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UNITARIZATION AND INVERSION FORMULAE FOR THE RADON TRANSFORM BETWEEN DUAL PAIRS*

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Abstract. We consider the Radon transform associated to dual pairs (X, Ξ) in the sense of Helgason, with $X = G/K$ and $\Xi = G/H$, where G is a locally compact group and K and H are closed subgroups of G. Under some technical assumptions, we prove that if the quasi-regular representations of G acting on $L^2(X)$ and $L^2(\Xi)$ are irreducible, then the Radon transform admits a unitarization intertwining the two representations. If, in addition, the representations are square-integrable, we provide an inversion formula for the Radon transform based on the voice transform associated to these representations.

Key words. Radon transform, dual pairs, square-integrable representations, inversion formula, wavelets, shearlets, homogeneous spaces, spherical means Radon transform

AMS subject classifications. 44A12, 42C40, 22D10

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1. Introduction.

In a remarkable series of papers (see, e.g., [19, 18]), for the most part subsumed in the monographs [21, 22, 20, 23], Helgason has developed a broad theory of Radon transforms in a differential geometric setup. In this paper we show that the above framework is particularly appropriate for treating in a unified way some results concerning unitarizability features and inversion formulae of various types of Radon transforms [5, 24] and permits one to handle a significant number of other interesting examples.

One of the central notions in Helgason's theory is that of dual pair $(G/K, G/H)$ of homogeneous spaces of the same locally compact group G, where K and H are closed subgroups of G. The transitive G-space $X = G/K$ is meant to describe the ambient in which the functions to be analyzed live, prototypically a space of constant curvature like the Euclidean plane, or the sphere S^2 or the hyperbolic plane $\mathbb{H}^2$. A large and important part of Helgason's work is devoted to the case when X is actually a symmetric space, whence the notation G/K that we retain. The second transitive G-space $\Xi = G/H$ is meant to parametrize the set of submanifolds of X over which one wants to integrate functions, for instance, hyperplanes in Euclidean space, great circles in S^2, geodesics or horocycles in $\mathbb{H}^2$. With this basic understanding in mind, the notion of incidence between $x \in X$ and $\xi \in \Xi$ translates the intuition that $x = g_1K$ is a point of $\xi = g_2H$ and amounts to the fact that $g_1K \cap g_2H \neq \emptyset$. In this way any element $\xi \in \Xi$ may be realized as a submanifold $\hat{\xi} \subseteq X$ simply by taking all the points $x \in X$ that are incident to ξ; conversely, one builds the "sheaf of manifolds" $\check{x}$ through the point $x \in X$ by taking all the points $\xi \in \Xi$ that are incident to x. If the maps $\xi \mapsto \hat{\xi}$ and $x \mapsto \check{x}$ are injective, then (X, Ξ) is called a dual pair. Under this assumption, the Radon transform $\mathcal{R}$ takes functions on X into functions on Ξ and is abstractly defined by

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$$\mathcal{R}f(\xi) = \int_{\hat{\xi}} f(x) \mathrm{d}m_\xi(x),$$

provided that, for all $\xi \in \Xi$, m_ξ is a suitable measure on the manifold $\hat{\xi}$ and the right-hand side is meaningful, possibly in some weak sense. The first requirement is achieved by observing that, denoting by $\xi_0 = eH$ the origin of Ξ, it is easy to check that $\widehat{\xi_0} \subseteq X$ is actually a transitive H-space, hence $\widehat{\xi_0}$ carries a measure m_0 which is quasi-invariant with respect to the H-action [33]. The idea is then to define the measures m_ξ on each ξ by pushing forward m_0 via the G-action. This is achieved by fixing a Borel section $\sigma\colon \Xi \to G$ (see (A3)), on the choice of which the whole construction therefore depends, including the Radon transform itself (see (A6)). Observe that if the classical but rather strong assumption that $\widehat{\xi_0}$ carries an H-invariant measure is removed, then the issue of selecting the family of measures $\{m_\xi\}$ is no longer canonical and yet is crucial (Lemma 2.1 in [20] breaks down). In general, $\widehat{\xi_0}$ can only be assumed to carry a quasi-invariant measure m_0 with respect to the H-action, so that assuming the existence of a relatively H-invariant measure and fixing a Borel section σ seems a reasonable way of making this selection.

As for the right space of functions $f\colon X \to \mathbb{C}$ for which the Radon transform makes sense, a natural choice is the L^2 setting. Indeed, both X and Ξ are transitive spaces, so that there exist quasi-invariant measures $\mathrm{d}x$ and $\mathrm{d}\xi$. In this context, a central issue is to prove that the Radon transform, up to a composition with a suitable pseudo-differential operator, can be extended to a unitary map $\mathcal{Q}$ from $L^2(X, \mathrm{d}x)$ to $L^2(\Xi, \mathrm{d}\xi)$ intertwining the quasi-regular representations π and $\hat{\pi}$ of G, which naturally act on $L^2(X, \mathrm{d}x)$ and $L^2(\Xi, \mathrm{d}\xi)$, respectively. The reader is again referred to Helgason's books for a thorough treatment, as well as for the broad problem of the operator properties of $\mathcal{R}$.

In this paper we address the special case in which the representations π and $\hat{\pi}$ are both irreducible. Under some technical assumptions that we describe below, we prove a unitarization result; see Theorem 3.9. The proof is based on the generalization of Schur's lemma provided by Duflo and Moore [12]. One of the novelties of our treatment consists in making weaker assumptions on m_0, $\mathrm{d}x$, and $\mathrm{d}\xi$, namely their relative invariance instead of invariance. This allows for considering a wider variety of cases, such as wavelets and shearlets. A well-known predecessor of Theorem 3.9 is Theorem 4.1 in [20], an alternative proof of which, tailored to our particular viewpoint, is to be found in [5].

If, in addition, we require that π is square-integrable (so that $\hat{\pi}$ is square-integrable, too), we derive a general inversion formula for $\mathcal{R}$ of the form

$$f = \int_G \chi(g) \langle \mathcal{R}f, \hat{\pi}(g)\Psi \rangle \, \pi(g)\psi \, \mathrm{d}\mu(g), \tag{1.1}$$

where χ is a character of G and $\psi \in L^2(X, \mathrm{d}x)$ and $\Psi \in L^2(\Xi, \mathrm{d}\xi)$ are suitable mother wavelets and the Haar integral is weakly convergent; see Theorem 4.1. We point out that the coefficients $\langle \mathcal{R}f, \hat{\pi}(g)\Psi \rangle$ depend on f only through its Radon transform $\mathcal{R}f$, so that the above equation allows us to reconstruct an unknown signal from its Radon transform by computing the family of coefficients $\{\langle \mathcal{R}f, \hat{\pi}(g)\Psi \rangle\}_{g \in G}$. As is clear from (1.1), $\hat{\pi}$ is used as an "analysis" transform applied to $\mathcal{R}f$ and π as a "synthesis" transform to reconstruct f. This kind of reconstruction formulae is already known for the classical Radon transform where G is the affine group of $\mathbb{R}^d$ associated with the multidimensional wavelets [24, 34, 29, 7, 27] and for the affine Radon transform where G is the shearlet group [5].

We illustrate the construction and the result with the examples where G is either the similitude group of the plane (with two different choices of Ξ) or the standard shearlet group [26], but other cases could also be covered, such as the generalized shearlet dilation groups [16, 2]. In all these examples the group G is a semidirect product of the form $\mathbb{R}^d \rtimes K$, where K is a closed subgroup of $\mathrm{GL}(d,\mathbb{R})$. This structure is not necessary in order for our construction to work, but it is important to observe that, under this assumption, $X = G/K$ is naturally identified with $\mathbb{R}^d$, a somewhat canonical space for applications. From the point of view of the geometry involved this case is thus different from the setup considered in much of Helgason's work, where G is a semisimple Lie group, X is a symmetric space, and $\{\hat{\xi}\}$ are, for example, horocycles or geodesics. We believe that our contribution may be further substantiated with several other examples and deepened in several directions. In particular, it would be interesting to relax the assumption of irreducibility of the representations. This would allow us to include many other examples, such as the class of groups studied in [1, 3].

For clarity, we list the main assumptions that are made along the way. Let G be a locally compact second countable (lcsc) group. We consider two lcsc transitive G-spaces, X and Ξ, where the continuous actions on $x \in X$ and $\xi \in \Xi$ are denoted by

$$(g,x) \mapsto g[x], \quad (g,\xi) \mapsto g.\,\xi.$$

We fix $x_0 \in X$ and $\xi_0 \in \Xi$ and we denote the corresponding stability subgroups by K and H. We assume that the following conditions hold true:

(A1) the spaces X and Ξ carry relatively G-invariant measures $\mathrm{d}x$ and $\mathrm{d}\xi$, respectively;

(A2) the H-transitive space

$$\hat{\xi}_0 = H[x_0] \subseteq X$$

carries a relatively H-invariant measure m_0 with character γ;

(A3) there exist a Borel section $\sigma\colon \Xi \to G$ and a character $\iota\colon G \to (0,+\infty)$ such that

$$\gamma\left(\sigma(\xi)^{-1} g \sigma\left(g^{-1}.\,\xi\right)\right) = \iota(g), \qquad g \in G,\, \xi \in \Xi;$$

(A4) the quasi-regular representation π of G acting on $L^2(X,\mathrm{d}x)$ is irreducible and square-integrable;

(A5) the quasi-regular representation $\hat{\pi}$ of G acting on $L^2(\Xi,\mathrm{d}\xi)$ is irreducible;

(A6) there exists a nontrivial π-invariant subspace $\mathcal{A} \subseteq L^2(X,\mathrm{d}x)$ such that for all $f \in \mathcal{A}$

$$f(\sigma(\xi)[\cdot]) \in L^1\left(\hat{\xi}_0, m_0\right) \quad \text{for almost all } \xi \in \Xi, \tag{1.2a}$$

$$\mathcal{R}f := \int_{\hat{\xi}_0} f(\sigma(\cdot)[x])\mathrm{d}m_0(x) \in L^2(\Xi,\mathrm{d}\xi), \tag{1.2b}$$

and the adjoint of the operator $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi,\mathrm{d}\xi)$ has nontrivial domain.

It is worth observing that the whole construction, and in particular the existence of the character ι and the construction of the Radon transform $\mathcal{R}$ itself, depends on the choice of the section σ. Note that we do not require the injectivity of the maps $\xi \mapsto \hat{\xi}$ and $x \mapsto \check{x}$, but our framework completely fits that of Helgason's dual pairs (X,Ξ). Apart from the cases considered below, the reader may consult [20] for numerous examples of dual pairs (X,Ξ). We add a few comments on assumption (A6). We will show that $\mathcal{R}$ is a semi-invariant densely defined operator (see Lemma 3.1), and as a consequence the assumption that the domain of $\mathcal{R}^*$ is nontrivial is equivalent to the closability of $\mathcal{R}$; see Corollary 3.4. By the irreducibility of π, the minimal choice for $\mathcal{A}$ is $\mathrm{span}\{\pi_g f_0 : g \in G\}$, where f_0 is a nonzero function in $L^2(X, \mathrm{d}x)$ satisfying conditions (1.2). In general the closure of an operator depends on its domain; however, if the Radon transform extends to a larger domain we will show that under some weak conditions also its extension is closable and the two closures coincide; see Corollary 3.6 and the comment below Lemma 3.5. This delicate issue is further discussed in Example 3.8.

2. Preliminaries.

2.1. Notation. We briefly introduce the notation. We set $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\mathbb{R}^+ = (0, +\infty)$. The Euclidean norm of a vector $v \in \mathbb{R}^d$ is denoted by $|v|$ and its scalar product with $w \in \mathbb{R}^d$ by $v \cdot w$. For any $p \in [1, +\infty]$ we denote by $L^p(\mathbb{R}^d)$ the Banach space of functions $f : \mathbb{R}^d \to \mathbb{C}$ that are p-integrable with respect to the Lebesgue measure $\mathrm{d}x$ and, if $p = 2$, the corresponding scalar product and norm are $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. If E is a Borel subset of $\mathbb{R}^d$, $|E|$ also denotes its Lebesgue measure. The Fourier transform is denoted by $\mathcal{F}$ both on $L^2(\mathbb{R}^d)$ and on $L^1(\mathbb{R}^d)$, where it is defined by

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-2\pi i\, \omega \cdot x} \mathrm{d}x, \qquad f \in L^1\left(\mathbb{R}^d\right).$$

If G is an lcsc group, we denote by $L^2(G)$ the Hilbert space of square-integrable functions with respect to a left Haar measure on G. If X is an lcsc transitive G-space with origin x_0, denoted by $g[x]$ the action of G on X, a Borel measure μ of X is relatively invariant if there exists a positive character α of G such that for any measurable set $E \subseteq X$ and $g \in G$ it holds that $\mu(g[E]) = \alpha(g)\mu(E)$. Furthermore, a Borel section is a measurable map $s : X \to G$ satisfying $s(x)[x_0] = x$ and $s(x_0) = e$, with e the neutral element of G; a Borel section always exists since G is second countable [33, Theorem 5.11]. We denote the (real) general linear group of size $d \times d$ by $\mathrm{GL}(d, \mathbb{R})$.

2.2. Dual homogeneous spaces. In this section we recall the basic construct due to Helgason [20] and, whenever possible, we keep the notation as in [20].

