Robust Moving Horizon Estimation for Lateral Vehicle Dynamics

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***Abstract*— This paper deals with the problem of robust stability analysis of Moving Horizon Estimator (MHE) for Linear Parameter Varying (LPV) systems. We introduced novel stability analysis tools which guarantee exponential robust convergence of the MHE under the incremental Exponential Input-Output-to-State Stability (i-EIOSS) assumption without observability condition. An application on a steering-controlled lateral vehicle model is provided to show the effectiveness of the proposed estimation scheme.**

I. Introduction

Mathematical modeling is a good tool for engineers to develop autonomous vehicles that meet the requirements of comfort, stability, and safety performance standards. In particular, the vehicle's cornering behavior, known as lateral dynamics, plays a crucial role in the design of autonomous vehicles. However, it remains critical to estimate and control. For this, engineers need fairly simple models to design control systems while guaranteeing sufficient information to capture all the essential characteristics of dynamics [1], [2], [3]. In this paper, we consider a steering-controlled lateral vehicle model with two degrees of freedom, lateral and yaw, borrowed from [2].

The precision of the vehicle dynamics state estimation determines how reliable vehicle control algorithms are. Several estimation methods have been developed in the literature to estimate the vehicle state. In this paper, our focus is on Moving Horizon Estimation (MHE), one of the estimation techniques commonly mentioned in the literature along with Kalman filtering and state observer techniques. MHE is based on the idea of minimizing an estimation cost function defined over a sliding window composed of a finite number of time steps [4], [5], [6], [7], [8]. To ensure that systems affected by noise, disturbances, and measurement errors remain stable, we rely on a more general notion of stability: the state input stability (ISS). ISS guarantees that the system remains globally exponentially stable up to a measured input-dependent error term via the essential supremum norm [9], [10], [11].

In our recent work [12], we provided new sufficient conditions to ensure the Exponential Robust Stability (ERS) of the MHE by assuming that the system is incremental Exponential Input Output-to-State Stability (i-EIOSS). In this work, we establish a simpler proof of such a sophisticated result. First, in Lemma (1), we present a key result that we will exploit to analyze the robustness of the MHE without observability conditions. This result can applied to various control design problems such as time-delay systems [13], or model-trajectory based approach . The main result is then stated in Theorem 1. We provide sufficient conditions to ensure the Exponential Robust Stability (ERS) of the MHE by only assuming that the system is i-EIOSS without observability condition as done in [14], [15]. Finally, we apply the proposed algorithms to a steering-controlled lateral vehicle model [2] represented with two degrees of freedom, lateral and yaw. In conclusion, the MHE estimation strategy developed in this paper provides an accurate estimation of the original states.

A. Problem Formulation and Preliminaries

The moving horizon estimation technique (MHE) is a state estimation method that is particularly useful for nonlinear or constrained dynamic systems. Consider the following LPV discrete-Time system:

$$\begin{cases} x_{t+1} = A(\rho_t)x_t + w_t \\ y_t = C(\rho_t)x_t + v_t \end{cases} \tag{1}$$

where $x_t \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state of the system; $\rho_t \subset \mathcal{X}_\rho \subset \mathbb{R}^n$ with $\mathcal{X}_\rho$ is a compact set; $y_t \in \mathbb{R}^n$ is the output vector and $w_t \in W \subseteq \mathbb{R}^n$ and $v_t \in V \subset \mathbb{R}^n$ are unknown external disturbances. The parameter ρ_t is assumed to be known in real-time. The estimation scheme can be obtained by defining $\hat{x}_{t-N+1|t}, \ldots, \hat{x}_{t|t}$ as estimates generated by $\hat{x}_{t-N|t}$ through the dynamics

$$\hat{x}_{i+1|t} = A(\rho_i)\hat{x}_{i|t}, \quad i = t-N, \ldots, t-1.$$

The MHE technique is based on the idea of minimizing a quadratic estimation cost function defined on a sliding window composed of a finite number of time stages. We consider the classic quadratic objective function,

