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Full Length Article

## A topology-based bounded rationality day-to-day traffic assignment model

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### ABSTRACT

This paper analyzes the day-to-day adjustment process of users' behaviors in a transport network which is affected by relevant alterations such as disruptions due to critical events which cause the impossibility to use one or more links. For representing the progressive adjustment of the flows on the network to reach a new equilibrium, a day-to-day discrete-time model is proposed, based on the idea that people are bounded rational in their choices, i.e. they often do not behave according to the optimal solution but they accept solutions they consider satisfying. Users, in their choice process, are influenced by the topological similarity between the route they are currently using and others. This means that they tend to prefer the solutions that are more similar to the one they are already using. In parallel, users exhibit a myopic behavior, i.e., they tend to overestimate the goodness of a route if, when using it, they suddenly experience a significant reduction in travel time compared to what they are used to. In the paper it is shown that such route choice behaviour implies that the steady state of the system corresponds to a Bounded Rational User Equilibrium, i.e., a state that does not diverge from the user equilibrium more than a certain value which increases when the relative importance given to the topological similarity grows. The model also assumes that these biases vanish, at least with respect to those routes that are most frequently used by users, after a sufficient amount of time. Under certain conditions, it is then shown that the steady state can eventually collapse into a User Equilibrium. The effectiveness of the proposed model is assessed via simulation results in which two test networks are analyzed in detail to show the evolution of the users' behaviour in a transport network after a disruption.

### 1. Introduction

Over the decades, transport systems have become increasingly complex and interdependent, and at the same time vital to our societies and economies. Their high degree of interdependence makes them very unpredictable and vulnerable to malfunctions, whose consequences can reverberate through the system for a long time while reaching areas that are quite far from the epicenter of the disruption. Among these systems, the road network is no exception and is possibly the most striking example. It is therefore of fundamental importance to be able to represent it with sufficient accuracy to capture its behaviour.

When a network is perturbed, i.e., when it experiences a significant change in demand or supply, some users end up on routes that have become more expensive (in terms of generalized costs) and are therefore motivated to switch in favor of less expensive ones. This users' attitude, first formulated by [Wardrop \(1952\)](#) in the transportation domain, is likely to result in a progressive redistribution of flows towards less expensive routes until a new steady state is reached. Among the various

methodologies used to study transportation networks, traffic assignment analyses have played a pivotal role. Taking as input the network topology, its features and the mobility demand of users in the form of an origin-destination (OD) matrix, traffic assignment models allow to estimate flow patterns moving on the network. In this field, the notion of User-Equilibrium (UE) has represented, since its first formulation by [Wardrop \(1952\)](#), a cornerstone of a whole line of research in the transportation area. According to UE, the users try to minimize their travel costs while moving on the network. Thus, the equilibrium is reached when all the paths actually used for each OD pair are at minimum cost. Since users are acting independently, none of them has any incentive to modify his choice when this state is reached. Subsequently, the UE concept has been extended to Stochastic User-Equilibrium (SUE), in [Sheffi \(1985\)](#), according to which the users operate so that their perceived (instead of the actual) travel cost is minimized. All these models are static and allow the estimation of the final equilibrium reached by the system but cannot describe how the system evolves towards such equilibrium. In other words, static models cannot represent

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the evolution of the users' behavior and the consequent progressive adjustment of the flows on the network.

Day-to-day assignments are considered, instead, as the most suitable models to represent how a transport system evolves from a non-equilibrium state to an equilibrium one. Such models attempt to capture traffic fluctuations resulting from progressive adjustments of flows over the network. The focus in these cases is on the dynamic process itself rather than on the final equilibrium, although, with the necessary considerations, they can also be used for mere equilibrium computations. In Watling (1999), a classification of day-to-day traffic assignment models is provided. All these models assume perfect user rationality and employ either deterministic or stochastic models to represent random aspects in the real process, e.g., random traffic phenomena.

In contrast with the view of completely rational users, Simon (1957) proposed the idea that people are bounded rational in their choice process and, rather than trying to optimize their condition, they tend to stick with solutions they consider satisfying. Such a behavior can be caused by several reasons related to the complexity of the scenarios in which they have to decide and the cognitive effort required to determine the optimal solution. Mahmassani and Chang (1987) introduced the concept of bounded rationality for the first time in the field of transportation science. Among the causes behind bounded rationality, researchers mention myopia, i.e., people's inability to assess the long-term consequences of their decisions, as well as the power of habits and the inertia to change.

The model presented in this paper is a *proportional-switch day-to-day discrete-time adjustment model*, inspired by the well-known Smith's model (Smith, 1984). The model describes how, in a generic day, some users reconsider their travel choices based, not only on the level of congestion of the routes experienced so far, but also on the level of topological similarity between the routes potentially improving their condition and the ones already used. Then, the perceived cost of a given route at a given time is not unique but relative to the type of user considering it. The more the route currently used by a given user overlaps with another, the lower will be the inertia to leave the first in favor of the second. In addition to this sort of "spatial inertia", a "temporal inertia" is also introduced. If, following a switch, the travel cost for the users has decreased significantly, it is more likely that a smaller proportion of users will decide to change routes again, convinced of the improvement already achieved.

The paper is structured as follows. In Section 2, the main literature related with the proposed model is reviewed and our contribution is highlighted. In Section 3, the main notation is introduced, the proposed day-to-day assignment model is described and some results are proven. Section 4 reports some details about the model implementation, while in Section 5 the model is applied to two test networks and the results are discussed. Finally, some concluding remarks are reported in Section 6.

## 2. Literature review

In this section, the related literature is reviewed in detail, referring to the two research lines that are more relevant for our work, i.e., day-to-day assignment models and bounded rationality concepts. At the end of the section, our contribution is highlighted and discussed compared with the state-of-the-art.

### 2.1. Day-to-day assignment models

Static models, as those mentioned in the Introduction, have been applied in many cases, also involving the analysis of traffic network response to changed conditions, as in Fatouche and Miller-Hooks (2014); Pasquale et al. (2021a); Pasquale et al. (2021b); Li et al. (2019). For their nature, these models allow to estimate the final equilibrium reached by the system but they do not describe the evolution of the users' behavior to such equilibrium. In other words, they are not able to dynamically represent the interaction between supply and demand when at least one of them undergoes significant fluctuations. These latter can be originated

by a multitude of events that it is possible to classify into four classes, based on their magnitude and duration (Cascetta, 2009): (1) localized incidents, which cause a temporary capacity drop on a road section; (2) momentary demand variations due to particular, localized events; (3) interventions on the infrastructure that may cause localized capacity reductions; (4) major and long-lasting network topology changes.

In order to model the network response to these scenarios, dynamic assignment models are usually used. The first and probably most relevant distinction among dynamic assignment models concerns the time horizon of study and the evolution rate of the phenomenon under study. When it is necessary to study phenomena with fast dynamics whose effects primarily occur locally (first and second type of events listed above), *fully-dynamic traffic assignment models* (DTA) are typically applied. They are called "fully-dynamic" because they make use of network loading models directly representing the dynamics of traffic flows moving on the network. Depending on the granularity level of the representation, these models can be further classified into macroscopic (METANET, CTM, LWR), mesoscopic (MatSim, DTALite), and microscopic (SUMO, Vissim). Macroscopic models represent traffic through aggregate variables such as density, average speed and accelerations, while microscopic models describe the interaction between single vehicles at a fine scale. In Wang et al. (2018), the reader can find a comprehensive review on DTA models.

On the other hand, when an event produces global and long-lasting effects on the network, *semi-dynamic models* are generally used, allowing to study the long-term average consequences of the events. The prefix 'semi' refers to the fact that these models attempt to capture the global evolution of traffic patterns as a series of static assignments associated with subsequent periods (Nakayama and Connors, 2014). For this reason, they are also referred to as *inter-period* or *day-to-day traffic assignment models* (DTD) because the considered period often represents a typical day. The focus, compared to DTA models, is not on vehicle dynamics but on the evolution from one period to the next one of users' choices in response to a changing environment that influences them and it is influenced by them, in a circular process that gradually evolves from a situation of disequilibrium to an equilibrium one. It is assumed that at the beginning of each period the users have the possibility to acquire awareness about the characteristics of the network in that given moment and can therefore act accordingly.

The early seminal work in DTD (Smith, 1984; Horowitz, 1984) aims to study the stability of equilibrium states as defined by static assignment models. More specifically, in Horowitz (1984), the stability of the SUE on a network of two links is analyzed by defining a day-to-day dynamics based on the theory of nonlinear discrete-time dynamical systems. In Smith (1984), the Smith's model is introduced whereby users "shift" from a path to cheaper paths that are available to the same origin-destination pair. The rate of this shift is proportional to the cost difference between the paths at a particular time. This flow swapping process is known in the literature as a "proportional-switch adjustment process", and its major strength lies in the fact that it allows an endless number of different user shift behaviors to be implemented with relative ease. Starting from these early works, research in DTD has developed considerably in the subsequent years into a solid body of literature.

Different classifications of DTDs can be made. First of all, DTDs can be divided into continuous-time models and discrete-time models. Continuous-time models (Smith, 1984; Smith and Wisten, 1995; Zhang and Nagurney, 1996; Bar-Gera, 2005; He et al., 2010; He and Peeta, 2016) are defined by differential equations and present the advantage that analytical or theoretical results can be obtained. Continuous dynamics can be considered as an approximation of the corresponding discrete models, and the accuracy of this approximation increases if the considered periods are shorter compared to the overall scale of the model. Discrete-time models (Watling, 1999; Horowitz, 1984; Bar-Gera, 2005; Cantarella and Cascetta, 1995; He and Liu, 2012), on the contrary, generate a sequence of snapshots of the network, one for each period, to capture its evolution. Depending on the implemented route choice behavior model, at the beginning of every period each user makes use of

the available information and formulates a decision. The result of the combined choices of all users characterizes the flow pattern of a specific period.

It is possible to make a distinction between deterministic and stochastic DTD processes, depending on whether the day-to-day process establishes a one-to-one or one-to-many relationship between the (link/path) flow vectors of two consecutive periods. Stochastic models (Cantarella and Cascetta, 1995; Cascetta, 1989; Watling and Cantarella, 2015; Watling and Hazelton, 2018; Hazelton, 2022) aim to intercept the uncertainty and the variability that characterizes real phenomena and usually represent the evolution of the fundamental system variables via Markovian processes. By contrast, deterministic models assign a unique (link/path) flow pattern to each period. Another important distinction needs to be made regarding deterministic models, by differentiating DTD models which encapsulate deterministic path choice models (Smith, 1984; He et al., 2010; Smith and Mounce, 2011; Kumar and Peeta, 2015a) and DTD models based on stochastic path choice models (Watling, 1999; Horowitz, 1984; Cascetta et al. 1996; Cantarella and Watling, 2016; Smith and Watling, 2016). The former models assume that users have perfect information at their disposal, i.e., at any given time, they know all the relevant network details (usually travel times) needed to formulate their mobility decisions. If the hypotheses about the users' behavior prefigure a rational path choice process (Yang and Zhang, 2009), then the stationary state of the system corresponds to a Wardrop equilibrium. The latter, on the contrary, assume that users lack perfect information and, therefore, their assessments of the network state are affected by an estimation error. As a result, perceived travel times, which vary across individuals, differ from actual travel times. The residuals are modeled by means of random variables given a certain distribution. Different assumptions about residuals distribution lead to different models, among which the most famous one is certainly the logit model (Watling, 1999; Guo et al., 2013; Smith and Watling, 2016), allowing to find the path choice probability solution in closed form. Other notable variants have been also proposed, such as the C-logit model (Cascetta et al. (1996), the path-size logit and the nested logit (Ben-Akiva and Bierlaire, 2003; Yu et al., 2020), to alleviate some of the weaknesses of logit-based models, as well as the more recent day-to-day weibit-based models (Castillo et al., 2008; Xu et al. 2021). The stationary state of these models leads to a SUE. Note that, despite incorporating a stochastic choice behavior, these models are fully deterministic, i.e., any given pattern of costs is associated with one and only one vector of path choice probabilities. The total flows at the next period are then distributed proportionally to the computed probabilities. Given the initial conditions, the trajectory of the system is unique.

