# SHIMURA VARIETIES IN THE TORELLI LOCUS VIA GALOIS COVERINGS

PAOLA FREDIANI, ALESSANDRO GHIGI AND MATTEO PENEGINI

Abstract. Given a family of Galois coverings of the projective line, we give a simple sufficient condition ensuring that the closure of the image of the family via the period mapping is a special (or Shimura) subvariety of $A_g$. By a computer program we get the list of all families in genus $g \leq 9$ satisfying our condition. There are no families with $g = 8, 9$, all of them are in genus $g \leq 7$. These examples are related to a conjecture of Oort. Among them we get the cyclic examples constructed by various authors (Shimura, Mostow, De Jong-Noot, Rohde, Moonen and others) and the abelian non-cyclic examples found by Moonen-Oort. We get 7 new non-abelian examples.

## Contents



## 1. Introduction

1.1. Denote by $\mathsf{A}_g$ the moduli space of principally polarized abelian varieties of dimension $g$ over $\mathbb{C}$, by $\mathsf{M}_g$ the moduli space of smooth complex algebraic curves of genus $g$ and by $j\colon \mathsf{M}_g \to \mathsf{A}_g$ the period mapping or Torelli mapping. We set $\mathsf{T}_g^0 := j(\mathsf{M}_g)$ and call it the open Torelli locus. The closure of $\mathsf{T}_g^0$ in $\mathsf{A}_g$ is called the *Torelli locus* (see e.g. [32]) and is denoted by $\mathsf{T}_g$. From the complex analytic point of view, $\mathsf{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathfrak{H}_g$, where $\mathfrak{H}_g$ is the Siegel upper half-space. Therefore $\mathsf{A}_g$ has a natural structure of complex analytic orbifold and the symmetric metric on $\mathfrak{H}_g$ descends to a locally symmetric orbifold metric on $\mathsf{A}_g$. We will always consider this metric on $\mathsf{A}_g$. It is an

2000 *Mathematics Subject Classification.* 14G35, 14H15, 14H40, 32G20 (primary) and 14K22 (secondary).

The first and second authors were partially supported by PRIN 2012 MIUR "Moduli, strutture geometriche e loro applicazioni". The first author was partially supported also by FIRB 2012 "Moduli spaces and applications" and by a grant of Max-Planck Institut für Mathematik, Bonn. The second author was supported also by FIRB 2012 "Geometria differenziale e teoria geometrica delle funzioni". The third author was partially supported by PRIN 2010 MIUR "Geometria delle Varietà Algebriche".

interesting problem to study the metric properties of the inclusion $\mathsf{T}_g^0 \subset \mathsf{A}_g$. The moduli space of curves also admits a natural structure of complex orbifold and the period mapping is an orbifold map. Moreover outside the hyperelliptic locus the period mapping is an orbifold immersion [36]. This allows to study $\mathsf{T}_g^0$ (outside the hyperelliptic locus) using Riemannian geometry, i.e. via the second fundamental form. This is the direction taken in [10], [8], [7], [9]. One expects that $\mathsf{T}_g^0$ be very curved inside $\mathsf{A}_g$. For example the second fundamental form should be in some sense non-degenerate and in particular $\mathsf{T}_g^0$ should contain very few totally geodesic submanifolds of $\mathsf{A}_g$. Among the results in this direction we mention the following ones. Let $\mathsf{Z}$ be a totally geodesic subvariety of $\mathsf{A}_g$ such that $\mathsf{Z} \subset \mathsf{T}_g$ and $\mathsf{Z} \cap \mathsf{T}_g^0 \neq \emptyset$. Toledo [43] considered the case when $\mathsf{Z}$ is a compact curve and obtained an upper bound for the area and some curvature restrictions for $\mathsf{Z}$. Hain [18] and later de Jong and Zhang [12] proved under some conditions, that if $\mathsf{Z}$ is a locally symmetric variety uniformized by an irreducible symmetric domain, this must be the complex ball. (Recall that a submanifold of $\mathsf{A}_g$ is totally geodesic if and only if it is a locally symmetric submanifold.) Very recently Liu, Sun, Yang and Yau [25] got the same result by differential geometric techniques, under the assumption that $\mathsf{Z}$ is contained in $\mathsf{T}_g^0$. In [9] Colombo and the first two authors used the second fundamental form to get an upper bound for the dimension of $\mathsf{Z}$ depending only on the genus. Other related papers include [26], [17].

1.2. The stack $\mathsf{A}_g$ (or equivalently its associated complex analytic orbifold) parametrizes Hodge structures of weight 1 on a lattice of rank $2g$. On $\mathsf{A}_g$ there is a natural variation of Hodge structure over $\mathbb{Q}$ (in the orbifold sense), whose fibre over $A$ is $H^1(A, \mathbb{Q})$. The Hodge loci for this variation of Hodge structure are called *special subvarieties* or *Shimura subvarieties*, see [32, §3.3]. The special varieties are totally geodesic and an important theorem of Moonen [30] says that an algebraic totally geodesic subvariety of $\mathsf{A}_g$ is special if and only if it contains a CM point. Arithmetical consideration led Oort [34] to the following expectation: for large $g$ there should be no positive-dimensional special subvariety $\mathsf{Z}$ of $\mathsf{A}_g$, such that $\mathsf{Z} \subset \mathsf{T}_g$ and $\mathsf{Z} \cap \mathsf{T}_g^0 \neq \emptyset$. See [32, §4] for more details. On the other hand, for low genus there are examples of such $\mathsf{Z}$ (see [41, 33, 11, 39, 31] and also the survey [32, §5].) All the examples known so far are in genus $\leq 7$ and are constructed using abelian Galois covers of the line.

1.3. The purpose of this paper is, first of all, to give a simple sufficient condition for a family of Galois covers of the line to yield a Shimura variety (see Theorem 1.4 below). This criterion simplifies and extends the previous arguments. Next, we apply it to construct new examples of such families for non-abelian Galois coverings. Moreover we analyze in detail the geometry of all examples, both with abelian and with non-abelian Galois group, giving the complete list of all the distinct Shimura families in genus $g \leq 9$ obtained using this criterion.

A Galois covering of $C \to \mathbb{P}^1$ is determined by the ramification data $\mathbf{m} := (m_1, \ldots, m_r)$, the Galois group $G$, an epimorphism $\theta : \Gamma_r \to G$ and the branching points $t_1, \dots, t_r \in \mathbb{P}^1$, (see §2 for the notation). Fixing the datum $(\mathbf{m}, G, \theta)$ and letting the points $t_j$ vary, one gets a family of curves and a corresponding family of Jacobians. Denote by $\mathsf{Z}(\mathbf{m}, G, \theta)$ the closure of this set of Jacobians in $\mathsf{A}_g$. It is an $(r-3)$–dimensional subvariety of $\mathsf{A}_g$. If $C \to \mathbb{P}^1$ is one of the coverings, consider the representation $\rho$ of $G$ on $H^0(C, K_C)$ and on its symmetric power $S^2 H^0(C, K_C)$. We set

$$N := \dim(S^2 H^0(C, K_C))^G.$$

Both the isomorphism class of $\rho$ and the number $N$ depend only on the datum $(\mathbf{m}, G, \theta)$, not on the particular element $C$ of the family.

**Theorem 1.4** (see Theorem 3.9). *Let $(\mathbf{m}, G, \theta)$ be a datum as above. Assume that*

$$(*) \qquad N = r - 3.$$

*Then $\mathsf{Z}(\mathbf{m}, G, \theta)$ is a special subvariety of PEL type of $\mathsf{A}_g$, such that $\mathsf{Z}(\mathbf{m}, G, \theta) \subset \mathsf{T}_g$ and $\mathsf{Z}(\mathbf{m}, G, \theta) \cap \mathsf{T}_g^0 \neq \emptyset$.*

Observe that when $r = 3$ and $N = 0$ this yields a criterion for a Jacobian to have complex multiplication, see Corollaries 3.10 and 3.11.

1.5. The condition in Theorem 1.4 already appears in [9, Prop. 5.4]. There it is shown that under this condition $\mathsf{Z}(\mathbf{m}, G, \theta)$ is totally geodesic. The proof uses the second fundamental form of the family of Jacobians. Since special subvarieties are totally geodesic, the theorem above strenghtens the result in [9] with a different proof.

We have used the criterion in Theorem 3.9 for a systematic search of special subvarieties of the form $\mathsf{Z}(\mathbf{m}, G, \theta)$. At the beginning, especially in genus 4, we used a classification of the groups acting on algebraic curves from the point of view of the representation on holomorphic 1-forms. This classification is available in genus $g \leq 5$, thanks to the efforts of Akikazu Kuribayashi, Izumi Kuribayashi and Hideyuki Kimura [21, 23, 24, 22]. See also [27]. Breuer [4] has made a systematic computation of the possible automorphism groups for all the curves of genus $g \leq 48$. For the calculations done in this paper we used the computer algebra program `MAGMA` [28]. Our script is available at:

`users.mat.unimi.it/users/penegini/`
`publications/PossGruppigFix_v2Hwr.m`

Using this script we determine all the families $\mathsf{Z}(\mathbf{m}, G, \theta)$ with genus $g \leq 9$ and we compute the number $N$, checking which families satisfy the condition of Theorem 3.9. Our results are summarized in the following.

**Theorem 1.6.** *For genus $g \leq 9$ there are exactly 40 data $(\mathbf{m}, G, \theta)$ such that $N = r-3 > 0$. For these 40 data the image $\mathsf{Z}(\mathbf{m}, G, \theta)$ is a special subvariety of $\mathsf{A}_g$ of positive dimension, which is contained in $\mathsf{T}_g$ and intersects $\mathsf{T}_g^0$. Among these data there are 20 cyclic ones and 7 abelian non-cyclic ones. The remaining 13 have non-abelian Galois group. All these data occur in genus $g \leq 7$.*

The 20 cyclic data have been found in [41, 33, 11, 39, 31] and the 7 abelian non-cyclic data have been found in [32, §5]. The 13 non-abelian data are new. Professor Xin Lu informed us that one of the non-abelian families has already been studied very recently and from a different point of view in [26, Ex. 7.2]. See Table 2 for the list of all the 40 data.

1.7. It should be remarked that as far as we know the condition $(*)$ is only sufficient, but not necessary for $\mathsf{Z}(\mathbf{m}, G, \theta)$ to be special. So one cannot exclude that some datum $(\mathbf{m}, G, \theta)$ with $N > r-3$ gives a special $\mathsf{Z}(\mathbf{m}, G, \theta)$. For the case of cyclic coverings this has been ruled out by Moonen [31] using deep results in arithmetic geometry. Thus in the case of cyclic groups the condition that $N = r - 3$ is both sufficient and necessary.

1.8. It can happen that two different data $(\mathbf{m}, G, \theta)$ and $(\mathbf{m}', G', \theta')$ give rise to the same subvariety in $\mathsf{A}_g$, i.e. $\mathsf{Z}(\mathbf{m}, G, \theta) = \mathsf{Z}(\mathbf{m}', G', \theta')$. In fact the 40 data we found do not give rise to 40 different subvarieties. In §5 we describe systematically this phenomenon and we get the complete list of the distinct subvarieties, which is summarized in the following theorem.