$$J_t^N(\hat{x}_{t-N}) = \mu|\hat{x}_{t-N} - \bar{x}_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|y_i - C(\rho_i)\hat{x}_i|^2 \tag{2}$$

where $\eta \in (0,1)$ and $\mu, \nu > 0$ under the constraints

$$\hat{x}_{t+1} = A(\rho_t)\hat{x}_t. \tag{3}$$

and

$$\bar{x}_{t-N} = A(\rho_{t-N-1})\hat{x}_{t-N-1|t-N-1}. \tag{4}$$

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*This work was partially supported by the ANR agency under the project ArtISMo ANR-20-CE48-0015. The authors would also like to acknowledge SEGULA Engineering for its financial support..

The system states are derived by solving the following optimization problem

$$\begin{cases} \hat{x}_{0|t} \in \left\{ \underset{\hat{x}_0 \in \mathcal{X}}{\text{argmin}}\, J_t^t\left(\hat{x}_0\right) \right. \\ \qquad\qquad\qquad \left. \text{s.t. (3) holds for} \quad t = 1, ..., N \right\} \\ \hat{x}_{t-N|t} \in \left\{ \underset{\hat{x}_{t-N} \in \mathcal{X}}{\text{argmin}}\, J_t^N\left(\hat{x}_{t-N}\right) \right. \\ \qquad\qquad\qquad \left. \text{s.t. (3) holds for} \quad t = N, N+1, \dots \right\}. \end{cases}$$

For further ease of presentation and for use later in the paper, note that the cost function $J_t^t(\hat{x}_0)$ for $t \leq N$ is clearly defined as follows:

$$J_t^t(\hat{x}_0) = \mu|\hat{x}_0 - \bar{x}_0|^2\eta^t + \nu\sum_{i=0}^{t-1}\eta^{t-1-i}|y_i - C(\rho_i)\hat{x}_i|^2, \ \forall\, t \leq N. \quad (5)$$

We rely on the **modified prediction** equation (4). This modified prediction is a copy of the original system where the computation of $\bar{x}_{t-N}$ depends on the estimated state at time $t-N-1$, namely $\hat{x}_{t-N-1|t-N-1}$, instead of $\hat{x}_{t-N-1|t-1}$ like with the classical prediction equation. This new prediction equation will play an important role in deriving the stability conditions in terms of required necessary assumptions while ensuring ERS of the MHE$_N$. For further ease of presentation and for use later in the paper, note that the cost function $J_t^t(\hat{x}_0)$ for $t \leq N$ is clearly defined as follows:

$$J_t^t(\hat{x}_0) = \mu|\hat{x}_0 - \bar{x}_0|^2\eta^t + \nu\sum_{i=0}^{t-1}\eta^{t-1-i}|y_i - C(\rho_i)\hat{x}_i|^2, \ \forall\, t \leq N. \quad (6)$$

In the rest of the paper, let $j \triangleq t-N-1$ to avoid cumbersome notations. We start by providing two key definitions used in this paper. We first introduce the following definition of the exponential robust stability of an estimator. For that, we focus only on MHE$_N$.

Definition 1: An MHE$_N$ is *robustly exponentially stable* (RES) if the following inequality holds:

$$|x_t - \hat{x}_{t|t}| \leq \alpha_1|x_0 - \bar{x}_0|\lambda^t + \alpha_2\sum_{i=0}^{t-1}\lambda^{t-1-i}|v_i|^2 + \alpha_3\sum_{i=0}^{t-1}\lambda^{t-1-i}|w_i|^2 \quad (7)$$

for some $\lambda \in (0,1)$ and $\alpha_i > 0, \forall i = 1,2,3$. Further, if inequality (7) is satisfied for all $t \geq \ell$, where $\ell \geq 1$ is a natural number, we say that the MHE$_N$ is $\ell-$RES.