Moreover, as argued by Tan et al., (2015), day-to-day user path choice behaviors have been declined into five major categories of dynamic processes: simplex gravity flow dynamics (Smith, 1983), network tonnement process (Jin, 2007; Guo and Huang, 2009), projected dynamical system (Zhang and Nagurney, 1996; Nagurney and Zhang, 1997), evolutionary traffic dynamics (Sandholm, 2001; Yang, 2005) and the aforementioned proportional-switch adjustment process (Smith, 1984; Smith and Wisten, 1995; Huang and Lam, 2002; Mounce and Carey, 2011). In particular, the proportional-switch adjustment process has been quite popular due to its simple formulation and intuitive behavioral interpretation. For instance, in Cho and Hwang (2005), a variation of the original Smith's model is presented based on stimulus-reactions whereby the diverting flow is proportional not only to travel time differences but also to users' sensitivity to them. In Smith and Mounce (2011), the adjustment process is formulated on split rates at nodes to mitigate some of the problems associated with path-based models, as noted by He et al., (2010). In Li et al., (2012), an excess cost dynamics is formulated where the path choice behavior depends on the difference between the costs experienced by users and a certain reference value. A non-linear pairwise proportional-switch-based model is defined by Zhang et al. (2015). Finally, the proportional-switch mechanism has been incorporated into a mixed equilibrium model by

Wang et al., (2019) for assessing the effects of autonomous vehicle penetration on the network, as well as into a dynamic assignment model which takes into account the presence of electric vehicles (Agrawal et al., 2016).

It is worth noting that the progressive adjustment mechanism of vehicular flows in response to demand-side or supply-side fluctuations is well suited to be represented by a path-flow dynamics, in which the state space is expressed by means of path flows. Nevertheless, starting from the work of He et al., (2010), a second stream of DTD models has been developed where the dynamics is described by means of link flows (He and Peeta, 2016; He and Liu, 2012; Guo et al., 2013; Siri et al., 2020). As pointed out by He et al., (2010), not having to rely on path flows avoids a substantial problem that afflicts path-based models (indeed only deterministic ones): the need to estimate an initial path flow pattern that, given the corresponding link flow pattern, is generally not unique and therefore not uniquely identifiable. The initial state of the system is in fact reasonably estimated using classical static assignments which, in the deterministic case, are known to generate unique solutions in the domain of link flows but not in the domain of path flows. Different initial path-flow patterns result in significantly different path-flow trajectories. Aside from that, link-based DTD models also have their shortcomings. They do not allow to estimate "how far" from a disturbed area the flows start shifting to avoid that area. This aspect limits the range of control strategies that can be used in link-based DTD models. But more importantly, link-based DTD models do not allow the adoption of path choice models which have straightforward behavioral interpretation Kumar and Peeta (2015a). For these reasons, several techniques have been developed to estimate, out of a link flow pattern, the most reasonable corresponding path flow pattern and thus making the use of path-based models plausible in these cases as well (Rossi et al., 1989; Larsson et al. 2001).

## 2.2. Bounded rationality

The perfect rationality of the decision maker is a widely leveraged assumption in decision-making models, and the transportation domain is no exception. Typically, for both modal and route choice, individuals are represented as utility maximizers (or disutility minimizers). In contrast to this view, an alternative is proposed by Simon (1957) suggesting that the individual is generally rationally bounded while making decisions and, rather than aiming for the optimal solution, he most often seeks to avoid the worst. The concept of bounded rationality was first adopted in the transportation field by Mahmassani and Chang (1987) to describe the pre-trip departure selection process.

The multiple reasons behind bounded rationality have been extensively studied by researchers and effectively detailed by Di and Liu (2016). They are briefly summarized here: 1) the choice process is inherently heuristic, leading to potentially diverging outcomes still under the same scenario (Conlisk, 1996); 2) any decision-making process requires cognitive energies, which are limited by nature, leading to "cognitively cheaper" choices (even if suboptimal) being favored (Hiraoka et al., 2002); 3) the "force of habit" favors previously adopted solutions (Samuelson and Zeckhauser, 1988); 4) decision makers are "myopic" and tend to underestimate long-term consequences of their actions (Conlisk, 1996).

The phenomenon of bounded rationality has been repeatedly detected in the field of transportation where it manifests in the form of a shortest path violation. In several studies it is estimated that only between 60% and 90% of drivers actually choose the shortest path (Bekhor et al., 2006; Prato and Bekhor, 2006; Zhu et al., 2010). The analysis of the residual gap in flow patterns between the pre-collapse situation of the I-35 W Mississippi River bridge and the post-reconstruction scenario has been a fundamental case study in this line of research Zhu et al., 2010. Furthermore, the series of studies conducted by Srinivasan and Mahmassani (1999) showed how the shortest route choice violation persists even in case drivers are provided with all the information they need to

formulate the decision. In that paper, the inertia shown by drivers is estimated in terms of an “indifference band”, which is a common way of representing this phenomenon. In short, the users are insensitive to tiny variations in route travel costs. Other aspects of the broader concept of bounded rationality have been formulated in the form of cognitive cost (Gao et al., 2011) and aspirational level (Zhao and Huang, 2016).

The definition of Bounded Rationality User Equilibrium (BRUE) (Mahmassani and Chang, 1987), the state in which each user cannot significantly decrease (i.e. beyond a certain threshold) his travel cost by unilaterally deviating, has been extensively addressed within the framework of static traffic assignments (Lou et al., 2010; Di et al., 2014). In recent years, the concept of bounded rationality has also spread to the research field of DTD models. Following the study on the shift in traffic patterns due to the collapse of the I-35 W Mississippi River bridge, some research works have pointed out that these phenomena are poorly represented by traditional day-to-day models (He and Liu, 2012; Guo and Liu, 2011; Di et al. 2015). More specifically, in Guo and Liu (2011), mutating the idea behind the work in He et al., (2010), a day-to-day link-based assignment model is proposed which embeds the features of bounded rationality and it is shown that, although the dynamics converges to a set of BRUEs, the system does not necessarily tend to the original one after a perturbation. In other words, due to bounded rationality, the equilibrium state of the network can change irreversibly following a disturbance. The same phenomenon is observed in Di et al. (2015), where stability properties for bounded rationality dynamics are provided. In Wu et al. (2013), the features of bounded rationality are incorporated into a cost updating mechanism originally proposed in Cantarella and Cascetta (1995) and the dynamics are applied to the railway context. In Ye and Yang (2017), bounded rationality through indifference band representation is embedded within day-to-day proportional-switch adjustment and tatonnement dynamics. In Shang et al. (2017), the impact of information sharing among travelers is investigated through the use of an agent-based day-to-day assignment model. The user’s adjustment process is therefore influenced also by the experience gathered by other users within the same “information-cluster”, whose generation is obtained by employing percolation theory. The process convergence to a steady state is then investigated under both scenarios with or without user bounded rationality. In Guo and Huang (2016), a discrete alternative of Smith’s original dynamic model is proposed which, subject to certain preconditions, evolves towards the classical UE and SUE or towards a BRUE. Furthermore, in Zhang et al. (2018), a non-linear pairwise adjustment process is suggested in both absolute and relative bounded rationality implementations. In the first case, the indifference band is constant and unique for each OD pair, while in the second case it scales with the actual travel costs and thus indirectly with the level of congestion and size of the network. Finally, in Zhang, et al. (2019), the model is extended by incorporating net marginal cost gain in the route choice process. This means that users, when considering a switch, not only take into account the current absolute cost differences between the route they are on and the one they will potentially switch to, but they also rely on the cost increase resulting from their choice influenced by path marginal costs.

### 2.3. Paper contribution

This paper presents a proportional-switch adjustment process inspired by the discrete implementation of Smith’s model Smith (1984) as shown in Guo and Huang (2016), specifically designed to represent the evolution of a network under substantial alterations. For any given day, the amount of net flow that shifts from one path to another in response to changing network conditions is computed. As highlighted before, a common way to interpret the concept of bounded rationality in the context of traffic assignments is by defining an indifference band. A driver is therefore stimulated to change route only when the travel time exceeds a certain threshold. In addition, Lotan (1997) shows that the

drivers familiar with the network are less likely to abandon roads they already use. One interpretation of this phenomenon may be found in the empirical study conducted by Hiraoka et al. (2002), where it is shown that drivers tend to prefer routes that are cognitively easier to determine. Choosing to adopt solutions that have already been previously drafted is likely to be cognitively cheaper compared to developing new ones. The empirical study by Vreeswijk et al. (2013) investigates users’ route choice behaviour and in particular the underlying reasons for their systematic travel time miscalculation. It is shown that only 41% of respondents choose the actual shortest route. The results of the interviews support the hypothesis according to which users are affected by “choice supportive bias” (Mather et al., 2003), i.e., users are inclined to associate positive attributes to the choices they have made while, conversely, they are more likely to associate negative attributes to the alternatives they have not chosen. Regarding route choice, a clear correlation is shown between travel time overestimation and whether a route was chosen or not. Travel time overestimation tends to be on average significantly higher for non-chosen routes. Meanwhile, it is well recognized in the literature that the extent of overlap between paths can affect their probability of being chosen. For instance, one of the greatest flaws attributed to logit-based stochastic models is that they present two paths as totally distinct alternatives even when they almost completely overlap (independence of irrelevant alternatives). Therefore it is reasonable to assume that the more two paths overlap, the more indifferent a traveller is in using one rather than the other.

The model proposed in this paper attempts to encapsulate this behavior with a dynamics in which the users reconsider their travel choices not only based on the congestion level on the routes they are experiencing but also on the degree of topological similarity between the roads potentially improving their condition and the one currently used. When pushed by new network conditions to change routes, users will favor those that more strongly overlap with the one they currently use. In other words, users on different routes perceive the cost of the same route differently. We call this phenomenon as “spatial inertia”. This tendency is formally expressed by an over-cost that users assign to the paths they are not using. This approach is similar to the one applied by Cantarella and Cascetta (1995) where, within a conditional path choice model, an extra utility is attributed to the path chosen on the previous day, representing a transition cost. In the present work, however, an extra disutility (extra travel cost) is attributed to the non-chosen routes which is not considered fixed but instead inversely proportional to the topological similarity between a route and the one chosen on the previous day. Additionally, a “temporal inertia” is also introduced to capture the myopia phenomenon identified by Conlisk (1996). The users with a travel cost substantially decreased as a result of a switch will be less inclined to eventually leave it in the future since they are confident of the improvements already achieved. In the article, we will show how these attitudes correspond to a bounded rationality behavior and we will prove that the stationary point reached by the dynamic process corresponds to a BRUE.

Recently, Zhang, et al. (2019) proposed a continuous-time DTD which takes into account the path overlap within the switch process. More in detail, both the cost difference between the paths moderated by a variable indifference band, which scales with respect to the costs, and the net marginal cost, intended as the marginal cost difference between the paths, are taken into account in the user decision process. According to Zhang, et al. (2019), the users prefer, given the same amount of actual travel cost savings, the routes that will result in a lower cost increase once the switch takes place. As a result, given the same travel cost savings, users tend to prefer those routes that overlap with the one currently used, since the overlapped portion does not contribute to the net marginal cost increase. In the present work, conversely, such a behavior is obtained from the topological characteristics of the network exclusively and not from the functional features of the arcs. The flow switch process is not directly influenced by marginal costs, although it is indirectly influenced once the costs are updated (this is a common feature of DTD models), but

it is instead clearly influenced by the overlapping percentage of the paths, evaluated taking into account the specific link lengths since links of different lengths have a different impact.