**Theorem 1.9.** *The 40 data satisfying* $(*)$ *yield exactly 30 distinct Shimura subvarieties, which are listed in Table 1. The numbers refer to the data listed in Table 2.*

Table 1. Distinct Shimura subvarieties

| $g$ | dim | families |
|---|---|---|
| 1 | 1 | (1) = (21) |
| 2 | 1 | (3) = (5) = (28) = (30)<br>(4) = (29) |
| 2 | 2 | (26) |
| 2 | 3 | (2) |
| 3 | 1 | (7) = (23) = (34)<br>(9)<br>(22)<br>(33) = (35) |
| 3 | 2 | (6)<br>(8)<br>(31)<br>(32) |
| 3 | 3 | (27) |

| $g$ | dim | families |
|---|---|---|
| 4 | 1 | (11)<br>(12)<br>(13) = (24)<br>(25) = (38)<br>(36)<br>(37) |
| 4 | 2 | (14) |
| 4 | 3 | (10) |
| 5 | 1 | (15)<br>(39) |
| 6 | 2 | (16) |
| 6 | 1 | (17) |
| 6 | 1 | (18) |
| 7 | 1 | (19)<br>(20)<br>(40) |

In particular in genus 2 all families are abelian. There are three non-abelian Shimura families in genus 3, two in genus 4, one in genus 5 and one in genus 7.

One of the two non–abelian families in genus 4 (family (36)) and the non–abelian family in genus 5 (family (39)) are contained in the hyperelliptic locus, see 4.6. All the other non–abelian Shimura families are not contained in the hyperellitic locus, see 4.3, 4.5 and the proof of Theorem 5.3.

1.10. The plan of the paper is the following.

In §2 we fix the notation and we recall some preliminary results on families of Galois coverings of the projective line.

In §3 we give a brief summary of definitions and results on special subavarieties of $\mathsf{A}_g$, especially those of PEL type. Next we prove Theorem 1.4.

Section §4 is devoted to the new examples. In two sample cases we do the computation of $N$ by hand. We also make some additional remarks on the hyperellipticity of the families and on inclusions between them.

Section §5 is devoted to the proof of Theorem 1.9.

In the Appendix we explain the computations performed by the script. Table 2 contains the list of all data satisfying $(*)$.

**Acknowledgements.** We wish to thank Elisabetta Colombo for many interesting conversations, which led us to attack this problem. We also wish to thank Bert van Geemen for crucial help with Proposition 3.7 and Ben Moonen for useful emails. We thank the anonymous referees for reading very carefully the manuscript and for many questions and suggestions that helped us to improve the paper a lot. The first and second authors wish to thank the Max-Planck Institut für Mathematik, Bonn for excellent conditions provided during their visit at this institution, where part of this work was prepared.

## 2. Galois coverings of the line

2.1. For any integer $r \geq 3$ let $\Gamma_r$ denote the group with presentation $\Gamma_r = \langle \gamma_1, \ldots, \gamma_r | \gamma_1 \cdots \gamma_r = 1 \rangle$.

Definition 2.2. *A* datum *is a triple* $(\mathbf{m}, G, \theta)$*, where* $\mathbf{m} := (m_1, \ldots, m_r)$ *is an* $r$*-tuple of integers* $m_i \geq 2$*,* $G$ *is a finite group and* $\theta : \Gamma_r \to G$ *is an epimorphism such that* $\theta(\gamma_i)$ *has order* $m_i$ *for each* $i$*.*

Let $t := (t_1, \ldots, t_r)$ be an $r$-tuple of distinct points in $\mathbb{P}^1$. Set $U_t := \mathbb{P}^1 - \{t_1, \ldots, t_r\}$ and choose a base point $t_0 \in U_t$. By elementary topology there exists an isomorphism $\pi_1(U_t, t_0) \cong \Gamma_r$ such that the element $\gamma_i$ corresponds to a simple closed loop winding around the point $t_i$ counterclockwise. If $f : C \longrightarrow \mathbb{P}^1$ is a Galois cover with branch locus $t$, set $V := f^{-1}(U_t)$. Then $f|_V : V \to U_t$ is an unramified Galois covering. Let $G$ denote the group of deck transformations of $f|_V$. Then there is a surjective homomorphism $\pi_1(U_t, t_0) \longrightarrow G$, which is well-defined up to composition by an inner automorphism of $G$. Since $\Gamma_r \cong \pi_1(U_t, t_0)$ we get an epimorphism $\theta : \Gamma_r \to G$. If $m_i$ is the local monodromy around $t_i$ and $\mathbf{m} = (m_1, \ldots, m_r)$, then $(\mathbf{m}, G, \theta)$ is a datum. Thus a Galois cover of $\mathbb{P}^1$ branched over $t$ gives rise – up to some choices – to a datum. The Riemann's existence theorem ensures that the process can be reversed: a branch locus $t$ and a datum determine a covering of $\mathbb{P}^1$ up to isomorphism (see e.g. [29, Sec. III, Corollary 4.10]). We wish to show that the process can be reversed also in families, namely that to any datum is associated a family of Galois covers of $\mathbb{P}^1$.

2.3. In fact let $(\mathbf{m}, G, \theta)$ be a datum. Set $Y_r := \{t = (t_1, \ldots, t_r) \in (\mathbb{P}^1)^r : t_i \neq t_j \text{ for } i \neq j\}$. Fix a point $t \in Y_r$, a base point $t_0 \in U_t$ and an isomorphism $\Gamma_r \cong \pi_1(U_t, t_0)$ (this is equivalent to choosing a point in the Teichmüller space $T_{0,r}$). By the above we get a $G$-cover $C_t \to \mathbb{P}^1$ branched at the points $t_i$ with local monodromies $m_1, \ldots, m_r$. This yields a monomorphism of $G$ into the mapping class group $\mathrm{Map}_g := \pi_0(\mathrm{Diff}^+(C_t))$. Denote by $T_g^G$ the fixed point locus of $G$ on the Teichmüller space $T_g$. It is a complex submanifold of dimension $r - 3$, isomorphic to the Teichmüller space $T_{0,r}$ (see e.g. [5, 16]). This isomorphism can be described as follows: if $(C, \varphi)$ is a curve with a marking such that $[(C, \varphi)] \in T_g^G$, the corresponding point in $T_{0,r}$ is $[(C/G, \psi, b_1, \ldots, b_r)]$, where $\psi$ is the induced marking (see [16]) and $b_1, \ldots, b_r$ are the critical values of the projection $C \to C/G$.

We remark that on $T_g^G$ we have a universal family $\mathcal{C} \to T_g^G$ of curves with a $G$–action. It is simply the restriction of the universal family on $T_g$.

The diagonal action of $\mathrm{PSL}(2, \mathbb{C})$ on $(\mathbb{P}^1)^r$ preserves $Y_r$. Thus $\mathrm{PSL}(2, \mathbb{C})$ acts on $Y_r$. The map $Y_r \to \mathsf{M}_g$, $t \mapsto [C_t]$ is $\mathrm{PSL}(2, \mathbb{C})$–invariant, so we get a map $Y_r / \mathrm{PSL}(2, \mathbb{C}) \to \mathsf{M}_g$. Since $Y_r / \mathrm{PSL}(2, \mathbb{C})$ is isomorphic to $\mathsf{M}_{0,r}$, there is a surjective map $T_{0,r} \to Y_r / \mathrm{PSL}(2, \mathbb{C})$. Since $T_g^G \subset T_g$ there is also a map $T_g^G \to \mathsf{M}_g$ which has discrete fibres. Recalling the description of the isomorphism $T_g^G \cong T_{0,r}$ one can easily check that the following diagram commutes:

$$\begin{array}{ccccc} T_{0,r} & \longrightarrow & M_{0,r} \cong Y_r / \mathrm{PSL}(2, \mathbb{C}) & \longrightarrow & \mathsf{M}_g . \\ & \nwarrow & & \nearrow & \\ & & T_g^G \cong & & \end{array}$$

We denote by $\mathsf{M}(\mathbf{m}, G, \theta)$ the image of $Y_r$ in $\mathsf{M}_g$, which is equal to the image of $T_g^G$. It is an irreducible algebraic subvariety of the same dimension as $T_g^G \cong T_{0,r}$, i.e. $r - 3$ (see e.g. [5, 16]). Applying the Torelli map to $\mathsf{M}(\mathbf{m}, G, \theta)$ one gets a subset of $\mathsf{A}_g$. We let $\mathsf{Z}(\mathbf{m}, G, \theta)$ denote the closure of this subset in $\mathsf{A}_g$. By the above it is an algebraic subvariety of dimension $r - 3$.

Different data $(\mathbf{m}, G, \theta)$ and $(\mathbf{m}, G, \theta')$ may give rise to the same subvariety of $\mathsf{M}_g$. This is related to the choice of the isomorphism $\Gamma_r \cong \pi_1(U_t, t_0)$. The change from one choice to another can be described using an action of the braid group $\mathbf{B_r} := \langle \sigma_1, \ldots, \sigma_r | \, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2, \, \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \rangle$. There is a morphism $\varphi : \mathbf{B_r} \to \mathrm{Aut}(\Gamma_r)$ defined as follows:

$$\begin{gathered} \varphi(\sigma_i)(\gamma_i) = \gamma_{i+1}, \quad \varphi(\sigma_i)(\gamma_{i+1}) = \gamma_{i+1}^{-1} \gamma_i \gamma_{i+1}, \\ \varphi(\sigma_i)(\gamma_j) = \gamma_j \quad \text{for } j \neq i, i+1. \end{gathered}$$

Thus we get an action of $\mathbf{B_r}$ on the set of data: $\sigma \cdot (\mathbf{m}, G, \theta) := (\sigma(\mathbf{m}), G, \theta \circ \varphi(\sigma^{-1}))$, where $\sigma(\mathbf{m})$ is the permutation of $\mathbf{m}$ induced by $\sigma$. Also the group $\mathrm{Aut}(G)$ acts on the set of data by $\alpha \cdot (\mathbf{m}, G, \theta) := (\mathbf{m}, G, \alpha \circ \theta)$. The orbits of the $\mathbf{B_r} \times \mathrm{Aut}(G)$–action are called *Hurwitz equivalence classes*. Data in the same class give rise to the same subvariety $\mathsf{M}(\mathbf{m}, G, \theta)$ and hence to the same subvariety $\mathsf{Z}(\mathbf{m}, G, \theta) \subset \mathsf{A}_g$. For more details see [38, 5, 2].