Let us recall the main objects introduced in section 1. We fix an lcsc group G, and we consider two lcsc transitive G-spaces X and Ξ, whose continuous actions on $x \in X$ and $\xi \in \Xi$ are denoted by

$$(g, x) \mapsto g[x], \quad (g, \xi) \mapsto g.\xi.$$

We fix $x_0 \in X$ and $\xi_0 \in \Xi$ and we denote the corresponding stability subgroups by K and H, so that $X \simeq G/K$ and $\Xi \simeq G/H$.

By assumption (A1), X and Ξ admit relatively invariant measures $\mathrm{d}x$ and $\mathrm{d}\xi$ with positive characters $\alpha : G \to (0, +\infty)$ and $\beta : G \to (0, +\infty)$, respectively, which may be expressed by the equalities

$$\int_X f(g^{-1}[x])\, \mathrm{d}x = \alpha(g) \int_X f(x)\, \mathrm{d}x, \qquad f \in L^1(X, \mathrm{d}x),\ g \in G, \tag{2.1a}$$

$$\int_\Xi f(g^{-1}.\xi)\, \mathrm{d}\xi = \beta(g) \int_\Xi f(\xi)\, \mathrm{d}\xi, \qquad f \in L^1(\Xi, \mathrm{d}\xi),\ g \in G. \tag{2.1b}$$

We define the spaces

$$\check{x}_0 = K.\xi_0 \subseteq \Xi, \qquad \hat{\xi}_0 = H[x_0] \subseteq X.$$

By definition, $\check{x}_0$ and $\hat{\xi}_0$ are K and H transitive spaces, respectively. In order to define the Radon transform we will make use of assumption (A2), namely that $\hat{\xi}_0$ carries a relatively H-invariant Radon measure m_0, that is

$$\int_{\hat{\xi}_0} f(h^{-1}[x])\,\mathrm{d}m_0(x) = \gamma(h)\int_{\hat{\xi}_0} f(x)\,\mathrm{d}m_0(x), \qquad f \in L^1(\hat{\xi}_0, \mathrm{d}m_0),\ h \in H, \tag{2.2}$$

where $\gamma\colon H \to (0,+\infty)$ is a positive character of H. This is a weaker assumption than in Helgason's approach, in which $\hat{\xi}_0$ is assumed to admit a bona fide invariant measure for the H-action.

We fix two Borel sections $s, X \to G$ and $\sigma\colon \Xi \to G$, such that (A3) holds true. With an equivalent approach to that of Helgason's, we define the sets

$$\hat{\xi} = \sigma(\xi)[\hat{\xi}_0] \subseteq X, \qquad \check{x} = s(x).\,\check{x}_0 \subseteq \Xi, \tag{2.3}$$

which are closed subsets by [20, Lemma 1.1]. It is worth observing that $\hat{\xi}$ and $\check{x}$ do not depend on the choice of the sections σ and s.

Remark 2.1 (semidirect product). Let us see how the construction of the measure $\mathrm{d}x$ and of the section s simplifies in the particular case when the group G is the semidirect product of the Euclidean space $\mathbb{R}^d$ with a closed subgroup K of $\mathrm{GL}(d,\mathbb{R})$. This is the setting of all our examples. Further, this structure is enjoyed by several groups of interest in applications, such as the similitude group studied by Antoine and Murenzi [4] and the generalized shearlet dilation groups introduced by Führ in [14, 16] for the purpose of generalizing the standard shearlet group introduced in [26, 10].

We recall that $G = \mathbb{R}^d \rtimes K$ is the manifold $\mathbb{R}^d \times K$ endowed with the group operation

$$(b_1,k_1)(b_2,k_2) = (b_1 + k_1 b_2, k_1 k_2), \qquad b_1, b_2 \in \mathbb{R}^d,\ k_1, k_2 \in K,$$

where kb is the natural linear action of the matrix k on the column vector b, so that G is a Lie group. The inverse of an element in G is given by $(b,k)^{-1} = (-k^{-1}b, k^{-1})$. A left Haar measure of G is

$$\mathrm{d}\mu(b,k) = |\det k|^{-1}\mathrm{d}b\mathrm{d}k, \tag{2.4}$$

where $\mathrm{d}b$ is the Lebesgue measure of $\mathbb{R}^d$ and $\mathrm{d}k$ is a left Haar measure on K. The transitive space we consider is $X = \mathbb{R}^d$, regarded as smooth G-space with respect to the canonical action

$$(b,k)[x] = b + kx, \qquad (b,k) \in G,\ x \in X.$$

The action is clearly transitive, the isotropy at the origin $x_0 = 0$ is the subgroup $\{(0,k) : k \in K\}$, which we identify with K, so that $X \simeq G/K$. Furthermore, the map

$$s\colon X \to G, \qquad s(x) = (x, \mathrm{I}_d),$$

is a Borel section and the Lebesgue measure $\mathrm{d}x$ on X is a relatively G-invariant measure, since for any measurable set $E \subseteq \mathbb{R}^d$ we have $|(b,k)[E]| = |b + kE| = |kE| = |\det k||E|$, and so

$$\alpha(b,k) = |\det k|. \tag{2.5}$$

The following example shows that the (classical) Radon transform can be obtained in this framework. Two other examples are illustrated in section 5.

Example 2.2. The (connected component of the identity of the) similitude group $SIM(2)$ of the plane is $\mathbb{R}^2 \rtimes K$, with $K = \{R_\phi A_a \in \mathrm{GL}(2,\mathbb{R}) : \phi \in [0, 2\pi),\ a \in \mathbb{R}^+\}$, where

$$R_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \qquad A_a = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

By the identification $K \simeq [0, 2\pi) \times \mathbb{R}^+$, we write (b, ϕ, a) for the elements in $SIM(2)$. With this identification the group law becomes

$$(b, \phi, a)(b', \phi', a') = (b + R_\phi A_a b', \phi + \phi' \bmod 2\pi, aa'),$$

and the inverse of (b, ϕ, a) is given by

$$(b, \phi, a)^{-1} = \left(-A_a^{-1} R_\phi^{-1} b, -\phi \bmod 2\pi, a^{-1}\right). \tag{2.6}$$

By (2.4), a left Haar measure of $SIM(2)$ is

$$\mathrm{d}\mu(b, \phi, a) = a^{-3} \mathrm{d}b\mathrm{d}\phi\mathrm{d}a, \tag{2.7}$$

where $\mathrm{d}b$, $\mathrm{d}\phi$, and $\mathrm{d}a$ are the Lebesgue measures on $\mathbb{R}^2$, $[0, 2\pi)$ and $\mathbb{R}_+$, respectively. The group $SIM(2)$ acts transitively on $X = \mathbb{R}^2$ by

$$(b, \phi, a)[x] = R_\phi A_a x + b.$$

By (2.5), we have $\alpha(b, \phi, a) = a^2$.

It remains to choose the space Ξ and the corresponding subgroup H of $SIM(2)$. The group $SIM(2)$ acts transitively on $\Xi = [0, \pi) \times \mathbb{R}$ by

$$(b, \phi, a).(\theta, t) = \left(\theta + \phi \bmod \pi, a\left(t + w(\theta) \cdot A_a^{-1} R_\phi^{-1} b\right)\right),$$

where $w(\theta) = {}^t(\cos\theta, \sin\theta)$, or equivalently

$$(b, \phi, a)^{-1}.(\theta, t) = \left(\theta - \phi \bmod \pi, \frac{t - w(\theta) \cdot b}{a}\right),$$

where we slightly abuse the notation and denote by $\bmod \pi$ the equivalence relation $(\theta, t) \sim (\theta', t')$ if $\theta' = \theta + k\pi$ and $t' = (-1)^k$ for some $k \in \mathbb{Z}$. The isotropy at $\xi_0 = (0, 0)$ is

$$H = \{((0, b_2), \phi, a) : b_2 \in \mathbb{R}, \phi \in \{0, \pi\}, a \in \mathbb{R}^+\}.$$

Thus, $[0, \pi) \times \mathbb{R} = SIM(2)/H$. An immediate calculation gives

$$\int_\Xi f\left((b, \phi, a)^{-1}.(\theta, t)\right) \mathrm{d}\theta\mathrm{d}t = a \int_\Xi f(\theta, t)\, \mathrm{d}\theta\mathrm{d}t, \qquad f \in L^1(\Xi, \mathrm{d}\theta\mathrm{d}t),$$

namely, (2.1b) is satisfied with the character $\beta(b, \phi, a) = a$. Thus, the Lebesgue measure $d\xi = \mathrm{d}\theta\mathrm{d}t$ is a relatively invariant measure on Ξ.

Consider now the sections $s\colon \mathbb{R}^2 \to SIM(2)$ and $\sigma\colon [0, \pi) \times \mathbb{R} \to SIM(2)$ defined by

$$s(x) = (x, 0, 1), \qquad \sigma(\theta, t) = (t\, w(\theta), \theta, 1).$$

It is easy to verify by direct computation that

$$\hat{\xi}_0 = H[x_0] = \{(0, b_2) : b_2 \in \mathbb{R}\} \simeq \mathbb{R},$$
$$\check{x}_0 = K.\xi_0 = \{(\theta, 0) : \theta \in [0, \pi)\} \simeq [0, \pi).$$

It is immediate to see that the Lebesgue measure $\mathrm{d}b_2$ on $\hat{\xi}_0$ is a relatively H-invariant measure with character $\gamma((0, b_2), \phi, a) = a$. Further, we have that

$$\widehat{(\theta, t)} = \sigma(\theta, t)[\hat{\xi}_0] = \{x \in \mathbb{R}^2 : x \cdot w(\theta) = t\},$$

which is the set of all points lying on the line of equation $x \cdot w(\theta) = t$ and

$$\check{x} = s(x).\check{x}_0 = \{(\theta, t) \in [0, \pi) \times \mathbb{R} : t - w(\theta) \cdot x = 0\},$$

which parametrizes the set of all lines passing through the point x.

Incidentally, it is worth observing that $X = \mathbb{R}^2$ and $\Xi = [0, \pi) \times \mathbb{R}$ are homogeneous spaces in duality. Indeed, the map $x \mapsto \check{x}$ which sends a point to the set of all lines passing through that point and the map $(\theta, t) \mapsto \widehat{(\theta, t)}$ which sends a line to the set of points laying on that line are both injective.

2.3. The representations. The group G acts unitarily on $L^2(X, \mathrm{d}x)$ via the quasi-regular representation defined by

$$\pi(g)f(x) = \alpha(g)^{-1/2} f(g^{-1}[x]),$$

with α given in (2.1a). By assumption (A4), π is irreducible and square-integrable. We stress that this latter condition is needed only in section 4.

The group G acts also on $L^2(\Xi, \mathrm{d}\xi)$ via the quasi-regular representation

$$\hat{\pi}(g)F(\xi) = \beta(g)^{-1/2} F(g^{-1}.\xi),$$

where β is defined in (2.1b). The representation $\hat{\pi}$ is irreducible by assumption (A4).

Example 2.3 (Example 2.2, continued). The group $SIM(2)$ acts on $L^2(\mathbb{R}^2)$ by means of the unitary irreducible representation π defined by

$$\pi(b, \phi, a)f(x) = a^{-1} f\left(A_a^{-1} R_\phi^{-1}(x - b)\right) \tag{2.8}$$

or, equivalently, in the frequency domain

$$\mathcal{F}[\pi(b, \phi, a)f](\omega) = a e^{-2\pi i b \cdot \omega} \mathcal{F}f\left(A_a R_\phi^{-1} \omega\right). \tag{2.9}$$

Furthermore, G acts on $L^2([0, \pi) \times \mathbb{R})$ by means of the quasi-regular representation $\hat{\pi}$ defined by

$$\hat{\pi}(b, \phi, a)F(\theta, t) = a^{-\frac{1}{2}} F\left(\theta - \phi \bmod \pi, \frac{t - w(\theta) \cdot b}{a}\right), \tag{2.10}$$

which is irreducible, too.

2.4. The Radon transform. As remarked earlier, the setup in which we are interested differs from that considered by Helgason because $\hat{\xi}_0$ is not assumed to carry an H-invariant measure but only relatively invariant. We thus use the section σ in order to push-forward the measure m_0 (see (2.2)) to the manifolds $\hat{\xi}$ given in (2.3) and we define the Radon transform of f as the map $\mathcal{R}f \colon \Xi \to \mathbb{C}$ given by

$$\mathcal{R}f(\xi) = \int_{\hat{\xi}} f(x)\mathrm{d}m_\xi(x) := \int_{\hat{\xi}_0} f(\sigma(\xi)[x])\mathrm{d}m_0(x), \tag{2.11}$$

provided that the integral converges. Note that this depends intrinsically on the choices of m_0 and σ and not only on the subset of integration $\hat{\xi}$. Let us investigate how the expression of $\mathcal{R}f(\xi)$ depends on the choice of σ (see also Example 2.4 below). Given another section σ', we have

$$\begin{aligned}
\mathcal{R}'f(\xi) &= \int_{\hat{\xi}_0} f(\sigma'(\xi)[x])\mathrm{d}m_0(x) \\
&= \int_{\hat{\xi}_0} f(\sigma(\xi)\sigma(\xi)^{-1}\sigma'(\xi)[x])\mathrm{d}m_0(x) \\
&= \gamma\left(\sigma'(\xi)^{-1}\sigma(\xi)\right)\int_{\hat{\xi}_0} f(\sigma(\xi)[x])\mathrm{d}m_0(x) \\
&= \gamma\left(\sigma'(\xi)^{-1}\sigma(\xi)\right)\mathcal{R}f(\xi),
\end{aligned}$$

since $\sigma(\xi)^{-1}\sigma'(\xi) \in H$. Thus, the dependence of the Radon transform on σ is only through a multiplicative factor depending only on ξ and not on f.

