HIGHER SYZYGIES ON SURFACES WITH NUMERICALLY TRIVIAL CANONICAL BUNDLE

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1. INTRODUCTION

The aim of this paper is to understand higher syzygies of polarized surfaces with trivial canonical bundles. More concretely, for a polarized surface (X, L) with $K_X = 0$ over the complex numbers, we establish a tight connection between property (N_p) for (X, L) and the (non)existence of certain forbidden subvarieties of X. Our results unify the main statement of [24], and the classical theorem of Saint-Donat [27] on complete linear series on K3 surfaces, which hints at a beautiful interplay between the local and global geometry of the surface.

In the current context syzygies of projective embeddings have entered the stage after the seminal work of Green [15], and have become an active area of research ever since. From a geometric point of view, it has been observed that projective normality of an embedding and the conditon of the homogeneous ideal of the embedding being generated by quadrics can be seen as the first elements of an increasing sequence of positivity properties, namely the cases $p = 0$ and $p = 1$ for property (N_p). The arising conjectures of Green and Lazarsfeld energized the subject, which then has become an important topic in algebraic geometry to this day. For some of the highlights we refer to [11, 29, 30] and the book [2]. The last years have witnessed a new line of research which focuses on the asymptotic behaviour of Koszul cohomology groups (see [12] and the references therein).

Our guiding principle is that in the case of a polarized surface (X, L) with $K_X \equiv 0$ and $(L^2) \gg 0$, one can expect an equivalence between the vanishing of certain Koszul cohomology groups of low degree for (X, L), and the nonexistence of elliptic curves of low L-degree on X passing through a very general point of X. In addition, we will link the equivalence to the local positivity of (X, L) at a very general point, which we will measure in terms of Seshadri constants.

With this in mind, the main result of this article is the following.

Theorem A. *Let p be a natural number, X a smooth projective surface over the complex numbers with trivial canonical bundle, and L an ample line bundle on X with $(L^2) \geqslant 5 \cdot (p+2)^2$. Then the following are equivalent.*

(1) (X, L) satisfies property (N_p).
(2) There is no smooth elliptic curve $F \subseteq X$ (through a very general point of X) with $1 \leqslant (L \cdot F) \leqslant p+2$ and $(F^2) = 0$.
(3) The Seshadri constant at a very general point x satisfies $\varepsilon(L; x) > p+2$.

Remark 1.1. The parentheses in condition (2) amount to the fact that if there is a single smooth elliptic curve with $(L \cdot F) \leqslant p+2$, then there is one passing through a very general point with the same property. This is clear for abelian surfaces since we can move the curve with translations, and for K3 surfaces it is a classic application of Riemann-Roch: see for example [4, Proposition VIII.13].

We will prove Theorem A in Section 3, where we will observe in addition that the constant 5 can be much improved in the K3 case. By setting $p = 0$ and $p = 1$, we will recover Saint-Donat's results when $(L^2) \geqslant 10$. Beside the by now classical theory of Koszul cohomology, our argument relies on recent work of Aprodu and

Farkas [1], where Green's conjecture for smooth curves on K3 surfaces is verified, and earlier work of Serrano [28].

Going further, we establish an similar result for Enriques surfaces; this time however our result is conditional on the Green–Lazarsfeld secant conjecture for curves (for a precise statement see Conjecture 5.1).

Theorem B. *Let p be a natural number, X an Enriques surface over the complex numbers, and L an ample and globally generated line bundle on X such that $(L^2) > 4(p+2)^2$. Assume that the Green–Lazarsfeld secant conjecture holds for a general curve in $|L|$. Then the following are equivalent*

(1) L has property N_p.
(2) there is no effective and reduced divisor F with $(F^2) = 0$ such that $1 \leqslant (E \cdot L) \leqslant p+2$.

We prove Theorem B as Theorem 5.4 in Section 5 by reducing to the one-dimensional case using Koszul cohomology. Our argument is supported by results of Knutsen and Lopez [22] on linear series on Enriques surfaces.

Remark 1.2. We believe that the lower bound $(L^2) \geqslant 5(p+2)^2$ in Theorem A can be improved to $(L^2) > 4(p+2)^2$. This is true for K3 surfaces as a consequence of the more precise estimates of Theorem 4.5, so that the question remains open for abelian surfaces. This would also be the optimal bound for abelian surfaces: if $p = 0$, then by work of Barth [3], a general polarized abelian surface (X, L) of type $(2, 4)$ (in particular with $(L^2) = 16$) does not have property N_0, at the same time such a surface does not contain elliptic curves.

The more accurate results of Theorem 4.5 and Proposition 3.5 establish Theorem A for polarized K3 surfaces. As an easy consequence of Theorem 4.5 we also get a characterization of property (N_p) for ample line bundles of type $L^{\otimes m}$, with $m \geqslant p$, along the lines of Mukai's conjecture. The case of abelian surfaces follows from [24, Theorem 1.1] and Proposition 3.5.

About the organization of the article: In Section 2 we establish the necessary preliminaries, collect a few useful statements about Koszul cohomology and property (N_p) in general. In Section 3 we discuss property (N_p) on surfaces, and among others we prove the equivalence of (2) and (3) of Theorem A. Section 4 is devoted to the proof of Theorem A and some consequences, note again that the results here usually provide estimates on (L^2) that are considerably stronger than the ones in Theorem A. Section 5 hosts the proof of Theorem B.

2. PRELIMINARIES

We work over the complex numbers, every variety is connected, smooth, and projective, unless otherwise mentioned. We recall that for a polarized variety (X, L) and an arbitrary point $x \in X$, the *Seshadri constant* of L at x is defined to be

$$\varepsilon(L; x) \overset{\text{def}}{=} \inf_{x \in C \subseteq X} \frac{(L \cdot C)}{\text{mult}_x(C)} \ .$$

If the infimum is achieved by some curve $C \subseteq X$, then we call C a *Seshadri exceptional curve*.