2.4. Given a positive integer $m$ set $\zeta_m = e^{2\pi i/m}$ and

$$I(m) := \{\nu \in \mathbb{Z} : 1 \leq \nu < m, \gcd(\nu, m) = 1\}.$$

If $(\mathbf{m}, G, \theta)$ is a datum, set $x_i := \theta(\gamma_i)$. The $r$-tuple $(x_1, \ldots, x_r)$ is called a *spherical system of generators.* If $C$ is a curve with a $G$–action with datum $(\mathbf{m}, G, \theta)$, then the cyclic subgroups $\langle x_i \rangle$ and their conjugates are the non-trivial stabilizers of the action of $G$ on $C$. The action of the stabilizers near the fixed points can be completely described in terms of the epimorphism $\theta$, see [19, Theorem 7]. In particular we need the following results. Suppose that an element $\mathsf{g} \in G$ fixes a point $P \in C$. Let $m$ be the order of $\mathsf{g}$. The differential $d\mathsf{g}_P$ acts on $T_P C$ by multiplication by an $m$-th root of unity $\zeta_P(\mathsf{g})$. The action can be linearized in a neighbourhood of $P$, i.e. there is a local coordinate $z$ centered in $P$, such that $\mathsf{g}$ acts as $z \mapsto \zeta_P(\mathsf{g})z$. Thus $\zeta_P(\mathsf{g})$ is a primitive $m$-th root of unity. (See also [29, Cor. III.3.5 p. 79].) Denote by $\mathsf{Fix}(\mathsf{g})$ the set of fixed points of $\mathsf{g}$. For $\nu \in I(m)$ set

$$\mathsf{Fix}_\nu(\mathsf{g}) := \{P \in C : \mathbf{g}P = P, \zeta_P(\mathsf{g}) = \zeta_m^\nu\}.$$

**Lemma 2.5.** *If $G \subseteq \mathrm{Aut}(C)$ and $\mathsf{g} \in G$ has order $m$, then*

$$|\mathsf{Fix}_\nu(\mathsf{g})| = |C_G(\mathsf{g})| \cdot \sum_{\substack{1 \leq i \leq r, \\ m | m_i, \\ \mathsf{g} \sim_G x_i^{m_i \nu / m}}} \frac{1}{m_i}.$$

(Here $C_G(\mathsf{g})$ denotes the centralizer of $\mathsf{g}$ in $G$ and $\sim_G$ denotes the equivalence relation given by conjugation in $G$.) This lemma follows from [19, Theorem 7], see also [4, Lemma 11.5].

2.6. Given a $G$-Galois cover $C \to \mathbb{P}^1$ let $\rho \colon G \longrightarrow \mathrm{GL}(H^0(C, K_C))$ denote the representation on holomorphic 1-forms and let $\chi_\rho$ be the character of $\rho$. Notice that up to equivalence the representation $\rho$ only depends on the data $(\mathbf{m}, G, \theta)$, not on the parameter $t \in Y_r$.

**Theorem 2.7** (Eichler Trace Formula)**.** *Let $\mathsf{g}$ be an automorphism of order $m > 1$ of a Riemann surface $C$ of genus $g > 1$. Then*

$$\chi_\rho(\mathsf{g}) = \mathrm{Tr}(\rho(\mathsf{g})) = 1 + \sum_{P \in \mathsf{Fix}(\mathsf{g})} \frac{\zeta_P(\mathsf{g})}{1 - \zeta_P(\mathsf{g})}. \tag{2.1}$$

(See e.g. [14, Thm. V.2.9, p. 264].) Collecting the terms with equal exponent and using the previous lemma one gets the following.

**Corollary 2.8.**

$$\chi_\rho(\mathsf{g}) = 1 + |C_G(\mathsf{g})| \sum_{\nu \in I(m)} \left\{ \sum_{\substack{1 \leq i \leq r, \\ m | m_i, \\ \mathsf{g} \sim_G x_i^{m_i \nu / m}}} \frac{1}{m_i} \right\} \frac{\zeta_m^\nu}{1 - \zeta_m^\nu}. \tag{2.2}$$

2.9. Another corollary of the Eichler Trace Formula is the well known Chevalley–Weil formula which gives the multiplicity of a given irreducible representation of $G$ in $H^0(X, K_C)$. More precisely, denote by $\mathrm{Irr}(G)$ the set of irreducible characters of $G$. For $\chi \in \mathrm{Irr}(G)$ let $\sigma_\chi$ be the corresponding irreducible representation and let $d_\chi$ be the degree of $\sigma_\chi$. Next, denote by $\mu_\chi$ the multiplicity of $\sigma_\chi$ inside $\rho$. Moreover, let $x_i$ be an element of order

$m_i$ in $G$ that represents the local monodromy of the covering $C \to \mathbb{P}^1$ at the branch point $P_i$ and let $E_{i,\alpha}$ denote the number of eigenvalues of $\sigma_\chi(x_i)$ that are equal to $\zeta_{m_i}^\alpha$, where $\zeta_{m_i} = e^{2\pi i/m_i}$ as usual.

**Theorem 2.10** (Chevalley–Weil [6])**.** *Let $C \to \mathbb{P}^1$ be a $G$-Galois cover branched at $r$ points. Let $m_i$ and $E_{i,\alpha}$ be as above. Then the multiplicity $\mu_\chi$ of a given irreducible character $\chi$ in $H^0(C, K_C)$ is*

$$\mu_\chi = -d_\chi + \sum_{i=1}^{r} \sum_{\alpha=0}^{m_i-1} E_{i,\alpha} \Big\langle -\frac{\alpha}{m_i} \Big\rangle + \varepsilon, \tag{2.3}$$

*where $\varepsilon = 1$ if $\chi$ is the trivial character and $\varepsilon = 0$ otherwise. Here we denote by $\langle q \rangle$ the fractional part of $q \in \mathbb{Q}$.*

2.11. Let $\sigma \colon G \to \mathrm{GL}(V)$ be any linear representation of $G$ with character $\chi_\sigma$. Denote by $S^2\sigma$ the induced representation on $S^2V$ and by $\chi_{S^2\sigma}$ its character. Then for $x \in G$

$$\chi_{S^2\sigma}(x) = \frac{1}{2}\big(\chi_\sigma(x)^2 + \chi_\sigma(x^2)\big). \tag{2.4}$$

(See e.g. [40, Proposition 3]).

2.12. We are only interested in the multiplicity $N$ of the trivial representation inside $S^2\rho$. We remark that since the representation $\rho$ only depends on the datum $(\mathbf{m}, G, \theta)$, the same happens for $N$. Using the orthogonality relations and (2.4), $N$ can be computed as follows:

$$N = (\chi_{S^2\rho}, 1) = \frac{1}{|G|} \sum_{x \in G} \chi_{S^2\rho}(x) = \frac{1}{2|G|} \sum_{x \in G} \big(\chi_\rho(x^2) + \chi_\rho(x)^2\big). \tag{2.5}$$

Since $\chi_\rho = \sum_{\chi \in \mathrm{Irr}(G)} \mu_\chi \chi$ we obtain

$$N = \frac{1}{2|G|} \sum_{x \in G} \Big( \big( \sum_{\chi \in \mathrm{Irr}(G)} \mu_\chi \chi(x) \big)^2 + \sum_{\chi \in \mathrm{Irr}(G)} \mu_\chi \chi(x^2) \Big) \tag{2.6}$$

where $\mathrm{Irr}(G)$ denotes the set of irreducible characters of $G$. Formula (2.6) is the one used in our `MAGMA` script. To computed directly the examples by hand, one can use (2.5) together with (2.2). This is the method used in the computation at the end of §4.

## 3. Special subvarieties

3.1. Fix a rank $2g$ lattice $\Lambda$ and an alternating form $E : \Lambda \times \Lambda \to \mathbb{Z}$ of type $(1, \dots, 1)$. For $F$ a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$, set $\Lambda_F := \Lambda \otimes_{\mathbb{Z}} F$. The Siegel upper half-space can be defined as follows [20, Thm. 7.4]:

$$\mathfrak{H}_g := \{J \in \mathrm{GL}(\Lambda_\mathbb{R}) : J^2 = -I, J^*E = E, E(x, Jx) > 0, \ \forall x \neq 0\}.$$

The group $\mathrm{Sp}(\Lambda, E)$ acts on $\mathfrak{H}_g$ by conjugation and $\mathsf{A}_g = \mathrm{Sp}(\Lambda, E) \backslash \mathfrak{H}_g$. This space has the structure of a smooth algebraic stack and also of a complex analytic orbifold. The orbifold structure is the one naturally associated with the properly discontinuous action of $\mathrm{Sp}(\Lambda, E)$ on $\mathfrak{H}_g$. Throughout the paper we will work with $\mathsf{A}_g$ with this orbifold structure. Denote by $A_J$ the quotient $\Lambda_\mathbb{R}/\Lambda$ provided with the complex structure $J$ and the polarization

$E$. On $\mathfrak{H}_g$ there is a natural variation of rational Hodge structure, with local system $\mathfrak{H}_g \times \Lambda_{\mathbb{Q}}$ and corresponding to the Hodge decomposition of $\Lambda_{\mathbb{C}}$ in $\pm i$ eigenspaces for $J$. This descends to a variation of Hodge structure on $\mathsf{A}_g$ in the orbifold or stack sense.

3.2. We refer to §2.3 in [32] for the definition of Hodge loci for a variation of Hodge structure. A *special subvariety* $\mathsf{Z} \subseteq \mathsf{A}_g$ is by definition a Hodge locus of the natural variation of Hodge structure on $\mathsf{A}_g$ described above. Special subvarieties contain a dense set of CM points and they are totally geodesic [32, §3.4(b)]. Conversely an algebraic totally geodesic subvariety that contains a CM point is a special subvariety [30, Thm. 4.3]. The simplest special subvarieties are the *special subvarieties of PEL type*, whose definition is as follows (see [32, §3.9] for more details). Given $J \in \mathfrak{H}_g$, set

$$\operatorname{End}_{\mathbb{Q}}(A_J) := \{f \in \operatorname{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) : Jf = fJ\}. \tag{3.1}$$

Fix a point $J_0 \in \mathfrak{H}_g$ and set $D := \operatorname{End}_{\mathbb{Q}}(A_{J_0})$. The *PEL type* special subvariety $\mathsf{Z}(D)$ is defined as the image in $\mathsf{A}_g$ of the connected component of the set $\{J \in \mathfrak{H}_g : D \subseteq \operatorname{End}_{\mathbb{Q}}(A_J)\}$ that contains $J_0$.

**Lemma 3.3.** *Let $(M, g)$ be a Riemannian symmetric space of the noncompact type. Let $G$ be a group acting isometrically on $(M, g)$. If $M^G$ is nonempty, then it is a smooth connected submanifold of $M$.*

*Proof.* Fix $x \in M^G$. Then $G$ acts on $T_xM$ via the differential (isotropy action). The exponential map $\exp_x : T_xM \to M$ is a global diffeomorphism and it is $G$-equivariant with respect to the isotropy action on $T_xM$ and the natural action on $M$. Thus $M^G = \exp_x((T_xM)^G)$. Since $(T_xM)^G$ is a linear subspace of $T_xM$, $M^G$ is a smooth connected submanifold. □

**Remark 3.4.** If $G$ is finite, then $M^G$ is always nonempty (Cartan fixed point theorem), see [13, p. 21].

**Corollary 3.5.** *Let $G \subseteq \operatorname{Sp}(\Lambda, E)$ be a finite subgroup. Denote by $\mathfrak{H}_g^G$ the set of points of $\mathfrak{H}_g$ that are fixed by $G$. Then $\mathfrak{H}_g^G$ is a connected complex submanifold of $\mathfrak{H}_g$.*

*Proof.* This follows from the Lemma since $\mathfrak{H}_g$ is a symmetric space of the noncompact type and $\operatorname{Sp}(\Lambda, E)$ acts isometrically and holomorphically on $\mathfrak{H}_g$. □

Another proof of the Corollary follows from [15, Lemma 5.2]. Set

$$D_G := \{f \in \operatorname{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}}) : Jf = fJ, \ \forall J \in \mathfrak{H}_g^G\}. \tag{3.2}$$

**Lemma 3.6.** *If $J \in \mathfrak{H}_g^G$, then $D_G \subseteq \operatorname{End}_{\mathbb{Q}}(A_J)$ and the equality holds for $J$ in a dense subset of $\mathfrak{H}_g^G$.*