Assumption (A6) states that there exists a nontrivial π-invariant subspace $\mathcal{A}$ of $L^2(X, \mathrm{d}x)$ such that $\mathcal{R}f$ is well defined for all $f \in \mathcal{A}$ and the adjoint of the Radon transform $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ has nontrivial domain.

Example 2.4 (Example 2.2, continued). We compute by (2.11) the Radon transform between the homogeneous spaces in duality $\mathbb{R}^2$ and $[0,\pi)\times\mathbb{R}$ and we obtain

$$\mathcal{R}^{\mathrm{pol}}f(\theta,t) = \int_{\mathbb{R}} f\left(t\cos\theta - y\sin\theta, t\sin\theta + y\cos\theta\right)\mathrm{d}y, \tag{2.12}$$

which is the so-called polar Radon transform.

If we choose the section $\sigma'(\theta,t) = (tw(\theta), \theta, c(t))$, where $c\colon \mathbb{R} \to (0,+\infty)$ is a Borel function, we obtain

$$\mathcal{R}'f(\theta,t) = c(t)^{-1}\mathcal{R}^{\mathrm{pol}}f(\theta,t),$$

which shows that the Radon transform depends nontrivially on the choice of the section σ.

Next we show that assumption (A6) holds true. We recall that, since $\mathcal{R}^{\mathrm{pol}}$ is a continuous map from $\mathcal{S}(\mathbb{R}^2)$ into $\mathcal{S}([0,\pi)\times\mathbb{R})$ (see [19]), given $g \in \mathcal{S}'([0,\pi)\times\mathbb{R})$, the tempered distribution $\mathcal{R}^{\#}g\colon \mathcal{S}(\mathbb{R}^2) \to \mathbb{C}$ is defined by

$$\langle \mathcal{R}^{\#}g, f\rangle = \langle g, \mathcal{R}^{\mathrm{pol}}f\rangle.$$

If $g \in \mathcal{S}([0,\pi)\times\mathbb{R})$, by Theorem 1.4 in [28, Chapter 2], the tempered distribution $\mathcal{F}\mathcal{R}^{\#}g$ is represented by the function

$$\mathcal{F}\mathcal{R}^{\#}g(\omega) = |\omega|^{-1}(I\otimes\mathcal{F})g\left(\frac{\omega}{|\omega|}, |\omega|\right),$$

which is in $L^2(\mathbb{R}^2)$ if we choose $g \in \mathcal{S}^*([0,\pi)\times\mathbb{R})$, namely $g \in \mathcal{S}([0,\pi)\times\mathbb{R})$ such that

$$\int_{\mathbb{R}} t^m g(\theta,t)\mathrm{d}t = 0,$$

for all $m \in \mathbb{N}$. Then, if we fix $g \in \mathcal{S}^*([0,\pi)\times\mathbb{R})$,

$$|\langle g, \mathcal{R}^{\mathrm{pol}} f\rangle| = |\langle \mathcal{R}^{\#} g, f\rangle| \le C\|f\|$$

for any $f \in \mathcal{S}(\mathbb{R}^2)$. Therefore, if we take $f_0 \in \mathcal{S}(\mathbb{R}^2)$, the vector subspace $\mathcal{A} = \mathrm{span}\{\pi(g)f_0 : g \in G\} \subseteq \mathcal{S}(\mathbb{R}^2)$ and any $g \in \mathcal{S}^*([0,\pi)\times\mathbb{R})$ belongs to the domain of the adjoint of the restriction $\mathcal{R}$ of $\mathcal{R}^{\mathrm{pol}}$ to $\mathcal{A}$ and we can conclude.

3. The unitarization theorem. Our construction is based on the following lemma, which shows that the Radon transform intertwines the representations π and $\hat{\pi}$ up to a positive character of G.

LEMMA 3.1. *The Radon transform $\mathcal{R}$ restricted to $\mathcal{A}$ is a densely defined operator from $\mathcal{A}$ into $L^2(\Xi, \mathrm{d}\xi)$ satisfying*

$$\mathcal{R}\pi(g) = \chi(g)^{-1}\hat{\pi}(g)\mathcal{R} \tag{3.1}$$

for all $g \in G$, where

$$\chi(g) = \alpha(g)^{1/2}\beta(g)^{-1/2}\gamma(g\sigma(g^{-1}.\xi_0))^{-1}. \tag{3.2}$$

For the classical Radon transform considered in Example 2.2, this result is a direct consequence of the behavior of $\mathcal{R}^{\mathrm{pol}}$ under linear actions [30, Chapter 2]. With a slight abuse of notation, $\mathcal{R}$ denotes both the Radon transform defined by (2.11) and its restriction to $\mathcal{A}$.

Proof. By assumption (A6), $\mathcal{R}$ is a well-defined operator from $\mathcal{A}$ into $L^2(\Xi, \mathrm{d}\xi)$. We now prove (3.1). By the π-invariance of $\mathcal{A}$, for $f \in \mathcal{A}$ and $g \in G$ we have

$$\begin{aligned}(\mathcal{R}\pi(g)f)(\xi) &= \alpha(g)^{-1/2}\int_{\hat{\xi}_0} f(g^{-1}\sigma(\xi)[x])\mathrm{d}m_0(x)\\ &= \alpha(g)^{-1/2}\int_{\hat{\xi}_0} f(\sigma(g^{-1}.\,\xi)\sigma(g^{-1}.\,\xi)^{-1}g^{-1}\sigma(\xi)[x])\mathrm{d}m_0(x)\\ &= \alpha(g)^{-1/2}\int_{\hat{\xi}_0} f(\sigma(g^{-1}.\,\xi)m(g,\xi)^{-1}[x])\mathrm{d}m_0(x),\end{aligned}$$

where $m(g,\xi)^{-1} := \sigma(g^{-1}.\,\xi)^{-1}g^{-1}\sigma(\xi)$. It is known that for any $g \in G$ and any $\xi \in \Xi$

$$m(g,\xi) = \sigma(\xi)^{-1}g\sigma(g^{-1}.\,\xi) \in H. \tag{3.3}$$

We show this property for the reader's convenience. Indeed

$$\sigma(\xi)^{-1}g\sigma(g^{-1}.\,\xi).\xi_0 = \sigma(\xi)^{-1}g.(g^{-1}.\,\xi) = \sigma(\xi)^{-1}.\xi = \xi_0,$$

so that $m(g,\xi) \in H$. Thus, using (2.2) we obtain

$$(\mathcal{R}\pi(g)f)(\xi) = \alpha(g)^{-1/2}\gamma(m(g,\xi))\int_{\hat{\xi}_0} f(\sigma(g^{-1}.\xi)[x])\mathrm{d}m_0(x).$$

Then,

$$\begin{aligned}(\mathcal{R}\pi(g)f)(\xi) &= \alpha(g)^{-1/2}\gamma(m(g,\xi))(\mathcal{R}f)(g^{-1}.\xi)\\ &= \alpha(g)^{-1/2}\beta(g)^{1/2}\gamma(m(g,\xi))\hat{\pi}(g)\mathcal{R}f(\xi).\end{aligned}$$

Thanks to assumption (A3), there exists a character $\iota\colon G \to (0,+\infty)$ such that $\gamma(m(g,\xi)) = \iota(g)$ for every $g \in G$ and $\xi \in \Xi$. In particular, $\gamma(m(g,\xi)) = \gamma(m(g,\xi_0)) = \gamma(g\sigma(g^{-1}.\xi_0))$, and (3.1) follows.

We finally prove that $\mathcal{A}$ is dense. By assumption (A6), the domain of $\mathcal{R}$ is π invariant, so that $\pi(g)\overline{\mathcal{A}} \subseteq \overline{\mathcal{A}}$ for every $g \in G$. Since $\mathcal{A} \neq \{0\}$ and π is irreducible, then $\overline{\mathcal{A}} = L^2(X, \mathrm{d}x)$. ◻

Observe that if γ extends to a positive character of G, then

$$\gamma(m(g,\xi)) = \gamma(\sigma(\xi))^{-1}\gamma(g)\gamma(\sigma(g^{-1}.\,\xi))$$

and the independence of ξ is implied by the stronger condition

$$\gamma(\sigma(g^{-1}.\,\xi)) = \gamma(\sigma(\xi))$$

that must be satisfied for all $g \in G$ and $\xi \in \Xi$. This is equivalent to requiring that $\gamma(\sigma(\xi)) = 1$ for all $\xi \in \Xi$, which is true in all our examples.

Lemma 3.1 is at the base of our construction and its validity strongly depends on the choice of the Borel section $\sigma\colon \Xi \to G$ (see (A3)). For instance, if we change the Borel section in Example 2.2 and we choose $\sigma'\colon \Xi \to G$ defined as $\sigma'(\theta,t) = (tw(\theta), \theta, e^t)$, then hypothesis (A3) does not hold true anymore and consequently Lemma 3.1 fails. This example underlines once again that our whole construction depends on the choice of σ in (A3).

Example 3.2 (Example 2.2, continued). By (3.1) and (3.2) we have that

$$\mathcal{R}^{\mathrm{pol}}\pi(b,\phi,a) = \chi(b,\phi,a)^{-1}\hat{\pi}(b,\phi,a)\mathcal{R}^{\mathrm{pol}}, \tag{3.4}$$

where $\chi(b,\phi,a) = a^{-1/2}$ since $\alpha(b,\phi,a) = a^2$ and $\beta(b,\phi,a) = \gamma(b,\phi,a) = a$.

Our approach is based on a classical result due to Duflo and Moore [12]. According to [12], a densely defined closed operator T from a Hilbert space $\mathcal{H}$ to another Hilbert space $\hat{\mathcal{H}}$ is called semi-invariant with weight ζ if it satisfies

$$\hat{\pi}(g)T\pi(g)^{-1} = \zeta(g)T, \qquad g \in G, \tag{3.5}$$

where ζ is a character of G and π and $\hat{\pi}$ are unitary representations of G acting on $\mathcal{H}$ and $\hat{\mathcal{H}}$, respectively.

Theorem 3.3 (see [12, Theorem 1]). *With the above notation, assume that π is irreducible. Let T be a densely defined closed nonzero operator from $\mathcal{H}$ to $\hat{\mathcal{H}}$, semi-invariant with weight ζ.*

(i) *Suppose that $\pi = \hat{\pi}$. If T' is another densely defined closed operator from $\mathcal{H}$ to $\mathcal{H}$, semi-invariant with weight ζ, then T' is proportional to T.*
(ii) *Let $T = \mathcal{Q}|T|$ be the polar decomposition of T. Then $|T|$ is a positive self-adjoint operator in $\mathcal{H}$ semi-invariant with weight $|\zeta|$, and $\mathcal{Q}$ is a partial isometry of $\mathcal{H}$ into $\hat{\mathcal{H}}$, semi-invariant with weight $\zeta/|\zeta|$.*

As shown by the next corollary, the assumption that T is closed can be removed.

Corollary 3.4. *Let π and $\hat{\pi}$ be two irreducible unitary representations of G acting on $\mathcal{H}$ and $\hat{\mathcal{H}}$, respectively, and let ζ be a character of G. Suppose that $T\colon \mathrm{dom}(T) \subseteq \mathcal{H} \to \hat{\mathcal{H}}$ is such that*

(a) $\mathrm{dom}(T)$ *and* $\mathrm{dom}(T^*)$ *are nontrivial;*
(b) $\mathrm{dom}(T)$ *is π-invariant and $\hat{\pi}(g)T\pi(g)^{-1} = \zeta(g)T$ for all $g \in G$.*

Then

(i) $\mathrm{dom}(T^*)$ *is* $\hat{\pi}$ *invariant and* $\mathrm{dom}(T)$ *and* $\mathrm{dom}(T^*)$ *are dense;*

(ii) T *is closable, the closure* $\overline{T}$ *of* T *is semi-invariant with weight* ζ *and* $\overline{T}$ *is the unique closed extension of* T*;*

(iii) $T^* = \overline{T}^*$ *is a densely defined closed operator and is semi-invariant with weight* $\overline{\zeta}$.

Since $\mathrm{dom}(T)$ is shown to be dense, the adjoint T^* is uniquely defined.

Proof. We prove the first claim. Given $\hat{f} \in \mathrm{dom}(T^*)$ and $g \in G$, for all $f \in \mathrm{dom}(T)$

$$
\begin{aligned}
|\langle \hat{\pi}(g)\hat{f}, Tf\rangle| &= |\overline{\zeta(g^{-1})}\langle \hat{f}, T\pi(g^{-1})f\rangle| = |\zeta(g^{-1})||\langle T^*\hat{f}, \pi(g^{-1})f\rangle| \\
&\leq |\zeta(g^{-1})|\|T^*\hat{f}\|_{\mathcal{H}}\|f\|_{\mathcal{H}},
\end{aligned}
$$

which implies that $\hat{\pi}(g)\hat{f} \in \mathrm{dom}(T^*)$. Since $\mathrm{dom}(T)$ and $\mathrm{dom}(T^*)$ are nontrivial and are G-invariant, the irreducibility of π and $\hat{\pi}$ implies that $\mathrm{dom}(T)$ and $\mathrm{dom}(T^*)$ are dense and, hence, T is closable; see [31, Theorem VIII.1].