2.1. **Koszul cohomology.** Let V be a vector space of dimension n and $S = \operatorname{Sym}^\bullet V$ the graded symmetric algebra over V. Then for any finitely generated graded S-module M we have an unique (up to isomorphism) *minimal graded free resolution*:

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where the F_p are graded free S-modules of finite rank. We can write

$$F_p = \bigoplus_{q \in \mathbb{Z}} K_{p,q}(M;V) \otimes_\mathbb{C} S(-p-q)$$

for certain finite dimensional vector spaces $K_{p,q}(M;V)$ which are called the *Koszul cohomology groups* of M w.r.t V. The name Koszul cohomology comes from the fact that they can be computed as the middle cohomology of the Koszul complex [15]:

$$\wedge^{p+1} V \otimes M_{q-1} \to \wedge^p V \otimes M_q \to \wedge^{p-1} V \otimes M_{q+1} \qquad v_1 \wedge \cdots \wedge v_p \otimes m \mapsto \sum_{i=1}^p (-1)^{i+1} v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p \otimes v_i \cdot m$$

We will need later to compare Koszul cohomology with respect to two different vector spaces. More specifically, suppose that we have a short exact sequence of vector spaces

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0 \,,$$

and a finitely generated graded $\operatorname{Sym}^\bullet(W)$-module M. Then M has also a $\operatorname{Sym}^\bullet(V)$-module structure, and we can compare the Koszul cohomologies computed with respect to W and V.

Lemma 2.1. *In the above situation, we have a non-canonical isomorphism*

$$K_{p,q}(M;V) \cong \bigoplus_{i=0}^p \wedge^{p-i} U \otimes K_{i,q}(M;W)$$

In particular $K_{p,q}(M;W) \subseteq K_{p,q}(M;V)$.

Proof. Fix a splitting $V = U \oplus W$. Then $\wedge^p V = \bigoplus_{i=0}^p \wedge^{p-i} U \otimes \wedge^i W$ and the Koszul complex behaves well w.r.t this splitting: indeed, since $U \subseteq \operatorname{Ann}(M)$ we see that

$$u_1 \wedge \cdots \wedge u_{p-i} \wedge w_1 \wedge \cdots \wedge w_i \otimes m \mapsto u_1 \wedge \cdots \wedge u_{p-i} \wedge \left(\sum_{k=0}^i (-1)^{k+1} \wedge w_1 \wedge \cdots \wedge \widehat{w_k} \wedge \cdots \wedge w_i \otimes w_k \cdot m \right)$$

thus, the Koszul complex of $(M;V)$ splits and Koszul cohomology splits as well. The last statement follows by taking $i = p$. $\qquad\square$

In the geometric setting, let X be an irreducible projective variety of positive dimension, $\mathscr{F}$ a coherent sheaf on X and L a globally generated ample line bundle on X. We denote by $S \overset{\text{def}}{=} \operatorname{Sym}^\bullet H^0(X,L)$ the homogeneous coordinate ring of the space $\mathbb{P}(H^0(X,L))$ and by $\Gamma_X(\mathscr{F},L) = \bigoplus_{q \in \mathbb{Z}} H^0(X, \mathscr{F} \otimes qL)$ the module of sections of $\mathscr{F}$. Then $\Gamma_X(\mathscr{F},L)$ has a natural structure of a finitely generated graded S-module so that we can take the Koszul cohomology groups

$$K_{p,q}(X,\mathscr{F},L) := K_{p,q}(\Gamma_X(\mathscr{F},L); H^0(X,L))$$
$$K_{p,q}(X,L) := K_{p,q}(X,\mathscr{O}_X,L)$$

Property (N_p) for polarized varieties (X,L) has been introduced by Green and Lazarsfeld in [17].

Definition 2.2 (Property (N_p))**.** In the above situation, we say that (X,L) has property N_p if $K_{i,q}(X,L) = 0$ for all $i \leqslant p$ and $q \geqslant 2$.

Remark 2.3. Property (N_p) means that the resolution is as simple as possible in the first p steps:

$$\cdots \to S(-p-1)^{\oplus k_p} \to S(-p)^{\oplus k_{p-1}} \to \cdots \to S(-2)^{\oplus k_1} \to S \to \Gamma_X(\mathscr{O}_X, L) \to 0$$

with $k_i \in \mathbb{N}$. In particular, property (N_0) coincides with projective normality, and property (N_1) means that the embedding is projectively normal and that the homogeneous ideal is generated by quadrics.

2.2. **Syzygies on curves.** Canonically polarized curves are an important and much studied case. In this situation, we expect that property (N_p) is determined by the Clifford index of the curve.

Definition 2.4 (Clifford index). Let C be a smooth, irreducible, projective curve of genus $g \geqslant 4$. We define the Clifford index of C as

$$\mathrm{Cliff}(C) = \min \left\{ \deg(L) - 2h^0(X, L) + 2 \,|\, L \in \mathrm{Pic}(C), h^0(C, L) \geqslant 2, h^1(C, L) \geqslant 2 \right\}$$

Moreover, if C is a smooth curve of genus 2 or 3, we define the Clifford index of C to be zero if the curve is hyperelliptic and one otherwise.

Generalizing some classical results about canonical curves, Green [15] proposed the following.

Conjecture 2.5 (Green). *Let C be a smooth, irreducible, projective curve of genus $g \geqslant 2$. Then (C, ω_C) has property N_p if and only if $p < \mathrm{Cliff}(C)$.*

This conjecture is open in general, but it has been proven for general curves by Voisin [29, 30] using K3 surfaces. Building on her work, Aprodu and Farkas [1] verified the conjecture for every smooth curve on any K3 surface.

