developing this LMI method is for the purpose of stability analysis in MHE. Indeed, the design of a robust MHE as developed in [5], [6] requires the i-EIOSS coefficients as tuning parameters. It is worth noticing that related results on ensuring the i-IOSS property by using matrix inequalities are established in [7], where the authors proposed general but not numerically tractable design conditions. Compared to [7], the advantages of our proposed method in this letter can be summarized as follows:

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- The method in [7] is based on the differential dynamics of the system and its linearization at a given point. However, our technique uses the generalized version of the differential mean value theorem for vector-valued functions to transform equivalently, without linearization, the nonlinear terms into a polytopic form.
- Our method is simpler than that of [7] due to the polytopic form of the error dynamics and the convexity principle to get a finite number of LMIs.
- Our method does not require any coordinate transformation to convert the original system into a new one with linear outputs as in [7].

II. Problem Formulation and Preliminaries

A. System Description and Assumptions

Consider the following nonlinear discrete-time system:

$$\begin{cases} x_{t+1} = f(x_t, w_t) \\ y_t = h(x_t, v_t) \end{cases} . \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state of the system, $y_t \in \mathbb{R}^p$ is the output vector, and w_t and v_t are unknown external disturbances of appropriate dimensions. The functions $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ satisfy $f(0,0)=0$, $h(0,0)=0$.

For convenience of calculation, and brevity as well and to avoid cumbersome notations, we consider the system (1) with the same disturbance input w_t in the output y_t. This is not a restriction. It is assumed without loss of generality since there are no constraints on the dependence of the functions $f(.)$ and $h(.)$ on w_t and v_t, respectively. To summarize, instead of system (1), we consider the following form:

$$\begin{cases} x_{t+1} = f(x_t, \boldsymbol{\omega}_t) \\ y_t = h(x_t, \boldsymbol{\omega}_t) \end{cases} . \tag{2}$$

where $\boldsymbol{\omega}_t \triangleq \left[w_t^\top \; v_t^\top \right]^\top \in \mathbb{R}^q$. Then, all the next definitions and results are based on the system (2). This form is convenient in the LMI context as usual in the literature [8], [9].

Before introducing the main definitions, we need to make some assumptions, which are necessary for the developed design methodology.

Assumption 1: The nonlinear functions $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are differentiable with respect to their arguments and satisfy the following conditions:

$$\sup_{\substack{x\in\mathbb{R}^n \\ \boldsymbol{\omega}\in\mathbb{R}^q}} \left| \frac{\partial f_i}{\partial x}(x,\boldsymbol{\omega}) \right| < +\infty, \quad \sup_{\substack{x\in\mathbb{R}^n, \\ w\in\mathbb{R}^q}} \left| \frac{\partial f_i}{\partial \boldsymbol{\omega}}(x,\boldsymbol{\omega}) \right| < +\infty \tag{3}$$

$$\sup_{\substack{x\in\mathbb{R}^n \\ \boldsymbol{\omega}\in\mathbb{R}^q}} \left| \frac{\partial h_i}{\partial x}(x,\boldsymbol{\omega}) \right| < +\infty, \quad \sup_{\substack{x\in\mathbb{R}^n \\ \boldsymbol{\omega}\in\mathbb{R}^q}} \left| \frac{\partial h_i}{\partial \boldsymbol{\omega}}(x,\boldsymbol{\omega}) \right| < +\infty \tag{4}$$

where the functions $f_i, i=1,\ldots,n$ and $h_i, i=1,\ldots,p$ are the i-th component of the functions f and h, respectively.

The above assumption means that the Jacobians $\frac{\partial f(x,\boldsymbol{\omega})}{\partial x}$, $\frac{\partial f(x,\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}$, $\frac{\partial h(x,\boldsymbol{\omega})}{\partial x}$, and $\frac{\partial h(x,\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}$ are bounded, then they belong to convex polytopic sets defined respectively by:

$$\mathcal{V}_{f,x} \triangleq \left\{ \sum_{j=1}^{n_x} \alpha_j \mathcal{F}_j^x, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{n_x} \alpha_j = 1 \right\} \tag{5}$$

$$\mathcal{V}_{f,\boldsymbol{\omega}} \triangleq \left\{ \sum_{j=1}^{n_w} \alpha_j \mathcal{F}_j^{\boldsymbol{\omega}}, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{n_{\boldsymbol{\omega}}} \alpha_j = 1 \right\} \tag{6}$$

$$\mathcal{V}_{h,x} \triangleq \left\{ \sum_{j=1}^{q_x} \alpha_j \mathcal{H}_j^x, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{q_x} \alpha_j = 1 \right\} \tag{7}$$

$$\mathcal{V}_{h,\boldsymbol{\omega}} \triangleq \left\{ \sum_{j=1}^{q_{\boldsymbol{\omega}}} \alpha_j \mathcal{H}_j^{\boldsymbol{\omega}}, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{q_{\boldsymbol{\omega}}} \alpha_j = 1 \right\} \tag{8}$$

where $\mathcal{F}_j^x$, $\mathcal{F}_j^{\boldsymbol{\omega}}$, $\mathcal{H}_j^x$, and $\mathcal{H}_j^{\boldsymbol{\omega}}$ are known constant matrices of appropriate dimensions and the known integers $n_x, n_{\boldsymbol{\omega}}, q_x, q_{\boldsymbol{\omega}}$ are the number of vertices of each convex set, respectively. The result is standard in the representation of elements in a convex set. Indeed, since the Jacobians are bounded, then the partial derivatives admit lower and upper bounds from which we can construct a convex polytopic set containing the Jacobians. We refer the reader to the classic book [10] on the representation of elements in a convex set and the books [11], [12] for convex decomposition of nonlinear functions.