*Proof.* Consider the variation of Hodge structure on $\mathfrak{H}_g$ defined in 3.1 and restrict it to $\mathfrak{H}_g^G$. There is an algebraic subgroup $M \subseteq \operatorname{CSp}(2g, \mathbb{Q})$ such that the Mumford-Tate group $\operatorname{MT}(A_J)$ is contained in $M$ for any $J \in \mathfrak{H}_g^G$ and $\operatorname{MT}(A_J) = M$ for $J$ in a dense subset $\Omega \subseteq \mathfrak{H}_g^G$. The complement of $\Omega$ is a countable union of analytic subsets. Recall that

$$\operatorname{End}_{\mathbb{Q}}(A_J) = \operatorname{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})^{\operatorname{MT}(A_J)}. \tag{3.3}$$

So $\operatorname{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})^M \subseteq \operatorname{End}_{\mathbb{Q}}(A_J)$ for any $J \in \mathfrak{H}_g^G$, with equality for $J \in \Omega$. It follows immediately from (3.1) and (3.2) that

$$D_G = \bigcap_{J \in \mathfrak{H}_g^G} \operatorname{End}_{\mathbb{Q}}(A_J) = \operatorname{End}_{\mathbb{Q}}(\Lambda_{\mathbb{Q}})^M$$

and that $D_G = \operatorname{End}_{\mathbb{Q}}(A_J)$ for any $J \in \Omega$. □

**Proposition 3.7.** *The image of $\mathfrak{H}_g^G$ in $\mathsf{A}_g$ coincides with the PEL subvariety $\mathsf{Z}(D_G)$.*

*Proof.* Set

$$Y := \{J \in \mathfrak{H}_g : D_G \subseteq \operatorname{End}_{\mathbb{Q}}(A_J)\} = \{J \in \mathfrak{H}_g : fJ = Jf, \forall f \in D_G\}.$$

Since $G \subseteq D_G$, we get immediately that $Y \subseteq \mathfrak{H}_g^G$. Conversely, if $J \in \mathfrak{H}_g^G$, then by definition $D_G \subseteq \operatorname{End}_{\mathbb{Q}}(A_J)$, i.e. $J \in Y$. Thus $Y = \mathfrak{H}_g^G$. By the previous lemma there is $J_0 \in Y$ such that $D_G = \operatorname{End}_{\mathbb{Q}}(A_{J_0})$. Thus the image of $Y$ in $\mathsf{A}_g$ is indeed the special subvariety $\mathsf{Z}(D_G)$. □

Recall that $N = \dim \left(S^2 H^0(C, K_C)\right)^G$ and that $\mathsf{Z}(\mathbf{m}, G, \theta)$ is defined in 2.3.

**Lemma 3.8.** *If $J \in \mathfrak{H}_g^G$, then $\dim \mathfrak{H}_g^G = \dim \mathsf{Z}(D_G) = \dim(S^2\Lambda_{\mathbb{R}})^G$ where $\Lambda_{\mathbb{R}}$ is endowed with the complex structure $J$.*

*Proof.* By Lemma 3.3 it is enough to compute the dimension of $(T_J\mathfrak{H}_g)^G$. But $T_J\mathfrak{H}_g = S^2\Lambda_{\mathbb{R}}$, where $\Lambda_{\mathbb{R}}$ is endowed with the complex structure $J$. □

**Theorem 3.9.** *Fix a datum $(\mathbf{m}, G, \theta)$ and assume that*

$$(*) \qquad N = r - 3.$$

*Then $\mathsf{Z}(\mathbf{m}, G, \theta)$ is a special subvariety of PEL type of $\mathsf{A}_g$ that is contained in $\mathsf{T}_g$ and such that $\mathsf{Z}(\mathbf{m}, G, \theta) \cap \mathsf{T}_g^0 \neq \emptyset$.*

*Proof.* Let $\mathcal{C} \to T_g^G$ be the universal family as in 2.3. For any $t \in T_g^G$, $G$ acts holomorphically on $C_t$, so it maps injectively into $\operatorname{Sp}(\Lambda, E)$, where $\Lambda = H_1(C_t, \mathbb{Z})$ and $E$ is the intersection form. Denote by $G'$ the image of $G$ in $\operatorname{Sp}(\Lambda, E)$. It does not depend on $t$ since it is purely topological. The Siegel upper half-space $\mathfrak{H}_g$ parametrizes complex structures on the real torus $\Lambda_{\mathbb{R}}/\Lambda = H_1(C_t, \mathbb{R})/H_1(C_t, \mathbb{Z})$ compatible with the polarization $E$. The period map associates to the curve $C_t$ the complex structure $J_t$ on $\Lambda_{\mathbb{R}}$ obtained from the splitting $H^1(C_t, \mathbb{C}) = H^{1,0}(C_t) \oplus H^{0,1}(C_t)$ and the isomorphism $H_1(C_t, \mathbb{R})^*_{\mathbb{C}} = H^1(C_t, \mathbb{C})$. Since $G$ acts holomorphically on $C_t$, the complex structure $J_t$ is invariant by $G'$. This shows that $J_t \in \mathfrak{H}_g^{G'}$, so the Jacobian $j(C_t)$ lies in $\mathsf{Z}(D_{G'})$. This shows that $\mathsf{Z}(\mathbf{m}, G, \theta) \subseteq \mathsf{Z}(D_{G'})$. Since $\mathsf{Z}(D_{G'})$ is irreducible (e.g. by Corollary 3.5), to conclude it is enough to check that they have the same dimension. The dimension of $\mathsf{Z}(\mathbf{m}, G, \theta)$ is $r-3$, see 2.3. By Lemma 3.8, if $J \in \mathfrak{H}_g^{G'}$, then $\dim \mathsf{Z}(D_{G'}) = \dim \mathfrak{H}_g^{G'} = \dim(S^2\Lambda_{\mathbb{R}})^{G'}$, where $\Lambda_{\mathbb{R}}$ is endowed with the complex structure $J$. If $J$ corresponds to the Jacobian of a curve $C$ in the family, then $(S^2\Lambda_{\mathbb{R}})^{G'}$ is isomorphic to the dual of $(S^2H^0(C, K_C))^G$. Thus $\dim \mathsf{Z}(D_{G'}) = N$ and $(*)$ yields the result. □

Although we are mainly interested in positive dimensional families, we note the following corollaries, which might be of independent interest.

**Corollary 3.10.** *Let $A$ be a principally polarized abelian variety. Let $G$ be a finite group of automorphisms of $A$ that preserve the polarization. If*

$$\big(S^2H^0(A,\Omega^1_A)\big)^G = \{0\}, \tag{3.4}$$

*then $A$ has complex multiplication.*

*Proof.* Set $\Lambda := H_1(A,\mathbb{Z})$. Assume that $A$ equals $\Lambda_\mathbb{R}/\Lambda$ provided with some $J_0 \in \mathfrak{H}_g$ and that $G \subset \mathrm{Sp}(\Lambda, E)$. As in the previous proof, (3.4) implies that $\mathfrak{H}_g^G = \{J_0\}$. The result follows immediately from Proposition 3.7, since special varieties contain CM points. Nevertheless a more direct argument can be given as follows. The elements of $\mathfrak{H}_g$ correspond to morphisms $h : \mathbb{S} \to \mathrm{CSp}(\Lambda, E)$. If $J$ corresponds to $h$, the Mumford-Tate group of $A_J$ is the smallest algebraic subgroup of $\mathrm{GL}(\Lambda_\mathbb{C})$ that is defined over $\mathbb{Q}$ and contains $h(\mathbb{S})$. Denote by $h_0$ the morphism corresponding to $J_0$ and by $M_0$ the Mumford-Tate group of $A_{J_0}$. We claim that the set of morphisms $h$ with $h(\mathbb{S}) \subseteq M_0$ reduces to $h_0$. Indeed if $h$ corresponds to $J$ and $h(\mathbb{S}) \subseteq M_0$, then $\mathrm{MT}(A_J) \subseteq M_0$. So using (3.3)

$$\mathrm{End}_\mathbb{Q}(A_{J_0}) = \mathrm{End}_\mathbb{Q}(\Lambda_\mathbb{Q})^{M_0} \subseteq \mathrm{End}_\mathbb{Q}(\Lambda_\mathbb{Q})^{\mathrm{MT}(A_J)} = \mathrm{End}_\mathbb{Q}(A_J).$$

Thus $G \subseteq \mathrm{End}_\mathbb{Q}(A_J)$, so $J \in \mathfrak{H}_g^G$, $J = J_0$ and $h = h_0$ as claimed. If $g \in M_0(\mathbb{R})$, the morphism $gh_0g^{-1}$ clearly maps $\mathbb{S}$ to $M_0$. By the above $gh_0g^{-1} = h_0$. So $h_0(\mathbb{S})$ is contained in the center of $M_0(\mathbb{R})$. Since $M_0(\mathbb{Q})$ is dense in $M_0(\mathbb{R})$ [3, Cor. 18.3 p. 220], this center coincides with the set of real points of the algebraic group $Z(M_0)$, which is defined over $\mathbb{Q}$. Thus $M_0 = Z(M_0)$, $M_0$ is a torus and $A$ is CM. □

**Corollary 3.11.** *Let $C$ be a curve and $G$ a subgroup of* $\mathrm{Aut}(C)$. *If $(S^2H^0(C, K_C))^G = \{0\}$, then $J(C)$ is an abelian variety of CM type.*

3.12. Using this criterion and the `MAGMA` script, we found some examples of CM Jacobians: 10 for $g = 2$, 19 for $g = 3$, 18 for $g = 4$, 17 for $g = 5$, 17 for $g = 6$, 23 for $g = 7$. If $C$ and $G$ satisfy the hypothesis of Corollary 3.11, then clearly the corresponding family is a point, so $C$ is a curve *with many automorphisms*, using the terminology of [35, Def. 5.17]. As remarked there it is expected that not every curve with many automorphisms be CM. The above criterion identifies a subclass of curves with many automorphisms where this is true. It would be interesting to check if the 0–dimensional examples with $N > 0$ (they do exist) are CM or not.

## 4. New examples

4.1. In this section we give the list of all new families of Galois covers and we explain some of them with more details. As explained in the Appendix the `MAGMA` script computes the list of all Hurwitz equivalence classes of data. Next it decomposes the representation on $H^0(C, K_C)$ using the Chevalley-Weil formula (2.3) and computes the number $N$. The complete list of all the families corresponding to the data $(\mathbf{m}, G, \theta)$ with genus $g \le 9$ such that $N = r - 3 > 0$ is given in Table 2 in the Appendix.

The new examples are the ones in Table 2 with numbers (28)–(40). For them we now give a presentation of the Galois group and an explicit description of a representative of an epimorphism $\theta$ (we use the same notation as in §2.1 and §2.4).

Genus 2

(28) $S_3 = \langle x, y : y^2 = x^3 = 1, y^{-1}xy = x^2 \rangle$.
$x_1 = y, x_2 = y, x_3 = x,\ x_4 = x^2$.
(29) $D_4 = \langle x, y : y^2 = x^4 = 1, y^{-1}xy = x^3 \rangle$.
$x_1 = x^3y, x_2 = x^2, x_3 = y,\ x_4 = x^3$.
(30) $D_6 = \langle x, y : y^2 = x^6 = 1, y^{-1}xy = x^5 \rangle$.
$x_1 = x^3y, x_2 = x^4y, x_3 = x^3,\ x_4 = x^4$.