3. PROPERTY (N_p) ON SURFACES

This section contains general results about property (N_p) on surfaces. Let X be a smooth projective surface, L an ample and globally generated line bundle on X. Then the evaluation map on global sections gives rise to the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathscr{O}_X \longrightarrow L \longrightarrow 0\,,$$

where M_L is a locally free, it is called the *syzygy bundle* of L.

As pointed out in the introduction, we intend to characterize property (N_p) on surfaces via the non-existence of certain low degree curves. In one direction we one can detect failure of property N_p on surfaces by restricting to curves.

Proposition 3.1. *Let X be a smooth projective surface, L an ample and globally generated line bundle on X and $p \geqslant 0$ be some nonnegative integer. Suppose that there is a reduced effective divisor $F \subseteq X$ of arithmetic genus $p_a(F) \geqslant 1$ such that $(L \cdot F) \leqslant p + 2$. Then (X, L) does not have property N_p.*

Proof. We can clearly suppose F to be connected, otherwise we can restrict to a subdivisor. First we claim that $h^0(F, \mathscr{O}_F(L)) \leqslant (L \cdot F)$: by Riemann-Roch, we get that

$$h^0(\mathscr{O}_F(L)) = (L \cdot F) + 1 - h^1(\mathscr{O}_F) + h^1(\mathscr{O}_F(L))$$

and recalling that $\mathscr{O}_F(K_X + F)$ is dualizing on F, we can rewrite this as

$$h^0(\mathscr{O}_F(L)) = (L \cdot F) + 1 - h^0(\mathscr{O}_F(K_X + F)) + h^0(\mathscr{O}_F(K_X + F - L))$$

so that the claim is equivalent to $h^0(\mathscr{O}_F(K_X + F - L)) \leqslant h^0(\mathscr{O}_F(K_X + F)) - 1$. To show this, we observe that $h^0(\mathscr{O}_F(K_X + F)) = p_a(F) \geqslant 1$, so that there exists a nonzero section in $H^0(F, \mathscr{O}_F(K_X + F))$: since F is reduced,

there exists a point $p \in F$ where the section does not vanish, and since L is globally generated, we can find a curve $C \in |L|$ that intersects F transversally and passes through p. Taking cohomology in the short exact sequence

$$0 \to \mathscr{O}_F(K_X + F - L) \to \mathscr{O}_F(K_X + F) \to \mathscr{O}_{C \cap F}(K_X + F) \to 0$$

we get an exact sequence

$$0 \to H^0(\mathscr{O}_D(K_X + F - L)) \to H^0(\mathscr{O}_F(K_X + F)) \to H^0(\mathscr{O}_{C \cap F}(K_X + F))$$

and since the last map is nonzero by construction we see that $h^0(\mathscr{O}_F(K_X + F - L)) \leqslant h^0(\mathscr{O}_F(K_X + F)) - 1$.

The rest of the proof goes as in the proof of [24]Theorem 4.1: suppose now that (X, L) has property N_p, then in particular L is very ample, so that we can consider X as an embedded surface $X \subseteq \mathbb{P}^N$. Let Λ be the linear space spanned by F. By definition Λ is the projectivization of the image of the restriction map $H^0(X, L) \to H^0(F, \mathscr{O}_F(L))$, hence

$$\dim \Lambda \leqslant h^0(F, \mathscr{O}_F(L)) - 1 \leqslant (L \cdot F) - 1 \leqslant p + 1$$

By sheafifying the minimal free resolution of $\Gamma_X(\mathscr{O}_X, L)$ we get a resolution of sheaves on $\mathbb{P}^N$:

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to \mathscr{O}_X \to 0 \qquad E_p = \bigoplus_q K_{p,q}(X, L) \otimes \mathscr{O}_{\mathbb{P}^N}(-p - q)$$

and if we restrict this resolution to Λ, we get a complex

$$0 \to E_{n|\Lambda} \to E_{n-1|\Lambda} \to \cdots \to E_{1|\Lambda} \to E_{0|\Lambda} \to \mathscr{O}_{X \cap \Lambda} \to 0 \qquad E_{p|\Lambda} = \bigoplus_q K_{p,q}(X, L) \otimes \mathscr{O}_\Lambda(-p - q)$$

which is exact at $\mathscr{O}_{X \cap \Lambda}$ and everywhere exact outside $X \cap \Lambda$, which has dimension at most one, as $X \subseteq \mathbb{P}^N$ is nondegenerate. Since (X, L) has property N_p, we know that $K_{i,q}(X, L) = 0$ for all $0 \leqslant i \leqslant p$, $q \geqslant 2$ and in particular we obtain that $H^i(\Lambda, E_{i|\Lambda}) = 0$ for all $i > 0$ and $0 \leqslant p \leqslant \dim \Lambda - 1$. Hence using [19, Lemma 1.6] we get that $H^1(X \cap \Lambda, \mathscr{O}_{X \cap \Lambda}) = 0$. Since F is a subscheme of $X \cap \Lambda$, we have an exact sequence

$$0 \to \mathscr{I}_{F/X \cap \Lambda} \to \mathscr{O}_{X \cap \Lambda} \to \mathscr{O}_F \to 0$$

and taking cohomology we see that $H^2(X \cap \Lambda, \mathscr{I}_{F/X \cap \Lambda}) = 0$, because $X \cap \Lambda$ has dimension one. Then, it follows that $H^1(F, \mathscr{O}_F) = 0$, which is absurd because $p_a(F) \geqslant 1$. $\qquad \square$

The following is a general condition under which property (N_p) for a polarized surface (X, L) is inherited by curves in X.

Proposition 3.2. *Let X be a surface, L an ample and globally generated line bundle on X. Let $F \subseteq X$ be a smooth curve such that*

(1) $H^1(X, L - F) = H^1(X, 2L - F) = 0$.
(2) $H^2(X, L - F) = 0$.
(3) $K_{i,0}(X, K_X + F, L) = H^0(X, \wedge^i M_L \otimes (K_X + F)) = 0$ for all $i = h^0(X, L) - p - 2$.