B. Definition of i-EIOSS Property

The next definition is important in this letter. We will introduce the main definition concerned by this letter, which is necessary in the developed conditions ensuring the incremental exponential input-to-state stability of the system (2).

Definition 1: System (2) is *incrementally Exponentially Input/Output-to-State Stable* (i-EIOSS) if there exist constants $c_x, c_v, c_w > 0$ and $\varrho \in (0,1)$ such that for each pair of initial conditions $x_0, \tilde{x}_0 \in \mathcal{X}$ and each two disturbance sequences $\boldsymbol{\omega}_t, \tilde{\boldsymbol{\omega}}_t \in \Omega$, the following holds:

$$\begin{aligned} |x_t\left(x_0, \boldsymbol{\omega}_0^{t-1}\right) - x_t\left(\tilde{x}_0, \tilde{\boldsymbol{\omega}}_0^{t-1}\right)|^2 &\leq c_x |x_0 - \tilde{x}_0|^2 \varrho^t \\ &+ c_v \sum_{i=0}^{t-1} \varrho^{t-1-i} |y_i\left(x, \boldsymbol{\omega}_0^{i-1}\right) - y_i\left(\tilde{x}, \tilde{\boldsymbol{\omega}}_0^{i-1}\right)|^2 \\ &+ c_w \sum_{i=0}^{t-1} \varrho^{t-1-i} |\boldsymbol{\omega}_i - \tilde{\boldsymbol{\omega}}_i|^2 \end{aligned} \tag{9}$$

where $x_t(x_0, \boldsymbol{\omega}_0^{t-1})$ means the solution of (2) generated from the initial state x_0 and $\boldsymbol{\omega}_0^{t-1} \triangleq \left[\boldsymbol{\omega}_0 \ldots \boldsymbol{\omega}_{t-1}\right]^\top$.

For more details on the above definition, we refer the reader to [4] and [13, Definition 2, and Lemma 7] for a more general formulation.

C. Mathematical Tools for Stability Analysis

In this section, we present some basic results, presented in three lemmas, which we will exploit in the next section to analyze the i-EIOSS property of the system (2) by using quadratic Lyapunov functions. Such lemmas are presented in a general framework so that they can be exploited in different cases for different control design problems.

Lemma 1 [5]: Let $(u_t)_{t\geq -\ell}$ be a sequence of non-negative real numbers and $\ell \geq 1$ such that

$$u_t \leq \alpha u_{t-\ell} + \beta z_t, \forall t \geq \ell,$$

where α and β are scalars such that $\beta \geq 0, 0 < \alpha < 1$. The sequence $(z_t)_{t\geq 0}$ is non negative. Then the following inequality holds for any $\kappa \in \mathbb{N}, \kappa \geq 2$:

$$u_t \leq \alpha^{\frac{t}{\kappa \ell}} \max_{-\ell \leq j \leq 0} u_j + \left(\frac{\beta}{1-\alpha^{\frac{\kappa-1}{\kappa}}}\right) \max_{\substack{t-s\ell \leq j \leq t \\ \frac{t-j}{\ell} \in \mathbb{N}}} \left(\alpha^{\frac{t-j}{\kappa \ell}} z_j\right). \quad (10)$$

Lemma 2 (Differential Mean Value Theorem): Let $\Psi : \mathbb{R}^n \mapsto \mathbb{R}^q$ be a differentiable function and two vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Then, there exists

$$z \triangleq \begin{bmatrix} z^1 \\ z^2 \\ \vdots \\ z^q \end{bmatrix} \in \mathbb{R}^{nq},\ z^i \in \mathbf{Co}(x, y), i = 1, \ldots, q \quad (11)$$

where $\mathbf{Co}(x, y)$ stands for the set of convex combinations of x and y, such that

$$\Psi(x) - \Psi(y) = \nabla_x^{\Psi}(z)(x-y) \quad (12)$$

where

$$\nabla_x^{\Psi}(z) \triangleq \left[\frac{\partial \Psi_1(z^1)}{\partial x}\ \frac{\partial \Psi_2(z^2)}{\partial x}\ \ldots\ \frac{\partial \Psi_q(z^q)}{\partial x}\right]^{\top}. \quad (13)$$

Lemma 3: Let $\Psi : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^q$ be a differentiable function and two vectors $(x, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(y, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$. Then, there exist $z \in \mathbb{R}^{nq}$ as in (11) and

$$v \triangleq \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^q \end{bmatrix} \in \mathbb{R}^{mq},\ v^i \in \mathbf{Co}(\bar{x}, \bar{y}), i = 1, \ldots, q \quad (14)$$

such that

$$\Psi(x, \bar{x}) - \Psi(y, \bar{y}) = \nabla_x^{\Psi}(z, \bar{x})(x-y) + \nabla_{\bar{x}}^{\Psi}(y, v)(\bar{x} - \bar{y}). \quad (15)$$