In this paper, we propose novel robust stability conditions of the MHE by considering only a particular i-IOSS notion, namely the incremental Exponential Input Output-to-State Stability (i-EIOSS) property. These definitions will be exploited in Section II for robust stability analysis of the MHE$_N$.

Definition 2: System (1) is *incrementally Exponentially Input Output-to-State-Stable* (i-EIOSS) if there exist constants $c_x, c_v, c_w > 0$ and $\eta \in (0,1)$ such that for each pair of initial conditions $x_0, \tilde{x}_0 \in \mathcal{X}$ and each two disturbance sequences $w_t, \tilde{w}_t \in \Omega$, the following holds:

$$|x_t(x, w_0^{t-1}) - \tilde{x}_t(\tilde{x}, \tilde{w}_0^{t-1})|^2 \leq c_x|x_0 - \tilde{x}_0|^2\varrho^t + c_v\sum_{i=0}^{t-1}\varrho^{t-1-i}|y_i(x, w_0^{i-1}, v_0^{i-1}) - y_i(\tilde{x}, \tilde{w}_0^{i-1}, \tilde{v}_0^{i-1})|^2 + c_w\sum_{i=0}^{t-1}\varrho^{t-1-i}|w_i - \tilde{w}_i|^2 \quad (8)$$

for some $\varrho \in (0,1)$ and positive reals c_x, c_v, and c_w.

Remark 1: Notice that the above definition is general and does not be applied only for states at time t and 0, respectively. It can be applied, for instance, to account for the exponential discount of the error on trajectories between t and $t-\ell$. Especially since for the MHE problem studied here, we will need to apply the definition for $t \geq \ell$, and between t and $t-\ell$, then we will use the following inequality:

$$|x_t(x, w_{t-\ell}^{t-1}) - \tilde{x}_t(\tilde{x}, \tilde{w}_{t-\ell}^{t-1})|^2 \leq c_x|x_{t-\ell} - \tilde{x}_{t-\ell}|^2\varrho^\ell + c_v\sum_{i=t-\ell}^{t-1}\varrho^{t-1-i}|y_i(x, w_{t-\ell}^{i-1}, v_{t-\ell}^{i-1}) - y_i(\tilde{x}, \tilde{w}_{t-\ell}^{i-1}, \tilde{v}_{t-\ell}^{i-1})|^2 + c_w\sum_{i=t-\ell}^{t-1}\varrho^{t-1-i}|w_i - \tilde{w}_i|^2 \quad (9)$$

which is straightforward from (8). For more details on this inequality, we refer the reader to [16, Eq. (28)] and [17, Definition 2, and Lemma 7] for a more general case.

II. Robust Stability Analysis of the MHE

In this section, we give necessary conditions that ensure the robust convergence of the MHE without needing the observability condition. To this end, we first present a key result that we will exploit to analyze the robustness of the MHE for system (1) **without observability conditions**. This result is derived in a subtle way, it can be exploited in different cases for different control design problems like time-delay systems [13], model-trajectory based approach. This novel stability analysis tool, combined with the new prediction equation (4), will lead to less conservative necessary conditions.

Lemma 1: Let $(u_t)_{t\geq -\ell}$ be a nonnegative sequence of real numbers and $\ell \geq 0$ is a natural number such that

$$u_t \leq \alpha u_{t-\ell} + \beta z_t, \forall t \geq \ell,$$

with

$$z_t = \sum_{i=t-\ell}^{t-1}\eta^{t-1-i}\,|d_i|^2 \quad (10)$$

where $\beta \geq 0, 0 < \alpha < 1$, and $(d_j)_{j\geq\ell}$ is any arbitrary bounded sequence with $d_j = 0, \forall j < \ell$, by definition. Then the following inequality holds:

$$u_t \leq \lambda^t \max_{-\ell\leq j\leq 0} u_j + \beta\sum_{i=0}^{t-1}\lambda^{t-1-i}\,|d_i|^2 \quad (11)$$

where

$$\lambda \triangleq \max\left(\eta, \alpha^{\frac{1}{\ell}}\right). \tag{12}$$

Further, if $u_j = u_0$ for $-\ell \leq j \leq 0$, we get

$$u_t \leq u_0 \lambda^t + \beta \sum_{i=0}^{t-1} \lambda^{t-1-i} |d_i|^2. \tag{13}$$