If (X, L) has property (N_p), then so does $(F, \mathscr{O}_F(L))$.

Proof. Observe first that conditions (1) and (2) together imply that $\mathscr{O}_X(-F)$ is 3-regular with respect to L in the sense of Castelnuovo-Mumford, it follows in particular that $H^1(X, qL - F) = H^2(X, qL - F) = 0$ for all $q \geqslant 1$. Assume that (X, L) satisfies property N_p, and consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_X(-F) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_F \longrightarrow 0 \,.$$

This induces to an exact sequence of graded $\text{Sym}^\bullet H^0(X,L)$-modules

$$0 \longrightarrow \Gamma_X(\mathscr{O}_X(-F),L) \longrightarrow \Gamma_X(\mathscr{O}_X,L) \longrightarrow M \longrightarrow 0$$

where M is a graded module such that $M_q = H^0(F,\mathscr{O}_F(qL))$ for all $q \geqslant 1$. In particular, from the description of the $K_{p,q}$ via the Koszul complex, we see that $K_{j,q}(F,\mathscr{O}_F(L);H^0(X,L)) = K_{j,q}(M;H^0(X,L))$ for all $q \geqslant 2$, so that, if we can prove $K_{j,q}(M;H^0(X,L)) = 0$ for all $j \leqslant p, q \geqslant 2$, we are done thanks to Lemma 2.1. Using the long exact sequence in Koszul cohomology [15, Corollary 1.d.4], arising from the above sequence and the assumption that (X,L) has property (N_p), we see that $K_{j,q}(M;H^0(X,L)) \subseteq K_{j-1,q+1}(X,-F,L)$ and we claim that this latter group is zero. When $q \geqslant 3$ this follows from the regularity of $\mathscr{O}_X(-F)$ with respect to L [2, Proposition 2.37]. If $q = 2$ we can use Green's duality theorem for Koszul cohomology (see [15, Theorem 2.c.6]), which gives

$$K_{j-1,3}(X,-F,L)^\vee = K_{h^0(X,L)-j-2,0}(X,K_X+F,L),$$

To conclude, we observe that $H^0(K_X+F-L) = H^2(L-F) = 0$ thanks to condition (2), and looking at the Koszul complex we immediately see that $K_{j,q}(X,K_X+F,L) = 0$ for all $j \leqslant 0$. Then [13, Proposition 1.9] shows that condition (3) implies the vanishing $K_{h^0(X,L)-j-2,0}(X,K_X+F,L)$ for all $j \leqslant p$. $\square$

Remark 3.3. To check condition (3) we can sometimes use the stability considerations for the bundle M_L. Assuming that M_L is slope stable with respect to L, the vanishing that we want is implied by

$$\left(c_1(\wedge^{h^0(X,L)-p-2} M_L \otimes (K_X+F)) \cdot L \right) < 0.$$

If we denote by $r = h^0(X,L)-1$ the rank of M_L and $i = h^0(X,L)-p-2$, then the intersection number on the left is computed as follows:

$$c_1(\wedge^i M_L \otimes (K_X+F)) \cdot L = c_1(\wedge^i M_L) \cdot L + \binom{r}{i}((K_X+F) \cdot L) = \binom{r-1}{i-1} c_1(M_L) \cdot L + \binom{r}{i}((K_X+F) \cdot L)$$

$$= -\binom{r-1}{i-1}(L^2) + \binom{r}{i}((K_X+F) \cdot L) = -\binom{r-1}{i-1}\left[(L^2) - \frac{r}{i}((K_X+F) \cdot L) \right]$$

$$= -\binom{r-1}{i-1}\left[(L^2) - \frac{h^0(X,L)-1}{h^0(X,L)-p-2}((K_X+F) \cdot L) \right]$$

In particular, condition (3) in Proposition 3.2 is satisfied, if M_L is slope stable with respect to L, and

$$(h^0(X,L)-p-2) \cdot (L^2) > (h^0(X,L)-1) \cdot ((K_X+F) \cdot L).$$

As an illustration of this technique, we prove the following extension of Proposition 3.1:

Proposition 3.4. *Let X be a K3 surface, an abelian surface or an Enriques surface, and L an ample and globally generated line bundle such that $(L^2) \geqslant 3p+6$ (if K3), $(L^2) \geqslant 3p+14$ (if abelian) or $(L^2) \geqslant 3p+7$ (if Enriques). Suppose there exists a smooth curve $F \subseteq X$ of genus two with $(L \cdot F) \leqslant p+4$, then (X,L) does not satisfy property (N_p).*

Proof. Suppose that (X,L) satisfies property (N_p), in particular L is very ample. We start with computing $h^0(F,\mathscr{O}_F(L))$. Since $\mathscr{O}_F(L)$ is very ample, we must have $(L \cdot F) \geqslant 3$, so that Riemann–Roch gives $h^0(F,\mathscr{O}_F(L)) = (F \cdot L) - 1 \leqslant p+3$. Since L is ample, we have $H^1(X,L) = 0$ by Kodaira vanishing. Consider then the exact sequence

$$0 \to H^0(X,L-F) \to H^0(X,L) \to H^0(F,\mathscr{O}_F(L)) \to H^1(X,L-F) \to 0.$$

We see that if $H^1(X,L-F) \neq 0$, then the image of the restriction map on global sections has dimension at most $h^0(F,s\mathscr{O}_F(L)) - 1 \leqslant p+2$, hence we are done using the same proof as in Proposition 3.1. Therefore, we can

assume without loss of generality that $H^1(X, L - F) = 0$. Now we try to check the conditions of Lemma 3.2: if $C \in |L|$ is a smooth curve then we can look at the short exact sequence

$$0 \to \mathcal{O}_X(L - F) \to \mathcal{O}_X(2L - F) \to \mathcal{O}_C(2L - F) \to 0 .$$

We see that $((2L - F) \cdot L) > (L^2)$, so that Riemann-Roch implies $H^1(C, \mathcal{O}_C(2L - F)) = 0$, and then $H^1(X, L - F) = 0$ gives $H^1(X, 2L - F) = 0$ as well. Serre duality and $(L \cdot (L - F)) > 0$ imply that we have $H^2(X, L - F) = 0$.