Proof: The proof is straightforward. It is based on the decomposition

$$\Psi(x, \bar{x}) - \Psi(y, \bar{y}) = \left[\Psi(x, \bar{x}) - \Psi(y, \bar{x})\right] + \left[\Psi(y, \bar{x}) - \Psi(y, \bar{y})\right] \quad (16)$$

and the application of (12) in Lemma 2 on both terms in the right hand side of (16). ■

D. Motivation

The i-EIOSS property is extensively used in the recent literature to develop robust estimators. Such robust estimators depend on some tuning parameters and those tuning parameters depend on the coefficients related to the i-EIOSS of the system, namely the parameters ϱ, c_x, c_v, and c_w. However, the computation of these parameters from the trajectories of the system (2) is not easy or even impossible in general. Subsequently, it becomes crucial to develop a straightforward and practical universal numerical method for achieving this goal. We will present a novel numerical approach based on LMI conditions, which are easily tractable by numerical software. Such a simple method is based on the use of the Lyapunov theory.

III. Lyapunov-Based Stability Criterion for i-EIOSS

A. Lyapunov Function Based i-EIOSS

In this section, we provide a general criterion based on Lyapunov theory to guarantee the i-EIOSS property of a given system. The result of this section is summarized in the following proposition.

Proposition 1: Let $(x_t, \tilde{x}_t)$ be two arbitrary solutions of (1) generated from two initial conditions $x_0, \tilde{x}_0 \in \mathcal{X}$ and two disturbance sequences $w_t, \tilde{w}_t \in \Omega$, respectively. Let $\vartheta(x_t, \tilde{x}_t)$ be a Lyapunov function and

$$\begin{aligned} \Delta^{\theta}\vartheta(x_t, \tilde{x}_t) &\triangleq \vartheta(x_{t+1}, \tilde{x}_{t+1}) - \theta\vartheta(x_t, \tilde{x}_t) \\ &\triangleq \Delta^{\theta}\vartheta_t = \vartheta_{t+1} - \theta\vartheta_t \end{aligned} \quad (17)$$

where $\theta > 0$. Define $\epsilon_t \triangleq x_t - \tilde{x}_t$ and assume that the following items hold:

(i) There exist two positive scalars $\vartheta_{\min}$ and $\vartheta_{\max}$ with $\vartheta_{\min} < \vartheta_{\max}$ satisfying:

$$\vartheta_{\min}|\epsilon_t|^2 \leq \vartheta(x_t, \tilde{x}_t) \leq \vartheta_{\max}|\epsilon_t|^2,\ \forall t \geq 0; \quad (18)$$

(ii) There exist $\theta < 1$, $c_y > 0$, $\bar{c}_w > 0$ such that

$$\Delta^{\theta}\vartheta_t \leq c_y|y_t - \tilde{y}_t|^2 + \bar{c}_w|w_t - \tilde{w}_t|^2, \forall t \geq 0, \quad (19)$$

where y_t and $\tilde{y}_t$ are the outputs generated by system (1) with x_t and $\tilde{x}_t$, respectively.

Then system (1) is i-EIOSS according to (9) in Definition 1 with the following coefficients $\forall \kappa \geq 2$:

$$\begin{cases} \varrho = \theta,\ c_v = \frac{c_y}{\vartheta_{\min}(1-\theta^{\kappa-1})} \\ c_x = \frac{\vartheta_{\max}}{\vartheta_{\min}},\ c_w = \frac{\bar{c}_w}{\vartheta_{\min}(1-\theta^{\kappa-1})} \end{cases} \quad (20)$$

with $\theta \in (0, 1)$.

Proof: From item (ii) and (17), we can write

$$\vartheta_t \leq \theta\vartheta_{t-1} + z_t, \forall t \geq 1, \quad (21)$$

where $z_t = c_y|y_{t-1} - \tilde{y}_{t-1}|^2 + \bar{c}_w|w_{t-1} - \tilde{w}_{t-1}|^2$. Therefore, by applying (10) of Lemma 1 with the parameters

$$\ell = 1, \beta = 1, s = t-1, \alpha = \theta^{\kappa}, \kappa \geq 2 \quad (22)$$

and from $\vartheta_j = \vartheta_0, \forall j \leq 0$, by convention and construction, we get

$$\begin{aligned} \vartheta_t &\leq \vartheta_0\theta^t + \left(\frac{1}{1-\theta^{\kappa-1}}\right) \max_{1\leq j\leq t}\left(\theta^{t-j}z_j\right) \\ &\leq \vartheta_0\theta^t + \frac{1}{\left(1-\theta^{\kappa-1}\right)}\sum_{j=1}^{t}\theta^{t-j}z_j \\ &\underbrace{=}_{i:=j-1} \vartheta_0\theta^t + \frac{1}{\left(1-\theta^{\kappa-1}\right)}\sum_{i=0}^{t-1}\theta^{t-1-i}z_{i+1} \\ &= \vartheta_0\theta^t + \frac{c_y}{\left(1-\theta^{\kappa-1}\right)}\sum_{i=0}^{t-1}\theta^{t-1-i}|y_i - \tilde{y}_i|^2 \\ &\quad + \frac{\bar{c}_w}{\left(1-\theta^{\kappa-1}\right)}\sum_{i=0}^{t-1}\theta^{t-1-i}|w_i - \tilde{w}_i|^2. \end{aligned} \quad (23)$$