Proof: Since we work in *Archimedian* space, then for any $t \geq \ell$, there exists an integer $s \geq 1$ so that $t \in I_s = \{s\ell, s\ell+1, \ldots, (s+1)\ell\}$. Then by backward substitution, we get

$$\begin{aligned} u_t &\leq \alpha^{s+1} u_{t-(s+1)\ell} + \beta \sum_{j=0}^{s} \alpha^j z_{t-j\ell} \\ &\leq \max_{-\ell \leq j \leq 0} (u_j)\, \alpha^{\frac{t}{\ell}} + \beta \sum_{j=0}^{s} \alpha^j z_{t-j\ell} \\ &= \max_{-\ell \leq j \leq 0} (u_j)\, \alpha^{\frac{t}{\ell}} + \beta \sum_{\substack{j=t-s\ell \\ \frac{t-j}{\ell} \in \mathbb{N}}}^{t} \alpha^{\frac{t-j}{\ell}} z_j. \end{aligned} \tag{14}$$

To conclude the proof of this lemma, we only need to compute the double sum coming from the second term in (14) by taking care with the index $t - js$ which jumps by ℓ steps for every $j = 0, \ldots, s$. In (14), we have

$$\begin{aligned} \sum_{\substack{j=t-s\ell \\ \frac{t-j}{\ell} \in \mathbb{N}}}^{t} \alpha^{\frac{t-j}{\ell}} z_j &= \sum_{\substack{j=t-s\ell \\ \frac{t-j}{\ell} \in \mathbb{N}}}^{t} \sum_{i=j-\ell}^{j-1} \alpha^{\frac{t-j}{\ell}} \eta^{j-1-i} |d_i|^2 \\ &\leq \sum_{\substack{j=t-s\ell \\ \frac{t-j}{\ell} \in \mathbb{N}}}^{t} \sum_{i=j-\ell}^{j-1} \lambda^{t-1-i} |d_i|^2 \\ &\leq \sum_{i=t-s\ell-\ell}^{t-s\ell-1} \lambda^{t-1-i} |d_i|^2 + \sum_{i=t-s\ell-\ell}^{t-s\ell-1} \lambda^{t-1-i} |d_i|^2 \\ &\quad + \cdots + \sum_{i=t-2\ell}^{t-\ell-1} \lambda^{t-1-i} |d_i|^2 + \sum_{i=t-\ell}^{t-1} \lambda^{t-1-i} |d_i|^2 \\ &= \sum_{i=t-(s+1)\ell}^{t-1} \lambda^{t-1-i} |d_i|^2 = \sum_{i=0}^{t-1} \lambda^{t-1-i} |d_i|^2 \end{aligned} \tag{15}$$

since $d_i = 0, i < 0$ by assumption/definition. Consequently, by substituting (15) in (14), we get (11) from (12). Finally, if $u_j = u_0$ for $-\ell \leq j \leq 0$, then the inequality (13) is straightforward. ■

Before providing the main theorem, it is worth noticing that due to the definition of the cost function $J_t^t(\hat{x}_0)$, for $t \leq N$, as in (6), we can write the inequality (21) in a unified way for any $t \geq 1$, as follows:

$$\begin{aligned} |e_t|^2 \leq & 2\mu \left|\bar{e}_{t-\min(t,N)}\right|^2 \eta^{\min(t,N)} \\ &+ \nu \sum_{i=t-\min(t,N)}^{t-1} \eta^{t-1-i} |v_i|^2 \\ &+ \omega \sum_{i=t-\min(t,N)}^{t-1} \eta^{t-1-i} |w_i|^2 \end{aligned} \tag{16}$$

which is nothing but the ERS condition (7) for $t \leq N$. Indeed, for $t \leq N$, the previous inequality (16) is reduced to

$$\begin{aligned} |e_t|^2 \leq & 2\mu |\bar{e}_0|^2 \eta^t \\ &+ \nu \sum_{i=0}^{t-1} \eta^{t-1-i} |v_i|^2 \\ &+ \omega \sum_{i=0}^{t-1} \eta^{t-1-i} |w_i|^2 \end{aligned} \tag{17}$$