Thus, we are left with checking that condition (3) of Lemma 3.2 holds. One way to verify this according to Remark 3.3 is to show that the syzygy bundle M_L is stable. Note that under the conditions in the statement we have the inequality

$$(h^0(X, L) - p - 2) \cdot (L^2) \; > \; (h^0(X, L) - 1) \cdot (F \cdot L) ,$$

The stability condition on M_L was proved in [6], hence, by Lemma 3.2 $(F, \mathcal{O}_F(L))$ has property (N_p). However a line bundle of degree $d \leqslant p + 4$ on a hyperelliptic curve of genus at least 2 can never satisfy (N_p) by [18, Theorem 2]. $\qquad \square$

Finally, we establish the equivalence between (2) and (3) of Theorem A on abelian and K3 surfaces under a light numerical condition.

Proposition 3.5. *Let X be a surface with $K_X = 0$ and L an ample line bundle on X with $(L^2) > \frac{8}{7}(p + 2)^2$ for a given natural number p. Then conditions* (2) *and* (3) *in Theorem A are equivalent.*

Proof. The implication (3) $\implies$ (2) follows immediately from the definition of the Seshadri constant; therefore, it remains to deal with the converse implication. Let $x \in X$ be a very general point. If the Seshadri constant is maximal, i.e. $\varepsilon(L; x) = \sqrt{(L^2)}$, then (3) follows from the given lower bound on (L^2). Assume that $\varepsilon(L; x)$ is submaximal, then there exists a curve $F \subseteq X$ with $m \overset{\text{def}}{=} \operatorname{mult}_x(F) \geqslant 1$ giving the Seshadri constant $\varepsilon(L; x)$. If $m = 1$ and $(L \cdot F) \leqslant p + 2$, then the Hodge index theorem yields $(F^2) \leqslant 0$. However, since $x \in X$ was chosen to be very general, it does not lie on any negative curve, in particular we obtain that $(F^2) = 0$. Now, we can assume F to be irreducible and reduced, and if X is an abelian surface F must be also smooth, since there are no rational curves on X. If instead X is a K3 surface, we can again suppose that F is smooth via [4, Proposition VIII.13]. In both cases, we get to a contradiction of condition (2), therefore, either $m = 1$ and $(L \cdot F) \geqslant p + 3$, or $m \geqslant 2$.

In the first case condition (3) is immediate, so we can assume that $m \geqslant 2$. First, note that X is not uniruled, and then [23, Theorem 2.1] implies that

$$(F^2) \geqslant m^2 - m + 2 .$$

Next, the definition of the Seshadri constant and the Hodge index theorem imply the sequence of inequalities

$$\varepsilon(L; x)^2 \; = \; \frac{(L \cdot F)^2}{m^2} \; \geqslant \; \frac{(L^2) \cdot (F^2)}{m^2} \; > \; \frac{8}{7}(p + 2)^2 \left(\frac{m^2 - m + 2}{m^2} \right) \; \geqslant \; (p + 2)^2 ,$$

where the last inequality is valid for any $m \geqslant 1$. In particular, $\varepsilon(L; x) > p + 2$, which yields (3). $\qquad \square$

4. HIGHER SYZYGIES ON K3 SURFACES

We devote this section to the proof of Theorem A. First we present a stronger variant of the equivalence between (1) and (2) in Theorem A for K3 surfaces. The fundamental principle here is that Koszul cohomology for K3 surfaces can be reduced to curves:

Lemma 4.1. *Let (X,L) be a polarized K3 surface, with L ample and globally generated. Let $C \in |L|$ be any smooth curve, then*

$$K_{p,q}(X,L) \cong K_{p,q}(C, \omega_C) \qquad \text{for all } p, q$$

Proof. This follows from a general Lefschetz theorem for Koszul cohomology (see [2, Theorem 2.20]) together with the fact that $H^1(X, L^q) = 0$ for all $q \geqslant 0$, since we are on a K3 surface. $\square$

Moreover, we know from results of Aprodu-Farkas that Green's conjecure holds for every smooth curve on any K3 surface, so that we have:

Corollary 4.2 (Aprodu–Farkas). *Let (X,L) be a polarized K3 surface with L ample and globally generated. Then (X,L) has property N_p if and only if for one (or any) smooth curve $C \in |L|$ one has $p < \mathrm{Cliff}(C)$.*

Then, it becomes important to study the Clifford index of curves on K3 surfaces. For example, we know that for a smooth curve C we have $\mathrm{Cliff}(C) \in \{\mathrm{gon}(C) - 2, \mathrm{gon}(C) - 3\}$. In the K3 case one can make a very precise distinction (see [21, Proposition 8.6]):

Proposition 4.3. *Let (X,L) be a K3 surface, with L big and globally generated. Suppose that for all smooth curves $C \in |L|$ one has $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 3$. Then $L \cong \mathscr{O}_X(2D + \Gamma)$, where $D \subseteq X$ is an effective divisor and $\Gamma \subseteq X$ is a smooth curve such that:*

$$(D^2) \geqslant 1 \qquad (\Gamma^2) = -2 \qquad (\Gamma \cdot D) = 1$$

In this case, for any smooth curve $C \in |L|$ we have

$$\mathrm{Cliff}(C) = (D^2) - 1 \qquad \mathrm{gon}(C) = (D^2) + 2$$

Remark 4.4. In particular, we see that the line bundle L described in the above proposition is not ample, since $(L \cdot \Gamma) = 2(D \cdot \Gamma) + (\Gamma^2) = 0$.