On the other hand, from item (i), we have $|\epsilon_t|^2 \leq \frac{1}{\vartheta_{\min}}\vartheta_t$ and $\vartheta_0 \leq \vartheta_{\max}|\epsilon_0|^2$. Hence, we deduce the following inequality for all $t \geq 1$:

$$|\epsilon_t|^2 \leq \frac{\vartheta_{\max}}{\vartheta_{\min}}|\epsilon_0|^2\theta^t + \frac{c_y}{\vartheta_{\min}\left(1-\theta^{\kappa-1}\right)}\sum_{i=0}^{t-1}\theta^{t-1-i}|y_i - \tilde{y}_i|^2 + \frac{\bar{c}_w}{\vartheta_{\min}\left(1-\theta^{\kappa-1}\right)}\sum_{i=0}^{t-1}\theta^{t-1-i}|w_i - \tilde{w}_i|^2 \tag{24}$$

which means that system (1) is i-EIOSS according to (9) with the coefficients given in (20). ■

Proposition 1 provides a criterion to guarantee the i-EIOSS of a given system, which is in general difficult to characterize. Without this Lyapunov-based characterization, computing the values of the i-EIOSS coefficients c_x, c_v, and c_w becomes a hard task. On the other hand, such coefficients are necessary to design the tuning parameters of any robust estimator of system (2). Then, in the next section, we propose an LMI-based design procedure, which is easily tractable by numerical software.

B. New LMI-Based i-EIOSS Criterion

By considering a particular Lyapunov function, we will obtain sufficient conditions, expressed in terms of LMIs, ensuring the property of i-EIOSS of the system (2). To this end, let us consider the following quadratic Lyapunov function, usually used in the literature in the LMI context:

$$\vartheta(x_t, \tilde{x}_t) \triangleq (x_t - \tilde{x}_t)^\top \mathbb{P}(x_t - \tilde{x}_t) \tag{25}$$

where $\mathbb{P} = \mathbb{P}^\top > 0$ and $(x_t, \tilde{x}_t)$ are two arbitrary solutions of (2) generated from two initial conditions $x_0, \tilde{x}_0 \in \mathbb{R}^n$ and two disturbance sequences $\boldsymbol{\omega}_t, \tilde{\boldsymbol{\omega}}_t \in \mathbb{R}^q$, respectively. Consider $\epsilon_t \triangleq x_t - \tilde{x}_t$, the error between the two trajectories, $\epsilon_{\boldsymbol{\omega}} \triangleq \boldsymbol{\omega}_t - \tilde{\boldsymbol{\omega}}_t$, $\epsilon_y \triangleq y_t - \tilde{y}_t$, and define ϑ_y as:

$$\vartheta_y \triangleq \Delta^\theta \vartheta(x_t, \tilde{x}_t) - c_y|\epsilon_y|^2 - \bar{c}_w|\epsilon_{\boldsymbol{\omega}}|^2. \tag{26}$$

First, we have

$$\epsilon_{t+1} = \nabla_x^f(z_f, \boldsymbol{\omega})\epsilon_t + \nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f)\epsilon_{\boldsymbol{\omega}} \tag{27}$$

$$\epsilon_y = \nabla_x^h(z_h, \boldsymbol{\omega})\epsilon_t + \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\epsilon_{\boldsymbol{\omega}}. \tag{28}$$

After developing $\Delta^\theta\vartheta(x_t, \tilde{x}_t)$ and from Lemma 2, we get:

$$\begin{aligned}\vartheta_y = {} & \epsilon_t^\top\Big[\Big(\nabla_x^f(z_f, \boldsymbol{\omega})\Big)^\top \mathbb{P}\nabla_x^f(z_f, \boldsymbol{\omega}). \\ & -c_y\Big(\nabla_x^h(z_h, \boldsymbol{\omega})\Big)^\top \nabla_x^h(z_h, \boldsymbol{\omega}) - \theta\mathbb{P}\Big]\epsilon_t \\ & + \epsilon_{\boldsymbol{\omega}}^\top\Big[\Big(\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f)\Big)^\top \mathbb{P}\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f). \\ & - c_y\Big(\nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\Big)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) - \bar{c}_w\mathbb{I}_q\Big]\epsilon_{\boldsymbol{\omega}} \\ & + 2\epsilon_t^\top\Big[\Big(\nabla_x^f(z_f, \boldsymbol{\omega})\Big)^\top \mathbb{P}\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f). \\ & - c_y\Big(\nabla_x^h(z_h, \boldsymbol{\omega})\Big)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\Big]\epsilon_{\boldsymbol{\omega}}\end{aligned} \tag{29}$$

which can be written under the matrix form

$$\begin{bmatrix}\epsilon_t \\ \epsilon_{\boldsymbol{\omega}}\end{bmatrix}^\top \mathbb{M}\big(\mathbb{P}, c_y, \bar{c}_w\big)\begin{bmatrix}\epsilon_t \\ \epsilon_{\boldsymbol{\omega}}\end{bmatrix} \tag{30}$$

where $\mathbb{M}\Big(\mathbb{P}, c_y, \bar{c}_w\Big)$ is defined in (31), shown at the bottom of the page.