The main features of the proposed proportional-switch day-to-day discrete-time adjustment model can be summarized as follows:

- spatial inertia, i.e., topological proximity dependent costs are introduced to represent that users have a high inertia to change route if the new route has little overlap with the one currently used;
- temporal inertia, i.e., users who have just achieved a good improvement with a switch present a high inertia to change route again;
- the model is based on the interpretation of users' choices and then it is proven that this behavior corresponds to a bounded rational attitude, i.e the stationary point reached by the system is a BRUE.

From a theoretical point of view, the present model proposes a different approach for representing users' bounded rationality. Relying on a unique indifference band, fixed or relative, it enables to represent the inertia involved in abandoning an option but it does not provide any insight about the option adopted instead. Answering this question generally results in picking the route that provides the user with the largest travel time/cost savings. This approach implicitly assumes that the user considers each alternative distinct and independent. But the works in (Lotan, 1997; Vreeswijk et al., 2013) suggest how this may not be true. It is therefore reasonable to assume that the inertia involved in abandoning one solution in favor of another should be influenced by the perceived similarity between the alternatives. In other words, small cost/time variations may trigger modest flow adjustments, i.e., between similar paths, while larger cost/time fluctuations may cause larger flow adjustments, leading to a switch even towards paths quite different from those previously used. By adopting this approach, it is possible to represent multiple indifference thresholds, whose size varies according to the extent of correlations between the alternatives (overlapping).

From an applicative point of view, the proposed model can be utilized for the representation of the dynamics emerging out of the interaction between users and the transport infrastructure, especially when the network undergoes significant planned or unintentional alterations. Therefore, by defining appropriate performance metrics, it is possible to assess, for example, the degree of resilience or vulnerability of a network relying not only on topological analysis, but also properly taking into account the impact of user reaction over time.

### 3. The proposed model

The transport network is represented by an oriented fully-connected graph  $\mathcal{G}(\mathcal{N}, \mathcal{A})$  with a finite set  $\mathcal{N}$  of nodes and a finite set  $\mathcal{A}$  of links. Time is discretized and represented by the variable  $t = t_0, t_1, \dots$ , indicating a generic day. Note that disruptions cannot occur at  $t_0$  but only on subsequent days. This allows to define a reference of the system fundamental quantities in a pre-disruption scenario, required in the definition of appropriate performance metrics, as will be illustrated later. The main notation of the proposed model is reported in Table 1.

Let  $\mathcal{H}$  be the set of all origin-destination node pairs. For each pair  $h \in \mathcal{H}$ , the set  $\mathcal{K}_h^t$  includes all the possible loop-free paths connecting the two nodes at day  $t$ . The travel demand is denoted by vector  $\mathbf{q} = \{q_h, h \in \mathcal{H}\}$ , where  $q_h$  is the demand for the origin-destination pair  $h \in \mathcal{H}$ . The demand is considered constant throughout the entire time horizon. Regarding the flows travelling on the network, the vector  $\mathbf{x}(t) = \{x_a(t), a \in \mathcal{A}\}$  represents the link flows, where  $x_a(t)$  is the amount of traffic on link  $a \in \mathcal{A}$  at day  $t$ . On the other hand, the vector of path flows is denoted as  $\mathbf{f}(t) = \{f_k^h(t), h \in \mathcal{H}, k \in \mathcal{K}_h^t\}$ , where  $f_k^h(t)$  identifies the traffic flow on path  $k \in \mathcal{K}_h^t$  at day  $t$ . The link-path incidence matrix is defined by  $\Delta = \{\delta_{ak}^h, a \in \mathcal{A}, h \in \mathcal{H}, k \in \mathcal{K}_h^t\}$ , where each binary element  $\delta_{ak}^h = 1$  if link  $a$  belongs to path  $k$  of OD pair  $h$  and 0 otherwise. Consequently, for each day, the following relationships can be defined:

**Table 1**

Main notation.

Sets		Costs	
$\mathcal{N}$	Set of nodes	$\mathbf{c}(\mathbf{x}(t))$	Link cost vector on day $t$
$\mathcal{A}$	Set of links	$c_a(x_a(t))$	Travel cost on path $a \in \mathcal{A}$ on day $t$
$t_0, t_1, \dots$	Days	$\mathbf{C}^h(t)$	Relative path cost matrix for OD pair $h$ on day $t$
$\mathcal{H}$	Set of origin-destination (OD) pairs	$C_{ks}^h(t)$	Relative cost of path $s$ compared with path $k$ of OD pair $h$ on day $t$
$\mathcal{K}_h^t$	Set of all loop-free paths of OD pair $h$ on day $t$	$S_{ks}^h(t)$	Cost of switching from path $k$ to path $s$ of OD pair $h$ on day $t$
$\Delta$	Link-path incidence matrix	$A_s^h(t)$	Actual travel cost on path $s$ of OD pair $h$ on day $t$
$\delta_{ak}^h$	Equal to 1 if link $a$ belongs to path $k$ of OD pair $h$	$D_{sk}^h(t)$	Cost-based swap-rate between path $k$ and path $s$ of OD pair $h$ at day $t$
$\Theta$	Link-path feasible set	$L^h(t)$	Myopia-based swap rate of OD pair $h$ on day $t$
Flow vectors		$\bar{C}^h(t)$	Average actual travel cost of OD pair $h$ on day $t$
$\mathbf{q}$	Travel demand vector	$E^h(t)$	Historic average travel cost of OD pair $h$ on day $t$
$q_h$	Travel demand of OD pair $h$	<b>Parameters</b>	
$\mathbf{x}(t)$	Link-flow vector on day $t$	$\Psi$	Switch coefficient
$x_a(t)$	Link-flow on link $a \in \mathcal{A}$ on day $t$	$\varphi$	Myopia coefficient
$\mathbf{f}(t)$	Path-flow vector on day $t$	$\psi$	Path usage threshold
$f_k^h(t)$	Path-flow on path $k$ on day $t$	$M$	Reluctance coefficient

$$x_a(t) = \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^t} f_k^h(t) \delta_{ak}^h, \quad a \in \mathcal{A} \quad (1)$$

$$q_h = \sum_{k \in \mathcal{K}_h^t} f_k^h(t) \quad h \in \mathcal{H} \quad (2)$$

$$f_k^h(t) \geq 0 \quad h \in \mathcal{H}, k \in \mathcal{K}_h^t \quad (3)$$

where (1)–(3) establish the link-path flows dependency, the mobility demand satisfaction and the non-negativity of flows, respectively.

Thus the set  $\Theta$  of feasible link and path flows can be defined as

$$\Theta = \{(\mathbf{x}, \mathbf{f}) : (1) - (3) \text{ hold}\} \quad (4)$$

Let then consider the case of separable, differentiable, monotonically increasing link cost functions  $\mathbf{c}(\mathbf{x}(t)) = \{c_a(x_a(t)), a \in \mathcal{A}\}$ , where  $c_a(x_a(t))$  is the travel cost that users experience at day  $t$  by travelling through link  $a$ . Regarding instead the paths, we introduce for each OD pair a matrix of relative-costs  $\mathbf{C}^h(t) = \{C_{ks}^h(t), h \in \mathcal{H}, k, s \in \mathcal{K}_h^t\}$ , where each element  $C_{ks}^h(t)$  represents how users travelling on path  $k$  perceive the costs of path  $s$  at day step  $t$ . The relative-cost  $C_{ks}^h(t)$  is defined by the sum of two cost components:

$$C_{ks}^h(t) = A_s^h(t) + S_{ks}^h(t) \quad h \in \mathcal{H}, k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t \quad (5)$$

where

$$A_s^h(t) = \sum_{a \in \mathcal{A}} c_a(x_a(t)) \delta_{as}^h \quad h \in \mathcal{H}, a \in \mathcal{A}, s \in \mathcal{K}_h^t \quad (6)$$

$$S_{ks}^h(t) = \frac{\Psi}{T_s^h(t)} \frac{\sum_{a \in \mathcal{A}} \delta_{ak}^h (1 - \delta_{as}^h) I_a}{\sum_{a \in \mathcal{A}} (\delta_{ak}^h I_a)} \quad h \in \mathcal{H}, a \in \mathcal{A}, s \in \mathcal{K}_h^t, k \in \mathcal{K}_h^t \quad (7)$$

Equation (6) allows to compute the actual travel cost of route  $s$  as the sum of the actual travel costs of each link within it. On the other hand, (7) is an additional cost component representing users' reluctance to change habits.  $S_{ks}^h(t)$  denotes the switching cost of users currently on path  $k$  willing to change their choice in favor of path  $s$ , where  $l_a$  represents a length measure of link  $a$ , while  $\Psi \geq 0$  determines the weight of this cost component compared with  $A_s^h(t)$ . As can be seen in (7), such cost depends on the overlapping percentage between the two paths, i.e. how much they share some portions of the network. The more the two paths overlap, the lower the switching cost becomes, reaching zero if  $k = s$ , i.e.:

$$S_{kk}^h(t) = \frac{\Psi}{T_s^h(t)} \frac{\sum_{a \in \mathcal{A}} \delta_{ak}^h (1 - \delta_{ak}^h) l_a}{\sum_{a \in \mathcal{A}} (\delta_{ak}^h l_a)} = 0 \quad h \in \mathcal{H}, k \in \mathcal{K}_h^t \quad (8)$$

since  $\delta_{ak}^h$  is binary, implying that  $\delta_{ak}^h (1 - \delta_{ak}^h) = 0$ .

This leads to the following relation:

$$A_k^h(t) = C_{kk}^h(t) \quad h \in \mathcal{H}, k \in \mathcal{K}_h^t \quad (9)$$

It is worth noting that if  $l_a = l, \forall a \in \mathcal{A}$ , the switching cost depends purely on the number of shared links between the two paths regardless of their length.

Referring again to (7),  $T_s^h(t)$  is responsible for the evolution of the switching cost over time. The main idea is that, when a route is significantly used by users of an OD pair, such a route becomes "familiar". Once a route is familiar to users, the perceived costs of switching to it decreases over time until they vanish. On the contrary, a route not significantly used by users is defined as "unfamiliar". In this case the cost of switching from other routes to it does not decrease over time. One way to represent this process is as follows:

$$T_s^h(t) = \begin{cases} t - t_s^h & \text{if } t > t_s^h \\ 1 & \text{otherwise} \end{cases} \quad (10)$$

where  $t_s^h$  is the day after which the route can be considered familiar. It must hold that  $f_s^h(t_s^h) \geq \psi q_h \wedge f_s^h(t) < \psi q_h, \forall t < t_s^h$ , with  $\psi \in [0, 1]$ . In other words,  $T_s^h(t)$  begins to increase when path  $s$  starts to be used by a consistent percentage of users, represented by parameter  $\psi$ . This implies that, for all familiar paths, the switching cost fades to zero over time resulting in  $C_{ks}^h(t) \rightarrow A_s^h(t)$ . That is, users consider only actual path costs in their assessments. It is worth noting, however, that  $t_s^h$  does not necessarily exist for each path of pair  $h$ . In that case,  $s$  is still considered unfamiliar and as a consequence  $T_s^h(t) = 1$  indefinitely.

Equations (5)–(9) imply that the relative path cost matrices are square matrices, having the actual costs on the diagonal as follows:

$$C^h(t) = \begin{bmatrix} A_1^h(t) & \dots & C_{1p}^h(t) \\ \vdots & \ddots & \vdots \\ C_{p1}^h(t) & \dots & A_p^h(t) \end{bmatrix} \quad (11)$$

where  $p$  is the cardinality of set  $\mathcal{K}_h^t$ .