Theorem 4.5. *Let X be a K3 surface and L be an ample line bundle on X. Then*

(a) *If $(L^2) > \frac{1}{2}(p+4)^2$ then conditions (1) and (2) in Theorem A are equivalent.*
(b) *If $(L^2) > \frac{1}{4}(p+6)^2$, then the conditions (1) and (2) in Theorem A are equivalent with the exception of the case when there exists a smooth genus two curve F on X such thata $1 \leqslant (L \cdot F) \leqslant p+4$.*

Remark 4.6. If $f: X \to \mathbb{P}^2$ is a cover of degree 2 branched over a generic sextic, then $\mathrm{Pic}(X) \simeq \mathrm{Pic}(\mathbb{P}^2)$, by [5]. In particular, there is no elliptic fibrations on X and condition (2) in theorem A holds for $L = f^*(O_{\mathbb{P}^2}(m+2))$. However, by Proposition 3.1, the pair (X,L) satisfies (N_{2m-1}) but not (N_{2m}).

Proof of Theorem 4.5. In both cases the implication $(1) \implies (2)$ in Theorem A is proved under less restrictive hypotheses in Proposition 3.1. Now we address the converse implication.

(a) Suppose that (X,L) does not have property N_p. Since L is ample, we know from Remark 4.4 that there exists a smooth curve $C \in |L|$ with $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2$. Then Corollary 4.2 implies that $\mathrm{Cliff}(C) \leqslant p$ which is the same as $\mathrm{gon}(C) \leqslant p+2$. Now, we could proceed in various ways: for example it is easy to conclude the proof using a result of Green–Lazarsfeld from [16], or Knutsen's [21, Theorem 1.3], but we would like to use an extension result due to Serrano, since in principle it could be applied also on other surfaces, whereas the other results we mentioned are specific to the K3 case.

So, by definition of gonality, there exists a map $f: C \to \mathbb{P}^1$ of degree $\deg f \leqslant p+2$. Then [28, Theorem 3.1] implies that f extends to a morphism $\overline{f}: X \to \mathbb{P}^1$: the general fiber F of $\overline{f}$ is a smooth elliptic curve and by construction we have that $(L \cdot F) = (C \cdot F) = \deg f \leqslant p+2$.

(b) As before, if (X,L) does not have property N_p, then there is a smooth curve $C \in |L|$ with a map $f \colon C \to \mathbb{P}^1$ with degree $d = \mathrm{gon}(C) \leqslant p+2$. The difference from case (a) is that due to the weaker numerical constraint, [28, Theorem 3.1] does not apply. Instead, we will make use of another result by Serrano [28] that requires some extra work. Let $x \in \mathbb{P}^1$ be a general point, then the fiber $f^{-1}(x) = \{P_1^x, \ldots, P_d^x\}$ consists of d distinct points. In the course of the proof of [28], Serrano shows that there exists a non-trivial effective divisor F_x and a $\mathbb{Q}$-effective P_x on X with $C = F_x + P_x$ satisfying

$$0 \leqslant a \stackrel{\text{def}}{=} (F_x^2) < b \stackrel{\text{def}}{=} (P_x^2), \qquad a \leqslant d \leqslant (p+2), \qquad 0 < e \stackrel{\text{def}}{=} (F_x \cdot P_x) \leqslant d,$$

and $P_1^x, \ldots, P_e^x \subseteq C \cap F_x$. Moreover, since $H^1(X, \mathscr{O}_X) = 0$, it follows from [28, Theorem 3.9] that the effective divisor F_x can be chosen to be irreducible and that infinitely many of them are in the same linear equivalence class, as $x \in \mathbb{P}^1$ vare and to remain in the same linear series while $x \in \mathbb{P}^1$ varies. In particular F_x is globally generated by [27, Theorem 3.1], and we can assume that F_x is smooth.

Then, if $a = 0$ we see that F_x is a smooth elliptic curve such that $(F_x \cdot L) = (F_x^2) + (F_X \cdot P_x) \leqslant p+2$. If instead $a > 0$, then the Hodge index theorem implies that $e^2 \geqslant ab$, thus by our hypotheses we get

$$\frac{1}{4}(p+6)^2 < (C^2) = a+b+2e \leqslant a + \frac{e^2}{a} + 2e \leqslant a + \frac{d^2}{a} + 2d \leqslant a + \frac{(p+2)^2}{a} + 2(p+2).$$

Now we observe that $a \leqslant d \leqslant (p+2)$ and the function $f(a) = a + (p+2)^2/a + 2(p+2)$ is decreasing for $0 < a \leqslant (p+2)$. Then, the above inequality implies $a \leqslant 3$ and since the intersection form on a K3 surface is even it must be $(F_x^2) = 2$. Then F_x is a smooth genus two curve such that $(C \cdot F_x) \leqslant p+4$. $\qquad \square$

Using this result it is straightforward to give a characterization of property (N_p) for ample line bundles of type $L^{\otimes m}$ with $m \geqslant p$, in the spirit of Mukai's conjecture. In particular this gives an extension of results of Gallego and Purnaprajina [14].

Corollary 4.7. *Let X be a K3 surface and L an ample line bundle, then $(p+3) \cdot L$ has property (N_p).*

Proof. Let $M = (p+3) \cdot L$: then M is globally generated [27] and $(M^2) = (p+3)^2(L^2)$ satisfies the condition of Theorem 4.5 (a). Hence M fails to have property (N_p) if and only if there exists a smooth elliptic curve $E \subseteq X$ such that $(E \cdot M) \leqslant p+2$, but $(E \cdot M) = (p+3)(L \cdot E) \geqslant p+3$. $\qquad \square$

For simplicity of exposition, in the next cases we restrict ourselves to the case when $p \geqslant 2$, since properties (N_0) and (N_1) are taken care of by Saint-Donat's Theorem [27].