Genus 3

(31) $S_3 = \langle x, y : y^2 = x^3 = 1, y^{-1}xy = x^2 \rangle$.
$x_1 = xy, x_2 = x^2y, x_3 = y,\ x_4 = xy,\ x_5 = x^2$.
(32) $D_4 = \langle x, y : y^2 = x^4 = 1, y^{-1}xy = x^3 \rangle$.
$x_1 = xy, x_2 = x^2y, x_3 = x^2,\ x_4 = x^2y,\ x_5 = x^3y$.
(33) $G = A_4$. Set $y_1 := (123),\ y_2 := (12)(34),\ y_3 := (13)(24)$.
$x_1 = y_3 = (13)(24), x_2 = y_2 = (12)(34), x_3 = y_1y_3 = (243),\ x_4 = y_1^2y_3 = (124)$.
(34) $((\mathbb{Z}/4) \times (\mathbb{Z}/2)) \rtimes \mathbb{Z}/2 = \langle y_1, y_2, y_3 : y_1^2 = y_2^2 = y_3^4 = 1, y_2y_3 = y_3y_2, y_1^{-1}y_2y_1 = y_2y_3^2, y_1^{-1}y_3y_1 = y_3 \rangle$.
$x_1 = y_1, x_2 = y_1y_2y_3^3, x_3 = y_2y_3^2,\ x_4 = y_3^3$.
(35) $G = S_4$. Set $y_1 := (12),\ y_2 := (123),\ y_3 := (13)(24),\ y_4 := (14)(23)$.
$x_1 = y_1y_2^2 = (13), x_2 = y_3y_4 = (12)(34), x_3 = y_1 = (12),\ x_4 = y_2^2y_4 = (143)$.

Genus 4

(36) $Q_8 = \langle y_1, y_2, y_3 \mid y_1^2 = y_2^2 = y_3, y_3^2 = 1, y_1^{-1}y_2y_1 = y_2y_3 \rangle$.
$x_1 = y_3, x_2 = y_2y_3, x_3 = y_1y_2,\ x_4 = y_1y_3$.
(37) $G = A_4$. Set $y_1 := (123),\ y_2 := (12)(34),\ y_3 := (13)(24)$.
$x_1 = y_3 = (13)(24), x_2 = y_1 = (123), x_3 = y_1 = (123),\ x_4 = y_1y_3 = (243)$.
(38) $(\mathbb{Z}/3) \times S_3 = \langle y_1, y_2, y_3 \mid y_1^2 = y_2^3 = y_3^3 = 1,\ y_1y_2y_1^{-1} = y_2, y_2y_3y_2^{-1} = y_3,\ y_1y_3y_1^{-1} = y_3^2 \rangle$.
$x_1 = y_1y_3^2, x_2 = y_1y_3, x_3 = y_2y_3,\ x_4 = y_2^2$.

Genus 5

(39) $(\mathbb{Z}/3) \rtimes \mathbb{Z}/4 = \langle y_1, y_3 \mid y_1^4 = y_3^3 = 1, y_1^{-1}y_3y_1 = y_3^2 \rangle$.
$x_1 = y_1^2, x_2 = y_3, x_3 = y_1^3y_3^2,\ x_4 = y_1^3y_3$.

Genus 7

(40) $y_1 = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}\ y_2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\ y_3 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\ y_4 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
$\mathrm{SL}(2, \mathbb{F}_3) = \langle y_1, y_2, y_3, y_4 | y_1^3 = y_4^2 = 1, y_2^2 = y_3^2 = y_4, y_1^{-1}y_2y_1 = y_3, y_1^{-1}y_3y_1 = y_2y_3, y_2^{-1}y_3y_2 = y_3y_4 \rangle$.

$$x_1 = y_4,\ x_2 = y_1^2 y_2 y_3 y_4 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ x_3 = y_1^2 y_2 y_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
$$x_4 = y_1^2 y_3 y_4 = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}.$$

Now we wish to make some remarks on the geometry of the various examples. First of all, we show how to check by hand that the examples give indeed special varieties. We do this by explaining in detail the computation in two sample examples, namely families (37) and (40), see 4.2 and 4.4. We also show that the families (37), (40) and (25) are not contained in the hyperelliptic locus (see 4.3, 4.5, 4.8), while (8), (22), (36) and (39) are hyperelliptic (see 4.6). We show that (25) and (38) have the same image in $\mathsf{M}_4$ and in $\mathsf{A}_4$, see 4.7. We also note that in the case of family (25) every Jacobian is reducible and it is possible to identify explicitely a CM point, see 4.9.

The following observation simplifies the computation of $N$. Denote by $G_0$ the set of elements of order 2 in $G$. The set of elements of order greater than 2 can be written as $G_1 \sqcup G_1^{-1}$ for some choice of $G_1 \subseteq G$. Then (2.5) becomes

$$(4.1)\quad N = \frac{\chi_\rho(1) + \chi_\rho(1)^2}{2|G|} + \frac{1}{2|G|} \sum_{x \in G_0} (\chi_\rho(x^2) + \chi_\rho(x)^2) +$$
$$+ \frac{1}{2|G|} \sum_{x \in G_1} (\chi_\rho(x^2) + \chi_\rho(x)^2) + \frac{1}{2|G|} \sum_{x \in G_1} (\chi_\rho(x^{-2}) + \chi_\rho(x^{-1})^2) =$$
$$= \frac{g + g^2 + |G_0| g}{2|G|} + \frac{1}{2|G|} \sum_{x \in G_0} \chi_\rho(x)^2 + \frac{1}{|G|} \sum_{x \in G_1} \mathrm{Re}\big(\chi_\rho(x^2) + \chi_\rho(x)^2\big)$$

4.2. Example (37). One easily checks that $\theta$ is an epimorphism. The conjugacy classes of $A_4$ are $\{1\}$, $A := \{y_1 = (123), (134), (142), (243)\}$ $B := \{(132), (143), (124), (234)\}$, $C = \{y_2 = (12)(34), (13)(24), (14)(23)\}$. We have $G_0 = C$ and we can set $G_1 := A$ so that $G_1^{-1} = B$. It suffices to compute $\chi_\rho(y_1)$ and $\chi_\rho(y_2)$. We have $|C_G(y_1)| = 3$, $|C_G(y_2)| = 4$. Moreover $y_1 \sim_G x_j^{m_j \nu/3}$ iff $\nu = 1$ and $j \in \{2, 3, 4\}$ and $y_2 \sim_G x_j^{m_j \nu/2}$ iff $\nu = 1$ and $j = 1$. Using (2.2) one gets $\chi_\rho(y_1) = \zeta_3$, $\chi_\rho(y_2) = 0$. Hence by (4.1)

$$24\,N = 4 + 16 + 3\big(\chi_\rho(y_2)^2 + \chi_\rho(y_2^2)\big) + 8\,\mathrm{Re}\big(\chi_\rho(y_1^2) + \chi_\rho(y_1)^2\big) =$$
$$= 32 + 8\,\mathrm{Re}(\bar{\zeta_3} + \zeta_3^2) = 24.$$

So $N = 1$ and by Theorem 3.9 we get a special curve in $\mathsf{T}_4$.

4.3. We claim that the above family (37) does not contain any hyperelliptic curve. In fact the hyperelliptic involution is central in $\mathrm{Aut}(C)$. Hence there is no hyperelliptic involution contained in $G$ since its center is trivial. If there is a hyperelliptic involution $\tau$ outside $G$, then $\mathsf{Fix}(\tau)$ consists of 10 points and it is $G$-invariant, so it is a union of $G$-orbits. The only possibility is that it consists of 2 orbits, the one over $t_1$, of cardinality 6, and another one of cardinality 4 over one of the critical values $t_2, t_3, t_4$. If $p \in \pi^{-1}(t_1)$, then the stabilizer $\mathrm{Aut}(C)_p$ contains $G_p \times \langle\tau\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. This is impossible since $\mathrm{Aut}(C)_p$ is cyclic.

4.4. Example (40). One easily checks that $\theta$ is an epimorphism. In $\mathrm{SL}(2,\mathbb{F}_3)$ there are 7 conjugacy classes: $\{1\}$, $G_0 = \{y_4\}$,

$$\begin{aligned}
&A = \{y_1 = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \alpha_1 := \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, \alpha_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \alpha_3 := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\} \\
&B = \{y_1^{-1}, \alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}\} \\
&C = \{y_2, y_3, y_2y_3, y_2^{-1}, y_3^{-1}, (y_2y_3)^{-1}\} \\
&D = \{a_1 := \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, a_2 := \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, a_3 := \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, a_4 := \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}\} \\
&F = \{a_1^{-1}, a_2^{-1}, a_3^{-1}, a_4^{-1}\}
\end{aligned}$$

The elements of $A$ and $B$ have order 3. The elements in $C$ have order 4. The elements in $D$ and $F$ have order 6. Using (2.2) one computes

$$\chi_\rho(y_4) = -5 \qquad \chi_\rho(y_1) = 2\zeta_3^2 - 1$$
$$\chi_\rho(y_2) = 1 \qquad \chi_\rho(a_1) = 1.$$

Using (4.1) one gets $N = 1$, hence this family yields a special curve in $\mathsf{T}_7$.

4.5. We claim that the above family (40) is not contained in the hyperelliptic locus. In fact the center of $G$ is generated by $y_4$, which has order 2, but its trace is $-5$. So it does not act as $-1$. Assume that $\tau$ is a hyperelliptic involution not contained in $G$. The set of fixed points of $\tau$ has order 16 and it is $G$-invariant. The orbits of $G$ have cardinality 24, 12 (only one orbit) or 8 (three orbits). Thus the only possibility is that $\tau$ fixes pointwise the fibres over say $t_2$ and $t_3$. We can assume that $t_1 = 1$, $t_2 = 0$ and $t_3 = \infty$. Then $\tau$ descends to an involution $\hat\tau$ of $\mathbb{P}^1$ fixing both 0 and $\infty$ and interchanging $t_1$ and $t_4$. But then necessarily $\hat\tau(z) = -z$ and $q_4 = -1$. Therefore there is at most one hyperelliptic curve in this family.

4.6. We now observe that some of the families in Table 2 are contained in the hyperelliptic locus. In fact, in example (36) one can check that $\chi_\rho(y_3) = -4$. This is enough to conclude that $y_3$ is the hyperelliptic involution. Indeed, fix on $H^0(C, K_C)$ a $G$-invariant Hermitian product and consider the Hermitian product $(A, B) = \mathrm{Tr}\, AB^*$ on $\mathrm{End}\, H^0(C, K_C)$. Since $\mathrm{Tr}\,\rho(y_3) = (\rho(y_3), \mathrm{id}) = -4$, and $\rho(y_3)$ is unitary, the Cauchy-Schwarz inequality yields $\rho(y_3) = -\mathrm{id}$. This shows that $y_3$ is the hyperelliptic involution and all the family is contained in the hyperelliptic locus.

The same applies to example (39) since $\chi_\rho(y_2) = -5$. In the same way one can check that families (8) and (22) of Table 1 and Table 2 in [32] are contained in the hyperelliptic locus.