Let T' be a closed extension of T. We claim that T' is semi-invariant with weight ζ. Given $f \in \mathrm{dom}(T')$, then there exists a sequence (f_n) in $\mathrm{dom}(T)$ such that it converges to f and (Tf_n) convergences to $T'f$. Given $g \in G$, clearly $(\pi(g)f_n)$ converges to $\pi(g)f$ and

$$
\lim_{n\to+\infty} T\pi(g)f_n = \lim_{n\to+\infty} \zeta(g^{-1})\hat{\pi}(g)Tf_n = \zeta(g^{-1})\hat{\pi}(g)T'f.
$$

Since T' is closed, then $\pi(g)f \in \mathrm{dom}(T')$ and

$$
\zeta(g^{-1})\hat{\pi}(g)T'f = T'\pi(g)f,
$$

so that T' is semi-invariant with weight ζ. It follows, in particular, that $\overline{T}$ is a densely defined closed operator semi-invariant with weight ζ. By item (ii) of Theorem 3.3, $|T'|$ and $|\overline{T}|$ are semi-invariant operators with weight $|\zeta|$, and by item (i) of the same theorem, $|T'| = c|\overline{T}|$ by a constant $c > 0$, hence $\mathrm{dom}(T') = \mathrm{dom}(\overline{T})$, so that $T' = \overline{T}$.

Finally, $T^* = \overline{T}^*$ is a densely defined closed operator by [31, Theorem VIII.1] and the semi-invariance of T^* follows straightforwardly. ∎

As a consequence of the above result and assumption (A6), we get the following property.

LEMMA 3.5. *The Radon transform* $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ *is closable and its closure* $\overline{\mathcal{R}}$ *is a densely defined operator satisfying*

$$
\overline{\mathcal{R}}\pi(g) = \chi(g)^{-1}\hat{\pi}(g)\overline{\mathcal{R}} \tag{3.6}
$$

for all $g \in G$, *where* χ *is given by* (3.2). *Furthermore,* $\overline{\mathcal{R}}$ *is the unique closed extension of* $\mathcal{R}$.

We note that by formula (2.11) $\mathcal{R}$ naturally extends to a densely defined operator $\mathcal{R}^{\max} : \mathcal{A}^{\max} \to L^2(\Xi, \mathrm{d}\xi)$, where $\mathcal{A}^{\max}$ is the π-invariant domain

$$
\mathcal{A}^{\max} = \left\{ f \in L^2(X, \mathrm{d}x) : f(\sigma(\xi)[\cdot]) \in L^1(\hat{\xi}_0, m_0) \text{ a.e. } \xi \in \Xi, \right.
$$
$$
\left. \int_{\hat{\xi}_0} f(\sigma(\cdot)[x])\mathrm{d}m_0(x) \in L^2(\Xi, \mathrm{d}\xi) \right\}.
$$

In general we are not able to show that $\mathcal{R}^{\max}$ is closable on $\mathcal{A}^{\max}$ and we need assumption (A6) to ensure that the Radon transform is closable at least on a smaller domain $\mathcal{A} \subseteq \mathcal{A}^{\max}$. Observe that if $\mathcal{A}'$ is another π-invariant vector space such that $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}^{\max}$ and the restriction $\mathcal{R}'$ of $\mathcal{R}^{\max}$ to $\mathcal{A}'$ is closable, then by Corollary 3.4, its closure $\overline{\mathcal{R}'}$ coincides with $\overline{\mathcal{R}}$. Hence, the choice of $\mathcal{A}$ in assumption (A6) is not crucial. The most delicate issue is to prove that the Radon transform is closable and, by the irreducibility of π, it is natural to choose $\mathcal{A}$ as a "minimal" domain, for example, $\mathcal{A} = \operatorname{span}\{\pi(g)f_0 : g \in G\}$, where $f_0 \in \mathcal{A}^{\max}$ is a suitable nonzero function. However, with this minimal choice, it would be nice to have a larger domain $\mathcal{A}'$ such that $\overline{\mathcal{R}}f = \mathcal{R}^{\max}f$ for all $f \in \mathcal{A}'$. The following result provides an equivalent characterization of this property, which is useful in the examples.

COROLLARY 3.6. *Let $\mathcal{A}'$ be a subspace of $L^2(X, \mathrm{d}x)$ such that*

$$\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}^{\max}$$

and $\mathcal{R}'$ denote the restriction of $\mathcal{R}^{\max}$ to $\mathcal{A}'$. Then $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ is closable and $\mathcal{A}$ is dense in $\mathcal{A}'$ with respect to the graph norm of $\mathcal{R}'$ if and only if $\mathcal{R}'\colon \mathcal{A}' \to L^2(\Xi, \mathrm{d}\xi)$ is closable and its closure coincides with $\overline{\mathcal{R}}$. In particular, $\overline{\mathcal{R}}f = \mathcal{R}'f$ for any $f \in \mathcal{A}'$.

The result is a direct consequence of the following Proposition 3.7, whose proof is standard and we include it for completeness.

LEMMA 3.7. *Assume two operators $T_0\colon \operatorname{dom}(T_0) \subseteq \mathcal{H} \to \mathcal{K}$ and $T\colon \operatorname{dom}(T) \subseteq \mathcal{H} \to \mathcal{K}$ from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$ such that $\operatorname{dom}(T_0)$ is dense and T extends T_0. Then T_0 is closable and $\operatorname{dom}(T_0)$ is dense in $\operatorname{dom}(T)$ with respect to the graph norm of T if and only if T is closable and $\overline{T} = \overline{T}_0$.*

Proof. We suppose that T_0 is closable and that $\operatorname{dom}(T_0)$ is dense in $\operatorname{dom}(T)$ with respect to the graph norm, namely

$$\|f\|_T^2 = \|f\|_{\mathcal{H}}^2 + \|Tf\|_{\mathcal{K}}^2, \qquad f \in \operatorname{dom}(T),$$

which gives to $\operatorname{dom}(T)$ a pre-Hilbertian structure. If we take $f \in \operatorname{dom}(T)$, then by hypothesis there exists a subsequence $(f_n)_n$ in $\operatorname{dom}(T_0)$ such that $f_n \to f$ in $\mathcal{H}$ and $Tf_n \to Tf$ in $\mathcal{K}$. Then, $f \in \operatorname{dom}(\overline{T}_0)$ and $\overline{T}_0 f = \lim_{n\to+\infty} T_0 f_n = Tf$. Therefore, $T \subseteq \overline{T}_0$ and so T is closable and $\overline{T} = \overline{T}_0$. Conversely, if we suppose that T is closable, then so is T_0. Furthermore, by hypothesis $\overline{T_0} = \overline{T}$ and in particular $\operatorname{dom}(\overline{T}) = \operatorname{dom}(\overline{T}_0)$. This implies that if we consider $f \in \operatorname{dom}(T)$, there exists a subsequence $(f_n)_n$ in $\operatorname{dom}(T_0)$ such that $f_n \to f$ in $\mathcal{H}$ and $Tf_n \to Tf$ in $\mathcal{K}$ and we can conclude that $\operatorname{dom}(T_0)$ is dense in $\operatorname{dom}(T)$ with respect to the graph norm. ◻

Example 3.8 (Example 2.2, continued). We already know that, fixed $f_0 \in \mathcal{S}(\mathbb{R}^2)$, the polar Radon transform $\mathcal{R}\colon \operatorname{span}\{\pi(g)f_0 : g \in G\} \to L^2([0,\pi) \times \mathbb{R})$ is closable and we denote its closure by $\overline{\mathcal{R}}$. By Fubini's theorem the integral (2.12) is well defined for any $f \in L^1(\mathbb{R}^2)$. Furthermore, we now verify that $\mathcal{R}^{\mathrm{pol}}f \in L^2([0,\pi) \times \mathbb{R})$ for every $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. This requires us to recall one of the fundamental results in Radon theory, the so-called Fourier slice theorem [20], which relates the Radon transform with the Fourier transform. For $f \in L^1(\mathbb{R}^2)$, the integral (2.12) converges for almost all $(\theta, t) \in [0,\pi) \times \mathbb{R}$ by Fubini's theorem and

$$(I \otimes \mathcal{F})(\mathcal{R}^{\mathrm{pol}}f)(\theta, \tau) = \mathcal{F}f(\tau w(\theta)) \tag{3.7}$$

for every $(\theta, \tau) \in [0,\pi) \times \mathbb{R}$, where I is the identity operator. Hence, we have

$$\begin{aligned}
&\int_{[0,\pi)\times\mathbb{R}} |\mathcal{R}^{\mathrm{pol}} f(\theta,t)|^2 \,\mathrm{d}\theta\mathrm{d}t = \int_{[0,\pi)\times\mathbb{R}} |\mathcal{F}f(\tau w(\theta))|^2 \mathrm{d}\theta\mathrm{d}\tau \\
&= \int_{[0,\pi)\times[-1,1]} |\mathcal{F}f(\tau w(\theta))|^2 \mathrm{d}\theta\mathrm{d}\tau + \int_{[0,\pi)\times[-1,1]^c} \frac{|\mathcal{F}f(\tau w(\theta))|^2}{|\tau|} |\tau| \mathrm{d}\theta\mathrm{d}\tau \\
&\leq 2\pi \|f\|_1^2 + \int_{|\omega|>1} \frac{|\mathcal{F}f(\omega_1,\omega_2)|^2}{|\omega|} \mathrm{d}\omega_1\mathrm{d}\omega_2 \\
&\leq 2\pi \|f\|_1^2 + \|f\|_2^2 < +\infty.
\end{aligned}$$

Therefore, $\mathcal{A}' = L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ comes out as a natural domain for the polar Radon transform and, denoting the restriction of $\mathcal{R}^{\mathrm{pol}}$ to $\mathcal{A}'$ by $\mathcal{R}'$, the question whether $\overline{\mathcal{R}} f = \mathcal{R}^{\mathrm{pol}} f$ for every $f \in \mathcal{A}'$ naturally rises. By Corollary 3.6, we need to show that $\mathrm{span}\{\pi(g)f_0 : g \in G\}$ is dense in $\mathcal{A}'$ with respect to the graph norm, namely

$$\|f\|_{\mathcal{R}'}^2 = \|f\|^2 + \|\mathcal{R}' f\|^2, \qquad f \in \mathcal{A}'.$$

We denote by $\langle \cdot, \cdot \rangle_{\mathcal{R}'}$ the respective scalar product. For this it suffices to prove that if $f \in \mathcal{A}'$ and satisfies

$$\langle f, \pi(g) f_0 \rangle_{\mathcal{R}'} = 0$$

for every $g \in G$, then $f = 0$. We choose $f_0(x) = e^{-\pi|x|^2}$. By the Fourier slice theorem and (2.9), for any $g = (b, \phi, a) \in SIM(2)$ we have that

$$\begin{aligned}
0 = \langle f, \pi(g) f_0 \rangle_{\mathcal{R}} &= \langle f, \pi(g) f_0 \rangle + \langle \mathcal{R} f, \mathcal{R} \pi(g) f_0 \rangle_{L^2([0,\pi)\times\mathbb{R})} \\
&= a \int_{\mathbb{R}^2} \mathcal{F} f(\omega) \left(1 + \frac{1}{|\omega|}\right) \overline{\mathcal{F} f_0(a R_\phi^{-1} \omega)} e^{2\pi i b \cdot \omega} \mathrm{d}\omega.
\end{aligned}$$

Then, fixed $(\phi, a) = (0, 1)$, by the injectivity of the Fourier transform, there exists a negligible set E such that

$$\mathcal{F} f(\omega) \left(1 + \frac{1}{|\omega|}\right) \overline{\mathcal{F} f_0(\omega)} = 0 \tag{3.8}$$

for every $\omega \notin E$. Since $\mathcal{F} f_0(\omega) > 0$ for every $\omega \in \mathbb{R}^2$, this implies $\mathcal{F} f(\omega) = 0$ for any $\omega \notin E$ and we conclude that $f = 0$. Therefore, we have that $\overline{\mathcal{R}} f = \mathcal{R}^{\mathrm{pol}} f$ for any $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Otherwise, it is possible to show directly that $\mathcal{R}'$ is closable following the same arguments as in section 5.1.3 and $\overline{\mathcal{R}'} = \overline{\mathcal{R}}$ by Corollary 3.4.

We are finally in a position to state and prove our main result.