Corollary 4.8. *Let X be a K3 surface, L an ample line bundle and $p \geqslant 2$ an integer. Then $(p+2) \cdot L$ has property (N_p) if and only if there is no smooth elliptic curve $E \subseteq X$ such that $(L \cdot E) = 1$.*

Proof. Let $M = (p+2) \cdot L$: then M is globally generated [27] and we see that the self-intersection $(M^2) = (p+2)^2(L^2)$ satisfies the condition of Theorem 4.5 (a). Hence M fails to have property (N_p) if and only if there exists an elliptic curve $E \subseteq X$ such that $(E \cdot M) \leqslant p+2$, which is equivalent to $(E \cdot L) = 1$. $\qquad \square$

Corollary 4.9. *Let X be a K3 surface, L an ample line bundle and $p \geqslant 2$ an integer. If $(L^2) \geqslant 4$, or $(L^2) = 2$ and $p > 2$, then $(p+1) \cdot L$ has property (N_p) if and only if there is no smooth elliptic curve $E \subseteq X$ such that $(E \cdot L) = 1$. Moreover, if $(L^2) = 2$, then $3 \cdot L$ does not have property (N_2).*

Proof. Suppose $(L^2) \geqslant 4$ or $(L^2) = 2$ and $p > 2$. Then $M = (p+1) \cdot L$ is globally generated [27], and (M^2) satisfies the condition of Theorem 4.5 (a). Hence M fails property (N_p) if and only if there is a smooth elliptic curve $E \subseteq X$ such that $(M \cdot E) \leqslant p+2$. But since $p > 0$, this is equivalent to $(L \cdot E) = 1$. Now, suppose that $(L^2) = 2$: we need

to show that $3 \cdot L$ does not have property N_2. Thanks to Lemma 4.1 and Corollary 4.2 this is equivalent to showing that for a smooth curve $C \in |3 \cdot L|$ we have $\mathrm{Cliff}(C) \leqslant 2$. So, let C be such a curve and consider the exact sequence

$$0 \to \mathscr{O}_X(-2L) \to \mathscr{O}_X(L) \to \mathscr{O}_C(L) \to 0.$$

Taking cohomology we see that $h^0(C, \mathscr{O}_C(L)) = h^0(X, L) = 3$ and $h^1(C, \mathscr{O}_C(L)) = h^2(X, 2L) = 6$, so that $\mathrm{Cliff}(C) \leqslant (C \cdot L) - 2h^0(C, \mathscr{O}_C(L)) + 2 = 3(L^2) - 4 = 6 - 4 = 2$. $\qquad\square$

Corollary 4.10. *Let X be a K3 surface and L an ample line bundle and $p \geqslant 2$ an integer.*

(1) if $(L^2) \geqslant 4$ and $p > 2$, or $(L^2) = 2$ and $p > 4$, then $p \cdot L$ has property (N_p) if and only if there is no smooth elliptic curve $E \subseteq X$ such that $(E \cdot L) = 1$.
(2) if $(L^2) \geqslant 6$ then $2 \cdot L$ has property (N_2) if and only if there is no smooth elliptic curve $E \subseteq X$ such that $(E \cdot L) \leqslant 2$.
(3) in the other cases $p \cdot L$ does not have property (N_p).

Proof. (1) Set $M = p \cdot L$. Then M is globally generated [27] and (M^2) satisfies the condition of Theorem 4.5 (a). Hence M fails property N_p if and only if there is a smooth elliptic curve $E \subseteq X$ such that $(E \cdot M) \leqslant p + 2$. But since $p > 2$, this is equivalent to $(E \cdot L) = 1$.

(2) Set $M = 2 \cdot L$. Then M is globally generated [27] and (M^2) satisfies the condition of Theorem 4.5 (a). Hence M fails property N_2 if and only if there is a smooth elliptic curve $E \subseteq X$ such that $(E \cdot M) \leqslant 4$, which is equivalent to $(E \cdot L) \leqslant 2$.

(3) The remaining cases are when $(L^2) = 4$ and $p = 2$ or when $(L^2) = 2$ and $p = 2, 3, 4$. In all these cases, we can show that $p \cdot L$ does not have property (N_p) reasoning as in the previous Corollary.

$\qquad\square$

Proof of Theorem A. The proof is an immediate consequence of Proposition 3.5, Theorem 3.4, and Theorem 4.5 in the K3, and [24, Theorem 1.1] in the case of abelian surfaces. $\qquad\square$

5. ENRIQUES SURFACES

We give here a result analogous to Theorem A on Enriques surfaces. To do this, we will assume the Green–Lazarsfeld secant conjecture for certain curves on Enriques surfaces, which we state for the sake of reference.

Conjecture 5.1 (Green–Lazarsfeld secant conjecture). *Let C be a smooth curve of genus g, L a line bundle of degree $\deg L \geqslant 2g + 1 + p - 2h^1(C, L) - \mathrm{Cliff}(C)$. Then L has property (N_p) if and only if it is $(p+1)$-very ample.*

On Enriques surfaces a result of Knutsen and Lopez [22] gives us some control over the Clifford index. In order to state their result, we introduce some notation. Let L be a globally generated line bundle on X, and set

$$\phi(L) \overset{\mathrm{def}}{=} \inf\{|(F \cdot L)| \mid F \in \mathrm{Pic}(X), (F^2) = 0, F \not\equiv 0\}$$

$$\mu(L) \overset{\mathrm{def}}{=} \inf\{(B \cdot L) - 2 \mid B \in \mathrm{Pic}(X), (B^2) = 4, \phi(B) = 2, B \not\equiv L\}.$$

Theorem 5.2 ([22], Corollary 1.2). *Let L be a globally generated line bundle on an Enriques surface X and let C be a general curve in $|L|$. Then*

$$\mathrm{Cliff}(C) = \min\left\{ 2\phi(L) - 2, \mu(L) - 2, \left\lfloor \frac{(L^2)}{4} \right\rfloor \right\}$$

Remark 5.3. In the definition of $\phi(L)$ we can actually suppose that F is effective. Indeed, if F is a divisor with $(F^2) = 0$ then Riemann-Roch gives $h^0(F) - h^1(F) + h^0(K_X - F) = 1$, hence if F is not effective then $K_X - F$ is effective, but then $|((K_X - F) \cdot L)| = |-(F \cdot L)| = |(F \cdot L)|$.