### 3.1. System dynamics

Making use of the notation and definitions exposed so far, the path flow dynamics can be expressed as

$$f_k^h(t+1) = f_k^h(t) + L^h(t) \sum_{s \in \mathcal{K}_h^t} \left[ f_s^h(t) D_{sk}^h(t) - f_k^h(t) D_{ks}^h(t) \right] \quad h \in \mathcal{H}, k \in \mathcal{K}_h^t \quad (12)$$

where:

$$D_{ks}^h(t) = \frac{[A_k^h(t) - C_{ks}^h(t)]_+}{\sum_{i \in \mathcal{K}_{h,j} \in \mathcal{K}_h^t} [A_i^h(t) - C_{ij}^h(t)]_+ + M} \quad k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t \quad (13)$$

$$L^h(t) = e^{\varphi[\bar{C}^h(t) - E^h(t-1)]_-} \quad h \in \mathcal{H} \quad (14)$$

with projection operators  $[\cdot]_+$  and  $[\cdot]_-$  defined as  $[x]_+ = \max\{x, 0\}$  and  $[x]_- = \min\{x, 0\}$ .

Referring to (13),  $D_{ks}^h(t)$  denotes a cost-based swap-rate between path  $k$  and path  $s$  of OD pair  $h$  at day  $t$ . It represents the percentage of flow  $f_k^h(t)$  which will swap towards  $s$  on the next day. According to the definition of projection operators, the equation implies that, if the travel cost of route  $k$  is higher than the one of route  $s$ , then a fraction  $D_{ks}^h(t)$  of the users will transfer to route  $s$ . The swapping percentage is equal to the cost difference between  $k$  and  $s$ , normalized over the total travel cost variation for OD pair  $h$  plus  $M$ , a reluctance coefficient, which represents the insensitivity of users to travel cost differences.

Referring to (14),  $L^h(t)$  further scales the flows of the pair  $h$  that will actually make the shift. This rate encapsulates users' myopia and subsequent optimism when they are facing an improving situation. On a day  $t$ , the more the average travel cost  $\bar{C}^h(t)$  experienced by the users has decreased compared to the historic average travel cost  $E^h(t-1)$  experienced up to the previous day, the lower the percentage of those who will decide to change route again on day  $t+1$ . The  $\varphi$  coefficient alters the intensity of this behavior. As defined in (15), the average cost for pair  $h$  is obtained by averaging the actual path costs over the relative flows. As time passes, these costs are embedded within the exponential smoothing defined in (16), representing users' past experiences, with  $\xi \in (0, 1]$ .

$$\bar{C}^h(t) = \frac{1}{q^h(t)} \sum_{k \in \mathcal{K}_h^t} f_k^h(t) \cdot A_k^h(t) \quad h \in \mathcal{H} \quad (15)$$

$$E^h(t) = \begin{cases} \bar{C}^h(t_0) & \text{if } t = t_0 \\ \xi \bar{C}^h(t) + (1 - \xi) E^h(t-1) & \text{otherwise} \end{cases} \quad h \in \mathcal{H} \quad (16)$$

Fig. 1 helps to better understand how users' myopic behavior is implemented and the impact of  $\varphi$  on it. On the x-axis we find the value, at day  $t$ , of the average cost  $\bar{C}^h(t)$ , while on the y-axis the corresponding value of  $L^h(t)$  is reported. By fixing  $\varphi$ , if  $\bar{C}^h(t) \geq E^h(t-1)$ , i.e. if the costs experienced by the users of pair  $h$  on day  $t$  are on average higher than those experienced in the past, then the percentage of users shifting is the highest and then  $L^h(t) = 1$ . On the contrary, if  $\bar{C}^h(t) < E^h(t-1)$  it results that  $L^h(t) < 1$ . This implies that if users, given their current choice configuration, have experienced a significant drop in travel costs, they

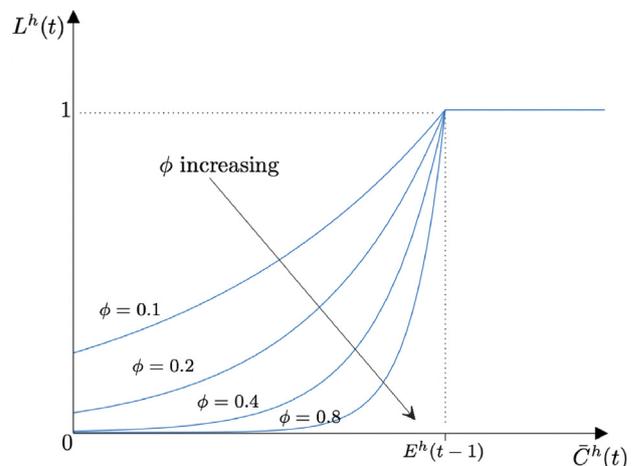


Fig. 1. The influence of  $\varphi$  over  $L^h(t)$ .

are more likely to replicate the same choices on the following day. In other words, a higher proportion of them will decide not to switch routes even if there are better opportunities. The greater the coefficient  $\varphi$ , the less users, given the same value of  $\bar{C}^h(t) - E^h(t-1)$ , will choose to switch to other paths.

According again to Fig. 1, it can be observed that, when  $\varphi \rightarrow \infty$ , the function  $L^h(t)$  tends to a step form in the proximity of  $E^h(t-1)$ . Users are extremely myopic and, as soon as they experience a drop in travel time, they replicate the same choice the next day until circumstances change. On the contrary,  $\varphi = 0$  represents users who are in no way affected by this behavior. It can be noticed that, if travel costs no longer exhibit significant variations as time passes, it holds that  $|E^h(t-1) - \bar{C}^h(t)| \rightarrow 0$  by the definition of exponential smoothing. Therefore,  $L^h(t) \rightarrow 1$  allowing flows to better arrange on the network. Finally, it is worth noting that, by applying (14),  $L^h(t) > 0$  always holds.

Going back to the dynamics in (12) and considering two paths  $k$  and  $s$ , every day the percentage of flow  $f_k^h(t)$  shifting from path  $k$  towards  $s$  or on the contrary the percentage of flow  $f_s^h(t)$  shifting from  $s$  to  $k$  depend on the products  $L^h(t)D_{ks}^h(t)$  and  $L^h(t)D_{sk}^h(t)$  respectively. However, it is worth noting that if  $D_{ks}^h(t) > 0 \Rightarrow D_{sk}^h(t) = 0, \forall s, k \in \mathcal{K}_h^t$ , and vice versa. Then, keeping in mind (9), it is possible to state the following proposition.

**Proposition 1.** *The exchange of flows between two paths of the same OD pair  $h \in \mathcal{H}$  is at most unidirectional.*

**Proof.** Let us consider the case in which it is convenient for users on path  $k$  to switch to path  $s$ , with  $k, s \in \mathcal{H}$ , which is equivalent to  $D_{ks}^h(t) > 0$ . By considering (13) and by applying (5)–(9) and the definition of projection operators  $[\cdot]_+$ , it follows that:

$$\begin{aligned} 0 &< C_{kk}^h(t) - C_{ks}^h(t) = C_{kk}^h(t) - A_s^h(t) - S_{ks}^h(t) \\ &< C_{kk}^h(t) + S_{sk}^h(t) - C_{ss}^h(t) \\ &= A_k^h(t) + S_{sk}^h(t) - C_{ss}^h(t) \\ &= C_{sk}^h(t) - C_{ss}^h(t) \end{aligned} \quad (17)$$

leading to:

$$[C_{ss}^h(t) - C_{sk}^h(t)]_+ = 0 \Rightarrow D_{sk}^h(t) = 0 \quad (18)$$

This result has an important implication, namely the elements  $D_{ks}^h(t)$  do not only represent the swap rate but also the direction of the trajectory at day  $t$  of the dynamical system defined in (12).

It is also important that the swapping process respects flow conservation. It is therefore possible to state the following proposition.

**Proposition 2.** *The dynamic model in (12) ensures the flow conservation.*

**Proof.** Considering each OD pair  $h \in \mathcal{H}$ , it follows from (12) that:

$$\begin{aligned} &\sum_{k \in \mathcal{K}_h^t} f_k^h(t+1) - \sum_{k \in \mathcal{K}_h^t} f_k^h(t) \\ &= L^h(t) \left[ \sum_{k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t} f_s^h(t) D_{sk}^h(t) - \sum_{k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t} f_k^h(t) D_{ks}^h(t) \right] \\ &= L^h(t) \left[ \sum_{k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t} f_s^h(t) D_{sk}^h(t) - \sum_{k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t} f_k^h(t) D_{sk}^h(t) \right] = 0 \end{aligned} \quad (19)$$

and this proves the flow conservation.

### 3.2. Results on the stationary point

In this section we show that the attitudes described in Section 3.1 imply a Bounded Rationality route choice behavior, i.e., the stationary state corresponds to a BRUE. As a necessary premise to Theorem 1, reported hereinafter, the definition of BRUE, as given in Guo and Liu (2011), is briefly provided below.

**Definition 1.** *A path flow  $f(t) \in \Theta$  is a BRUE flow pattern if the following holds:*

$$f_k^h(t) > 0 \Rightarrow A_k^h(t) \leq \mu^h + \varepsilon^h \quad k \in \mathcal{K}_h^t, h \in \mathcal{H} \quad (20)$$

where  $A_k^h(t)$  is the actual travel cost on path  $k \in \mathcal{K}_h^t$  and  $\mu^h$  is the minimum travel path cost between OD pair  $h \in \mathcal{H}$  given the flow pattern  $f^t$ , while  $\varepsilon^h \geq 0$  is the bounded rationality threshold of users of OD pair  $h \in \mathcal{H}$ .

The definition of BRUE implies that, at the equilibrium, the travel cost of all routes used by the users of an OD pair is not larger than the absolute minimum plus some margin. The reason for this is that the users are assumed to be indifferent to subtle variations in travel times. Definition 1 has an important implication, namely a BRUE equilibrium is usually not unique. In fact, there may be different patterns of flows that satisfy the condition.

On the contrary, when the threshold falls to zero, the definition reverts to the classical UE, i.e.,:

$$f_k^h(t) > 0 \Rightarrow A_k^h(t) \leq \rho^h \quad k \in \mathcal{K}_h^t, h \in \mathcal{H} \quad (21)$$

which implies that, for all actually used paths, the travel costs must be equal to the minimum one. It is then worth noting that the UE actually falls within the BRUE set of multiple equilibria.

The following theorem can now be formulated.

**Theorem 1** *If  $f(t)$  is a stationary point, then  $f(t)$  is a BRUE.*

**Proof.** Let  $f$  be a stationary point for the system. Then, by definition, the following equality must hold:

$$\begin{aligned} 0 &= \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^t} C_{kk}^h [f_k^h(t+1) - f_k^h(t)] \\ &= \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^t} C_{kk}^h(t) L^h(t) \sum_{s \in \mathcal{K}_h^t} [f_s^h(t) D_{sk}^h(t) - f_k^h(t) D_{ks}^h(t)] \\ &= \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t} C_{kk}^h(t) L^h(t) f_s^h(t) D_{sk}^h(t) - \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t} C_{kk}^h(t) L^h(t) f_k^h(t) D_{ks}^h(t) \\ &= \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t} C_{ss}^h(t) L^h(t) f_k^h(t) D_{ks}^h(t) - \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t} C_{kk}^h(t) L^h(t) f_k^h(t) D_{ks}^h(t) \\ &= - \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^t, s \in \mathcal{K}_h^t} L^h(t) f_k^h(t) D_{ks}^h(t) (C_{kk}^h(t) - C_{ss}^h(t)) \end{aligned} \quad (22)$$

It can be noticed that each element of the sum is non-negative. In fact by considering (3) and (14), we can see that  $L^h(t) > 0$  and  $f_k^h(t) \geq 0$  respectively. Therefore once considered (13),  $D_{ks}^h(t) \geq 0$  holds by the definitions of projector operator. The only term apparently able to change in sign is  $C_{kk}^h(t) - C_{ss}^h(t)$ . Considering (5)–(7) and the associated result in (9), it can be observed that, whenever this happens, the projection operator term is equal to zero, as shown below:

$$\begin{aligned} 0 &> C_{kk}^h(t) - C_{ss}^h(t) > C_{kk}^h(t) - C_{ss}^h(t) - S_{ks}^h(t) \\ &= C_{kk}^h(t) - C_{ks}^h(t) \Rightarrow [C_{kk}^h(t) - C_{ks}^h(t)]_+ = 0 \end{aligned} \quad (23)$$

which implies that  $D_{ks}^h(t) = 0$ . Thus each element must be individually equal to zero. This is true exclusively when  $f_k^h(t) = 0$  or when  $D_{ks}^h(t) = 0$ .