THEOREM 3.9. *There exists a unique positive self-adjoint operator*

$$\mathcal{I} \colon \mathrm{dom}(\mathcal{I}) \supseteq \mathrm{Im}\, \overline{\mathcal{R}} \to L^2(\Xi, \mathrm{d}\xi),$$

semi-invariant with weight $\zeta = \chi^{-1}$ *with the property that the composite operator* $\mathcal{I}\overline{\mathcal{R}}$ *extends to a unitary operator* $\mathcal{Q} \colon L^2(X, \mathrm{d}x) \to L^2(\Xi, \mathrm{d}\xi)$ *intertwining* π *and* $\hat{\pi}$*, namely*

$$\hat{\pi}(g) \mathcal{Q} \pi(g)^{-1} = \mathcal{Q}, \qquad g \in G. \tag{3.9}$$

Furthermore, π *and* $\hat{\pi}$ *are equivalent representations.*

The above result is a generalization of Helgason's theorem on the unitarization of the classical Radon transform, [20, Theorem 4.1], because by definition of extension it holds that

$$\mathcal{I}\mathcal{R}f = \mathcal{Q}f, \qquad f \in \mathcal{A}. \tag{3.10}$$

The isometric extension problem for the Radon transform was actually addressed and implicitly solved by Helgason in the general context of symmetric spaces; see [22, Corollary 3.11].

It is worth observing that the irreducibility of $\hat{\pi}$ enters only in the surjectivity of $\mathcal{Q}$. The first part of the statement holds true without this assumption but does need that $\mathcal{R}$ is closable. For this reason in the proof that follows we use assumption (A5) only to show that $\mathcal{Q}$ is surjective.

Proof. The unitarization of $\mathcal{R}$ is based on the polar decomposition $\overline{\mathcal{R}} = \mathcal{Q}|\overline{\mathcal{R}}|$ of $\overline{\mathcal{R}}$. By Lemma 3.5 and Theorem 3.3, item (ii), $|\overline{\mathcal{R}}|\colon \operatorname{dom}(\overline{\mathcal{R}}) \to L^2(X, \mathrm{d}x)$ is a positive self-adjoint operator semi-invariant with weight $|\chi| = \chi$, where χ is defined by (3.2), i.e.,

$$\pi(g)|\overline{\mathcal{R}}|\pi(g)^{-1} = \chi(g)|\overline{\mathcal{R}}|, \qquad g \in G, \tag{3.11}$$

and $\mathcal{Q}\colon L^2(X, \mathrm{d}x) \to L^2(\Xi, \mathrm{d}\xi)$ is a partial isometry with

$$\ker \mathcal{Q} = \ker \overline{\mathcal{R}}, \qquad \operatorname{Im} \mathcal{Q} = \overline{\operatorname{Im}(\overline{\mathcal{R}})},$$

and is semi-invariant with weight $\chi/|\chi| \equiv 1$, i.e., (3.9) is satisfied. Since π is irreducible, $\ker \mathcal{Q} = \{0\}$ and it follows that $\mathcal{Q}$ is an isometry. Furthermore, since $\hat{\pi}$ is irreducible and $\operatorname{Im}(\mathcal{Q})$ is a $\hat{\pi}$-invariant closed subspace of $L^2(\Xi, \mathrm{d}\xi)$ by (3.9), it follows that $\mathcal{Q}$ is surjective, so that $\mathcal{Q}$ is unitary and π and $\hat{\pi}$ are equivalent.

Define $W = \mathcal{Q}|\overline{\mathcal{R}}|\mathcal{Q}^*$ with $\hat{\pi}$-invariant domain

$$\operatorname{dom} W = \{f \in L^2(\Xi, \mathrm{d}\xi) : \mathcal{Q}^* f \in \operatorname{dom} \overline{\mathcal{R}}\} = \mathcal{Q}(\operatorname{dom} \overline{\mathcal{R}}) \oplus \overline{\operatorname{Im}(\overline{\mathcal{R}})}^{\perp},$$

which is a densely defined positive operator in $L^2(\Xi, \mathrm{d}\xi)$, semi-invariant with weight χ. Indeed, $\mathcal{Q}(\operatorname{dom} \overline{\mathcal{R}})$ is dense in $\mathcal{Q}(L^2(X, \mathrm{d}x)) = \overline{\operatorname{Im}(\overline{\mathcal{R}})}$ since $\overline{\mathcal{R}}$ is densely defined by Lemma 3.5. Observe that the $\hat{\pi}$-invariance of $\operatorname{dom} W$ follows from the π-invariance of $\operatorname{dom} \overline{\mathcal{R}}$. Further, by (3.9) and (3.11) and using that $\pi(g)$ is a unitary operator we readily derive

$$\begin{aligned}
\hat{\pi}(g)W\hat{\pi}(g)^{-1}f &= \hat{\pi}(g)\mathcal{Q}|\overline{\mathcal{R}}|\mathcal{Q}^*\hat{\pi}(g)^{-1}f \\
&= \left(\hat{\pi}(g)\mathcal{Q}\pi(g)^{-1}\right)\left(\pi(g)|\overline{\mathcal{R}}|\pi(g)^{-1}\right)\left(\pi(g)\mathcal{Q}^*\hat{\pi}(g)^{-1}\right)f \\
&= \mathcal{Q}\left(\chi(g)|\overline{\mathcal{R}}|\right)\mathcal{Q}^* f \\
&= \chi(g)Wf
\end{aligned}$$

for every $f \in \operatorname{dom} W$.

Since $\mathcal{Q}^*\mathcal{Q} = \mathrm{Id}$, then $\overline{\mathcal{R}} = W\mathcal{Q}$ and $\operatorname{Im} \overline{\mathcal{R}} \subseteq \operatorname{Im} W$. We denote by $\mathcal{I}$ the Moore–Penrose inverse of W [6, Chapter 9, section 3, Theorem 2] with densely defined domain given by

$$\operatorname{Im} W \oplus \operatorname{Im} W^{\perp} \supseteq \operatorname{Im} W\mathcal{Q} = \operatorname{Im} \overline{\mathcal{R}}.$$

Since W is a positive operator in $L^2(\Xi, \mathrm{d}\xi)$, then $\mathcal{I}$ is positive, too, and

$$\begin{aligned}
\mathcal{I}Wf &= f, \qquad && f \in \operatorname{dom} W \cap \ker W^{\perp}, \\
W\mathcal{I}f &= f, \qquad && f \in \operatorname{Im} W.
\end{aligned}$$

We claim that $\mathcal{I}$ is semi-invariant with weight χ^{-1} and

$$\mathcal{I}\overline{\mathcal{R}}f = \mathcal{Q}f, \qquad f \in \operatorname{dom} \overline{\mathcal{R}}.$$

Indeed, if $f \in \operatorname{Im} W$, by definition $\mathcal{I}f = h$ with $h \in \operatorname{dom} W \cap \ker W^{\perp}$ and $Wh = f$. Thus, by the semi-invariance of W we have that

$$\begin{aligned}
\hat{\pi}(g)\mathcal{I}\hat{\pi}(g)^{-1}f &= \hat{\pi}(g)\mathcal{I}\hat{\pi}(g)^{-1}Wh \\
&= \chi(g)^{-1}\hat{\pi}(g)\mathcal{I}W\hat{\pi}(g)^{-1}h \\
&= \chi(g)^{-1}\mathcal{I}f, \qquad (3.12)
\end{aligned}$$

where we used that $\hat{\pi}(g)^{-1}h \in \ker W^{\perp}$, which follows from the $\hat{\pi}$-invariance of $\ker W$. If $f \in \operatorname{Im} W^{\perp}$, by definition of $\mathcal{I}$ the semi-invariance property (3.12) is trivial.

Finally, since by (3.6) $\overline{\mathcal{R}}$ is an injective operator, we have that $\ker W = \ker \mathcal{Q}^*$ and hence $\ker W^{\perp} = \overline{\operatorname{Im}\mathcal{Q}} \supseteq \operatorname{Im}\mathcal{Q}$, whence $\mathcal{Q}f \in \operatorname{dom} W \cap \ker W^{\perp}$ for any $f \in \operatorname{dom} \overline{\mathcal{R}}$. Therefore $\mathcal{I}\overline{\mathcal{R}}f = \mathcal{I}W\mathcal{Q}f = \mathcal{Q}f$, as desired. ◻

Example 3.10 (Example 2.2, continued). Applying Lemma 3.5 and Corollary 3.6 to $\mathcal{R}$, by (3.4) its closed extension $\overline{\mathcal{R}}$ is a semi-invariant operator from $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ to $L^2([0,\pi)\times\mathbb{R})$ with weight $\chi(b,\phi,a) = a^{-1/2}$. By Theorem 3.9 there exists a positive self-adjoint operator $\mathcal{I}\colon \operatorname{dom}(\mathcal{I}) \supseteq \operatorname{Im}(\overline{\mathcal{R}}) \to L^2([0,\pi)\times\mathbb{R})$, semi-invariant with weight $\chi(g)^{-1} = a^{1/2}$, such that $\mathcal{I}\overline{\mathcal{R}}$ extends to a unitary operator $\mathcal{Q}\colon L^2(\mathbb{R}^2) \to L^2([0,\pi)\times\mathbb{R})$ intertwining the quasi-regular (irreducible) representations π and $\hat{\pi}$. Hence

$$\begin{aligned}
&\mathcal{I}\mathcal{R}^{\mathrm{pol}}f = \mathcal{Q}f, && f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \\
&\mathcal{Q}^*\mathcal{Q}f = f, && f \in L^2(\mathbb{R}^2), \\
&\mathcal{Q}\mathcal{Q}^*F = F, && F \in L^2([0,\pi) \times \mathbb{R}), \\
&\hat{\pi}(g)\,\mathcal{Q}\,\pi(g)^{-1} = \mathcal{Q}, && g \in SIM(2).
\end{aligned} \qquad (3.13)$$

We can provide an explicit formula for $\mathcal{I}$. Consider the subspace

$$\mathcal{D} = \left\{ f \in L^2([0,\pi) \times \mathbb{R}) : \int_{[0,\pi)\times\mathbb{R}} |\tau| |(I \otimes \mathcal{F})f(\theta,\tau)|^2 \, \mathrm{d}\theta\mathrm{d}\tau < +\infty \right\}$$

and define the operator $\mathcal{J}\colon \mathcal{D} \to L^2([0,\pi) \times \mathbb{R})$ by

$$(I \otimes \mathcal{F})\mathcal{J}f(\theta,\tau) = |\tau|^{\frac{1}{2}}(I \otimes \mathcal{F})f(\theta,\tau),$$

a Fourier multiplier with respect to the last variable. A direct calculation shows that $\mathcal{J}$ is a densely defined positive self-adjoint injective operator and is semi-invariant with weight $\zeta(g) = \chi(g)^{-1} = a^{1/2}$. By Theorem 3.3, item (i), there exists $c > 0$ such that $\mathcal{I} = c\mathcal{J}$ and we now show that $c = 1$. Consider a nonzero function $f \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$. Then, by the Plancherel theorem and the Fourier slice theorem (3.7) we have that

$$\begin{aligned}\|f\|^2 = \|\mathcal{I}\mathcal{R}^{\mathrm{pol}}f\|^2_{L^2([0,\pi)\times\mathbb{R})} &= c^2\|(I\otimes\mathcal{F})\mathcal{J}\mathcal{R}^{\mathrm{pol}}f\|^2_{L^2([0,\pi)\times\mathbb{R})}\\ &= c^2\int_{[0,\pi)\times\mathbb{R}}|\tau||(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{pol}}f(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau\\ &= c^2\int_{[0,\pi)\times\mathbb{R}}|\tau||\mathcal{F}f(\tau w(\theta))|^2\mathrm{d}\theta\mathrm{d}\tau\\ &= c^2\|f\|^2.\end{aligned}$$

Thus, we obtain $c=1$.

4. Inversion of the Radon transform. In this section, we make explicit use of the assumption that π is square-integrable to invert the Radon transform. We recall that, under this assumption, there exists a self-adjoint operator

$$C\colon\ \mathrm{dom}\, C\subseteq L^2(X,\mathrm{d}x)\to L^2(X,\mathrm{d}x),$$

semi-invariant with weight $\Delta^{\frac{1}{2}}$, where Δ is the modular function of G, such that for all $\psi\in\mathrm{dom}\, C$ with $\|C\psi\|=1$, the voice transform $\mathcal{V}_\psi$

$$(\mathcal{V}_\psi f)(g)=\langle f,\pi(g)\psi\rangle,\qquad g\in G,$$

is an isometry from $L^2(X,\mathrm{d}x)$ into $L^2(G)$ and we have the weakly convergent reproducing formula

$$f=\int_G(\mathcal{V}_\psi f)(g)\pi(g)\psi\ \mathrm{d}\mu(g), \tag{4.1}$$

where μ is the Haar measure (see, for example, [15, Theorem 2.25]). The vector ψ is called admissible vector.

As shown in the previous section, there exists a positive self-adjoint operator $\mathcal{I}$ semi-invariant with weight χ^{-1} such that $\mathcal{I}\mathcal{R}$ extends to a unitary operator $\mathcal{Q}$, which intertwines the quasi-regular representations π and $\hat\pi$ of G on $L^2(X,\mathrm{d}x)$ and $L^2(\Xi,\mathrm{d}\xi)$, respectively.