4.7. We claim that families (38) and (25) have the same image in $\mathsf{M}_4$ and in $\mathsf{A}_4$. In other words we claim that each curve in (25) admits another $\mathbb{Z}/2$–action. To do this we first show that (25) is contained in family (10) of Table 1 in [32]. This is the family of cyclic covers of $\mathbb{P}^1$ with group $G = \mathbb{Z}/3$ and ramification data $(3,3,3,3,3,3)$. An affine equation for this cyclic family is the following:

$$y^3 = \prod_{i=1}^{6}(x - t_i) \tag{4.2}$$

where the group action is $(x, y) \mapsto (x, \zeta_3 y)$. Choose the critical values as follows: $t_1 = 1,\ t_2 = \zeta_3,\ t_3 = \zeta_3^2,\ t_4 = t,\ t_5 = \zeta_3 t,\ t_6 = \zeta_3^2 t$. We obtain the one dimensional family

$$y^3 = (x^3 - 1)(x^3 - t^3) \tag{4.3}$$

as $t$ varies in $\mathbb{C} - \{0, 1\}$. This is in fact family (25) with group $\mathbb{Z}/3 \times \mathbb{Z}/3$ and ramification data $(3,3,3,3)$, where the action of the second generator is $(x, y) \mapsto (\zeta_3 x, \zeta_3 y)$. Now consider the maps $h_i : C_t \to C_t$

$$h_1(x, y) := (t/x, ty/x^2) \qquad h_2(x, y) = (x, \zeta_3 y) \qquad h_3(x, y) = (\zeta_3 x, \zeta_3 y).$$

$h_2$ and $h_3$ give the above action of $\mathbb{Z}/3 \times \mathbb{Z}/3$. Together with $h_1$ they yield an action of the group $\mathbb{Z}/3 \times S_3 = \langle h_1, h_2, h_3 \mid h_1^2 = h_2^3 = h_3^3 = 1,\ h_1 h_2 h_1^{-1} = h_2, h_2 h_3 h_2^{-1} = h_3,\ h_1 h_3 h_1^{-1} = h_3^2 \rangle$. This action has ramification data $(2,2,3,3)$. This proves that (25)=(38).

4.8. We note in passing that family (25) does not intersect the hyperelliptic locus. Assume that an element of the family admits a hyperelliptic involution $\tau$. Since $G = \mathbb{Z}/3 \times \mathbb{Z}/3$ does not contain elements of order 2, $\tau \notin G$. Since $\tau$ commutes with $G$, the fixed points of $\tau$ form a $G$-invariant set of cardinality 10. Since all the orbits of $G$ have cardinality either 9 or 3, this is impossible.

4.9. Note that using the method of [39, pp. 68-69] it is easy to identify a CM point in family (25). Setting $t = -1$ in (4.3), we get $C_{-1} := \{y^3 = (x^6 - 1) = \prod_{i=0}^{5}(x - \xi_6^i)\}$. Let $V$ be the Fermat curve with affine equation $x^{18} - y^{18} - 1 = 0$. Then $f(x, y) = (x^3, y^6)$ is a well-defined non-constant map $f : V \to C_{-1}$. By Lemma 2.4.3 in [39] $C_{-1}$ has complex multiplication. One can also check that the subgroup of order 3 generated by $h_3$ acts freely on $C_t$, with quotient a curve of genus 2. Hence the Jacobian of $C_t$ is isogenous to a product for every $t$.

## 5. Proof of Theorem 1.9

We already saw in 4.7 that distinct data can give rise to the same locus in $\mathsf{M}_g$ and in $\mathsf{A}_g$. In this section we explore this fact systematically and we prove Theorem 1.9.

**Lemma 5.1.** *Assume that two data $(\mathbf{m}, G, \theta)$ and $(\mathbf{m}', G', \theta')$ satisfying condition $(*)$ give rise to the same Shimura variety $\mathsf{Z}(\mathbf{m}, G, \theta) = \mathsf{Z}(\mathbf{m}', G', \theta')$. Then there is a third datum $(\mathbf{m}'', G'', \theta'')$, also satifying $(*)$, such that $\mathsf{Z}(\mathbf{m}, G, \theta) = \mathsf{Z}(\mathbf{m}', G', \theta') = \mathsf{Z}(\mathbf{m}'', G'', \theta'')$ and such that there are monomorphisms $f : G \to G''$ and $f' : G' \to G''$. Moreover if $|f(G) \cap f'(G')| \leq k$, then*

$$|G''| \geq \frac{|G| \cdot |G'|}{k}.$$

*Proof.* Let $C$ be a generic curve in the family defined by $(\mathbf{m}, G, \theta)$ or $(\mathbf{m}'$, $G'$, $\theta')$. Let $G''$ denote the automorphism group of $C$. The quotient $C/G''$ has genus zero and the action of $G''$ on $C$ defines a datum $(\mathbf{m}'', G'', \theta'')$. By construction there are monomorphisms $f$ and $f'$ as required corresponding to the actions of $G$ and $G'$ on $C$. So we can consider $G$ and $G'$ as subgroups of $G''$. The family defined by $(\mathbf{m}'', G'', \theta'')$ contains the one defined by $(\mathbf{m}, G, \theta)$, which coincides with the one defined by $(\mathbf{m}', G', \theta')$. Therefore $r-3 = r'-3 \leq r''-3$. Since $N'' := \dim(S^2H^0(C, K_C))^{G''}$ and $G \subseteq G''$, we have

$$r'' - 3 \leq N'' \leq N = r - 3 \leq r'' - 3.$$

Hence $N = N' = N'' = r-3 = r'-3 = r''-3$. This shows that $\mathsf{Z}(\mathbf{m}, G, \theta) = \mathsf{Z}(\mathbf{m}', G', \theta') = \mathsf{Z}(\mathbf{m}'', G'', \theta'')$. The last statement follows by considering the inclusions of sets $G/G \cap G' \hookrightarrow G''/G'$. □

**Theorem 5.2.** *In genus 2 the data satisfying* $(*)$ *yield the following four Shimura subvarieties:*

$N = 1$ (3) = (5) = (28) = (30), (4) = (29).
$N = 2$ (26).
$N = 3$ (2).

*In particular in genus 2 all families are abelian.*

*Proof.* Family (2) coincides with $\mathsf{M}_2$. It is different from all other families just by dimension reasons. Similarly (26) is different from all other families.

It remains to deal with the 1-dimensional families. First we show that (3) = (30). The Galois group of (30) is $D_6$. Using the notation of 4.1 set $H := \langle x^2 \rangle = \{1, x^2, x^4\}$. Since the only elements of order 3 in $D_6$ are $x^2$ and $x^4$, $H$ is a normal subgroup of $D_6$. For a given element $C$ of the family (30), set $B := C/H$. We claim that $B = \mathbb{P}^1$. Denote by $\pi : C \to \mathbb{P}^1 = C/D_6$ the original covering that defines family (30). The fixed points of $H$ are exactly the points of the fibre $\pi^{-1}(t_4)$. So there are exactly 4 points of $C$ that are fixed by $H$. By the Riemann-Hurwitz formula $g(B) = 0$. Therefore the family (30) is contained in a family of cyclic coverings of the line with Galois group $\mathbb{Z}/3$. This family does not necessarily satisfy $(*)$. Nevertheless the `MAGMA` script gives the list of all families, not only the ones satisfying $(*)$, see A.2. Looking at this list we conclude that this family must be (3), so (3) = (30).

Since every genus 2 curve is hyperelliptic, the family (3) must be contained in some family satisfying $(*)$ with Galois group $\mathbb{Z}/3 \times \mathbb{Z}/2 = \mathbb{Z}/6$. There is only one such family, namely (5). Therefore (3) =(5). The same reasoning shows that (28) must be contained in a family with group $S_3 \times \mathbb{Z}/2 = D_6$. Again there is only one family with this Galois group, so (28) = (30). We have proven (3) = (5) = (28) = (30).

The group $D_6$ is maximal in the list of possible groups. If we had (4) = (30), then by Lemma 5.1 there should exist a monomorphism $\mathbb{Z}/4 \hookrightarrow D_6$. Since this is not possible, (4) $\neq$ (30).

Finally we check that (4) = (29). Inside $D_4$ consider the subgroup $H := \langle x \rangle \cong \mathbb{Z}/4$. Let $C$ be an element of the family (29) with covering map $\pi : C \to \mathbb{P}^1 = C/D_4$. Set $B := C/H$. The ramification of the projection $C \to B$ is given by $\pi^{-1}(t_2) \cup \pi^{-1}(t_4)$. The first fiber consist of 4 points with stabilizer $\langle x^2 \rangle$. The second fiber consists of 2 points with stabilizer $H$. By Riemann-Hurwitz we get that $g(B) = 0$. Thus (29) is contained in a family with structure group $\mathbb{Z}/4$. This family does not necessarily satisfy $(*)$, nevertheless using the list of all families obtained using the `MAGMA` script (see A.2), we conclude that there is only one such family, namely (4). Thus we get (29) = (4). □

**Theorem 5.3.** *In genus 3 the data satisfying* $(*)$ *yield the following 9 distinct Shimura subvarieties:*

$N = 1$ (7) = (23) = (34), (9), (22), (33) = (35).
$N = 2$ (6), (8), (31), (32).
$N = 3$ (27).

*In particular there are 3 new non-abelian Shimura families.*

*Proof.* Since (27) is the only family of dimension 3, it is clearly distinct from all the others.

There are 4 families of dimension 2: (6), (8), (31) and (32). We want to prove that they are all different from each other. Since $S_3$ and $D_4$ are maximal within groups in these families, Lemma 5.1 implies that (31) $\neq$ (32). Similarly (6) $\neq$ (8) since there is no group $G''$ appearing in these 4 families with $|G''| \geq 12$. Moreover $D_4$ does not contain a subgroup isomorphic to $\mathbb{Z}/3$, so (6) $\neq$ (32). And similarly (8) $\neq$ (31).

We now check that (31) $\neq$ (6). If $G$ acts on a curve $C$ and $H \subseteq G$ is a subgroup, then the representation of $G$ on $H^0(C, K_C)$ obviously restricts to the reprensentation of $H$ on $H^0(C, K_C)$, so $\mathrm{tr}(\rho(H)) \subseteq \mathrm{tr}(\rho(G))$. The computation using the `MAGMA` script gives the full character of $\rho$ for both families (see A.2) and one can check that this does not happen. Another way of seeing this would be to check that the unique subgroup $H \subset S_3$ of order 3, which is $H = \langle x \rangle$, acts on an element $C$ in the family (31), in such a way that $C/H$ has genus 1.

Finally we check that (32) $\neq$ (8). One can just observe that (8) is hyperelliptic, while from the character of $\rho(D_4)$ it follows that $D_4$ does not contain any hyperlliptic involution. Since there is no familiy with group containing $D_4 \times \mathbb{Z}/2$, it follows that (32) is not hyperelliptic, hence (8) $\neq$ (32). By the same argument one shows that also family (31) is not hyperelliptic. This completes the analysis of 2-dimensional families.

There are 7 data yielding families of dimension 1 and we want to prove that they yield exactly four distinct families as follows:

$$(7) = (23) = (34) \qquad (33) = (35) \qquad (9) \qquad (22).$$

First observe that (34) is not hyperelliptic. By looking at the character of $\rho$ one can see that there is no hyperelliptic involution contained in $G = ((\mathbb{Z}/4) \times (\mathbb{Z}/2)) \rtimes \mathbb{Z}/2$. Since $G$ is maximal among the groups of the genus 3 families, there is no family with group $G \times \mathbb{Z}/2$. Thus (34) is not hyperelliptic.