In fact, under similar considerations as in (23), if the projection operator is strictly greater than zero, then  $C_{kk}^h(t) - C_{ss}^h(t) > 0$  necessarily.

It is therefore possible to state the following:

$$f_k^h(t) > 0 \Rightarrow D_{ks}^h(t) = 0 \quad k, s \in \mathcal{K}_h^t \quad (24)$$

implying that  $[C_{kk}^h(t) - C_{ks}^h(t)]_+ = 0$ . By applying once again the definition of the projection operator,  $D_{ks}^h(t) = 0$  only when  $C_{kk}^h(t) \leq C_{ss}^h(t) + S_{ks}^h(t)$ , i.e.,:

$$A_k^h(t) \leq A_s^h(t) + S_{ks}^h(t) \quad k, s \in \mathcal{K}_h^t \quad (25)$$

Equation (25) must hold for every used path  $k$  with respect to any

other path  $s$ , whether used or not. Chosen  $q$  such that  $A_q^h(t) = \min_{s \in \mathcal{K}_h^t} \{A_s^h(t)\}$  and  $p$  such that  $S_{pq}^h(t) = \max_{k \in \mathcal{K}_h^t} \{S_{kq}^h(t)\}$ , then the following holds.

$$A_k^h(t) \leq A_q^h(t) + S_{pq}^h(t) \quad p, q, k \in \mathcal{K}_h^t \quad (26)$$

Given that the system is stationary at day  $t$  and  $S_{pq}^h(t)$  is certainly not increasing over time by the definition in (7), let us fix  $\mu^h = A_q(t)$  and  $\epsilon^h(t) = S_{pq}^h(t)$ . It should be noted that, since all the switching costs are not increasing functions over time, the relation would remain valid from  $t$  on. It is then sufficient to pick any threshold such that  $\epsilon^h > \epsilon^h(t)$ . Thus,  $\mathbf{f}^{(0)}$  is a BRUE.

In the following, the type of equilibrium that the system actually reaches and how to estimate  $\epsilon^h$  in a reasonable way are further analyzed by means of two remarks. In the former, we consider the case in which  $T_s^h(t) = 1, \forall t$ , i.e., when the switching cost between two paths depends only on the overlapping percentage and does not change over time. In the latter case, instead, we consider the general case with no assumptions on  $T_s^h(t)$ .

**Remark 1.** *If the switching costs are fixed (i.e.,  $S_{ks}^h = \Psi \frac{\sum_{a \in A} (\delta_{ak}^h \delta_{as}^h l_a)}{\sum_{a \in A} (\delta_{ak}^h l_a)}$ ), then the stationary point is a BRUE, where  $\epsilon^h = \max_{ks} \{S_{ks}^h\}$ .*

**Proof.** The result of Theorem 1 holds even if the switching costs are constant. It is sufficient to require that  $T_s^h(t) = 1$  indefinitely for every path  $s$  of every OD pair  $h$ . Having indicated with  $q$  the path with the minimum objective cost for the pair  $h$  and given  $S_{pq}^h = \max_k \{S_{kq}^h\}$ , the following inequality holds:

$$S_{pq}^h \leq \max_{ks} \{S_{ks}^h\} \quad (27)$$

where the term on the right is the maximum switching cost among all paths of pair  $h$ . Rather intuitively, the largest switching cost with respect to the cheapest path cannot be greater than the largest switching cost regardless of the path for which it is calculated. This allows us to define the threshold as follows.

$$\epsilon^h = \max_{ks} \{S_{ks}^h\} \quad (28)$$

The switching costs can be rewritten by extracting the overlapping percentage as

$$S_{ks}^h = \Psi \left( 1 - \frac{\sum_{a \in A} (\delta_{ak}^h \delta_{as}^h l_a)}{\sum_{a \in A} (\delta_{ak}^h l_a)} \right) \quad (29)$$

For the sake of readability let us define the overlapping percentage as  $O_{ks}^h = \frac{\sum_{a \in A} (\delta_{ak}^h \delta_{as}^h l_a)}{\sum_{a \in A} (\delta_{ak}^h l_a)}$ , then we can conclude that (28) is equivalent to:

$$\epsilon^h = \Psi (1 - \min_{ks} \{O_{ks}^h\}) \quad (30)$$

The result in (30) states that we can estimate in advance a maximum threshold for pair  $h$  taking into account the swap coefficient  $\Psi$  and the two least overlapping paths for pair  $h$ .

**Remark 2.** *If the switching costs are time-dependent (i.e.,  $S_{ks}^h(t) = \frac{\Psi}{T_s^h(t)} \frac{\sum_{a \in A} (\delta_{ak}^h \delta_{as}^h l_a)}{\sum_{a \in A} (\delta_{ak}^h l_a)}$ ), then the result of Remark 1 represents an upper bound for thresholds  $\epsilon^h(t)$ .*

**Proof.** The  $S_{ks}^h(t)$  are non-increasing functions, moreover, it can be noted that  $S_{ks}^h(t_0) = S_{ks}^h$  where  $S_{ks}^h(t_0)$  is the switching cost for swapping from  $k$  to  $s$  at  $t_0$  and  $S_{ks}^h$  is a time-independent switching cost as defined in Remark 1. Then the following applies:

$$S_{ks}^h(t) \leq S_{ks}^h \quad \forall t > t_0, s \in \mathcal{K}_h^t, k \in \mathcal{K}_h^t \quad (31)$$

Consequently the following also holds:

$$\max_{ks} S_{ks}^h(t) \leq \max_{ks} S_{ks}^h \quad \forall t > t_0, s \in \mathcal{K}_h^t, k \in \mathcal{K}_h^t \quad (32)$$

Finally, by fixing  $\epsilon^h(t) = \max_{ks} S_{ks}^h(t)$  and  $\epsilon^h = \max_{ks} S_{ks}^h$ , it follows that:

$$\epsilon^h(t) \leq \epsilon^h \quad (33)$$

Remark 2 states that, since the switching costs cannot increase over time, the threshold  $\epsilon^h(t)$  will still be no larger than the one defined in (28)–(30).

To summarize, as previously defined in (10), at a given day  $t$  the paths can potentially be divided into two sets: paths labeled as familiar and paths labeled as unfamiliar. For the former, the switching costs decrease over time, while for the latter they remain constant. Given  $\mathbf{f}(t)$  as a stationary point of the system, there are two possible scenarios:

1. If the shortest path for pair  $h$  ends up being a familiar one, then  $\epsilon^h(t) \rightarrow 0$ . If this happens for every pair, then the system converges to an UE;
2. If, on the contrary, the shortest path is not sufficiently used and therefore unfamiliar, then  $\epsilon^h(t)$  does not fade to zero. If this is the case for at least one pair, then the system does not converge to an UE. In fact, said  $q$  the shortest path for pair  $h$ , (26) must hold. In this case, however,  $S_{pq}^h(t)$  does not fade to zero over time and so  $\mathbf{f}(t)$  remains a proper BRUE.

In conclusion, the system, once stationary, is in a BRUE with an upper bounded threshold as demonstrated in Remark 2. If the shortest paths are also familiar for all OD pairs, then all thresholds fade to zero and  $\mathbf{f}(t)$  is an UE.

#### 4. Model implementation

In this section a viable implementation of the proportional adjustment process described in this paper is discussed with a particular attention on the initialization phase. Indeed, as mentioned in the Introduction and outlined in He et al., (2010), this is a critical issue for deterministic path-based DTD models.

In order to determine the initial state of the network, and assuming that the boundary conditions have remained unaltered for a sufficiently long time to allow the dynamics of the system to stabilize, it is convenient to make use of traditional static assignment models. It is well known that the solution of deterministic models, under mild conditions, is unique with respect to link flows but not with respect to path flows. In other words, the pattern of link flows corresponding to the computed equilibrium is unique but at the same time it can be associated with multiple path flow patterns. The choice of the initial path flow pattern is therefore discretionary. This would not be a problem if different initial states implied identical dynamics, but this is not necessarily true. We can therefore employ a path flow estimation technique of the kind described in Larsson et al. (2001). In particular, in this work, we make use of the entropy maximization approach proposed in Rossi et al. (1989), which we briefly outline here.

Let us consider every single trip as a non-fungible entity, i.e. with its own identity and therefore uniquely identifiable. Given a path flow pattern  $\mathbf{f}$ , we define as “state” a specific allocation of each trip consistent with  $\mathbf{f}$ . Assuming that each state is equally like, then the most likely flow pattern  $\mathbf{f}$  will be the one that allows the highest number of admissible states, i.e. the one corresponding to the highest level of disorder. The resulting entropy function to be maximized is as follows:

$$\max g(\mathbf{f}) = \prod_h \frac{q_h!}{\prod_k f_k^h!} \quad (34)$$

After applying few algebraic manipulations and the Stirling's formula (see Rossi et al., 1989 for details),  $\max g(\cdot)$  is approximated by  $\min h(\cdot)$  and the resulting optimization problem becomes:

$$\min h(\mathbf{f}) = \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^0} f_k^h \cdot \ln(f_k^h) \quad (35)$$

subject to

$$\sum_{k \in \mathcal{K}_h^0} f_k^h = q_h \quad h \in \mathcal{H} \quad (36)$$

$$x_a^{\text{UE}} = \sum_{h \in \mathcal{H}} \sum_{k \in \mathcal{K}_h^0} f_k^h \cdot \delta_{ak}^h \quad a \in \mathcal{A} \quad (37)$$

where  $\mathcal{K}_h^{t_0}$  is the path set for OD pair  $h$  when  $t = t_0$ , i.e., at the equilibrium before the occurrence of the disruption.

Note that function  $h(\mathbf{f})$  is strictly convex. Moreover, let  $\mathbf{f}^{\text{UE}}$  be an optimal flow pattern for the optimization problem. Constraints (36) and (37) imply that  $\mathbf{f}^{\text{UE}}$  belongs to  $\Theta$  given  $\mathbf{x}^{\text{UE}}$ , the link flow pattern at the equilibrium. The link-path feasible set  $\Theta$  is convex, therefore  $\mathbf{f}^{\text{UE}}$  is unique. Finally, to initialize the DTD dynamics, it is sufficient to set:

$$\mathbf{f}(t_0) = \mathbf{f}^{\text{UE}} \quad (38)$$

Once the initial flow pattern is computed, the network is perturbed. In the numerical examples discussed in Section 5, the perturbation is represented by a network disruption resulting in users being prevented from using one or more links. Following the removal of a link, all directly impacted flows are reassigned on the cheapest routes according to (5)–(7). Thus,  $\mathbf{f}(t_1)$  is obtained. At this point the dynamics can take place. At each iteration, the following quantities are updated: path cost matrices  $\mathbf{C}^h(t)$  are computed according to (5)–(7); cost-based swap rates are updated by (13); average pair travel costs  $\bar{C}^h(t)$  and the exponential smoothing  $E^h(t)$  are estimated by applying (15) and (16); finally myopia-based swap rates are computed based on (14). It is now possible to update the system's dynamics in (12) thus obtaining  $\mathbf{f}(t+1)$ . Then, if a certain convergence criteria is met, the algorithm stops, otherwise the time variable  $t$  is updated and the process continues.