Then, we have $\vartheta_y < 0$ for all $\left[\epsilon_t^\top\ \epsilon_{\boldsymbol{\omega}}^\top\right]^\top \neq 0$ if the inequality $\mathbb{M}\Big(\mathbb{P}, c_y, \bar{c}_w\Big) < 0$ is satisfied. Hence, from Schur Lemma, the previous inequality is equivalent to (32), shown at the bottom of the page. Before stating the main theorem, we need to introduce some convex polytopic sets. As in (5)–(8), from Assumption 1, the Jacobians $\left(\nabla_x^h(z_h, \boldsymbol{\omega})\right)^\top \nabla_x^h(z_h, \boldsymbol{\omega})$, $\left(\nabla_x^h(z_h, \boldsymbol{\omega})\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)$, and $\left(\nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\right)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)$ are bounded. Therefore, by using the convex decomposition technique [10], they belong to the convex polytopic sets defined respectively as:

$$\mathcal{V}_h^\ell \triangleq \left\{\sum_{j=1}^{n_\ell}\alpha_j\mathcal{H}_j^\ell,\ \text{such that } \alpha_j \geq 0, \sum_{j=1}^{n_\ell}\alpha_j = 1\right\} \tag{33}$$

for $\ell = 1, 2, 3$, respectively. The matrices $\mathcal{H}_j^\ell, j = 1, 2, 3$ are known and constant with appropriate dimensions. As for the known integers n_ℓ, they represent the number of vertices of $\mathcal{V}_h^\ell$, for $\ell = 1, 2, 3$.

Now we are ready to state the main theorem, which provides LMIs ensuring the i-EIOSS property of the system (2).

$$\mathbb{M}\big(\mathbb{P}, c_y, \bar{c}_w\big) \triangleq \begin{bmatrix}\mathbb{N}\big(\mathbb{P}, c_y\big) - \theta\mathbb{P} & \Big(\nabla_x^f(z_f, \boldsymbol{\omega})\Big)^\top \mathbb{P}\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f) - c_y\big(\nabla_x^h(z_h, \boldsymbol{\omega})\big)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) \\ (\star) & \Big(\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f)\Big)^\top \mathbb{P}\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f) - c_y\big(\nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\big)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) - \bar{c}_w\mathbb{I}_q\end{bmatrix}$$

$$\mathbb{N}\big(\mathbb{P}, c_y\big) \triangleq \Big(\nabla_x^f(z_f, \boldsymbol{\omega})\Big)^\top \mathbb{P}\nabla_x^f(z_f, \boldsymbol{\omega}) - c_y\Big(\nabla_x^h(z_h, \boldsymbol{\omega})\Big)^\top \nabla_x^h(z_h, \boldsymbol{\omega}) \tag{31}$$

$$\begin{bmatrix}-c_y\big(\nabla_x^h(z_h, \boldsymbol{\omega})\big)^\top \nabla_x^h(z_h, \boldsymbol{\omega}) - \theta\mathbb{P} & -c_y\big(\nabla_x^h(z_h, \boldsymbol{\omega})\big)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) & \Big(\nabla_x^f(z_f, \boldsymbol{\omega})\Big)^\top \mathbb{P} \\ (\star) & -c_y\big(\nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h)\big)^\top \nabla_{\boldsymbol{\omega}}^h(\tilde{x}, \boldsymbol{v}_h) - \bar{c}_w\mathbb{I}_q & \Big(\nabla_{\boldsymbol{\omega}}^f(\tilde{x}, \boldsymbol{v}_f)\Big)^\top \mathbb{P} \\ (\star) & (\star) & -\mathbb{P}\end{bmatrix} < 0 \tag{32}$$

Theorem 1: Assume that there exists a positive definite and symmetric matrix $\mathbb{P}$, positive scalars $c_y, \bar{c}_w$, and $\theta \in (0,1)$ such that the following matrix inequalities are satisfied:

$$\begin{bmatrix} -c_y\mathcal{H}_i^1 - \theta\mathbb{P} & -c_y\mathcal{H}_j^2 & \left(\mathcal{F}_l^x\right)^\top\mathbb{P} \\ (\star) & -c_y\mathcal{H}_k^3 - \bar{c}_w\mathbb{I}_q & \left(\mathcal{F}_m^{\boldsymbol{\omega}}\right)^\top\mathbb{P} \\ (\star) & (\star) & -\mathbb{P} \end{bmatrix} < 0 \quad (34)$$

for all $i \in \{1,\ldots,n_1\}$, $j \in \{1,\ldots,n_2\}$, $k \in \{1,\ldots,n_3\}$, $l \in \{1,\ldots,n_x\}$, and $m \in \{1,\ldots,n_{\boldsymbol{\omega}}\}$. Then the system (2) is i-EIOSS according to (9) in Definition 1 with the coefficients defined in (20) with $\vartheta_{\max} = \lambda_{\max}(\mathbb{P})$ and $\vartheta_{\min} = \lambda_{\min}(\mathbb{P})$.