Theorem 5.4. *Let X be an Enriques surface and L an ample and globally generated line bundle on X such that $(L^2) > 4(p+2)^2$. Assume that Conjecture 5.1 holds for a general curve in $|L|$. Then the following are equivalent*

(1) L has property N_p.
(2) there is no effective and reduced divisor F with $(F^2) = 0$ such that $1 \leqslant (F \cdot L) \leqslant p+2$.

Proof. Assume first that there exists an effective and reduced divisor F with $(F^2) = 0$ such that $1 \leqslant (F \cdot L) \leqslant p+2$. Then we know that L does not have property (N_p) by Proposition 3.2.

Conversely, suppose that there is no reduced effective divisor F with $(F^2) = 0$ and $1 \leqslant (F \cdot L) \leqslant p+2$ and let C be a general smooth curve in $|L|$. It is easy to see that the restriction maps $H^0(X, qL) \to H^0(C, \mathcal{O}_C(qL))$ are surjective for all $q \geqslant 0$, and then it follows as in Lemma 4.1 that (X, L) has property (N_p) if and only if $(C, \mathcal{O}_C(L))$ does. We will assume the Green–Lazarsfeld secant conjecture on C.

First we show that the line bundle $\mathcal{O}_C(L)$ satisfies Green–Lazarsfeld condition on the degree. It is easy to check that $\mathcal{O}_C(L) = \mathcal{O}_C(K_C + \eta)$, where η is torsion bundle of orderexactly two. In particular, if we write g for the genus of C, we have $\deg(\mathcal{O}_C(L)) = 2g - 2$ and $h^1(C, \mathcal{O}_C(L)) = 0$. Hence, the Green–Lazarsfeld condition is equivalent to $\mathrm{Cliff}(C) \geqslant p+3$.

Suppose now that $\mathrm{Cliff}(C) \leqslant p+2$. Since $(L^2) > 4(p+2)^2$, we observe that $\lfloor \frac{(L^2)}{4} \rfloor \geqslant (p+2)^2 > (p+2)$, hence, by Theorem 5.2, there must exist either an effective and nontrivial divisor F with $(F^2) = 0$ and $(F \cdot L) \leqslant \frac{p+4}{2}$, or a divisor B with $(B^2) = 4$ and $(L \cdot B) \leqslant p+4$. However, the second possibility does not happen because if B is a divisor with $(B^2) = 4$ then

$$(B \cdot L) \geqslant \sqrt{(B^2)}\sqrt{(L^2)} > 4(p+2) \geqslant p+4 \, .$$

For the first possibility, suppose that there is an effective divisor such that $(F^2) = 0$ and $(F \cdot L) \leqslant \frac{p+4}{2}$. Observe that $\frac{p+4}{2} \leqslant (p+2)$, so that if we can take F to be reduced, we are done. First we observe that for every effective divisor $F' \leqslant F$, it must be that $(F'^2) \leqslant 0$: indeed, if $(F'^2) \geqslant 1$, then

$$\left(\frac{p+4}{2}\right)^2 \geqslant (L \cdot F)^2 \geqslant (L \cdot F')^2 \geqslant (L^2)(F'^2) > 4(p+2)^2 \, ,$$

which is impossible. Hence, if F has a reduced subdivisor $F' \leqslant F$ with $(F'^2) \geqslant 0$, we are done.

If this does not happen, then every connected component of the support of F is a tree of smooth rational curves. We can clearly reduce to the case when the support is connected, so that we can write $F = n_1 C_1 + \cdots + n_r C_r$ where the n_i are positive integers and the C_i are smooth rational curves with $(C_i^2) = -2$, and, if $j \neq i$ then $(C_i \cdot C_j) = 1$ if $j = i-1, i+2$, and $(C_i \cdot C_j) = 0$ otherwise. Then we see that

$$-(F^2) = \sum_{i=1}^{r} 2n_i^2 - 2n_1 n_2 - 2n_2 n_3 + \cdots - 2n_{r-1}n_r = n_1^2 + \sum_{i=1}^{r-1}(n_i - n_{i+1})^2 + n_r^2 > 0 \, ,$$

so that $(F^2) = 0$ cannot happen.

Next, we show that $\mathcal{O}_C(L)$ is $(p+1)$-very ample. Suppose that it is not, and let $Z \subseteq C$ be a zero-dimensional subscheme of minimum length $\ell \leqslant p+2$ such that $H^0(C, \mathcal{O}_C(L)) \to H^0(C, \mathcal{O}_C(L) \otimes \mathcal{O}_Z)$ is not surjective. In particular, since L is globally generated, we must have $\ell \geqslant 2$. Then, by [21, Proposition 3.7], there exists an effective divisor D on X, containing Z for which $(L \cdot D) \leqslant (D^2) + \ell \leqslant 2\ell$.

However since D contains Z, we see that $(L \cdot D) = (C \cdot D) \geqslant \ell$, hence $(D^2) \geqslant 0$. Suppose that $(D^2) \geqslant 1$. Then we have

$$4(p+2)^2 \geqslant 4\ell^2 \geqslant (L \cdot D)^2 \geqslant (L^2)(D^2) > 4(p+2)^2 \,,$$

which is impossible, so it must be $(D^2) = 0$ and consequently, $(L \cdot D) \leqslant \ell \leqslant p+2$.

$\square$

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