A viable implementation of the algorithm is summarized as follows:

**Input:** Network, Demand, Disrupted links  
 $\mathbf{x}^{\text{UE}} \leftarrow$  UE traffic assignment;  
 $\mathbf{f}(t_0) \leftarrow$  Entropy Maximization by (35)–(37);  
 $\mathbf{f}(t_1) \leftarrow$  Disruption;  
**repeat**  
    $\mathbf{C}^h(t) \leftarrow$  relative path cost matrices update by (5)–(7);  
    $\mathbf{D}^h(t) \leftarrow$  swap-rates update by (13);  
    $\bar{C}^h(t), E^h(t) \leftarrow$  average costs and exponential smoothing update by (15)–(16);  
    $L^h(t) \leftarrow$  dynamics rate update by (14);  
    $\mathbf{f}(t+1) \leftarrow$  next path flow vector estimation by (12)  
**until** convergence is met;

From a computational point of view, updating the dynamic process described by (12) requires the execution, at each iteration, of simple algebraic computations and for this reason, in line with most DTD models in the literature, the performance scales well even on large networks. It should also be noted that the main goal of this type of models is to mimic the behavior of an associated real network and, for this reason, the speed of convergence, i.e. number of iterations, should be evaluated only with respect to the accuracy of the representation. On the other hand, the initialization phase can be computationally demanding because two optimization problems need to be solved. By solving the well-known Beckmann's Problem (Beckmann, et al., 1956), the deterministic User-Equilibrium link flow pattern can be estimated, while solving the Entropy Maximization problem in (35)–(37) allows the most likely associated path flow pattern to be univocally determined.

The Frank and Wolfe (1956) algorithm (F-W) has been one of the most widely adopted approaches to solve the deterministic traffic

assignment problem. Its strengths are that it requires low memory, no path enumeration is needed, and the implementation procedure is straightforward, characterized by a sequence of all-or-nothing assignments. At the same time, it has a significant drawback. Once in the proximity of the optimum, the algorithm asymptotically converges sub-linearly, since the descent directions tend to become normal to the gradient resulting in a zig-zagging behavior. To overcome these limitations, several link-based algorithms have been proposed to enhance local convergence (Fukushima, 1984; Hearn et al., 1985; Florian, et al., 1987), as well as more recent path-based algorithms (Bar-Gera, 2002; Dial, 2006; Florian, et al., 2009; Kumar and Peeta, 2010) which have become a suitable alternative given the memory capabilities of modern computers. In particular, bush-based algorithms (Bar-Gera, 2002; Dial, 2006) sequentially decomposing flows by origins exploit the consequence that flows by origins constitute acyclic sub-networks. Flows are transferred from the longest path onto the shortest path, and without cycles the related computations become highly efficient, enabling their use on large-scale networks.

At the same time, the entropy maximization problem described in (35)–(37), making use of the Stearling's approximation, enables its use on networks of conspicuous size. However, the method is not insensitive to the scale of the problem. Alternatively, Bar-Gera (2006) proposed a primal method that exploits the proportionality condition (Bar-Gera and Boyce, 1999) associated with an entropy maximization problem. Such condition states that, when facing two alternative sections, travelers will distribute equally regardless of their origin and destination. Bar-Gera's algorithm shows good performance even on large-scale networks particularly when link flow are computed by a bush-based algorithm. More recently in Kumar and Peeta (2015b), extending the concept of proportionality, the single path flow pattern is obtained from the entropy weighted average of all possible path flow patterns. The algorithm is easy to implement and requires little computational effort to generate the solution which makes it an excellent alternative for large-scale networks.

The above mentioned methods, providing the same initial solution, can be employed indifferently within the initialization phase required by the proposed model depending solely on the requirements dictated by the scale of the specific case study.

## 5. Numerical examples

In this section, the proposed model is applied to two example networks. The former (Network 1) is a simple network and the main goal of the tests is to illustrate the different results with different values of the switching cost coefficient  $\Psi$ . The latter (Network 2) is a larger network and the tests are aimed to show the system behavior in a more complex scenario.

Some metrics are introduced, referred to a generic day  $t$ , to better describe the performance of the system state:

$$P_a(t) = \frac{c_a(t_0)}{c_a(t)} \quad a \in \mathcal{A} \quad (39)$$

$$P_k^h(t) = \frac{C_{kk}^h(t_0)}{C_{kk}^h(t)} \quad h \in \mathcal{H}, k \in \mathcal{K}_h^t \quad (40)$$

$$P^h(t) = \frac{\bar{C}^h(t_0)}{\bar{C}^h(t)} \quad h \in \mathcal{H} \quad (41)$$

$$P(t) = \frac{\bar{C}(t_0)}{\bar{C}(t)} \quad (42)$$

Equations (39)–(42) define the link performance, the path performance, the average performance for an OD pair, and finally the overall average performance of the system at any given day  $t$ , respectively. In (41), the average costs  $\bar{C}^h(t)$  are those defined in (15), while the global

average costs  $\bar{C}(t)$  are, quite similarly, defined as

$$\bar{C}(t) = \frac{\sum_{h \in \mathcal{H}} \bar{C}^h(t) \cdot q^h}{\sum_{h \in \mathcal{H}} q^h} \quad h \in \mathcal{H} \quad (43)$$

Remembering that  $t_0$  represents the pre-disruption setting, it is worth noting that each defined metric identifies the current performance of each component of the system by comparing the current travel costs with those computed in  $t_0$ . In other words, on a given day  $t$ , the performance degrades proportionally to travel cost increase at day  $t$  (whether computed for a link, a path, an OD pair, or for an entire network) compared with the same cost at day  $t_0$ . This is a rather standard way of defining the performance of a transportation network. In Zhou, et al., 2019 an extensive literature on the topic can be found, with a focus on the metrics that are most used in network resilience evaluations.

5.1. Network 1

The considered network is shown in Fig. 2, it is composed of 7 nodes and 8 links. Each link is characterized by the same performance function, relating the travel cost to the amount of congestion, which is shown below:

$$c_a(x_a) = 10^{-3} \cdot x_a + 10^{-1} \quad a \in \mathcal{A} \quad (44)$$

This network presents only one origin-destination pair 1–2. According to the network topology, 3 possible paths join origin node 1 with destination node 2. Table 2 shows the incidence relationships between routes and network arcs. The transportation demand associated with the OD pair 1–2 is 200.

Referring to this network, two different disruptions are tested:

- removal of link 6–2;
- removal of link 1–3.

The tests on Network 1 aim to show the evolution of the system after a disruption, depending on the switch coefficient  $\Psi$ , corresponding to the reluctance of users to use paths topologically dissimilar from those currently used. All the other parameters are kept constant and assume the following values:  $\psi = 0.01$ ,  $\varphi = 50$ ,  $\xi = 0.6$  and  $M = 3$ .

Fig. 2 shows 4 snapshots of the network state taken in 3 different moments of the system evolution after the removal of link 6–2 (the number reported close to each link represents the flow). In particular, in Fig. 2(a) the pre-disruption equilibrium is shown, which does not depend on  $\Psi$ . Reminding that all the arcs are qualitatively identical, the demand is divided between paths 1 and 2, consisting of 3 links, while path 3, composed of 4 links, is not used at all. After the removal of link 6–2, we see a forced reassignment of flows previously present on path 2: in Fig. 2(b), the result of such assignment is shown in case the switch coefficient is set to  $\Psi = 0.1$ , while Fig. 2(c) reports the assignment results with  $\Psi = 0$ , i.e., when the flows of the users are rearranged on the network based exclusively on actual path costs. As it can be noticed, when  $\Psi = 0.1$  all the flow is assigned to path 3, the one sharing a relatively wide portion of network with path 2, now unusable. By adopting

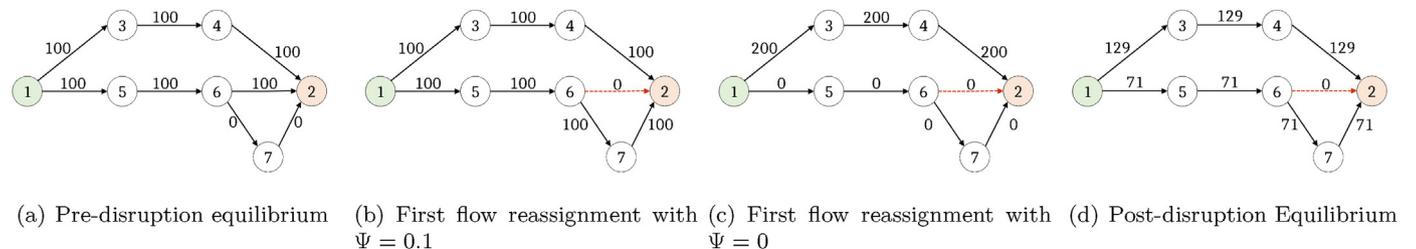


Fig. 2. Network flows in Network 1 with removal of link 6-2.

this solution, the assignment immediately after the disruption consists of an overall moderate redistribution of flows, reflecting a more conservative user behavior. When  $\Psi = 0$ , instead, the disruption is followed by a substantial redistribution of flows, now all loaded on path 1. In this case, the users evaluate only the actual path costs, regardless of the degree of similarity between the new chosen path and the abandoned one. Both in case  $\Psi = 0.1$  and  $\Psi = 0$ , the new equilibrium reached by the system is the same, as shown in Fig. 2(d). Whether path 3 or path 1 is preferred at first, the system evolves toward the same equilibrium, which is in this specific case a UE. This happens because in both cases the two paths end up being used significantly by users (i.e., they are considered familiar paths) and as described in (10) their switch costs  $S_{31}^1(t)$  and  $S_{13}^1(t)$  fade to 0 over time.

Fig. 3 shows the dynamics of the system in terms of actual path cost, flows and overall average system performance. In particular, Figs. 3(a), 3(c) and 3(e) show these trends in case  $\Psi = 0.1$ , while Figs. 3(b), 3(d) and 3(f) report the same variables when  $\Psi = 0$ . Confirming that the stationary state reached by the system is in fact a UE, in Figs. 3(a) and 3(b) it is possible to observe that the actual path costs converge to the same value of about 0.68. However, the way in which the system converges is significantly different in the two cases, with the former showing considerably smaller cost fluctuations.

Again in Fig. 3, it is possible to see a similar trend characterizing the path flow evolution. In case  $\Psi = 0.1$ , from a situation in which the transportation demand is equally distributed on the two remaining paths, path 3 begins to lose flow in favor of path 1 until the equilibrium is reached (see Fig. 3(c)). When  $\Psi = 0$ , instead, path 1 is the one that initially carries all the demand before it redistributes (see Fig. 3(d)). These two different dynamics have quite different impacts on the average system performance, as shown in Figs. 3(e) and 3(f). Even if in both cases the performance level settles around 87% of its original level, the values reached in the worst moment after the disruption differ considerably. In the former case, the system performance does not fall below 85% while in the second case the network performance collapses to about 66% of the original value at the most critical instant. The reason for this is that modeling users who prefers swapping between topologically similar paths results in a much smoother flow redistribution. This, especially right after the disruption, implies less performance loss.