Theorem 1: [12] Assume that system (1) is i-EIOSS according to (8) with the prediction equation (4) and the exponential discount parameter ϱ. Then, the MHE$_N$ is ERS according to the following inequality:

$$\begin{aligned} \left|x_t - \hat{x}_{t|t}\right|^2 \leq & \max(2\mu, 1) |x_0 - \bar{x}_0|^2 \lambda^t \\ &+ \nu \sum_{i=0}^{t-1} \lambda^{t-1-i} |v_i|^2 \\ &+ \max(4\mu, \omega) \sum_{i=0}^{t-1} \lambda^{t-1-i} |w_i|^2 \end{aligned} \tag{18}$$

with the exponential discount parameter

$$\lambda \triangleq \max\left(\eta, \left[4\mu\, \sigma_A \eta^N\right]^{\frac{1}{(N+1)}}\right), \tag{19}$$

if μ, ν, ω, η, and $N \geq 1$ satisfy the following conditions:

(i) $\varrho \leq \eta < 1$;
(ii) $\mu \geq 2c_x$;
(iii) $\nu \geq c_v$;
(iv) $\omega \geq 2c_w$;
(v) $4\mu\, \sigma_A \eta^N < 1$, which means that $N \geq 1 + \frac{\ln(4\mu\sigma_A)}{\ln(\frac{1}{\eta})}$.

Proof: We start by providing an upper bound on the estimation error $e_t \triangleq x_t - \hat{x}_{t|t}$. For that, we will exploit the minimization of the cost function and the i-EIOSS property of system (1). The upper bound on the error e_t depends on the prediction error $\bar{e}_t \triangleq x_t - \bar{x}_t$ (or $\bar{e}_t \triangleq x_{t-N} - \bar{x}_{t-N}$, at time $t - N$). From the definition of minimizer in

$$J_t^N(\hat{x}_{t-N|t}) \leq J_t^N(x_{t-N}),$$

we get

$$\mu|\hat{x}_{t-N|t} - \bar{x}_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|y_i - h(\hat{x}_{i|t})|^2$$
$$+ \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2$$
$$\leq \mu|x_{t-N} - \bar{x}_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|v_i|^2$$
$$+ \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2. \quad (20)$$

Since we always have

$$\frac{1}{2}|e_{t-N}|^2 \leq |\bar{e}_{t-N}|^2 + |\bar{x}_{t-N} - \hat{x}_{t-N|t}|^2$$

it follows that

$$\frac{\mu}{2}|e_{t-N}|^2\eta^N + \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|y_i - h(\hat{x}_{i|t})|^2$$
$$+ \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2$$
$$\leq 2\mu|\bar{e}_{t-N}|^2\eta^N + \nu \sum_{t=k-N}^{t-1} \eta^{t-i-i}|v_i|^2$$
$$+ \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2.$$

Since the system (1) is i-EIOSS according to Definition 2, then by applying inequality (9) with convenient parameters μ, ν, ω, and η such that

$$\begin{cases} \varrho \leq \eta < 1 \\ \mu \geq 2c_x, \ \nu \geq c_v, \ \omega \geq c_w \end{cases}$$

we obtain the following inequality:

$$|e_t|^2 \leq 2\mu|\bar{e}_{t-N}|^2\eta^N$$
$$+ \nu \sum_{i=t-N}^{t-1} \eta^{t-1-i}|v_i|^2$$
$$+ \omega \sum_{i=t-N}^{t-1} \eta^{t-1-i}|w_i|^2. \quad (21)$$