Proof: From Schur Lemma we have equivalence between $\mathbb{M}\left(\mathbb{P}, c_y, \bar{c}_w\right) < 0$ and (32). On other hand, the left hand side of (32) is affine (then convex) with respect to all the Jacobian matrices $\frac{\partial f(x,\boldsymbol{\omega})}{\partial x}$, $\frac{\partial f(x,\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}$, $\frac{\partial h(x,\boldsymbol{\omega})}{\partial x}$, $\frac{\partial h(x,\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}$, $\left(\nabla_x^h(z_h,\boldsymbol{\omega})\right)^\top\nabla_x^h(z_h,\boldsymbol{\omega})$, $\left(\nabla_x^h(z_h,\boldsymbol{\omega})\right)^\top\nabla_{\boldsymbol{\omega}}^h(\tilde{x},\boldsymbol{\nu}_h)$, and $\left(\nabla_{\boldsymbol{\omega}}^h(\tilde{x},\boldsymbol{\nu}_h)\right)^\top\nabla_{\boldsymbol{\omega}}^h(\tilde{x},\boldsymbol{\nu}_h)$. In addition, from (5)-(8) and (33), these Jacobians can be decomposed into a convex form by using the convex decomposition technique [10]. Hence, from the convexity principle [11], the inequality (32) is satisfied for any element on the convex sets defined by (5)-(8) and (33) if it is satisfied on the vertices $\mathcal{H}_i^1, \mathcal{H}_j^2, \mathcal{H}_k^3$, $\mathcal{F}_l^x$, and $\mathcal{F}_m^{\boldsymbol{\omega}}$. Since (34) are exactly (32) evaluated on the vertices, then from the convexity principle, we have $\mathbb{M}\left(\mathbb{P}, c_y, \bar{c}_w\right) < 0$, which leads to $\vartheta_y \leq 0$. Hence, from Proposition 1 it follows that the system (2) is i-EIOSS with the coefficients given in (20) with $\vartheta_{\max} = \lambda_{\max}(\mathbb{P})$ and $\vartheta_{\min} = \lambda_{\min}(\mathbb{P})$. This ends the proof. ■

To optimize the values of ϱ, c_x, c_ν, and c_w while solving the LMIs (34), we can minimize the values of c_y and $\bar{c}_w$ by fixing θ a priori in (0, 1) by using some gridding method. However, an ill-conditioned $\mathbb{P}$ would make $\lambda_{\min}(\mathbb{P})$ small, which leads to large values of c_ν and c_ω even if c_y and $\bar{c}_\omega$ are small. To avoid this issue, one can resort to additional constraints on $\mathbb{P}$ such as

$$\mathbb{P} \geq \mathbb{I}_n \quad (35)$$

by taking advantage of homogeneity. Nevertheless, the constraint (35) may increase the value of $c_x = \frac{\lambda_{\max}(\mathbb{P})}{\lambda_{\min}(\mathbb{P})} \leq \lambda_{\max}(\mathbb{P})$. To minimize c_x, we need the additional constraint:

$$\mathbb{P} \leq \alpha\mathbb{I}_n, \quad (36)$$

while minimizing α. To sum-up, to minimize the values of c_x, c_ν, and c_ω, we propose the following optimization problem

$$\min_{c_y,\bar{c}_w,\mathbb{P},\alpha}\left(\gamma_1\alpha + \gamma_2 c_y + \gamma_3\bar{c}_\omega\right) \text{ subject to } (35),\ (36),\ (34)$$

where $\gamma_i, i = 1,2,3$ are constants to be fixed by the user.

C. Case of a Particular Family of Nonlinear Systems

The systems described by equation (2) in this letter are quite broad, leading to a significant number of LMI conditions that need to be solved in (34). However, several real-world applications models are simpler than (2), namely the following class of systems is often used in the literature, especially in the LMI context:

$$\begin{cases} x_{t+1} = f(x_t) + E\boldsymbol{\omega}_t \\ y_t = h(x_t) + D\boldsymbol{\omega}_t \end{cases}. \quad (37)$$

In this case, the LMI (34) is reduced to the following one:

$$\begin{bmatrix} -c_y\mathcal{H}_i^1 - \theta\mathbb{P} & -c_y\mathcal{H}_j^x D & \left(\mathcal{F}_l^x\right)^\top\mathbb{P} \\ (\star) & -c_y D^\top D - \bar{c}_w\mathbb{I}_q & E^\top\mathbb{P} \\ (\star) & (\star) & -\mathbb{P} \end{bmatrix} < 0 \quad (38)$$

for all $i \in \{1,\ldots,n_1\}$, $j \in \{1,\ldots,q_x\}$, and $l \in \{1,\ldots,n_x\}$. In addition, if we consider systems with linear outputs, i.e.: $y_t = Cx_t + D\boldsymbol{\omega}_t$, then the LMI condition is much simplified as follows:

$$\begin{bmatrix} -c_y C^\top C - \theta\mathbb{P} & -c_y C^\top D & \left(\mathcal{F}_j^x\right)^\top\mathbb{P} \\ (\star) & -c_y D^\top D - \bar{c}_w\mathbb{I}_q & E^\top\mathbb{P} \\ (\star) & (\star) & -\mathbb{P} \end{bmatrix} < 0 \quad (39)$$

for all $j \in \{1,\ldots,n_x\}$.