The results obtained considering link 1–3 removal on the same network are shown in Fig. 4, reporting again 4 snapshots of the network. Specifically, the traffic pattern after the disruption is depicted in Fig. 4(b), while Figs. 4(c) and 4(d) show the new equilibrium reached by the system in case the switch coefficient is set to 0.1 and 2.7 respectively. As can be seen, they differ significantly. In particular, in the latter case, the system does not evolve anymore after the disruption and the only path to be used is the second one. This happens because the switching

Table 2 Incidence relations in Network 1.

OD pair	Path	Nodes sequence
(1,2)	1	[1,3,4,2]
	2	[1,5,6,2]
	3	[1,5,6,7,2]

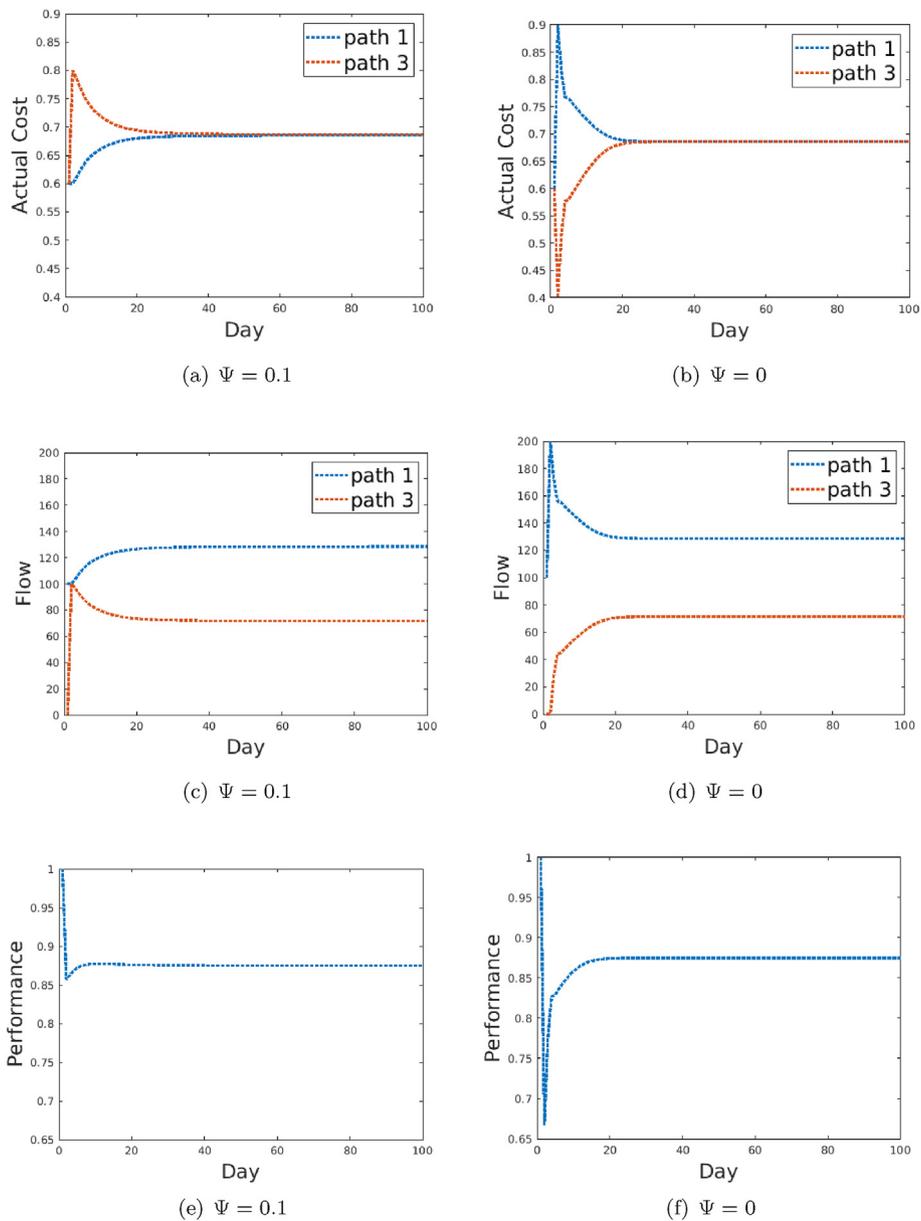


Fig. 3. Path actual cost, flow, performance evolution in Network 1 with removal of link 6-2.

cost  $S_{23}^1(t)$  is relatively high compared to the actual path cost  $A_2^1(t)$  and this means that, even if the cost on path 2 is far from being small, the users are reluctant to use other solutions. More specifically, as long as  $A_2^1(t) < S_{23}^1(t)$ , cost-based swap rate  $D_{23}^1(t) = 0$  by the definition in (13).

5.2. Network 2

Network 2 is shown in Fig. 5, composed of 19 links, qualitatively identical and with the same performance function reported in (44), and 13 nodes. In this network, the origin nodes are 1 and 4 while the destination nodes are 2 and 3. The mobility demand is the same for each of the four OD pairs [1-2, 1-3, 4-2, 4-3] and is equal to 200 units. In this network only one disruption is tested: removal of link 1-12.

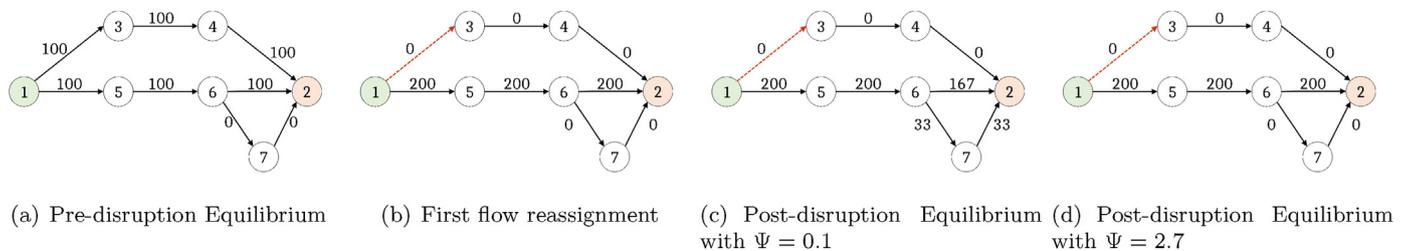


Fig. 4. Network flows in Network 1 with removal of link 1-3.

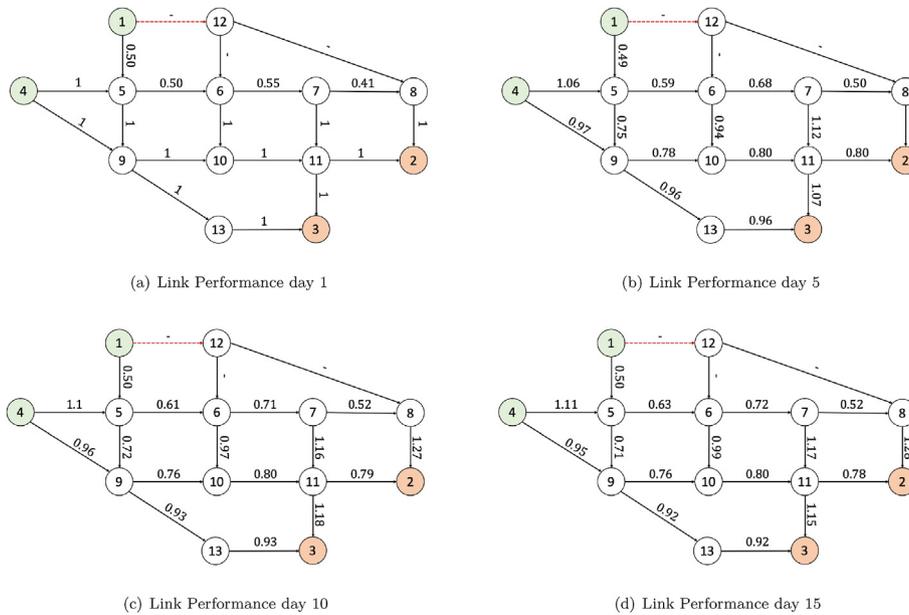


Fig. 5. Link performance in Network 2.

The incidence relationships between paths and links are shown in Table 3, where the underlined paths are those unavailable after the disruption. Moreover, the flows in the pre-disruption scenarios are reported.

In Fig. 5, the evolution of the network in the initial fifteen days after the disruption is shown. This is the time interval during which the most intense flow rearrangements take place. The performance values, as defined in (39), are reported and depicted close to each link.

As can be seen in Fig. 5(a), once the disruption has taken place, the most stressed links are those in its proximity. As the days go by, as shown in Figs. 5(b)–5(d), we can see a generalized performance degradation that spreads like a wave throughout the network, while the previously overloaded links recover part of their lost efficiency. As we move away from the disruption site, the magnitude of the disruption decreases. It is interesting to note that some links perform better in comparison to the pre-disruption scenario. This is the case of links 8–2, 7–11 and 11–3. Not surprisingly, all those links belong to paths that are no longer available and at the same time the new flow pattern makes less use of them, resulting in a reduction in the level of congestion on these links and, consequently, an increase in performance.

The same network evolution is reflected by looking at the path flow, actual cost and performance trajectories presented, respectively, in Figs. 6–8. As it is shown in Fig. 6(a), path 2 is heavily loaded just after the occurrence of the disruption. Over the following days part of these flows

shift to the remaining available paths, as mentioned above, affecting the flow patterns in the remaining links of the network. The OD pair 4-3 shows the smallest flow adjustment before reaching a new stationary state which is consistent with the fact that it is the pair furthest from disruption and whose paths are least affected by flow rearrangement. The new equilibrium, as illustrated in Fig. 7, is a proper UE for every OD pair  $h$  but with significant differences. For the 1–3 pair, the distribution of demand among the routes is more homogeneous than for the other OD pairs where instead one route in particular is significantly preferred over the others.

The trend of path performance illustrated in Fig. 8 further confirms that the closer a path is to the disruption epicenter, the more it is affected by it. At the critical moment, the performance degradation is in fact maximum for the paths of the OD pair 1–2 (Fig. 8(a)), where path 2 is the one suffering most from the new network configuration, with performance reaching 57% of the original value. Regarding OD pairs 1–3 and 4–2 (respectively shown in Figs. 8(b) and 8(c)), paths 2 and 3 suffer the greatest degradation in performance with 63% and 65% of pre-disruption values respectively. Lastly, the paths of OD pair 4-3 do not experience any significant performance loss (Fig. 8(d)). Path 4, the only one showing significant performance degradation, is in fact basically unused, as shown in Fig. 6(d).

Finally, the average performance of each OD pair as defined in (41) are reported in Fig. 9 which once again confirms the tendency whereby

Table 3  
Link-path incidence relationship and pre-disruption assignment in Network 2.