With the prediction (4), the term $\bar{e}_{t-N}$ in (21) can be upper bounded as follows:

$$|\bar{e}_{j+1}|^2 \leq 2|A(\rho_j)|^2|e_j|^2 + 2|w_j|^2. \quad (22)$$

By substituting (22) in (21), we obtain

$$|e_t|^2 \leq 4\mu\sigma_A|e_{t-(N+1)}|^2\eta^N$$
$$+ \nu \sum_{i=t-(N+1)}^{t-1} \eta^{t-1-i}|v_i|^2$$
$$+ \max(4\mu,\omega) \sum_{i=t-(N+1)}^{t-1} \eta^{t-1-i}|w_i|^2. \quad (23)$$

where

$$\sigma_A \triangleq \sup_{j\geq 0}|A(\rho_j)|^2.$$

Without expanding the computations, we get (18) by applying Lemma 1 with

$$d_i = \begin{bmatrix} \sqrt{\nu}\, v_i \\ \sqrt{\max(4\mu,\omega)} w_i \end{bmatrix}$$

and

$$\alpha = 4\mu\sigma_A\eta^N, \ \beta = 1, \ell = N+1,$$

we obtain easily (18) since the condition (v) in Theorem 1 allows applying Lemma 1. In addition by considering the initial bounds (17) for $t \leq N-1$, the relation (18) is inferred. ■

III. Application to Bicycle Model of Lateral Vehicle Dynamics

Lateral dynamics is concerned with the vehicle's turning behavior. A bicycle model of the vehicle with two degrees of freedom is considered, as shown in Figure 1. The two

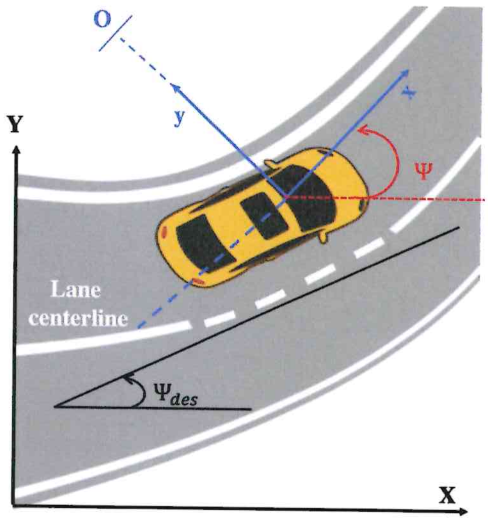

Fig. 1. Lateral vehicle dynamics

degrees of freedom are represented by the vehicle lateral position y and the vehicle yaw angle. The vehicle's lateral position is measured along the lateral axis of the vehicle to point O which is the center of rotation of the vehicle. The vehicle yaw angle is measured with respect to the global axis X. The longitudinal velocity of the vehicle at the c.g. is denoted by V_x. The influence of road bank angle will be considered later. Ignoring road bank angle for now and applying Newton's second law for motion along the axis [18]

$$ma_y = F_{yf} + F_{yr} \quad (24)$$

where $a_y = \left(\frac{d^2y}{dt^2}\right)_{\text{inertial}}$ is the inertial acceleration of the vehicle at the c.g. in the direction of the y axis and F_{yf} and

F_{yr} are the lateral tire forces of the front and rear wheels respectively. Two terms contribute to a_y: the acceleration $\ddot{y}$ which is due to motion along the y axis and the centripetal acceleration $V_x\dot{\psi}$. Hence

$$a_y = \ddot{y} + V_x\dot{\psi} \tag{25}$$

$$m\left(\ddot{y} + \dot{\psi}V_x\right) = F_{yf} + F_{yr} \tag{26}$$

$$I_z\ddot{\psi} = \ell_f F_{yf} - \ell_r F_{yr} \tag{27}$$

where ℓ_f and ℓ_r are the distances of the front tire and the rear tire respectively from the c.g. of the vehicle.