Next we show that (7) = (34). Set $H := \langle y_3 \rangle = Z(G)$. For $C$ an element of the family (34), one can check as above that $g(C/H) = 0$. So (34) is included in a family with Galois group $\mathbb{Z}/4$. This family does not necessarily satisfy $N = r - 3$. From the complete list of data in genus 3, one sees that there are 3 such families. One can check that two of them are hyperelliptic. The third one is (7). Since (34) is not hyperelliptic it follows that (34) = (7).

The same argument shows that (23) = (34). In fact take $H := \langle y_2, y_3 \rangle \subseteq G$. One can check that $C/H = \mathbb{P}^1$. So (34) is also contained in a family with group $\mathbb{Z}/4 \times \mathbb{Z}/2$. There are 3 such families and 2 of them are hyperelliptic. So (34) must coincide with the third, which is (23).

The same argument as above shows that (33) = (35). Indeed, if $C$ in an element of (35), then $C/A_4 = \mathbb{P}^1$, so (35) is contained in another family (not necessarily with $N = r - 3$) with group $A_4$. Since (33) is the unique such family, we conclude (35) = (33).

(34) $\neq$ (35) since both groups are maximal.

(35) $\neq$ (9) since $S_4$ is maximal and contains no elements of order 6.

(7) $\neq$ (9) since the only groups with order a multiple of 12 are $A_4$ and $S_4$, but (33) = (35) $\neq$ (9).

(34) $\neq$ (22) since (22) is hyperelliptic and (34) is not.

(35) is not hyperelliptic, since $S_4$ is centerless and maximal. So (35) $\neq$ (22).

Finally (9) $\neq$ (22). Otherwise by Lemma 5.1 they would be equal to a family with a Galois group $G''$ of order at least 24. The only possibility would be (9) = (22) = (35), which is false. □

**Theorem 5.4.** *In genus 4 the data satisfying* $(*)$ *yield the following 8 distinct Shimura subvarieties:*

$N = 1$ (11), (12), (13) = (24), (25) = (38), (36), (37).
$N = 2$ (14).
$N = 3$ (10).

*In particular there are 2 non-abelian Shimura families.*

*Proof.* (14) is the only 2–dimensional family and (10) is the only 3–dimensional one.

We analyze the 1–dimensional data. (11) is the only one with Galois group of order divisible by 5. So by Lemma 5.1 it is different from all the other families.

We already know that (38)=(25) from 4.7.

The group $Q_8$ is maximal among the ones appearing as Galois groups in the genus 4 families. (36) is hyperelliptic by 4.6. So it is different from (37) and (25) by 4.3, 4.8. By Lemma 5.1 it is also different from all the abelian ones: these have either an element of order 5 or an element of order 3. Thus (36) is different from all other families.

By 5.1 (37) and (38) are different, since both groups are maximal. We claim that (37) is different from all the abelian ones. If (37) is equal to some abelian family, the abelian group must be contained in $A_4$. This never happens. Thus also (37) is different from all other families.

Now consider (24). The group $G = \mathbb{Z}/2 \times \mathbb{Z}/6$ is maximal. An epimorphism for this family is given by

$$x_1 = (1, 0) \qquad x_2 = (0, 3) \qquad x_3 = (0, 2) \qquad x_4 = (1, 1).$$

(Compare with Table 2 in [32].) The subgroups of $G$ isomorphic to $\mathbb{Z}/6$ are

$$H_1 = \langle (0, 1) \rangle, \qquad H_2 = \langle (1, 1) \rangle \qquad H_3 = \langle (1, 2) \rangle.$$

One can check that for any element $C$ of the family (24) and for any $i = 1, 2, 3$ we have $C/H_i = \mathbb{P}^1$. Moreover the map $C \to C/H_2$ has ramification data $\mathbf{m} = (3, 3, 6, 6)$ and has monodromy $a = (1, 1, 2, 2)$ (notation as in [32]). Hence we conclude that (24)=(13). We note in passing that the other maps $C \to C/H_i$ for $i = 1, 3$ show that (24) $\subset$ (14). One needs to use the fact that in genus 4 there is a unique family – not necessarily satisfying ($*$) – with group $\mathbb{Z}/6$ and ramification (2,2,3,3,3); this is family (14).

Since the group $G$ is maximal, and none of the 3 subgroups $H_i$ yields a family with ramification $(2, 6, 6, 6)$ we conclude that (12) $\neq$ (24).

By maximality (24) $\neq$ (38).

Finally we show that (12) $\neq$ (38). There are three subgroups $H_i \subseteq \mathbb{Z}/3 \times S_3$, $i = 1, 2, 3$ isomorphic to $\mathbb{Z}/6$. One can check that $C/H_i = \mathbb{P}^1$ for any $i$ and for any $C$ in the family (38). But for all $i$ the ramification of the map $C \to C/H_i$ is of type (2,2,3,3,3). Since (38) is maximal, this shows that (12) $\neq$ (38) and also that (38) $\subset$ (14).

□

**Theorem 5.5.** *In genus 5 there are exactly two distinct 1-dimensional Shimura subvarieties, (15) and (39). The second one is non–abelian. In genus 6 there are exactly two distinct 1-dimensional Shimura subvarieties, (17) and (18) and a 2-dimensional one (16). They are all cyclic. In genus 7 there are exactly 3 distinct 1-dimensional Shimura subvarieties, (19), (20) and (40). The last one is non–abelian.*

*Proof.* In genus 5 there are only two data satisfying ($*$): (15) with group $\mathbb{Z}/8$ and (39) with group $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$. By Lemma 5.1 they are distinct. In genus 6 there are three data satisfying ($*$): (16) with group $\mathbb{Z}/5$, (17) with group $\mathbb{Z}/7$ and (18) with group $\mathbb{Z}/10$. The first one is 2-dimensional the others are 1-dimensional. By Lemma 5.1 the subvariety corresponding to (17) is different from the one of (18), while (16) is distinct from the others by dimension. In genus 7 there are three data satisfying ($*$): (19) with group $\mathbb{Z}/9$, (20) with group $\mathbb{Z}/12$ and (40) with group $\mathrm{SL}(2, \mathbb{F}_3)$. Again by Lemma 5.1 they are all distinct. □

Finally we notice that the results in this section give the proof of Theorem 1.9.

## Appendix A.

A.1. This appendix gives the relevant information on the script and contains a table of all the data $(\mathbf{m}, G, \theta)$ with genus $g \leq 9$ and $N = r - 3 > 0$ up to Hurwitz equivalence.

A.2. To perform our calculations we wrote a `MAGMA` [28] script, which is available at:

`users.mat.unimi.it/users/penegini/`
`publications/PossGruppigFix_v2Hwr.m`.

The program performs the following calculations. The first two steps correspond to the algorithm already described in [1].

(1) For a given group order and genus $g \geq 2$ the first routine of the program returns all the local monodromies $\mathbf{m} := (m_1, \dots, m_r)$ compatible with the Riemann–Hurwitz formula

$$2g(C) - 2 = |G| \left( -2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) \right).$$

These are finite. In fact the value of $\sum_{i=1}^{r}(1-1/m_i)$ is fixed and $m_i \geq 2$. Therefore $r$ is bounded. Since $m_i \leq |G|$, there is a finite number of possibilities. The function in the script that performs this calculation is `Signature`.

(2) After that, the program calculates all groups $G$ of a fixed order and all spherical systems of generators (SSG) for $G$ of a fixed type $\mathbf{m}$ up to Hurwitz equivalence. For more details on this see e.g. [38]. Here we borrow some parts of the script given in [1] (function `FindAllComponents`). One can find the tables of all inequivalent pairs ($G$, SSG) at the web page

`http://users.mat.unimi.it/users/penegini/publications.html`

(3) For each pair ($G$, SSG) it calculates the multiplicity of each irreducible representation of $G$ inside $\rho \colon G \longrightarrow \mathrm{GL}(H^0(C, K_C))$ using the Chevalley-Weil formula (2.3). Here we borrowed parts of the script given in [42]. The function that performs this calculation is `CW`.

(4) It calculates the number $N$ using (2.6) (function `S2rreFormula`).

(5) The final out-come, obtained by the main routines `SSGT` and `AllCases`, is the list of all data $(\mathbf{m}, G, \theta)$ with genus $g \leq 9$ up to Hurwitz equivalence, together with the number $N$, plus some additional information, e.g., if the group is cyclic or not, the decomposition of the representation on $H^0(C, K_C)$, etc.. The program points out those examples for which $N = r - 3$. One can find this information for the families with $r \geq 4$ at the web page

`http://users.mat.unimi.it/users/penegini/publications.html`

A similar script for `GAP4` was used in [37].

A.3. We got 40 data $(\mathbf{m}, G, \theta)$ with genus $g \leq 9$ and $N = r - 3 > 0$. In Table 2 we list them in order of increasing genus. The numbers in the last column are given to label the families following the numeration already assigned in Table 1 and 2 in [32]. For $\mathbf{m}$ we use a compact notation, for example $(2^2, 3^2) = (2, 2, 3, 3)$. The column dim lists the dimension of $\mathsf{Z}(\mathbf{m}, G, \theta)$. The column `Id` lists the `IdSmallGroup` name of the group in the `MAGMA` database.

Examples (1) and (21) are classical. Examples (2) – (20) have cyclic Galois group and are already listed in [39, p. 136-137], [31] and [32]. Examples (22) – (27) have already been found in [32]. Professor Xin Lu informed us that (36) has already been studied from a different point of view in [26, Ex. 7.2]. Notice that we get new data only for genus $g = 2, 3, 4, 5, 7$.

A.4. For the description of the monodromy (or equivalently of an epimorphism) in the non-abelian cases see §4. For the abelian cases we refer to Tables 1 and 2 in [32].