OD pair	Path	Nodes sequence	$f(t_0)$	OD pair	Path	Nodes sequence	$f(t_0)$	
(1–2)	<u>1</u>	[1,12,8,2]	200	(4–2)	1	[4,9,10,11,2]	104	
	2	[1,5,6,7,8,2]	0		2	[4,5,6,7,8,2]	36	
	<u>3</u>	[1,12,6,7,11,2]	0		3	[4,5,9,10,11,2]	0	
	4	[1,5,9,10,11,2]	0		4	[4,5,6,10,11,2]	23	
	5	[1,5,6,10,11,2]	0		5	[4,5,6,7,11,2]	37	
	6	[1,5,6,7,11,2]	0					
	<u>7</u>	[1,12,6,10,11,2]	0					
	8	[1,12,6,7,8,2]	0					
(1–3)	1	[1,5,9,13,3]	90	(4–3)	1	[4,9,13,3]	173	
	2	[1,12,6,10,11,3]	18		2	[4,5,9,13,3]	7	
	3	[1,5,6,7,11,3]	37		3	[4,9,10,11,3]	20	
	4	[1,5,9,10,11,3]	0		4	[4,5,6,7,11,3]	0	
	5	[1,12,6,7,11,3]	33		5	[4,5,9,10,11,3]	0	
	6	[1,5,6,10,11,3]	22		6	[4,5,6,10,11,3]	0	

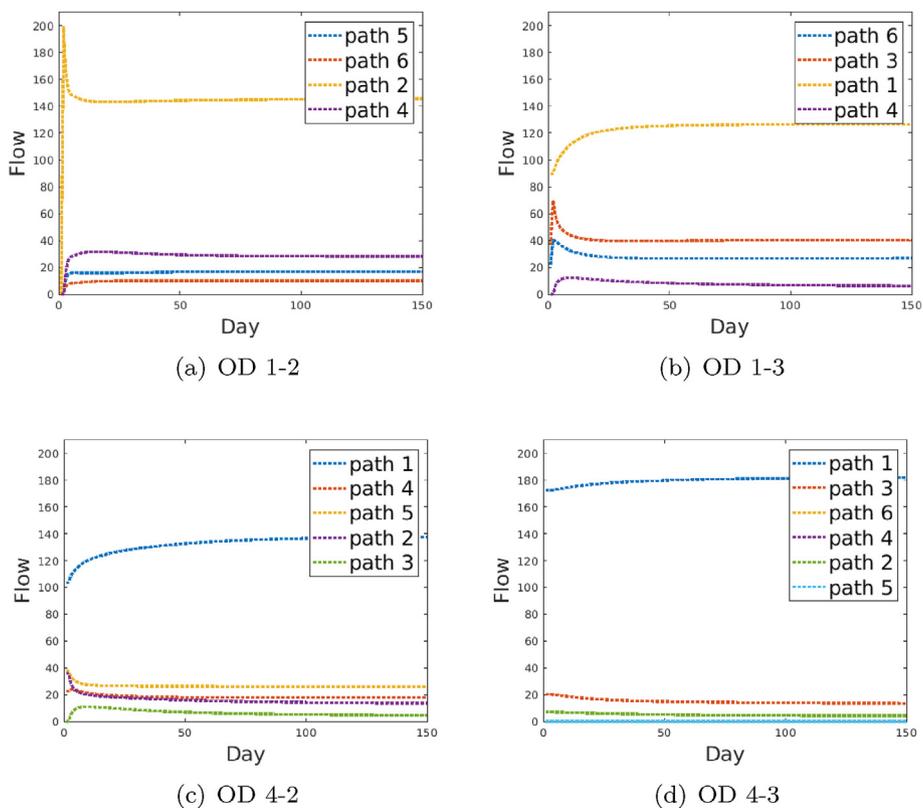


Fig. 6. Path flow evolution in Network 2.

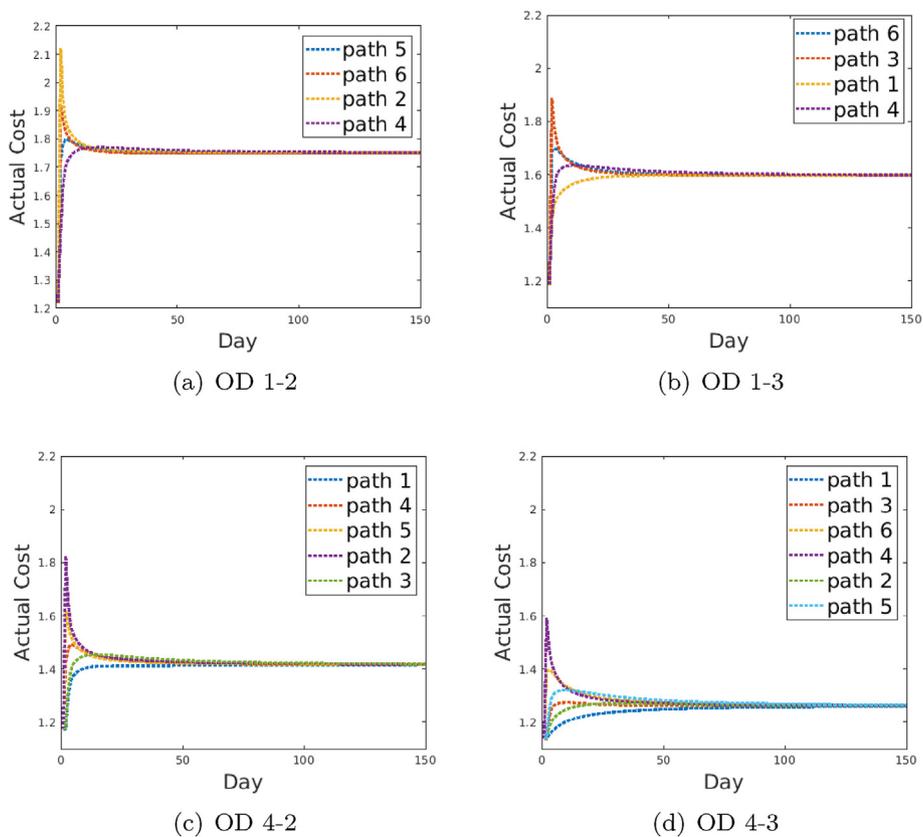


Fig. 7. Path actual cost evolution in Network 2.

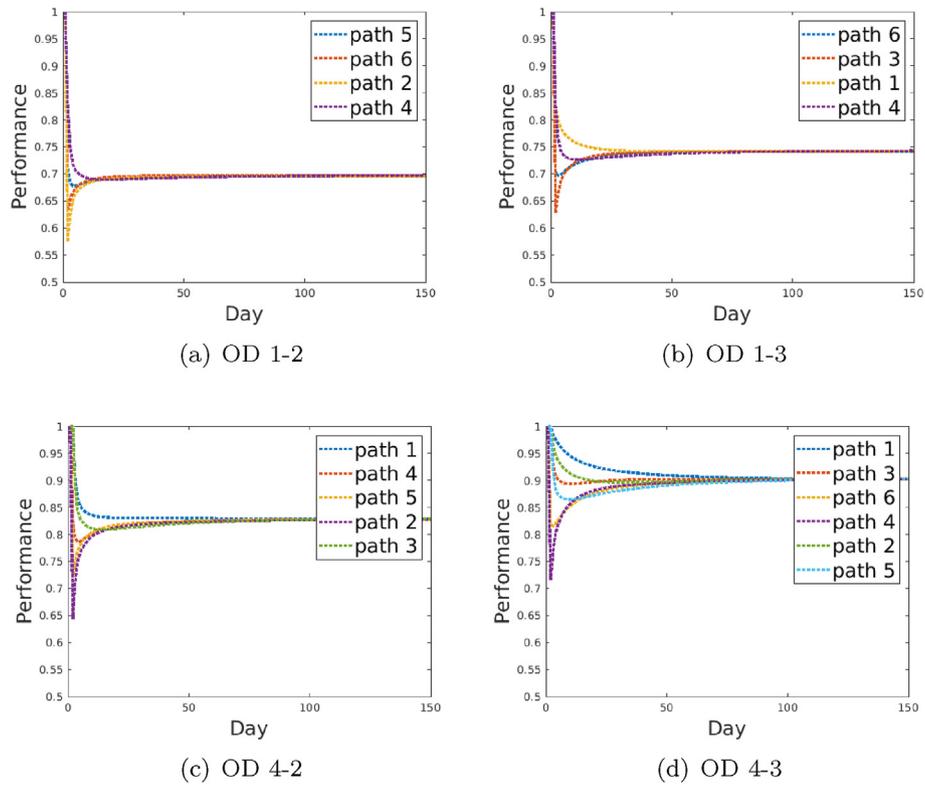


Fig. 8. Path performance evolution in Network 2.

the OD pairs close to the site of the disruption are the most affected by it. In particular, the OD pairs 1-2, 1-3 and 4-2 all exhibit, although to different magnitudes, a drastic drop in performance immediately after the disruption, before gradually recovering. The OD 1-2 pair is the one that suffers the most severe performance drop of roughly over 50%. In contrast, the OD 4-3 pair is not only the least affected by the disruption but is also the only one that does not suffer an immediate performance loss. Instead, its performance degrades as the performance of the others improves due to a better redistribution of flows.

6. Conclusions

This paper proposes a day-to-day discrete-time assignment model

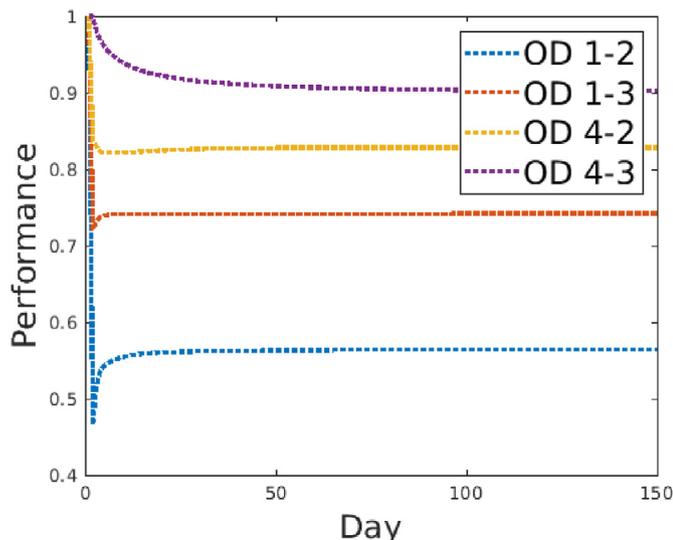


Fig. 9. O-D performance evolution in Network 2.

based on a proportional switch adjustment process. The model defines the amount of users who, unsatisfied with their current status, decide for the next day to satisfy their mobility needs by switching paths. The users' route choice behavior makes the routes that have the potential to reduce travel costs preferable. The greater the decrease in costs, the higher the percentage of users who switch. The model incorporates some cognitive biases within the users' route choice process. The first concerns a spatial inertia, whereby users overestimate the cost of potential paths in relation to the level of topological similarity they present with respect to the path they are currently using. The more the path they are currently on differs topologically from a potential candidate, the higher the perceived cost of abandoning the former in favor of the latter. This behavior is incorporated into the cost structure of the paths themselves. In addition to the additive component, related to the actual state of congestion on the network links, there is a second component defined as a switching cost, which are higher if two paths overlap for small portions. The cost of a route is therefore not unique but depends on where the users currently evaluating it are located. The adjustment process describes users who prefer switching between topologically similar paths. In parallel, users also exhibit a myopic behavior. After a switch, if the travel cost experienced is significantly lower than the one they are used to, they will tend to stick with it for the following days even though the conditions may have changed in the meantime. Finally, it is assumed that these biases may fade over time with respect to those routes that the users are more familiar with.

In the paper, we have shown that the new equilibrium state must necessarily fall into a BRUE, a set of states that, while not necessarily an UE, do not diverge from it more than a certain value which increases when the relative importance given to the topological similarity within the choice process grows. The examples reported in the paper show how, depending on the circumstances, the switching cost can significantly affect both the trajectory of the system and the equilibrium it reaches. Moreover, the example applied on the larger network shows how the model can represent the cascading effects in performance degradation that would be expected in the presence of a disruption. The paths to be

more influenced by the disruption tend to be those nearer to it, while those farther away are significantly less affected by the perturbations that involve the network. This model allows, as in this case, to consider the tendency of users to prefer the routes they use most against other solutions potentially superior but somewhat unusual. That said, the proposed model can be also used to represent, appropriately redefining the switching cost component, a variety of other scenarios where it may be necessary to characterize the mobility choice consequences based on different group of users.

### Replication and data sharing

The data and the complete code for reproducing the simulations reported in the present paper are available at <https://ets-data.sciope.n.com/detail/ETS2022072600001>.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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