The next step is to model the lateral tire forces F_{yf} and F_{yr} that act on the vehicle. Experimental results show that the lateral tire force of a tire is proportional to the "slip-angle" for small slip-angles. The slip angle of the front wheel is

$$\alpha_f = \delta - \theta_{Vf} \tag{28}$$

where θ_{vf} is the angle that the velocity vector makes with the longitudinal axis of the vehicle and δ is the front wheel steering angle. The rear slip angle is given by

$$\alpha_r = -\theta_{Vr} \tag{29}$$

$$F_{yf} = 2C_{\alpha f}\left(\delta - \theta_{Vf}\right) \tag{30}$$

where the proportionality constant $C_{\alpha f}$ is called the cornering stiffness of each front tire, δ is the front wheel steering angle and θ_{Vf} is the front tire velocity angle. Similarly, the lateral tire for the rear wheels can be written as

$$F_{yr} = 2C_{\alpha r}\left(-\theta_{Vr}\right) \tag{31}$$

where $C_{\alpha r}$ is the cornering stiffness of each rear tire and θ_{Vr} is the rear tire velocity angle. The following relations can be used to calculate θ_{Vf} and θ_{Vr} :

$$\begin{aligned} \tan\left(\theta_{V_f}\right) &= \frac{V_y + \ell_f\dot{\psi}}{V_x} \\ \tan\left(\theta_{V_r}\right) &= \frac{V_y - \ell_r\dot{\psi}}{V_x} \end{aligned} \tag{32}$$

Using small angle approximations and using the notation $V_y = \dot{y}$,

$$\begin{aligned} \theta_{Vf} &= \frac{\dot{y} + \ell_f\dot{\psi}}{V_x} \\ \theta_{Vr} &= \frac{\dot{y} - \ell_r\dot{\psi}}{V_x} \end{aligned} \tag{33}$$

Substituting from Eqs. ((28),(29), (33) into Eqs.(26) and(27), the state space model can be written in the continuous-time as follows [2]:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{2(C_f+C_r)\rho(t)}{M} & -\frac{2(C_f l_f+C_r l_r)\rho(t)}{M} & \frac{2(C_f+C_r)}{M} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{2(l_f C_f - l_r C_r)\rho(t)}{I_z} & -\frac{2(l_f{}^2 C_f + l_r{}^2 C_r)\rho(t)}{I_z} & \frac{2(l_f C_f - l_r C_r)}{I_z} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{2C_f}{M} \\ 0 \\ \frac{2l_f C_f}{I_z} \end{bmatrix} \delta$$

$$C = \begin{bmatrix} 1 & 0 & d_s & 0 \end{bmatrix} \tag{34}$$

The LPV lateral vehicle model is described under the form (1) after *Euler discretization* with sampling period $T_e = 0.01$. The values of the front wheel steering angle δ and the LPV parameter $\rho(t) = \frac{1}{V_x}$ are computed by

$$\rho(t) = 0.06 + \frac{1}{20}\left|\sin(2t)\right|, \ \delta(t) = 0.2\sin\left(\frac{\pi t}{15}\right).$$

For more details on the lateral model, we refer the reader to [2], [19].

Symbol	Nomenclature	Value
M	mass	1529.98 kg
I_z	Yaw moment of inertia	4607.47 kg/m^2
C_r	Rear tire cornering stiffness	1.02×10^5 N/rad
C_f	Front tire cornering stiffness	1.02×10^5 N/rad
l_f, l_r	length of the front-end and rear-end to the c.g, of the vehicle respectively	$l_f = 1.13906$, $l_r = 2.77622 - l_f$
δ	Front wheel steering angle	rad
V_x	Longitudinal velocity at the c.g.	$V_x = \frac{1}{\rho(t)}$

TABLE I

SUMMARY OF LATERAL MODEL PARAMETERS

The minimization of the cost function can be carried out by means of a descent method. The optimization was performed by using the general-purpose Matlab routine *fmincon* with a cost function with the parameters $\mu = 0.4$, $\nu = 1$, and $\eta = 0.9$, for different values of N and tolerance in the stopping criterion. The initial and estimated states given by $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^\top$ and $\begin{bmatrix}-1 & -1 & -1 & -1\end{bmatrix}^\top$, respectively. Figure 2 illustrates the results obtained in simulation runs with system and measurement noises generated according to zero-mean Gaussian distributions with covariances equal to 0.01. The MHE scheme developed in this paper provides an accurate estimation of the original states.