Since $\mathcal{Q}$ is unitary and satisfies (3.9), the voice transform reads

$$\mathcal{V}_\psi f(g)=\langle f,\pi(g)\psi\rangle=\langle\mathcal{Q}f,\mathcal{Q}\pi(g)\psi\rangle=\langle\mathcal{Q}f,\hat\pi(g)\mathcal{Q}\psi\rangle,\qquad g\in G, \tag{4.2}$$

for all $f\in L^2(X,\mathrm{d}x)$. Furthermore, the assumption that π is square-integrable ensures that any $f\in L^2(X,\mathrm{d}x)$ can be reconstructed from its unitary Radon transform $\mathcal{Q}f$ by means of the reconstruction formula (4.1), which becomes

$$f=\int_G\langle\mathcal{Q}f,\hat\pi(g)\mathcal{Q}\psi\rangle\,\pi(g)\psi\ \mathrm{d}\mu(g).$$

Moreover, if we can choose ψ in such a way that $\mathcal{Q}\psi$ is in the domain of the operator $\mathcal{I}$, by (4.2), for all $f\in\mathrm{dom}\,\overline{\mathcal{R}}$, we have

$$\begin{aligned}\mathcal{V}_\psi f(g)&=\langle\mathcal{Q}f,\hat\pi(g)\mathcal{Q}\psi\rangle\\ &=\langle\mathcal{I}\overline{\mathcal{R}}f,\hat\pi(g)\mathcal{Q}\psi\rangle\\ &=\langle\overline{\mathcal{R}}f,\mathcal{I}\hat\pi(g)\mathcal{Q}\psi\rangle\\ &=\chi(g)\langle\overline{\mathcal{R}}f,\hat\pi(g)\mathcal{I}\mathcal{Q}\psi\rangle,\end{aligned} \tag{4.3}$$

where we use that $\mathcal{I}$ is a self-adjoint operator, semi-invariant with weight χ^{-1}.

By (4.3) the voice transform $\mathcal{V}_\psi f$ depends on f only through its Radon transform $\overline{\mathcal{R}}f$. Therefore, (4.3) together with (4.1) allow us to reconstruct an unknown signal $f \in \operatorname{dom}\overline{\mathcal{R}}$ from its Radon transform. Explicitly, we have derived the following inversion formula for the Radon transform.

THEOREM 4.1. *Let $\psi \in L^2(X, \mathrm{d}x)$ be an admissible vector for the representation π such that $\mathcal{Q}\psi \in \operatorname{dom}\mathcal{I}$, and set $\Psi = \mathcal{I}\mathcal{Q}\psi$. Then, for any $f \in \operatorname{dom}\overline{\mathcal{R}}$,*

$$f = \int_G \chi(g)\langle \overline{\mathcal{R}}f, \hat{\pi}(g)\Psi\rangle\, \pi(g)\psi\, \mathrm{d}\mu(g), \tag{4.4}$$

where the integral is weakly convergent, and

$$\|f\|^2 = \int_G \chi(g)^2 |\langle \overline{\mathcal{R}}f, \hat{\pi}(g)\Psi\rangle|^2 \mathrm{d}\mu(g). \tag{4.5}$$

If, in addition, $\psi \in \operatorname{dom}\overline{\mathcal{R}}$, then $\Psi = \mathcal{I}^2\overline{\mathcal{R}}\psi$.

Note that the datum $\overline{\mathcal{R}}f$ is analyzed by the family $\{\hat{\pi}(g)\Psi\}_{g\in G}$ and the signal f is reconstructed by a different family, namely $\{\pi(g)\psi\}_{g\in G}$.

Example 4.2 (Example 2.2, continued). It is known that π is square-integrable and the corresponding voice transform gives rise to two-dimensional directional wavelets [4]. An admissible vector is a function $\psi \in L^2(\mathbb{R}^2)$ satisfying the admissibility condition [4]

$$\int_{[0,2\pi)\times\mathbb{R}^+} |\mathcal{F}\psi(A_a R_\phi^{-1}\omega)|^2 \mathrm{d}\phi \frac{\mathrm{d}a}{a} = 1 \qquad \text{for all } \omega \in \mathbb{R}^2/\{0\}, \tag{4.6}$$

which is equivalent to

$$\int_{\mathbb{R}^2} \frac{|\mathcal{F}\psi(\omega_1, \omega_2)|^2}{\omega_1^2 + \omega_2^2} \mathrm{d}\omega_1 \mathrm{d}\omega_2 = 1. \tag{4.7}$$

Given $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, define $\mathcal{G}(b, \phi, a) = a^{\frac{1}{2}}\langle \mathcal{R}^{\mathrm{pol}} f, \hat{\pi}(b, \phi, a)\Psi\rangle$, i.e., by (2.10)

$$\mathcal{G}(b, \phi, a) = \int_{[0,\pi)\times\mathbb{R}} \mathcal{R}^{\mathrm{pol}} f(\theta, t)\, \overline{\Psi\left(\theta - \phi \bmod \pi, \frac{t - b\cdot w(\theta)}{a}\right)} \mathrm{d}\theta \mathrm{d}t.$$

Then, taking into account that $\chi(b, \phi, a) = a^{-\frac{1}{2}}$, (4.4) reads

$$f(x) = \int_{\mathbb{R}^2 \rtimes ([0,2\pi)\times\mathbb{R}^+)} \mathcal{G}(b, \phi, a)\, \psi\left(R_\phi^{-1}\frac{x - b}{a}\right) \mathrm{d}b \mathrm{d}\phi \frac{\mathrm{d}a}{a^5}. \tag{4.8}$$

By (4.5), reconstruction formula (4.8) is equivalent to

$$\|f\|^2 = \int_{\mathbb{R}^2 \rtimes ([0,2\pi)\times\mathbb{R}^+)} |\mathcal{G}(b, \phi, a)|^2 \mathrm{d}b \mathrm{d}\phi \frac{\mathrm{d}a}{a^5}. \tag{4.9}$$

The idea to exploit the theory of the continuous wavelet transform to derive inversion formulae for the Radon transform is not new; we refer to [24, 7, 27, 34, 29, 32, 35], to name a few.

It is possible to obtain a version of (4.9) in which the scale parameter a varies only in a compact set. Consider a smooth function $\Phi \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ such that

$$|\mathcal{F}\Phi(\omega)|^2 + \int_{[0,2\pi)\times(0,1)} |\mathcal{F}\psi(A_a R_\phi^{-1}\omega)|^2 \mathrm{d}\phi \frac{\mathrm{d}a}{a} = 1. \tag{4.10}$$

By Plancherel's theorem, we have that

$$\begin{aligned}\int_{\mathbb{R}^2} |\langle f, T_b\Phi\rangle|^2 \ \mathrm{d}b &= \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \mathcal{F}f(\omega)\overline{\mathcal{F}\Phi(\omega)} e^{2\pi i b\cdot\omega} \ \mathrm{d}\omega \right|^2 \ \mathrm{d}b \\ &= \int_{\mathbb{R}^2} |\mathcal{F}^{-1}(\mathcal{F}f \ \overline{\mathcal{F}\Phi})(b)|^2 \ \mathrm{d}b \\ &= \int_{\mathbb{R}^2} |\mathcal{F}f(\omega)|^2 |\mathcal{F}\Phi(\omega)|^2 \ \mathrm{d}\omega. \end{aligned} \tag{4.11}$$

Using an analogous computation, by Plancherel's theorem, (2.9), and Fubini's theorem we have

(4.12)

$$\begin{aligned}&\int_{\mathbb{R}^2\rtimes([0,2\pi)\times(0,1))} |\mathcal{G}(b,\phi,a)|^2 \mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a^5} = \int_{\mathbb{R}^2\rtimes([0,2\pi)\times(0,1))} |\langle f, \pi(b,\phi,a)\psi\rangle|^2 \mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a^3} \\ &= \int_{\mathbb{R}^2\rtimes([0,2\pi)\times(0,1))} \left| \int_{\mathbb{R}^2} \mathcal{F}f(\omega)\overline{\mathcal{F}\psi(A_a R_\phi^{-1}\omega)} e^{2\pi i b\cdot\omega} \ \mathrm{d}\omega \right|^2 \mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a} \\ &= \int_{[0,2\pi)\times(0,1)} \left(\int_{\mathbb{R}^2} |\mathcal{F}^{-1}(\mathcal{F}f\overline{\mathcal{F}\psi(A_a R_\phi^{-1}\cdot)})(b)|^2 \mathrm{d}b \right) \mathrm{d}\phi\frac{\mathrm{d}a}{a} \\ &= \int_{\mathbb{R}^2} |\mathcal{F}f(\omega)|^2 \left(\int_{[0,2\pi)\times(0,1)} |\mathcal{F}\psi(A_a R_\phi^{-1}\omega)|^2 \mathrm{d}\phi\frac{\mathrm{d}a}{a} \right) \mathrm{d}\omega. \end{aligned}$$

Thus, combining (4.10), (4.11), and (4.12) we obtain the reconstruction formula

$$\|f\|^2 = \int_{\mathbb{R}^2} |\langle f, T_b\Phi\rangle|^2 \mathrm{d}b + \int_{\mathbb{R}^2\rtimes([0,2\pi)\times(0,1))} |\mathcal{G}(b,\phi,a)|^2 \mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a^5}. \tag{4.13}$$

It is worth observing that there always exists a function Φ satisfying (4.10) provided that the admissible vector ψ has fast Fourier decay. Indeed, if we require $\mathcal{F}\psi$ to satisfy a decay estimate of the form

$$|\mathcal{F}\psi(\omega)| = O\left(|\omega|^{-L}\right) \qquad \text{for every } L > 0,$$

then by (4.6) we have that

$$\begin{aligned} z(\omega) &:= 1 - \int_{[0,2\pi)\times(0,1)} |\mathcal{F}\psi(A_a R_\phi^{-1}\omega)|^2 \mathrm{d}\phi\frac{\mathrm{d}a}{a} \\ &= \int_{[0,2\pi)\times[1,+\infty)} |\mathcal{F}\psi(A_a R_\phi^{-1}\omega)|^2 \mathrm{d}\phi\frac{\mathrm{d}a}{a} \\ &\lesssim \int_{[0,2\pi)\times[1,+\infty)} a^{-2L}|\omega|^{-2L}\frac{\mathrm{d}a}{a}\mathrm{d}\phi \\ &\lesssim |\omega|^{-2L}. \end{aligned}$$

Therefore, there exists a smooth function Φ such that $\mathcal{F}\Phi(\omega) = \sqrt{z(\omega)}$, so that (4.10) holds true.

Finally, let us show that the first term in the right-hand side of (4.13) may be expressed in terms of $\mathcal{R}^{\mathrm{pol}}f$ only. We readily derive

$$\begin{aligned}\langle f, T_b\Phi\rangle = \langle f, \pi(b,0,1)\Phi\rangle &= \langle \mathcal{Q}f, \mathcal{Q}\pi(b,0,1)\Phi\rangle \\ &= \langle \mathcal{Q}f, \hat{\pi}(b,0,1)\mathcal{Q}\Phi\rangle \\ &= \langle \mathcal{I}\mathcal{R}^{\mathrm{pol}}f, \hat{\pi}(b,0,1)\mathcal{I}\mathcal{R}^{\mathrm{pol}}\Phi\rangle \\ &= \langle \mathcal{R}^{\mathrm{pol}}f, \hat{\pi}(b,0,1)\mathcal{I}^2\mathcal{R}^{\mathrm{pol}}\Phi\rangle, \end{aligned} \tag{4.14}$$

where we observe that $\mathcal{I}\mathcal{R}^{\mathrm{pol}}\Phi$ is always in the domain of the operator $\mathcal{I}$ since

$$\begin{aligned}\int_{[0,\pi)\times\mathbb{R}} |\tau||(I\otimes\mathcal{F})\mathcal{I}\mathcal{R}^{\mathrm{pol}}\Phi(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau &= \int_{[0,\pi)\times\mathbb{R}} |\tau|^2|(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{pol}}\Phi(\theta,\tau)|^2\mathrm{d}\theta\mathrm{d}\tau \\ &= \int_{[0,\pi)\times\mathbb{R}} |\tau|^2|\mathcal{F}\Phi(\tau w(\theta))|^2\mathrm{d}\theta\mathrm{d}\tau \\ &= \int_{\mathbb{R}^2} |\omega||\mathcal{F}\Phi(\omega)|^2\mathrm{d}\omega < +\infty,\end{aligned}$$

since by definition Φ is a smooth function. Therefore, reconstruction formula (4.13) reads

$$\|f\|^2 = \int_{\mathbb{R}^2} |\langle \mathcal{R}^{\mathrm{pol}}f, \hat{\pi}(b,0,1)\mathcal{I}^2\mathcal{R}^{\mathrm{pol}}\Phi\rangle|^2\mathrm{d}b + \int_{\mathbb{R}^2\rtimes([0,2\pi)\times(0,1))} |\mathcal{G}(b,\phi,a)|^2\mathrm{d}b\mathrm{d}\phi\frac{\mathrm{d}a}{a^5},$$

where all the coefficients depend on f only through its polar Radon transform.