Table 2. All data

| $g(C)$ | $\lvert G\rvert$ | $G$ | Id | $\mathbf{m}$ | dim | |
|---|---|---|---|---|---|---|
| 1 | 2 | $\mathbb{Z}/2$ | G(2,1) | $(2^4)$ | 1 | (1) |
| 1 | 4 | $(\mathbb{Z}/2)\times(\mathbb{Z}/2)$ | G(4,2) | $(2^4)$ | 1 | (21) |
| 2 | 2 | $\mathbb{Z}/2$ | G(2,1) | $(2^6)$ | 3 | (2) |
| 2 | 3 | $\mathbb{Z}/3$ | G(3,1) | $(3^4)$ | 1 | (3) |
| 2 | 4 | $\mathbb{Z}/4$ | G(4,1) | $(2^2,4^2)$ | 1 | (4) |
| 2 | 6 | $\mathbb{Z}/6$ | G(6,2) | $(2^2,3^2)$ | 1 | (5) |
| 2 | 4 | $(\mathbb{Z}/2)\times(\mathbb{Z}/2)$ | G(4,2) | $(2^5)$ | 2 | (26) |
| 2 | 6 | $S_3$ | G(6,1) | $(2^2,3^2)$ | 1 | (28) |
| 2 | 8 | $D_4$ | G(8,3) | $(2^3,4)$ | 1 | (29) |
| 2 | 12 | $D_6$ | G(12,4) | $(2^3,3)$ | 1 | (30) |
| 3 | 3 | $\mathbb{Z}/3$ | G(3,1) | $(3^5)$ | 2 | (6) |
| 3 | 4 | $\mathbb{Z}/4$ | G(4,1) | $(4^4)$ | 1 | (7) |
| 3 | 4 | $\mathbb{Z}/4$ | G(4,1) | $(2^3,4^2)$ | 2 | (8) |
| 3 | 6 | $\mathbb{Z}/6$ | G(6,2) | $(2,3^2,6)$ | 1 | (9) |
| 3 | 4 | $(\mathbb{Z}/2)\times(\mathbb{Z}/2)$ | G(4,2) | $(2^6)$ | 3 | (27) |
| 3 | 6 | $S_3$ | G(6,1) | $(2^4,3)$ | 2 | (31) |
| 3 | 8 | $(\mathbb{Z}/2)\times(\mathbb{Z}/4)$ | G(8,2) | $(2^2,4^2)$ | 1 | (22) |
| 3 | 8 | $(\mathbb{Z}/2)\times(\mathbb{Z}/4)$ | G(8,2) | $(2^2,4^2)$ | 1 | (23) |
| 3 | 8 | $D_4$ | G(8,3) | $(2^5)$ | 2 | (32) |
| 3 | 12 | $A_4$ | G(12,3) | $(2^2,3^2)$ | 1 | (33) |
| 3 | 16 | $(\mathbb{Z}/4\times\mathbb{Z}/2)\rtimes(\mathbb{Z}/2)$ | G(16,13) | $(2^3,4)$ | 1 | (34) |
| 3 | 24 | $S_4$ | G(24,12) | $(2^3,3)$ | 1 | (35) |
| 4 | 3 | $\mathbb{Z}/3$ | G(3,1) | $(3^6)$ | 3 | (10) |
| 4 | 5 | $\mathbb{Z}/5$ | G(5,1) | $(5^4)$ | 1 | (11) |
| 4 | 6 | $\mathbb{Z}/6$ | G(6,2) | $(2,6^3)$ | 1 | (12) |
| 4 | 6 | $\mathbb{Z}/6$ | G(6,2) | $(3^2,6^2)$ | 1 | (13) |
| 4 | 6 | $\mathbb{Z}/6$ | G(6,2) | $(2^2,3^3)$ | 2 | (14) |
| 4 | 8 | $Q_8$ | G(8,4) | $(2,4^3)$ | 1 | (36) |
| 4 | 9 | $(\mathbb{Z}/3)\times(\mathbb{Z}/3)$ | G(9,2) | $(3^4)$ | 1 | (25) |
| 4 | 12 | $(\mathbb{Z}/6)\times(\mathbb{Z}/2)$ | G(12,5) | $(2^2,3,6)$ | 1 | (24) |
| 4 | 12 | $A_4$ | G(12,3) | $(2,3^3)$ | 1 | (37) |
| 4 | 18 | $(\mathbb{Z}/3)\times S_3$ | G(18,3) | $(2^2,3^2)$ | 1 | (38) |
| 5 | 8 | $\mathbb{Z}/8$ | G(8,1) | $(2,4,8^2)$ | 1 | (15) |
| 5 | 12 | $(\mathbb{Z}/3)\rtimes(\mathbb{Z}/4)$ | G(12,1) | $(2,3,4^2)$ | 1 | (39) |
| 6 | 5 | $\mathbb{Z}/5$ | G(5,1) | $(5^5)$ | 2 | (16) |
| 6 | 7 | $\mathbb{Z}/7$ | G(7,1) | $(7^4)$ | 1 | (17) |
| 6 | 10 | $\mathbb{Z}/10$ | G(10,2) | $(2,5^2,10)$ | 1 | (18) |
| 7 | 9 | $\mathbb{Z}/9$ | G(9,1) | $(3,9^3)$ | 1 | (19) |
| 7 | 12 | $\mathbb{Z}/12$ | G(12,2) | $(2,3,12^2)$ | 1 | (20) |
| 7 | 24 | $\mathrm{SL}(2,\mathbb{F}_3)$ | G(24,3) | $(2,3^3)$ | 1 | (40) |

## References

[1] I. Bauer, F. Catanese, F. Grunevald, R. Pignatelli *Quotients of products of curves, new surfaces with $p_g=0$ and their fundamental groups*, American J. of Math. **134**, (2012), 993–1049.

[2] J. S. Birman. *Braids, links, and mapping class groups.* Princeton University Press, Princeton, N.J., 1974. Annals of Mathematics Studies, No. 82.
[3] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
[4] T. Breuer. *Characters and automorphism groups of compact Riemann surfaces*, volume 280 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.
[5] F. Catanese, M. Lönne, and F. Perroni. Irreducibility of the space of dihedral covers of the projective line of a given numerical type. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 22(3):291–309, 2011.
[6] C. Chevalley, A. Weil, *Über das Verhalten der Intergrale 1. Gattung bei Automorphismen des Funktionenkorpers.* Abhand. Math. Sem. Hamburg **10** (1934), 358–361.
[7] E. Colombo and P. Frediani. Some results on the second G aussian map for curves. *Michigan Math. J.*, 58(3):745–758, 2009.
[8] E. Colombo and P. Frediani. Siegel metric and curvature of the moduli space of curves. *Trans. Amer. Math. Soc.*, 362(3):1231–1246, 2010.
[9] E. Colombo, P. Frediani, and A. Ghigi. On totally geodesic submanifolds in the Jacobian locus. `arXiv:1309.1022 [math.AG]`, 2013. Preprint.
[10] E. Colombo, G. P. Pirola, and A. Tortora. Hodge-Gaussian maps. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 30(1):125–146, 2001.
[11] J. de Jong and R. Noot. Jacobians with complex multiplication. In *Arithmetic algebraic geometry (Texel, 1989)*, volume 89 of *Progr. Math.*, pages 177–192. Birkhäuser Boston, Boston, MA, 1991.
[12] J. de Jong and S.-W. Zhang. Generic abelian varieties with real multiplication are not Jacobians. In *Diophantine geometry*, volume 4 of *CRM Series*, pages 165–172. Ed. Norm., Pisa, 2007.
[13] P. B. Eberlein. *Geometry of nonpositively curved manifolds*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.
[14] H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.
[15] B. van Geemen. Projective models of Picard modular varieties. In *Classification of irregular varieties (Trento, 1990)*, volume 1515 of *Lecture Notes in Math.*, pages 68–99. Springer, Berlin, 1992.
[16] G. González Díez and W. J. Harvey. Moduli of Riemann surfaces with symmetry. In *Discrete groups and geometry (Birmingham, 1991)*, volume 173 of *London Math. Soc. Lecture Note Ser.*, pages 75–93. Cambridge Univ. Press, Cambridge, 1992.
[17] S. Grushevsky and M. Moeller. Shimura curves within the locus of genus 3 hyperelliptic curves. `arXiv:1308.5155 [math.AG]`, 2013.
[18] R. Hain. Locally symmetric families of curves and Jacobians. In *Moduli of curves and abelian varieties*, Aspects Math., E33, pages 91–108. Friedr. Vieweg, Braunschweig, 1999.
[19] W. J. Harvey. On branch loci in Teichmüller space. *Trans. Amer. Math. Soc.*, 153:387–399, 1971.
[20] G. R. Kempf. *Complex abelian varieties and theta functions*. Universitext. Springer-Verlag, Berlin, 1991.
[21] A. Kuribayashi. On analytic families of compact Riemann surfaces with non-trivial automorphisms. *Nagoya Math. J.*, 28:119–165, 1966.
[22] A. Kuribayashi and H. Kimura. Automorphism groups of compact Riemann surfaces of genus five. *J. Algebra*, 134(1):80–103, 1990.
[23] I. Kuribayashi. On automorphism groups of a curve as linear groups. *J. Math. Soc. Japan*, 39(1):51–77, 1987.
[24] I. Kuribayashi and A. Kuribayashi. Automorphism groups of compact Riemann surfaces of genera three and four. *J. Pure Appl. Algebra*, 65(3):277–292, 1990.
[25] K. Liu, X. Sun, X. Yang, and S.-T. Yau. Curvatures of moduli spaces of curves and applications. `arXiv:1312.6932 [math.DG]`, 2013. Preprint.
[26] X. Lu and K. Zuo. On shimura curves in the torelli locus of curves. `arXiv:1311.5858 [math.AG]`, 2013. Preprint.

[27] K. Magaard, T. Shaska, S. Shpectorov, and H. Völklein. The locus of curves with prescribed automorphism group. *Sūrikaisekikenkyūsho Kōkyūroku*, (1267):112–141, 2002. Communications in arithmetic fundamental groups (Kyoto, 1999/2001).

[28] MAGMA Database of Small Groups; `http://magma.maths.usyd.edu.au/magma/htmlhelp/text404.htm`.

[29] R. Miranda. *Algebraic curves and Riemann surfaces*, volume 5 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1995.

[30] B. Moonen. Linearity properties of Shimura varieties. I. *J. Algebraic Geom.*, 7(3):539–567, 1998.

[31] B. Moonen. Special subvarieties arising from families of cyclic covers of the projective line. *Doc. Math.*, 15:793–819, 2010.

[32] B. Moonen and F. Oort. The Torelli locus and special subvarieties. In *Handbook of Moduli: Volume II*, pages 549–94. International Press, Boston, MA, 2013.

[33] G. D. Mostow. On discontinuous action of monodromy groups on the complex $n$-ball. *J. Amer. Math. Soc.*, 1(3):555–586, 1988.

[34] F. Oort. Canonical liftings and dense sets of CM-points. In *Arithmetic geometry (Cortona, 1994)*, Sympos. Math., XXXVII, pages 228–234. Cambridge Univ. Press, Cambridge, 1997.

[35] F. Oort. Moduli of abelian varieties in mixed and in positive characteristic. In *Handbook of Moduli: Volume III*, pages 75–134. International Press, Boston, MA, 2013.

[36] F. Oort and J. Steenbrink. The local Torelli problem for algebraic curves. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 157–204. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[37] M. Penegini. The classification of isotrivially fibred surfaces with $p_g = q = 2$. *Collect. Math.*, 62(3):239–274, 2011. With an appendix by Sönke Rollenske.

[38] M. Penegini. Surfaces isogenous to a product of curves, braid groups and mapping class groups. `arXiv:1311.5671 [math.AG]`. Preprint to appear in *The Proceedings of the Conference "Beauville surfaces and groups", Newcastle*, 2013.

[39] J. C. Rohde. *Cyclic coverings, Calabi-Yau manifolds and complex multiplication*, volume 1975 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[40] J.-P. Serre. *Représentations linéaires des groupes finis*. Hermann, Paris, revised edition, 1978.

[41] G. Shimura. On purely transcendental fields automorphic functions of several variable. *Osaka J. Math.*, 1(1):1–14, 1964.

[42] D. Swinarski; http://www.math.uga.edu/ davids/ivrg/CWv2.1.txt

[43] D. Toledo. Nonexistence of certain closed complex geodesics in the moduli space of curves. *Pacific J. Math.*, 129(1):187–192, 1987.

Università di Pavia
*E-mail address*: `paola.frediani@unipv.it`

Università di Milano Bicocca
*E-mail address*: `alessandro.ghigi@unimib.it`

Università di Milano
*E-mail address*: `matteo.penegini@unimi.it`