IV. CONCLUSION AND PERSPECTIVES

The main result of this paper was to give sufficient conditions guarantying exponential robust convergence of the MHE under the i-EIOSS assumption without observability condition. In future work, we aim to work on an extension of the results proposed in this paper to nonlinear systems. We also aim to make a link between LMI-based LPV/nonlinear observer design and MHE by using new prediction equations.

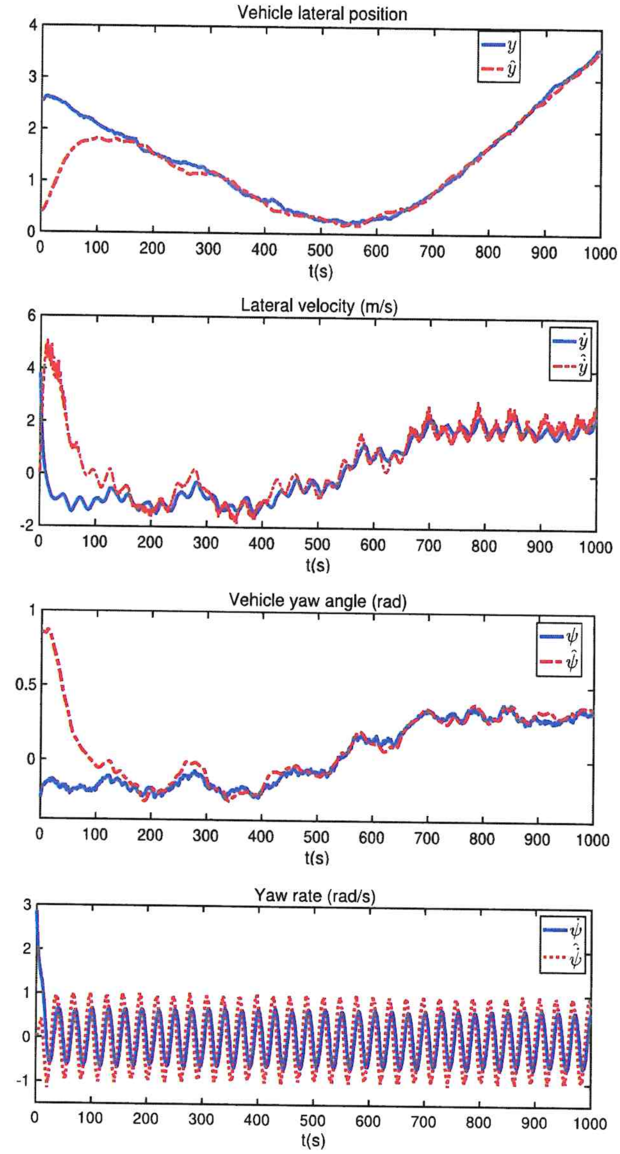

Fig. 2. States and their estimates

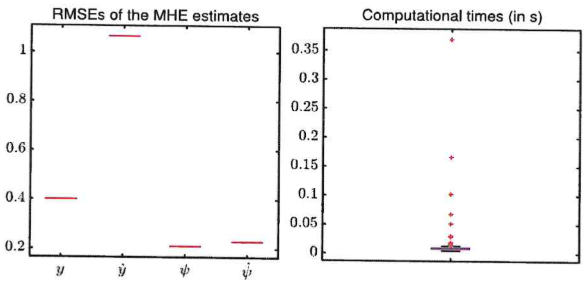

Fig. 3. The RMSE performance and the computational time

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