It is worth observing that the domain of $\overline{\mathcal{R}}$ is related to the domain of C, which defines the admissible vectors of π. By Theorem 3.3 (ii), the operator $|\overline{\mathcal{R}}|$ is a positive self-adjoint operator semi-invariant with weight $\chi(b,\phi,a) = a^{-1/2}$, which is a power of the modular function $\Delta(b,\phi,a) = a^{-2}$, i.e., $\chi(b,\phi,a) = \Delta(b,\phi,a)^{1/4}$. On the other hand, C is a positive self-adjoint operator semi-invariant with weight $\Delta^{1/2}$ and is such that $\psi \in L^2(\mathbb{R}^2)$ is an admissible vector of the square-integrable representation π if and only if $\psi \in \operatorname{dom} C$ and $\|C\psi\| = 1$. Therefore, $|\overline{\mathcal{R}}|$ and C are both positive self-adjoint operators on $L^2(\mathbb{R}^2)$ semi-invariant with a power of the modular function of $SIM(2)$ as weight. Finally, consider the subspace

$$\mathcal{D}_s = \left\{ f \in L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\omega|^{2s}|\mathcal{F}f(\omega)|^2\mathrm{d}\omega < +\infty \right\}$$

of $L^2(\mathbb{R}^2)$. It is not difficult to verify that the Fourier multiplier $A_s\colon \mathcal{D}_s \to L^2(\mathbb{R}^2)$ defined by

$$\mathcal{F}A_sf(\omega) = |\omega|^s\mathcal{F}f(\omega) \tag{4.15}$$

is a densely defined positive self-adjoint operator and is semi-invariant with weight $\chi_s(b,\phi,a) = \Delta(b,\phi,a)^{-s/2} = a^s$. Thus, by Theorem 3.3 (i), the operators $|\overline{\mathcal{R}}|$ and C are given, up to a constant, by (4.15) with $s = -1/2$ and $s = -1$, respectively. The above argument explains why the domain of $\overline{\mathcal{R}}$ and the domain of C, and thus the admissibility condition (4.6) of π, are strictly related. A similar result can be proved for the examples illustrated in section 5.

5. Examples. In this section, we illustrate two additional examples.

5.1. The affine Radon transform and the shearlet transform.

5.1.1. Groups and spaces. The (parabolic) shearlet group $\mathbb{S}$ is the semidirect product of $\mathbb{R}^2$ with the closed subgroup $K = \{N_s A_a \in \mathrm{GL}(2,\mathbb{R}) : s \in \mathbb{R}, a \in \mathbb{R}^\times\}$, where

$$N_s = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}, \qquad A_a = a \begin{bmatrix} 1 & 0 \\ 0 & |a|^{-1/2} \end{bmatrix}.$$

We can identify the element $N_s A_a$ with the pair (s,a) and we write (b,s,a) for the elements in $\mathbb{S}$. With this identification the product law amounts to

$$(b,s,a)(b',s',a') = \left(b + N_s A_a b', s + |a|^{1/2} s', aa'\right)$$

and the inverse of (b,s,a) is given by

$$(b,s,a)^{-1} = \left(-A_a^{-1} N_s^{-1} b, -|a|^{-1/2} s, a^{-1}\right).$$

A left Haar measure of $\mathbb{S}$ is

$$\mathrm{d}\mu(b,s,a) = |a|^{-3} \mathrm{d}b \mathrm{d}s \mathrm{d}a$$

with $\mathrm{d}b$, $\mathrm{d}s$, and $\mathrm{d}a$ the Lebesgue measures on $\mathbb{R}^2$, $\mathbb{R}$, and $\mathbb{R}^\times$, respectively. The group $\mathbb{S}$ acts transitively on $X = \mathbb{R}^2$ by

$$(b,s,a)[x] = N_s A_a x + b.$$

By (2.5), we have $\alpha(b,s,a) = |a|^{3/2}$.

Furthermore, the shearlet group acts transitively on $\Xi = \mathbb{R} \times \mathbb{R}$ by the action

$$(b,s,a)^{-1}.(v,t) = \left(|a|^{-1/2}(v-s), \frac{t - n(v)\cdot b}{a}\right),$$

where $n(v) = {}^t(1,v)$. The isotropy at $\xi_0 = (0,0)$ is

$$H = \{((0,b_2),0,a) : b_2 \in \mathbb{R}, a \in \mathbb{R}^\times\},$$

so that $\Xi = \mathbb{S}/H$. It is immediate to verify that the Lebesgue measure $\mathrm{d}\xi = \mathrm{d}v\mathrm{d}t$ is a relatively invariant measure on Ξ with positive character $\beta(b,s,a) = |a|^{3/2}$. Now, we consider the sections $s\colon \mathbb{R}^2 \to \mathbb{S}$ and $\sigma\colon \mathbb{R}\times\mathbb{R} \to \mathbb{S}$ defined by

$$s(x) = (x,0,1), \qquad \sigma(v,t) = ((t,0),v,1).$$

Thus, we have that

$$\hat{\xi}_0 = H[x_0] = \{(0,b_2) : b_2 \in \mathbb{R}\} \simeq \mathbb{R},$$
$$\check{x}_0 = K.\xi_0 = \{(s,0) : s \in \mathbb{R}\} \simeq \mathbb{R}.$$

It is easy to check that the Lebesgue measure $\mathrm{d}b_2$ on $\hat{\xi}_0$ is a relatively H-invariant measure with $\gamma((0,b_2),0,a) = |a|^{1/2}$. Further, we can compute

$$\widehat{(v,t)} = \sigma(v,t)[\hat{\xi}_0] = \{x \in \mathbb{R}^2 : x \cdot n(v) = t\},$$

which is the set of all points lying on the line of equation $x \cdot n(v) = t$ and

$$\check{x} = s(x).\check{x}_0 = \{(v,t) \in \mathbb{R}\times\mathbb{R} : t - n(v)\cdot x = 0\},$$

which parametrizes the set of all lines passing through the point x except the horizontal one. Thus, the maps $x \mapsto \check{x}$ and $(v,t) \mapsto \widehat{(v,t)}$ are both injective. Therefore, $X = \mathbb{R}^2$ and $\Xi = \mathbb{R}\times\mathbb{R}$ are homogeneous spaces in duality.

5.1.2. The representations. The (parabolic) shearlet group $\mathbb{S}$ acts on $L^2(\mathbb{R}^2)$ via the shearlet representation, namely

$$\pi(b,s,a)f(x) = |a|^{-3/4} f(A_a^{-1}N_s^{-1}(x-b)). \tag{5.1}$$

It is well known that the shearlet representation is irreducible [9].

Furthermore, since $\beta(b,s,a) = |a|^{3/2}$, the group $\mathbb{S}$ acts on $L^2(\mathbb{R}\times\mathbb{R})$ by means of the quasi-regular representation $\hat{\pi}$ defined by

$$\hat{\pi}(b,s,a)F(v,t) = |a|^{-\frac{3}{4}} F\left(|a|^{-1/2}(v-s), \frac{t-n(v)\cdot b}{a}\right). \tag{5.2}$$

By the Mackey imprimitivity theorem [13], one can show that also $\hat{\pi}$ is irreducible.

5.1.3. The Radon transform. By (2.11), the Radon transform between the homogeneous spaces in duality $\mathbb{R}^2$ and $\mathbb{R}\times\mathbb{R}$ is defined as

$$\mathcal{R}^{\mathrm{aff}} f(v,t) = \int_{\mathbb{R}} f(t-vy, y)\mathrm{d}y, \tag{5.3}$$

which is the so-called affine Radon transform [8, 17].

Following [5], we define

$$\mathcal{A} = \left\{ f \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} \frac{|\mathcal{F}f(\omega)|^2}{|\omega_1|}\mathrm{d}\omega < +\infty \right\},$$

where $\omega = (\omega_1,\omega_2) \in \mathbb{R}^2$, which is π-invariant and is such that $\mathcal{R}^{\mathrm{aff}} f \in L^2(\mathbb{R}\times\mathbb{R})$ for all $f \in \mathcal{A}$ (we refer to [5] for more details). Furthermore, it is easy to show that the restriction $\mathcal{R}$ of $\mathcal{R}^{\mathrm{aff}}$ to $\mathcal{A}$ is closable. Suppose that $(f_n)_n \subseteq \mathcal{A}$ is a sequence such that $f_n \to f$ in $L^2(\mathbb{R}^2)$ and $\mathcal{R}^{\mathrm{aff}} f_n \to g$ in $L^2(\mathbb{R}\times\mathbb{R})$. Since $I\otimes\mathcal{F}$ is unitary from $L^2(\mathbb{R}\times\mathbb{R})$ onto $L^2(\mathbb{R}\times\mathbb{R})$, we have that $(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{aff}} f_n \to (I\otimes\mathcal{F})g$ in $L^2(\mathbb{R}\times\mathbb{R})$. Since $f_n \in \mathcal{A}$, by the Fourier slice theorem adapted to the affine setting [5], for every $(v,\tau) \in \mathbb{R}\times\mathbb{R}$

$$(I\otimes\mathcal{F})\mathcal{R}^{\mathrm{aff}} f_n(v,\tau) = \mathcal{F}f_n(\tau n(v)).$$

Hence, passing to a subsequence if necessary,

$$\mathcal{F}f_n(\tau n(v)) \to (I\otimes\mathcal{F})g(v,\tau)$$

for almost every $(v,\tau)\in\mathbb{R}\times\mathbb{R}$. Therefore, for almost every $(v,\tau)\in\mathbb{R}\times\mathbb{R}$

$$(I\otimes\mathcal{F})g(v,\tau) = \lim_{n\to+\infty} \mathcal{F}f_n(\tau n(v)) = \mathcal{F}f(\tau n(v)),$$

where the last equality holds true using a subsequence if necessary. Therefore, if $(h_n)_n \in \mathcal{A}$ is another sequence such that $h_n \to f$ in $L^2(\mathbb{R}^2)$ and $\mathcal{R}^{\mathrm{aff}} h_n \to h$ in $L^2(\mathbb{R}\times\mathbb{R})$, then, for almost every $(v,\tau)\in\mathbb{R}\times\mathbb{R}$

$$(I\otimes\mathcal{F})h(v,\tau) = \mathcal{F}f(\tau n(v)).$$

Therefore,

$$(I\otimes\mathcal{F})g(v,\tau) = (I\otimes\mathcal{F})h(v,\tau)$$

for almost every $(v,\tau)\in\mathbb{R}\times\mathbb{R}$. Then $\lim_{n\to+\infty}\mathcal{R}^{\mathrm{aff}} f_n = \lim_{n\to+\infty}\mathcal{R}^{\mathrm{aff}} h_n$, the operator $\mathcal{R}$ is closable and we denote its closure by $\overline{\mathcal{R}}$. However, it is also possible to prove the closability of the operator $\mathcal{R}$ reasoning as in section 3.8 by choosing a "minimal" domain of the form $\mathrm{span}\{\pi(g)f_0 : g\in G\}$.

5.1.4. The unitarization theorem. Since $\alpha(b,s,a)=|a|^{3/2}$, $\beta(b,s,a)=|a|^{3/2}$, and $\gamma(b,s,a)=|a|^{1/2}$ the affine Radon transform satisfies the intertwining property

$$\mathcal{R}^{\mathrm{aff}}\pi(b,s,a)=\chi(b,s,a)^{-1}\hat{\pi}(b,s,a)\mathcal{R}^{\mathrm{aff}},$$

where $\chi(b,\phi,a)^{-1}=|a|^{1/2}$.

By Lemma 3.5, the closure $\overline{\mathcal{R}}$ of the affine Radon transform is a semi-invariant operator with weight $\chi(b,s,a)=|a|^{-1/2}$. Therefore, by Theorem 3.9, there exists a positive self-adjoint operator $\mathcal{I}\colon \mathrm{dom}(\mathcal{I})\subseteq L^2(\mathbb{R}\times\mathbb{R})\to L^2(\mathbb{R}\times\mathbb{R})$ semi-invariant with weight $\zeta(g)=\chi(g)^{-1}=|a|^{1/2}$ such that $\mathcal{I}\mathcal{R}^{\mathrm{aff}}$ extends to a unitary operator $\mathcal{Q}$ from $L^2(\mathbb{R}^2)$ onto $L^2(\mathbb{R}\times\mathbb{R})$, which intertwines the quasi-regular (irreducible) representations π and $\hat{\pi}$. Reasoning as in Example 2.2, it is possible to show that the operator $\mathcal{I}$ is defined by

$$(I\otimes\mathcal{F})\mathcal{I}f(v,\tau)=|\tau|^{\frac{1}{2}}(I\otimes\mathcal{F})f(v,\tau),\qquad f\in\mathcal{D},$$

where

$$\mathcal{D}=\left\{f\in L^2(\mathbb{R}\times\mathbb{R}):\int_{\mathbb{R}\times\mathbb{R}}|\tau||(I\otimes\mathcal{F})f(v,\tau)|^2\,\mathrm{d}v\mathrm{d}\tau<+\infty\right\}.$$

We refer to [5] for technical details.

5.1.5. The inversion formula. It is known that the shearlet representation π is square-integrable and its admissible vectors are the functions ψ in $L^2(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2}\frac{|\mathcal{F}\psi(\omega)|^2}{|\omega_1|^2}\mathrm{d}\omega=1, \tag{5.4}$$

where $\omega=(\omega_1,\omega_2)\in\mathbb{R}^2$ [9]. The shearlet transform is $\mathcal{S}_\psi f(b,s,a)=\langle f,\pi(b,s,a)\psi\rangle$ and is a multiple of an isometry from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{S})$ provided that ψ satisfies the admissible condition (5.4). By Theorem 4.1, for any $f\in\mathcal{A}$ we have the reconstruction formula

$$f=\int_{\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^\times}\mathcal{S}_\psi f(b,s,a)\,\pi(b,s,a)\psi\;\frac{\mathrm{d}b\mathrm{d}s\mathrm{d}a}{|a|^3}, \tag{5.5}$$

where the coefficients $\mathcal{S}_\psi f(b,s,a)$ are given by

$$\mathcal{S}_\psi f(b_1,b_2,s,a)=|a|^{-5/4}\int_{\mathbb{R}\times\mathbb{R}}\mathcal{R}^{\mathrm{aff}}f(v,t)\overline{\Psi\left(\frac{v-s}{|a|^{1/2}},\frac{t-n(v)\cdot b}{a}\right)}\mathrm{d}v\mathrm{d}t.$$

If we choose Ψ such that $\Psi(v,t)=\Psi_2(v)\Psi_1(t)$, then

$$\mathcal{S}_\psi f(b_1,b_2,s,a)=|a|^{-3/4}\int_{\mathbb{R}}\mathcal{W}_{\Psi_1}(\mathcal{R}^{\mathrm{aff}}f(v,\cdot))(n(v)\cdot b,a)\overline{\Psi_2\left(\frac{v-s}{|a|^{1/2}}\right)}\mathrm{d}v, \tag{5.6}$$

provided that Ψ_1 is a one-dimensional wavelet.