D. On the Conservatism and Feasibility of (34)

The conservatism related to the proposed approach lies, first, in converting (32) into (34) by using the convexity principle [11], [12]. Indeed, it is reported in [14] that using the polytopic approach based on the convexity principle always provides less conservative LMI conditions compared to other strong upper bounding techniques, namely the use of Lipschitz inequality or the Young inequality instead of the convexity principle. Furthermore, the use of a constant Lyapunov matrix is conservative; however, it provides a systematic numerical procedure applicable to a wide class of nonlinear systems. A more general Lyapunov function and matrices instead of the scalars $c_y, \bar{c}_\omega$ may be used; however, we will lose getting a systematic synthesis procedure, or even the linearity of the synthesis conditions.

The decision variables in (34) are the matrix $\mathbb{P}$ and the positive scalars c_y and $\bar{c}_w$, while θ is fixed a priori. All these decision variables are free solutions returned by (34) and have not been fixed a priori by the gridding method. Indeed, the gridding method on $\theta \in]0,\ 1[$ consists in subdividing the interval $]0,\ 1[$ into ℓ subintervals and solving the LMI (34) for each value $\theta_j = \frac{j}{\ell}$ until a solution is returned. All the other matrices are known and specific to the system at hand. Especially, the matrices $\mathcal{H}_i^1, \mathcal{H}_j^2, \mathcal{H}_k^3, \mathcal{F}_l^x$, and $\mathcal{F}_m^{\boldsymbol{\omega}}$ are known and result from the convex decomposition of the Jacobian matrices of the nonlinear functions. These matrices implicitly depend on the Lipschitz constant and the structure of the functions f and h. Therefore, the feasibility of (34) depends strongly on the structure of those matrices [8].

IV. Illustrative Example

In this section, we illustrate the proposed LMI-based technique to design the i-EIOSS coefficients, which are necessary for tuning the parameters of the MHE algorithm proposed in [5]. Indeed, it has been shown in [5, Th. 1] that the MHE algorithm associated to the cost function

$$\begin{aligned} J_t^N\left(\hat{x}_{t-N}\right) &= \mu|\hat{x}_{t-N} - \bar{x}_{t-N}|^2\eta^N \\ &\quad + \nu\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i - C\hat{x}_i|^2 \end{aligned} \quad (40)$$

with $\eta \in (0,1)$ and $\mu, \nu > 0$, is robustly exponentially convergent if the following conditions are satisfied: $\eta \geq \varrho$; $\mu \geq 2c_x$; $\nu \geq c_\nu$; $2\mu\eta^N < 1$, where ϱ, c_x, c_ν, and c_w are the i-EIOSS related coefficients to be computed by applying

Theorem 1. To this end, we consider the following nonlinear discrete-time system:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 - ax_1^2(t) + x_2(t) \\ bx_1(t) \end{bmatrix} + \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \end{bmatrix}$$
$$y_t = x_1(t) + \nu_t \tag{41}$$

with $a = 1.4$ and $b = 0.3$. The system exhibits a chaotic behavior and its state belongs to the bounded compact set $[-1, 1] \times [-1, 1]$, which is a bounded invariant compact set on which the nonlinearity is globally Lipschitz. Both system and measurement noises are generated according to zero-mean Gaussian distributions with covariances equal to 0.01.

We need only to determine the matrices $\mathcal{F}_j^x$ since we have linear outputs and the system depends linearly on the disturbance $\boldsymbol{\omega}_t$. To compute $\mathcal{F}_j^x$, we have to decompose the Jacobian matrix into a convex form. We have

$$\frac{\partial f}{\partial x}(z_f, \boldsymbol{\omega}) = \begin{bmatrix} -2az_f(t) & 1 \\ b & 0 \end{bmatrix}$$

where $z_f(t)$ comes from the differential mean value theorem in Lemma 2. Since $z_f(t) \in [-1, \ 1]$, then we have $-2a \leq -2az_f(t) \leq 2a$. By using the convex decomposition technique, there exists $0 < \alpha(t) \leq 1$ such that

$$-2az_f(t) = -2a\alpha(t) + 2a(1 - \alpha(t))$$

which means that $\alpha(t) = \frac{z_f(t)+1}{2} < 1$ since $z_f(t) \in [-1, \ 1]$. Hence, we can write the Jacobian matrix under the form

$$\frac{\partial f}{\partial x}(z_f, \boldsymbol{\omega}) = \alpha(t) \overbrace{\begin{bmatrix} -2a & 1 \\ b & 0 \end{bmatrix}}^{\mathcal{F}_1^x} + (1-\alpha(t)) \overbrace{\begin{bmatrix} 2a & 1 \\ b & 0 \end{bmatrix}}^{\mathcal{F}_2^x}.$$

It follows that $\mathcal{V}_{f,x}$ in (5) is given by:

$$\begin{aligned} \mathcal{V}_{f,x} &\triangleq \{\sum_{j=1}^{2} \alpha_j \mathcal{F}_j^x, \text{ such that } \alpha_j \geq 0, \sum_{j=1}^{2} \alpha_j = 1\} \\ &= \left\{ \frac{(z+1)}{2} \begin{bmatrix} -2a & 1 \\ b & 0 \end{bmatrix} + \frac{(1-z)}{2} \begin{bmatrix} 2a & 1 \\ b & 0 \end{bmatrix}, \right. \\ &\qquad \left. \text{such that } z \in [-1, \ 1] \right\}. \end{aligned} \tag{42}$$

In this case, we have $n_x = 2$ instead of $n_x = 2^{n^2} = 16$, since we have only one nonlinear component; the other components in the system are linear. Then we have two LMI conditions to solve in (39) to obtain the i-EIOSS related coefficients.