This argument gives an alternative proof of Theorems 8 and 10 in [5], where it is also proved that formula (5.6) can actually be extended to the whole $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$. Formula (5.5) is a continuous version of the reconstruction formula presented in [8, Theorem 3.3].

5.2. The spherical means Radon transform.

5.2.1. Groups and spaces. Take the same group $G = SIM(2)$ as in Example 2.2, namely $G = \mathbb{R}^2 \rtimes K$, with $K = \{R_\phi A_a \in \mathrm{GL}(2,\mathbb{R}) : \phi \in [0, 2\pi),\ a \in \mathbb{R}^+\}$. First, we choose $X = \mathbb{R}^2$ and, for what concerns this space, we keep the notation as in Example 2.2. Then, we consider the space $\Xi = \mathbb{R}^2 \times \mathbb{R}^+$, which we think of as parametrizing centers and radii of circles in $\mathbb{R}^2$, with the action

$$(b, \phi, a).(c, r) = (b + aR_\phi c, ar). \tag{5.7}$$

An immediate calculation shows that the isotropy at $\xi_0 = ((1, 0), 1)$ is

$$H = \{\big((1 - \cos\phi, -\sin\phi), \phi, 1\big) : \phi \in [0, 2\pi)\}.$$

By direct computation, recalling that $x_0 = 0$,

$$\hat{\xi}_0 = H[x_0] = \{(1 - \cos\phi, -\sin\phi) : \phi \in [0, 2\pi)\}$$

is the circle with center $(1, 0)$ and radius 1 and

$$\check{x}_0 = K.\xi_0 = \{((a\cos\phi, a\sin\phi), a) : \phi \in [0, 2\pi),\ a \in \mathbb{R}^+\}$$

is the family of circles passing through the origin. The measure $\mathrm{d}m_0 = \mathrm{d}\phi$ is H-invariant on $\hat{\xi}_0$, since the action of H on $\hat{\xi}_0$ is given by a simple rotation of a fixed angle. This gives $\gamma(h) \equiv 1$.

We define the section $\sigma\colon \Xi \to SIM(2)$ by $\sigma(c, r) = (c - (r, 0), 0, r)$. Thus, for $\xi = (c, r) \in \Xi$ we have

$$\hat{\xi} = \sigma(c, r)[\hat{\xi}_0] = \{c - rw(\phi) : \phi \in [0, 2\pi)\}, \tag{5.8}$$

namely, the circle with center c and radius r, and for $x \in \mathbb{R}^2$ we have

$$\check{x} = s(x).\check{x}_0 = \{(x + (a\cos\phi, a\sin\phi), a) : \phi \in [0, 2\pi),\ a \in \mathbb{R}^+\},$$

that is, the family of circles passing through the point x. It is easy to see that the maps $x \mapsto \check{x}$ and $\xi \mapsto \hat{\xi}$ are both injective. Thus, $X = \mathbb{R}^2$ and $\Xi = \mathbb{R}^2 \times \mathbb{R}^+$ are homogeneous spaces in duality.

We now determine a class of relatively invariant measures on Ξ. Given $\alpha \in \mathbb{R}$, we have

$$\int_{\mathbb{R}^2\times\mathbb{R}^+} f\left((b, \phi, a)^{-1}.(c, r)\right) \mathrm{d}c \frac{\mathrm{d}r}{r^\alpha} = a^{3-\alpha} \int_{\mathbb{R}^2\times\mathbb{R}^+} f(c, r) \mathrm{d}c \, \frac{\mathrm{d}r}{r^\alpha},$$

so that the measure $\mathrm{d}\xi = \mathrm{d}c \frac{\mathrm{d}r}{r^\alpha}$ is a relatively invariant measure on Ξ with character $\beta(b, \phi, a) = a^{3-\alpha}$. As shown in the next section, not all choices of α are equally good for our purposes.

5.2.2. The representations. Since the group G is the same as in Example 2.2, the representation π is given by (2.8), whereas we have to compute the quasi-regular representation $\hat{\pi}$ acting on $L^2(\Xi, \mathrm{d}\xi)$. Since $\beta(b, \phi, a) = a^{3-\alpha}$, by (2.6) and (5.7) we have

$$\begin{aligned} \hat{\pi}(b, \phi, a)F(c, r) &= a^{\frac{\alpha-3}{2}} F\left(\left(-A_a^{-1}R_\phi^{-1}b, -\phi \bmod 2\pi, a^{-1}\right).(c, r)\right) \\ &= a^{\frac{\alpha-3}{2}} F(a^{-1}R_{-\phi}(c - b), a^{-1}r), \end{aligned} \tag{5.9}$$

which is irreducible by Mackey imprimitivity theorem [13].

5.2.3. The Radon transform. By (5.8) and (2.11), the Radon trasform in this case is given by

$$\mathcal{R}^{\mathrm{cir}} f(c, r)=\int_{0}^{2\pi} f(c-rw(\phi))\mathrm{d}\phi,$$

namely, the integral of f over the circle of center c and radius r. This is the so-called spherical means Radon transform [25]. It is worth observing that more interesting problems arise when the available centers and radii are restricted to some hypersurface: this does not easily fit into our assumptions, and it is left for future investigation.

Let us now determine a suitable π-invariant subspace $\mathcal{A}$ of $L^2(\mathbb{R}^2)$ as in (A6). In order to do that, it is useful to derive a Fourier slice theorem for $\mathcal{R}^{\mathrm{cir}}$. For any $f\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, by Fubini's theorem and using [11, equation 10.9.1], we have

$$\begin{aligned}(\mathcal{F}\otimes I)\mathcal{R}^{\mathrm{cir}} f(\tau, r)&=\int_{0}^{2\pi}\int_{\mathbb{R}^2} f(c-rw(\phi))e^{-2\pi i c\cdot\tau}\mathrm{d}c\mathrm{d}\phi\\&=\int_{0}^{2\pi} e^{-2\pi i r w(\phi)\cdot\tau}\mathrm{d}\phi\int_{\mathbb{R}^2} f(c)e^{-2\pi i c\cdot\tau}\mathrm{d}c\\&=2\pi J_0(2\pi|\tau|r)\mathcal{F}f(\tau),\end{aligned}$$

where J_0 is the Bessel function of the first kind. As a consequence, by the Plancherel theorem, recalling that $\mathrm{d}\xi=\mathrm{d}c\frac{\mathrm{d}r}{r^\alpha}$ we obtain

$$\left\|\mathcal{R}^{\mathrm{cir}} f\right\|^2_{L^2(\Xi,\mathrm{d}\xi)}=\left\|(\mathcal{F}\otimes I)\mathcal{R}^{\mathrm{cir}} f\right\|^2_{L^2(\Xi,\mathrm{d}\tau\frac{\mathrm{d}r}{r^\alpha})}=c_\alpha\int_{\mathbb{R}^2}|\mathcal{F}f(\tau)|^2|\tau|^{\alpha-1}\,\mathrm{d}\tau,$$

where

$$c_\alpha=(2\pi)^{\alpha+1}\int_{\mathbb{R}^+}\frac{|J_0(r)|^2}{r^\alpha}\mathrm{d}r. \tag{5.10}$$

Observe that c_α is finite if and only if $\alpha\in(0,1)$, so that from now on we assume that $\alpha\in(0,1)$ and we set

$$\mathcal{A}_\alpha=\left\{f\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2):\int_{\mathbb{R}^2}|\mathcal{F}f(\tau)|^2|\tau|^{\alpha-1}\,\mathrm{d}\tau<+\infty\right\},$$

which is π-invariant and is such that $\mathcal{R}^{\mathrm{cir}} f\in L^2(\Xi,\mathrm{d}\xi)$ for all $f\in\mathcal{A}_\alpha$. Furthermore, as in Example 5.1, it is easy to show that $\mathcal{R}^{\mathrm{cir}}$, regarded as operator from $\mathcal{A}_\alpha$ to $L^2(\Xi,\mathrm{d}\xi)$, is closable. We stress that, if $\alpha\notin(0,1)$, the set

$$\{f\in L^2(X,\mathrm{d}x):\mathcal{R}^{\mathrm{cir}} f\in L^2(\Xi,\mathrm{d}\xi)\}=\{0\},$$

i.e., it is trivial. This motivates the role of assumption (A6) in our construction.

5.2.4. The unitarization theorem. By (3.1) and (3.2) we have that

$$\mathcal{R}^{\mathrm{cir}}\pi(b,\phi,a)=a^{\frac{1-\alpha}{2}}\hat{\pi}(b,\phi,a)\mathcal{R}^{\mathrm{cir}},$$

since $\alpha(b,\phi,a)=a^2$, $\beta(b,\phi,a)=a^{3-\alpha}$, and $\gamma(b,\phi,a)=1$, and so $\chi(b,\phi,a)=a^{\frac{\alpha-1}{2}}$.

By Theorem 3.9, there exists a positive self-adjoint operator $\mathcal{I}$, semi-invariant with weight $a^{\frac{1-\alpha}{2}}$, such that $\mathcal{I}\mathcal{R}^{\mathrm{cir}}$ extends to a unitary operator $\mathcal{Q}\colon L^2(\mathbb{R}^2)\to L^2(\Xi,\mathrm{d}\xi)$. Moreover, $\mathcal{Q}$ intertwines π and $\hat{\pi}$, namely

$$\hat{\pi}(b,\phi,a)\,\mathcal{Q}\,\pi(b,\phi,a)^{-1}=\mathcal{Q},\qquad (b,\phi,a)\in SIM(2).$$

As in the other examples, by using Theorem 3.3, part (i), it is possible to show that there exists a constant $k_\alpha \in \mathbb{R}^+$ such that $\mathcal{I} = k_\alpha \mathcal{J}$ with

$$(\mathcal{F} \otimes I)\mathcal{J} f(\tau, r) = |\tau|^{\frac{1-\alpha}{2}} (\mathcal{F} \otimes I) f(\tau, r), \qquad f \in \mathcal{D},$$

where

$$\mathcal{D} = \left\{ f \in L^2(\Xi, \mathrm{d}\xi) : \int_{\mathbb{R}^2 \times \mathbb{R}^+} |\tau|^{1-\alpha} |(\mathcal{F} \otimes I) f(\tau, r)|^2 \, \mathrm{d}\tau \frac{\mathrm{d}r}{r^\alpha} < +\infty \right\}.$$

Using the same argument as in Example 2.2, it is possible to determine the constant k_α. Take a function $f \in \mathcal{A}_\alpha \setminus \{0\}$. By Plancherel theorem and the Fourier slice theorem obtained for $\mathcal{R}^{\mathrm{cir}}$ we have that

$$\begin{aligned} \|f\|^2 = \left\| \mathcal{I} \mathcal{R}^{\mathrm{cir}} f \right\|^2_{L^2(\mathbb{R}^2 \times \mathbb{R}^+)} &= k_\alpha^2 \left\| (\mathcal{F} \otimes I) \mathcal{J} \mathcal{R}^{\mathrm{cir}} f \right\|^2_{L^2(\mathbb{R}^2 \times \mathbb{R}^+)} \\ &= k_\alpha^2 \int_{\mathbb{R}^2 \times \mathbb{R}^+} |\tau|^{1-\alpha} |(\mathcal{F} \otimes I) \mathcal{R}^{\mathrm{cir}} f(\tau, r)|^2 \mathrm{d}\tau \frac{\mathrm{d}r}{r^\alpha} \\ &= k_\alpha^2 c_\alpha \|f\|^2, \end{aligned}$$

where c_α is given by (5.10). Thus, we obtain that $k_\alpha = c_\alpha^{-1/2}$.

5.2.5. The inversion formula. Applying Theorem 4.1 to $\mathcal{R}^{\mathrm{cir}}$ we obtain the inversion formula for $f \in \mathcal{A}_\alpha$

$$f = \int_{SIM(2)} a^{\frac{\alpha-9}{2}} \langle \mathcal{R}^{\mathrm{cir}} f, \hat{\pi}(b, \phi, a) \Psi \rangle_{L^2(\Xi, \mathrm{d}\xi)} \, \psi \left(R_{-\phi} \frac{x - b}{a} \right) \, \mathrm{d}b \mathrm{d}\phi \mathrm{d}a,$$

where we used that $\chi(b, \phi, a) = a^{\frac{\alpha-1}{2}}$, the expression for the Haar measure of $SIM(2)$ given in (2.7), and the expression for the representation π given in (2.8).

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