Utilizing *MATLAB Yalmip* toolbox, the LMI (39) provides the following parameters associated with i-EIOSS property (9): $\theta = 0.8, c_x = 124.6827, c_\nu = 55.2984, c_\omega = 22.1197$. Then, the tuning parameters in (40) can be fixed to $\varrho = 0.8, \mu = 249.3654, \nu = 55.2984$, and the size of the window can be fixed to $N = 4$ due to the condition $2\mu\eta^N < 1$. The initial values of the actual and estimated states are $\begin{bmatrix} 0 \ 0 \end{bmatrix}^\top$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, respectively. Figure 1 illustrates the simulation results, which show that the MHE successfully estimates the actual states.

The minimization of the cost function is carried out by means of a descent method. The optimization was performed by using the general-purpose MATLAB routine *fmincon*.

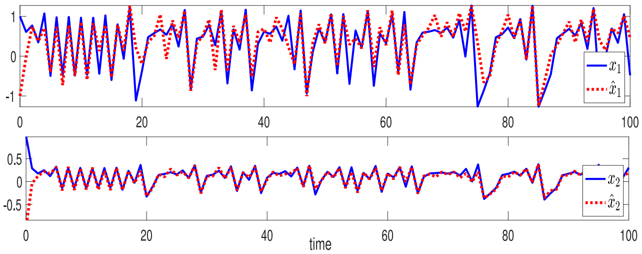

Fig. 1. Behavior of the system states and their estimates.

V. Conclusion and Future Work

In this letter, we provided a simple but useful LMI-based design method to check the i-EIOSS property of nonlinear systems. This method may be used easily without needing to compute its trajectories. To develop such an LMI method, we proposed a mathematical tool, which allows to develop a general method based on Lyapunov functions. The established outcome is significant within the field of robust estimation techniques, as it simplifies the process of tuning the parameters of the estimator. This is primarily because these tuning directly relies on the coefficients associated with the i-EIOSS property. In future work, we aim to combine the proposed design methodology with robust estimators, in particular, to solve MHE problems. We will also investigate a novel method of state observer design based on the use of the i-EIOSS property.

References

[1] E. D. Sontag and Y. Wang, "Output-to-state stability and detectability of nonlinear systems," *Syst. Control Lett.*, vol. 29, no. 5, pp. 279–290, 1997.

[2] D. A. Allan, *A Lyapunov-Like Function for Analysis of Model Predictive Control and Moving Horizon Estimation*. Madison, WI, USA: Univ. Wisconsin, 2020.

[3] D. A. Allan, J. Rawlings, and A. R. Teel, "Nonlinear detectability and incremental input/output-to-state stability," *SIAM J. Control Optim.*, vol. 59, no. 4, pp. 3017–3039, 2021.

[4] J. D. Schiller, S. Knüfer, and M. A. Müller, "Robust stability of suboptimal moving horizon estimation using an observer-based candidate solution," *IFAC-PapersOnLine*, vol. 54, no. 6, pp. 226–231, 2021.

[5] H. Arezki, A. Alessandri, and A. Zemouche, "Robust stability analysis of moving-horizon estimator for LPV discrete-time systems," in *Proc. 22nd IFAC World Congr.*, 2023, pp. 12424–12429.

[6] H. Arezki, A. Alessandri, and A. Zemouche, "Robust moving-horizon estimation for quasi-LPV discrete-time systems," in *Proc. 22nd IFAC World Congr.*, 2023, pp. 7353–7358.

[7] J. D. Schiller, S. Muntwiler, J. Köhler, M. N. Zeilinger, and M. A. Müller, "A Lyapunov function for robust stability of moving horizon estimation," *IEEE Trans. Autom. Control*, early access, May 26, 2023, doi: 10.1109/TAC.2023.3280344.

[8] A. Zemouche and M. Boutayeb, "On LMI conditions to design observers for Lipschitz nonlinear systems," *Automatica*, vol. 49, no. 2, pp. 585–591, 2013.

[9] A. Zemouche, R. Rajamani, H. Trinh, and M. Zasadzinski, "A new LMI based $\mathcal{H}_\infty$ observer design method for Lipschitz nonlinear systems," in *Proc. IEEE Eur. Control Conf.*, Aalborg, Denmark, 2016, pp. 2011–2016.

[10] R. T. Rockafellar, *Convex Analysis*, vol. 11. Princeton, NJ, USA: Princeton Univ. Press, 1997.

[11] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory* (SIAM Studies in Applied Mathematics), vol. 15. Philadelphia, PA, USA: Soc. Ind. Appl. Math., 1994.

[12] O. Sename, P. Gaspar, and J. Bokor, *Robust Control and Linear Parameter Varying Approaches: Application to Vehicle Dynamics*. vol. 437. Berlin, Germany, Springer, 2013.

[13] M. A. Müller, "Nonlinear moving horizon estimation in the presence of bounded disturbances," *Automatica*, vol. 79, pp. 306–314, May 2017.

[14] A. Zemouche, M. Boutayeb, and I. Bara, "Observers for a class of Lipschitz systems with extension to $\mathcal{H}_\infty$ performance analysis," *Syst. Control Lett.*, vol. 57, no. 1, pp. 18–27, 2008.